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APPLIED WELFARE ECONOMICS WITH DISCRETE CHOICE MODELS[1]

By Kenneth A. Small and Harvey S. Rosen

Economists have been paying increasing attention to the study of situations in which consumers face a discrete rather than a continuous set of choices. Such models are potentially very important in evaluating the impact of government programs upon consumer welfare. But little has been said in general regarding the tools of applied welfare economics in discrete choice situations.

This paper shows how the conventional methods of applied welfare economics can be modified to handle such cases. It focuses on the computation of the excess burden of taxation, and the evaluation of quality change. The results are applied to stochastic utility models, including the popular cases of probit and logit analysis. Throughout, the emphasis is on providing rigorous guidelines for carrying out applied work.

1. INTRODUCTION

ECONOMISTS ARE PAYING increasing attention to the study of situations in which consumers face a discrete rather than a continuous set of choices. A considerable literature has grown covering econometric methodology and applications to a wide variety of situations.[2] In a few specific cases, estimates of discrete choice models have been employed for welfare analysis.[3] Travel demand researchers have developed "composite cost" and "accessibility" measures to evaluate the desirability of various states (Williams [34], Ben-Akiva and Lerman [4]), but the relationship of these to conventional welfare criteria has not been thoroughly explored. Indeed, very little has been said in general regarding the validity of the tools of applied welfare economics in discrete choice situations.

Nevertheless, in at least two areas in which discrete choice models are applied, welfare judgments are of paramount interest. The first is the analysis of taxes and subsidies. Econometric studies of the impacts of taxes on labor supply, savings, and housing decisions have permitted calculation of the welfare cost or "excess burden" of the tax feature under consideration.[4] Traditionally, such studies have assumed that the taxed activity will be pursued regardless of the existence of the tax.

However, instances in which taxes influence discrete choices are numerous and important. For example, it has been suggested that the federal tax treatment of owner-occupied housing not only increases the quantity demanded of owner-occupied housing services, but also influences whether or not an individual owns a home at all (Rosen [27]). For secondary earners, taxes affect not only the number

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[2] For examples and references, see McFadden [22].

[3] For example, Domencich and McFadden [12], Small [30].

[4] See, for example, Aaron [1], Harberger [16], Boskin [6], and Laidler [20].

of hours of work, but also whether or not participation in the labor market takes place (Heckman [18]). Even for primary workers, taxes may influence not only the hours spent at a given occupation, but the choice between occupations as well.

A second important area is the analysis of quality changes in goods and services. Numerous studies in such areas as transportation, education, health care, child care, and pollution contain explicit or implicit welfare evaluations of public policies which change the quality of some publicly or privately supplied good. In many cases, the good affected is subject to discrete choice: mode of transportation, public or private schools, type of health care, type of child care, and location of recreational activities all serve as examples.

The purpose of the present paper is to demonstrate that the conventional methods of applied welfare economics can be generalized to handle cases in which discrete choices are involved. We think it important for several reasons to set out the generalization carefully. First, there has been some suggestion that conventional welfare change computations may have no relevance in discrete choice cases.[5] Second, as the econometric estimation of discrete choice models of many types gains in popularity, it is useful to have a clear statement regarding their use in applied welfare economics. Finally, the presence of discrete choice introduces some differences in the relation between ordinary and compensated demand curves which alter the way in which empirical approximations should be carried out.

In Section 2 we review the computation of the Hicksian compensated variation for the "standard" case in which there are price-induced changes in quantity demanded by each individual. We also note its relation to the Harberger "excess burden" and to the Marshallian consumer's surplus. In Section 3 it is shown how the computations are performed for circumstances in which discrete choice is involved, and Section 4 extends the result to quality changes. Section 5 discusses in detail the application of the results to stochastic utility models, including the cases of probit and logit analysis. Throughout, the emphasis is on providing rigorous guidelines for carrying out applied work.

2. CONTINUOUS CHOICE

In this section we review the justification for measuring price-induced utility changes as areas to the left of the appropriate compensated demand curves. The basic analytical tool is the expenditure function, which is treated concisely and elegantly in Diamond and McFadden [11].

Suppose that a consumer has a twice differentiable, strictly quasi-concave utility function u defined over the commodities x_n and x_1, where x_n is taken to be

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[5] For example, Ben-Porath and Bruno [5] state that in Israel, the pre-1975 tax structure was ". . . less a deterrent to work as such than an inducement to search for untaxable *types* of work" (our emphasis). They suggest that "The concept of excess burden is not particularly well defined in this case . . ." (p. 289).

the numeraire. The two good case is chosen for convenience; the results general-
ize easily to an arbitrary number of commodities. The consumer maximizes

(2.1) $u = u(x_n, x_1)$

subject to the budget and nonnegativity constraints

(2.2) $x_n + p_1 x_1 = y,$

(2.3) $x_j \geqslant 0, \quad j = n, 1,$

where p_1 is the price of good 1, and y is income.

   Maximization of (2.1) subject to (2.2) yields the ordinary demand functions

(2.4) $x_j = x_j(p_1, y), \quad j = n, 1.$

We assume an interior solution ($x_j > 0$) with respect to both goods. Substituting
(2.4) into (2.1) we define the consumer's indirect utility function

(2.5) $v(p_1, y) = u(x_n(p_1, y), x_1(p_1, y))$

which is known to satisfy Roy's Identity:[6]

(2.6) $-\dfrac{\partial v / \partial p_1}{\partial v / \partial y} = x_1.$

Provided that the direct utility function is strictly increasing in x_n and is nonde-
creasing in x_1, v is monotonic increasing in y and can therefore be inverted to yield
the expenditure function

(2.7) $y = e(p_1, u).$

The function e therefore indicates how much income is required to achieve the
utility level u when the price of good 1 is p_1; it satisfies

(2.8) $u = v(p_1, e(p_1, u)).$

   Now suppose that the price of the first good changes from p_1^0 to p_1^f. By
definition, the compensating variation associated with the price change is

(2.9) $\Delta e = e(p_1^f, u^0) - e(p_1^0, u^0).$

This expression shows the amount of income the consumer must be given to make
him as well off at price p_1^f as at p_1^0. Mohring [24] has forcefully argued that it is the
appropriate measure for most problems of applied welfare economics.[7]

   The problem now is to express (2.9) in terms of the compensated demand
function for x_1, which is defined by

(2.10) $x_1^c(p_1, u) = x_1(p_1, e(p_1, u)).$

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   [6] Roy's Identity follows from comparing the total differentials of (2.2) and (2.5) and using the
first-order maximization conditions.
   [7] It is employed, for example, by Hause [17] and Diamond and McFadden [11], both of which
demonstrate that the compensating variation measure leads to the well-known Harberger formula for
excess burden. Nevertheless, some controversy surrounds its use; see Chipman and Moore [9] and
Hause [17].

This is done by applying Shephard's Lemma:

$$(2.11) \quad \frac{\partial e(p_1, u)}{\partial p_1} = x_1^c(p_1, u),$$

where u is an arbitrarily selected utility level.[8] This equation gives the compensating variation for an infinitesimal change in p_1; to find the compensating variation for a finite change, (2.11) is integrated:

$$(2.12) \quad e(p_1^f, u^0) - e(p_1^0, u^0) = \int_{p_1^0}^{p_1^f} x_1^c(p_1, u^0)\, dp_1.$$

This gives the basic result mentioned at the start of this section; the compensating variation of a price change is the area to the left of the compensated demand curve.

This result is easily aggregated. Define the aggregate compensated demand function for good 1, X_1^c, as the sum of the individual compensated demand functions. X_1^c depends upon prices and upon the utility levels of *all* consumers. Similarly, define the aggregate expenditure function, E, as the sum of the individual expenditure functions at these same utility levels. Then by adding the compensating variations (2.12) for all consumers, we obtain

$$(2.13) \quad \Delta E = \int_{p_1^0}^{p_1^f} X_1^c(p_1, \{u^i\})\, dp_1,$$

where $\{u^i\}$ is the set of utility levels before the change.

For practical purposes, the area to the left of the compensated demand curve in (2.13) is usually calculated from an econometric estimate of the ordinary or Marshallian demand curve.[9] Although the compensated demand function for an *individual* can, under certain conditions, be computed exactly from an econometric estimate of the Marshallian demand (Seade [28], Willig [35]), the same is not true of *aggregates*. This had led to an approximation procedure, employed by a number of investigators such as Harberger [16] and Feldstein [13], in which the Slutsky equation is applied to the ordinary aggregate demand curve in order to obtain estimates of compensated price elasticities. This procedure is exact if (a) $\partial x_1 / \partial y$ is the same for everyone, and (b) each individual's share of aggregate income is fixed regardless of price and income changes; in this case the Slutsky equation can be aggregated. To the extent that individuals' income slopes are similar, then, this would appear to be a reasonable approximation procedure.

For the special case in which the price change is induced by a tax (or subsidy), (2.13) can be used to calculate the "excess burden" of the tax, i.e., the welfare

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[8] Shephard's Lemma follows from differentiating (2.8) with respect to p_1, and applying Roy's Identity.

[9] It has been shown that the area under the ordinary demand curve is itself an exact measure of the utility change under very restrictive conditions on the demand function. See, for example, Chipman and Moore [10], Rader [25], or Bruce [8].

loss brought about by distortions in relative prices. Following Diamond and McFadden [11], we define excess burden as

$$(2.14) \quad \chi = \Delta E - T,$$

where T denotes tax receipts. The exact form which equation (2.14) takes in practice depends on the income effects generated by the uses of the tax revenues (or, alternatively, by a set of hypothetical lump-sum taxes of equal yield), and therefore depends on the distribution of these revenue uses or taxes. Although this point is well-known (see Friedman [15], Bailey [2], Harberger [16], and Hause [17]), it is worth restating here because its implications for discrete goods, discussed in the next section, are somewhat surprising.

Let α and α^c be the own-price slopes of the aggregate Marshallian and compensated demand functions, respectively; and let ω be the income-slope of the former, viewed as a function of aggregate income $Y \equiv \Sigma_i y^i$. Then the Slutsky approximation discussed above,

$$(2.15) \quad \alpha^c = \alpha + X_1 \omega,$$

can be intuitively justified from the observation that $X_1 \, dp_1$ is the income compensation required to ameliorate the effect of a price increase dp_1, and hence $\omega X_1 \, dp_1$ is the additional consumption induced if income is compensated.

A linear approximation to the compensated demand curve yields

$$(2.16) \quad \Delta E \simeq X_1^0 \Delta p_1 + \tfrac{1}{2} \alpha^c (\Delta p_1)^2$$

$$(2.17) \quad \quad = X_1^0 \Delta p_1 + \tfrac{1}{2} [\alpha + X_1^0 \omega] (\Delta p_1)^2,$$

where X_1^0 is the aggregate consumption at initial price p_1^0, and where $\Delta p_1 = p_1^f - p_1^0$. Equation (2.17) can be viewed as the Taylor series approximation to ΔE, of second-order in Δp_1. (Throughout this paper we will be using such second-order approximations. In order to keep the exposition as simple as possible, we will refrain from writing out the expressions for remainder terms that exceed second order. But care is taken to compute fully the second order terms prior to truncation.)

Now consider two alternative assumptions about the uses of the tax revenues T: (a) the *partial* equilibrium assumption that they are used in such a way as to generate no repercussions on demand for good 1, and (b) the general equilibrium assumption that they are distributed in lump sum transfers. We again compute approximations of second order in the price change. Under assumption (a), tax revenues are computed along the Marshallian demand curve:

$$(2.18) \quad T^P \simeq [X_1^0 + \alpha \Delta p_1] \Delta p_1.$$

Under assumption (b), on the other hand, final demand contains an income effect ωT^G, where T^G are the tax revenues, so that

$$(2.19) \quad T^G \simeq [X_1^0 + \alpha \Delta p_1 + \omega T^G] \Delta p_1.$$

Equation (2.19) can be solved for T^G and written as a Taylor series in Δp_1. If the series is truncated after terms that are quadratic in Δp_1, we find[10]

$$(2.20) \quad T^G \simeq [X_1^0 + (\alpha + X_1^0\omega)\Delta p_1]\Delta p_1$$

$$(2.21) \qquad = [X_1^0 + \alpha^c \Delta p_1]\Delta p_1.$$

But this is equivalent to computing taxes along the *compensated* demand curve.[11]

Subtracting tax revenues from the compensating variation in (2.16) and rearranging yields the following alternative formulae for excess burden:

(a) Partial equilibrium:

$$(2.22) \quad \chi^p \simeq \tfrac{1}{2}[\alpha^c - 2\alpha](\Delta p_1)^2$$

$$(2.23) \qquad = \tfrac{1}{2}[-\alpha + X_1\omega](\Delta p_1)^2;$$

(b) General equilibrium:

$$(2.24) \quad \chi^G \simeq \tfrac{1}{2}[-\alpha^c](\Delta p_1)^2$$

$$(2.25) \qquad = \tfrac{1}{2}[-\alpha - X_1\omega](\Delta p_1)^2.$$

The general equilibrium version (2.24) is the famous excess burden triangle usually associated with Harberger.

Note that, whereas the compensating variation is of first order in Δp_1, the excess burden is itself of second-order in Δp_1. Thus, the conditions given by Willig [35], under which the compensating variation is adequately approximated by the ordinary consumer's surplus, do not necessarily justify using the ordinary instead of the compensated demand curve for excess burden calculations. Indeed, if the general equilibrium assumptions underlying equation (2.24) prevail, then it is the best approximation (up to second-order terms in Δp_1) of the excess burden, and the fractional error from using α instead of α^c in equation (2.24) is $(\alpha - \alpha^c)/\alpha^c = -X_1\omega/\alpha^c = -\theta\eta/\varepsilon^c$, where θ is the budget share of good 1, η is its income elasticity, and ε^c is its compensated own-price elasticity. This error can be substantial for a good, such as housing, which absorbs a large portion of consumers' incomes.

For a normal good, the excess burden is positive and is smaller in the general than in the partial equilibrium context. This is because the consumption cutback caused by the price distortion is lessened by the income effects arising from the uses of the tax revenues. We will see in the next section that some of the

[10] Proof: From (2.19),

$$T^G \simeq \frac{1}{1 - \omega\Delta p_1}(X_1^0 + \alpha\Delta p_1)\Delta p_1.$$

The term $1/(1 - \omega\Delta p_1)$ can be approximated by $1 + \omega\Delta p_1 + (\omega\Delta p_1)^2$. Multiplying this by $(X_1^0 + \alpha\Delta p_1)\Delta p_1$ and truncating terms that are cubic or higher in order yields (2.20).

[11] See Hause [17, pp. 1156–1160] for a fuller discussion of the rationale for using the compensated demand curve in computing tax revenue under general equilibrium assumptions. Note also that we are assuming a constant cost technology, i.e., introduction of the tax does not change the producer's price over the relevant region. See also Rohlfs' [26] discussion of the adequacy of surplus welfare measures under partial equilibrium assumptions.

ameliorating income effect in the general equilibrium case is lost in discrete good models, making the potential welfare losses from taxes on such goods greater than conventional analysis would imply.

3. DISCRETE CHOICE

In this section, we prove the basic theorem which establishes the validity of using the area to the left of a compensated demand curve for a *discrete* good as a measure of the compensating variation. We take advantage of the notation and general plan of attack used in Section 2 for the continuous goods case. We then show how the use of empirical approximations for computing welfare losses is modified.

Discreteness in demand can be modelled in at least three ways. (a) Commodities may be available in continuous quantities, but in only one of a small number of mutually exclusive varieties. An example is housing. It can be purchased in either the rental or owner mode, but given the tenure choice, a continuous quantity of housing services can be consumed. (b) Goods may be available in discrete units which are so large that most consumers choose only one or two units. Examples include college degrees, transportation mode for work trips, and many consumer durables. (c) Goods may be purchased in discrete units because nonconcavities in utility functions lead the consumer to choose among alternative corner solutions. For example, two television shows aired simultaneously can be consumed in any fractions, but most viewers will prefer to consume all of one and none of the other. Note that many examples could be modelled equally well as cases (a) or (c).

In addition, even with concave utility functions, particular prices and incomes may lead to corner solutions at which a good is not consumed at all. Empirically, this has led to econometric techniques such as Tobit analysis. Although this is not usually considered discrete choice, we mention it here because the issues involved are formally very similar to those of the other cases.

The problem raised by all these cases for the theoretical derivation of the compensating variation is that discreteness introduces a point of discontinuity into the demand functions, and/or a point of nondifferentiability into the indirect utility and expenditure functions. Thus, the derivative in equation (2.11) may be undefined at some "crossover" price. The key to the result in this section is that it is still possible to integrate over (2.11), since the expenditure function is continuous and has right and left derivatives at the point in question.

The formal models for the various cases differ in detail, but the proofs for all are essentially similar. We therefore proceed by considering case (a) in a three-good model in which two of the goods are mutually exclusive. The proofs for the other cases and extensions to more than three goods are outlined in the Appendix.

Consider, then, a consumer with utility function

$$(3.1) \qquad u = u(x_n, x_1, x_2),$$

where x_n is the numeraire good. In order for the possibility of zero consumption of good 1 or 2 to make sense, we assume that utility is finite when either x_1 or x_2 is

zero. As in the standard case, utility is maximized subject to

(3.2) $x_n + p_1 x_1 + p_2 x_2 = y,$

(3.3) $x_j \geq 0 \quad (j = 1, 2, n).$

To prevent both x_1 and x_2 from being consumed in positive amounts, we impose the constraint

(3.4) $x_1 x_2 = 0.$

As in the previous section, we assume interior solutions[12] rather than dealing explicitly with nonnegativity constraints: thus the numeraire and *either* x_1 or x_2 are consumed in positive quantities.

THEOREM 1: *Suppose a consumer maximizes, subject to constraints* (3.2) *through* (3.4), *a twice differentiable, strictly quasi-concave utility function of the form* (3.1). *Assume u is finite whenever* x_1 *or* x_2 *is zero, is strictly increasing in* x_n, *and is nondecreasing in* x_1 *and* x_2. *Let* $e(p_1, p_2, u)$ *be the minimum expenditure required to achieve utility u, and let* $u^0 = v(p_1^0, p_2, y)$ *be the value of the indirect utility function*[13] *at initial prices and income. Then the compensating variation for a change in price* p_1 *from* p_1^0 *to* p_1^f *is*

(3.5) $e(p_1^f, p_2, u^0) - e(p_1^0, p_2, u^0) = \displaystyle\int_{p_1^0}^{p_1^f} x_1^c(p_1, p_2, u^0)\, dp_1,$

where $x_1^c(\cdot)$ *is the compensated demand function for good* 1 *defined by*

(3.6) $x_1^c(p_1, p_2, u^0) = x_1(p_1, p_2, e(p_1, p_2, u^0)).$

PROOF: Constraint (3.4) requires that consumption of either good 1 or good 2 be zero. If $x_2 = 0$, the optimal amount of x_1 is found by maximizing $u(x_n, x_1, 0)$ subject to $x_n + p_1 x_1 = y$. The function u is well-behaved when viewed as a function only of x_n and x_1, so this sub-problem is formally identical to the problem of the previous section, and leads to well-defined continuously differentiable indirect utility, expenditure, and compensated demand functions, each conditional on $x_2 = 0$. Denoting these by $\tilde{v}_1(p_1, y)$, $\tilde{e}_1(p_1, u)$, and $\tilde{x}_1^c(p_1, u)$, respectively, we know from (2.11) that

(3.7) $\partial \tilde{e}_1 / \partial p_1 = \tilde{x}_1^c.$

Similarly, if $x_1 = 0$, the problem of maximizing $u(x_n, 0, x_2)$ yields functions $\tilde{v}_2(p_2, y)$, $\tilde{e}_2(p_2, u)$, and $\tilde{x}_2^c(p_2, u)$ with the property that

(3.8) $\partial \tilde{e}_2 / \partial p_2 = \tilde{x}_2^c.$

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[12] Corner solutions can also be dealt with using the framework developed in this section. See the Appendix for a demonstration.

[13] So long as the numeraire is perfectly divisible, the assumption that u is strictly increasing in x_n and nondecreasing in x_1 and x_2 guarantees that the indirect utility function exists and is strictly increasing in y, and therefore can be inverted to find the expenditure function.

For any income $\bar{y}$, the choice between good 1 and good 2 will be made by comparing $\tilde{v}_1(p_1, \bar{y})$ with $\tilde{v}_2(p_2, \bar{y})$. Let $k(p_1, p_2, \bar{y})$ index the larger of the two, with $k = 1$ in case of a tie. Then the utility achieved is

$$(3.9) \qquad \bar{u} = \tilde{v}_k(p_k, \bar{y}) \equiv \max\{\tilde{v}_1(p_1, \bar{y}), \tilde{v}_2(p_2, \bar{y})\}.$$

The quantity $\bar{y}$ is therefore the income required to achieve utility $\bar{u}$, both unconditionally and conditionally on choice k:

$$(3.10) \qquad \bar{y} = e(p_1, p_2, \bar{u}) = \tilde{e}_k(p_k, \bar{u}).$$

Letting j index *either* good ($j = 1, 2$), the fact that $\tilde{v}_j(p_j, \bar{y}) \leqslant \bar{u}$ implies

$$(3.11) \qquad \tilde{e}_j(p_j, \bar{u}) \geqslant \bar{y}, \quad j = 1, 2,$$

since $\tilde{v}_j$ is strictly increasing in $\bar{y}$, and $\tilde{e}_j$ is simply its inverse. But (3.10) and (3.11) together imply that

$$(3.12) \qquad e(p_1, p_2, \bar{u}) = \min_j \tilde{e}_j(p_j, \bar{u}).$$

Since each of the functions $\tilde{e}_j$ is continuously differentiable in price, so is e, except at those prices for which $\tilde{e}_1 = \tilde{e}_2$; and at these points, e is continuous and right and left differentiable.[14] Its price derivatives are therefore bounded and piecewise continuous, and hence integrable.[15] Furthermore,

$$(3.13) \qquad \partial e / \partial p_1 = \begin{cases} \partial \tilde{e}_1/\partial p_1 \equiv \tilde{x}_1^c, & \text{if } k = 1, \\ \partial \tilde{e}_2/\partial p_1 \equiv 0, , & \text{if } k \neq 1, \end{cases}$$

provided the upper and lower expressions are interpreted as the left and right derivatives, respectively, at the point of indifference between the two goods.

It remains only to identify the right-hand side of (3.13) as the compensated demand function, $x_1^c(p_1, p_2, \bar{u})$. When good 1 is chosen (given $p_1, p_2, \bar{u}$), $x_1^c = \tilde{x}_1^c$; when good 2 is chosen, $x_1^c = 0$. Thus, for an infinitesimal change in p_1, the compensating variation (3.13) reduces to

$$(3.14) \qquad \partial e(p_1, p_2, \bar{u})/\partial p_1 = x_1^c(p_1, p_2, \bar{u}).$$

Evaluating (3.14) at utility $u^0 \equiv \max\{\tilde{v}_1(p_1^0, y), \tilde{v}_2(p_2, y)\}$ and integrating over p_1 yields (3.5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ *Q.E.D.*

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[14] For given p_2 and u, let $y^* = e_2(p_2, u)$. Let p_1^* be the single value of p_1 (if any) at which the consumer switches goods. Thus, $\tilde{e}_1(p_1^*, u) = \tilde{e}_2(p_2, u) = e(p_1^*, p_2, u)$. For $p_1 < p_1^*$, good 1 is chosen and $e = \tilde{e}_1$; for $p_1 > p_1^*$, good 2 is chosen and $e = \tilde{e}_2$. Clearly e is continuous at p_1^*, since $\tilde{e}_1 = \tilde{e}_2$ there. The left derivative of e with respect to p_1, evaluated at p_1^*, is

$$\lim_{p_1 \uparrow p_1^*} \frac{e(p_1, p_2, u) - e(p_1^*, p_2, u)}{p_1 - p_1^*} = \lim_{p_1 \uparrow p_1^*} \frac{\tilde{e}_1(p_1, u) - \tilde{e}_1(p_1^*, u)}{p_1 - p_1^*}$$

$$= (\partial \tilde{e}_1/\partial p_1)|_{p_1^*},$$

since $\tilde{e}_1$ is differentiable. Similarly, the right derivative exists and is equal to $\partial \tilde{e}_2/\partial p_1$.

[15] See any calculus text, e.g., Thomas [**32**, p. 214].

A more heuristic discussion of the theorem may be useful at this point. Suppose an infinitesimal change, dp_1, occurs in the price of good 1. If a positive amount x_1 of good 1 is being consumed, then the welfare loss is $x_1\,dp_1$. If good 1 is not being consumed, i.e., $x_1 = 0$, then the welfare loss is again $x_1\,dp_1$, which in this case equals zero. If the consumer is just at the point of indifference between goods 1 and 2, no welfare loss arises from switching from one good to the other. Therefore, integrating over this point of discontinuity adds nothing to the total for a finite change. In every case, then, the welfare loss from an incremental price change is described by the incremental area to the left of the demand curve. If income is compensated as the price changes, this is equivalent to measuring the area to the left of the compensated demand curve. This intuitive discussion applies equally well to cases (b) and (c), mentioned earlier,[16] and to corner solutions.

As in the ordinary goods case, we define an aggregate compensated demand function for good 1, X_1^c, as the sum of the individual compensated demand functions. It depends upon the prices and the utilities of all consumers. Similarly, define the aggregate expenditure function E as the sum of the individual expenditure functions at arbitrarily selected utility levels. Then, by adding over consumers, we obtain the result that the sum of the individual compensating variations is given by the area to the left of the aggregate compensated demand curve:

$$(3.15) \quad \Delta E = \int_{p_1^0}^{p_1^f} X_1^c(p_1, p_2, \{u^i\})\,dp_1$$

where u^i is the initial utility level of the ith consumer.

Applying the Theorem

For purposes of applying Theorem 1, it is important to notice that the demand curve in (3.15) is the unconditional demand for good 1, and is not conditional on the choice of good 1. Both the discrete choice itself *and* the quantity demanded conditional on that choice are incorporated in X_1^c.

To make this distinction clear, write the compensated demand function in equation (3.5), for individual i, as

$$(3.16) \quad x_1^{ci}(p_1, p_2, u^{i0}) = \delta_1^{ci}(p_1, p_2, u^{i0})\tilde{x}_1^{ci}(p_1, p_2, u^{i0})$$

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[16] Mäler [**23**, pp. 131–136] demonstrates an even stronger result for case (b), in which the good in question must be purchased in a given quantity or not at all. He shows that, for a *fall* in the price of such a good, the compensating variation is given by the area to the left of its *ordinary* demand curve. This follows from our result because income effects are absent, and thus the ordinary and compensated demand curves are identical. So long as the price remains above the "critical price" p^* at which the consumer purchases the discrete good, no compensation is required to keep him at the initial utility level; and below the critical price, the compensation cannot affect the amount of the discrete good purchased. However, R. Willig has pointed out to us that Mäler is incorrect in stating that the result also holds when the individual is consuming the good, and the price *rises* above p^*. This asymmetry occurs because for the price rise, the compensating income alters the critical price itself, and hence the ordinary demand curve and the relevant compensated demand curve are not identical.

where δ_1^{ci} is a compensated discrete choice index, i.e., δ_1^{ci} is 1 if good 1 is chosen and 0 otherwise, assuming that price changes are compensated in order to permit achievement of initial utility level u^{i0}. The compensated demand *conditional* on choice of good 1, $\tilde{x}_1^{ci}$, is defined as in the above proof. Adding the right-hand sides of equation (3.16) for the N individuals in the economy yields an alternative version of (3.15):

$$(3.17) \quad \Delta E = \int_{p_1^0}^{p_1^f} \sum_{i=1}^{N} \delta_1^{ci}(p_1, p_2, u^{i0})\tilde{x}_1^{ci}(p_1, p_2, u^{i0})\, dp_1.$$

There are two important problems which must be solved in order to make equation (3.17) useful for applied work. First, an investigator cannot know with certainty whether or not a representative individual will consume a given discrete good: All that can be assigned is the *probability* π_j^i that the good will be purchased, conditional upon a set of measurable characteristics (race, sex, etc.) as well as prices and income. An attractive way to model this is to assume that each individual's utility function is deterministic (as in the preceding section), but that because of factors unobservable to the investigator, the outcome appears random.[17] Thus as we move to applications of our theorem and attach probabilistic interpretations to variables that were heretofore deterministic, we are not moving to a different framework, but are dealing with a special case of the model developed above. Replacing the choice index in (3.17) with this choice probability converts (3.17) to an equation for the *expected* welfare change.[18]

The second problem is that δ_1^{ci} is a *compensated* discrete choice index; therefore, it should be replaced with a compensated choice probability. We provide in the next subsection an approximation procedure, analogous to that for continuous goods, for doing this.

Our method is therefore fully operational provided we can find econometric techniques for estimating the required demand curves. A straightforward procedure would be to estimate the discrete choice and conditional demand functions jointly using maximum likelihood techniques. Alternatively, a consistent recursive procedure has been developed by Heckman [19].

Slutsky Effect

A satisfactory approximation procedure for converting econometric estimates into *compensated* demand relations must take into account the fact that neither the Slutsky equation nor the computation of tax revenues is the same for discrete

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[17] This seems to have acquired the name "random utility" approach, in contrast to the "constant utility" approach, in which the utility function itself (or equivalently, the choice mechanism) is viewed as having a random component. Operationally, these two formulations are equivalent, i.e. lead to the same kind of estimating equations. Their interpretations are discussed at length by Williams [34].

[18] Note that only if the choice probability is independent of price p_1 will the correct answer be obtained by calculating areas for *conditional* demand curves and multiplying them by the probabilities. Thus, for example, Feldstein and Friedman [14] are correct in using the latter procedure to calculate welfare effects of changes in the price of health insurance, because in their model the probability of illness is assumed to be exogenous. However, one must not be misled into thinking their procedure is generalizable to endogenous probabilities.

as for continuous goods, because the nature of the income effects is fundamentally changed. In the continuous case, it is at least reasonable to suppose that all consumers have identical income slopes for good 1; this permits the Slutsky equation to be aggregated, and makes plausible the general equilibrium assumption that tax revenues are redistributed so as to return all consumers to their initial utility levels. In the discrete case, however, we explicitly recognize a group whose consumption of good 1 is zero throughout a range of incomes, and thus the analysis must be modified.

In order to focus on the issue of discreteness *per se*, it is useful to abstract from other differences among individuals by assuming N consumers who differ only in their preferences between goods 1 and 2.[19] Thus, they have identical incomes y, and identical *conditional* Marshallian and compensated demand functions $\tilde{x}_1(p_1, y)$ and $\tilde{x}_1^c(p_1, u)$, respectively. This is, of course, a very restrictive case. However, it is useful in that it highlights the important differences that appear when discreteness is explicitly taken into account. If the assumptions of this section seem too restrictive for a given problem, they can be relaxed, although the results will be less intuitive and computationally more complicated.

The Slutsky equation for this model is

$$(3.18) \qquad \frac{\partial \tilde{x}_1^c}{\partial p_1} = \frac{\partial \tilde{x}_1}{\partial p_1} + \tilde{x}_1 \frac{\partial \tilde{x}_1}{\partial y}.$$

Now let $\pi_1(p_1, p_2, \{y^i\})$ be the fraction choosing good 1 at the given prices and incomes. In analogy to the compensated demand function, define the compensated choice probability

$$(3.19) \qquad \pi_1^c(p_1, p_2, \{u^i\}) = \pi_1(p_1, p_2, \{e^i(p_1, p_2, u^i)\})$$

where u^i and e^i represent the utility level and the expenditure function, respectively, for consumer i.

Define the aggregate quantities

$$(3.20) \qquad X_1 = \sum_i x_1^i = N\pi_1 \tilde{x}_1,$$

$$(3.21) \qquad X_1^c = \sum_i x_1^{ic} = N\pi_1^c \tilde{x}_1^c,$$

$$(3.22) \qquad Y = \sum_i y^i = Ny.$$

Note that since all incomes y^i are assumed equal, X_1 and π_1 depend on them only through aggregate income Y. Hence we may write

$$(3.23) \qquad \pi_1 = \pi_1(p_1, p_2, Y),$$

$$(3.24) \qquad X_1 = X_1(p_1, p_2, Y) = N\pi_1(p_1, p_2, Y)\tilde{x}_1(p_1, p_2, Y/N).$$

[19] Assume, for example, that utility for the ith individual is given by
$$u^i(x_n, x_1, x_2) = f(x_n, x_1, x_2) + \varepsilon_1^i \delta_1 + \varepsilon_2^i \delta_2,$$
where $\delta_j = 1$ if x_j is chosen and $\delta_j = 0$ otherwise ($j = 1, 2$), and where ε_1^i and ε_2^i are constants.

Differentiating (3.19) at an initial point for which $x_1^c = x_1$ and $\pi_1^c = \pi_1$, and applying our previous result (3.14), yields the choice-probability analogue to the Slutsky equation:

$$(3.25) \quad \frac{\partial \pi_1^c}{\partial p_1} = \frac{\partial \pi_1}{\partial p_1} + \sum_i \frac{\partial \pi_1}{\partial y^i} \frac{\partial e^i}{\partial p_1}$$

$$= \frac{\partial \pi_1}{\partial p_1} + X_1 \frac{\partial \pi_1}{\partial Y}.$$

Equations (3.18) through (3.25) taken together yield a Slutsky-like equation for the aggregate demand functions:[20]

$$(3.26) \quad \frac{\partial X_1^c}{\partial p_1} = \frac{\partial X_1}{\partial p_1} + \frac{X_1}{\pi_1}\left[\frac{\partial X_1}{\partial Y} - (1 - \pi_1)\frac{X_1}{\pi_1}\frac{\partial \pi_1}{\partial Y}\right].$$

In the special case $\partial \pi_1/\partial Y = 0$, equation (3.26) is the usual aggregate Slutsky equation, except the income effect is divided by π_1 and is thus larger than in the continuous case. The intuition behind this follows from the conceptual experiment implicit in the compensating variation measure. Were it required to maintain utility after a price rise dp_1, it would be necessary to provide compensation

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[20] Proof: Differentiate (3.21) to find

$$(3.25a) \quad \frac{\partial X_1^c}{\partial p_1} = N\left(\pi_1^c \frac{\partial \tilde{x}_1^c}{\partial p_1} + \tilde{x}_1^c \frac{\partial \pi_1^c}{\partial p_1}\right).$$

Substituting (3.18) and (3.25) into (3.25a) yields

$$(3.25b) \quad \frac{\partial X_1^c}{\partial p_1} = N\left[\pi_1^c\left(\frac{\partial \tilde{x}_1}{\partial p_1} + \tilde{x}_1 \frac{\partial \tilde{x}_1}{\partial y}\right) + \tilde{x}_1^c\left(\frac{\partial \pi_1}{\partial p_1} + X_1 \frac{\partial \pi_1}{\partial Y}\right)\right].$$

Note that from (3.20), $\tilde{x}_1 = X_1/(N\pi_1)$, so that $\partial \tilde{x}_1/\partial p_1$ can be written

$$(3.25c) \quad \frac{\partial \tilde{x}_1}{\partial p_1} = \frac{1}{N}\left[\frac{1}{\pi_1}\frac{\partial X_1}{\partial p_1} - \frac{X_1}{\pi_1^2}\frac{\partial \pi_1}{\partial p_1}\right].$$

Similarly,

$$(3.25d) \quad \frac{\partial \tilde{x}_1}{\partial y} = \frac{1}{N}\left[\frac{1}{\pi_1}\frac{\partial X_1}{\partial y} - \frac{X_1}{\pi_1^2}\frac{\partial \pi_1}{\partial y}\right].$$

Recalling that we are differentiating at an initial point where $X_1 = X_1^c$ and $\pi_1 = \pi_1^c$, and substituting (3.25c), (3.25d), (3.20), and (3.21) into (3.25b) yields

$$(3.25e) \quad \frac{\partial X_1^c}{\partial p_1} = \frac{\partial X_1}{\partial p_1} - \frac{X_1}{\pi_1}\frac{\partial \pi_1}{\partial p_1} + \frac{X_1}{N}\left[\frac{1}{\pi_1}\frac{\partial X_1}{\partial y} - \frac{X_1}{\pi_1^2}\frac{\partial \pi}{\partial y}\right] + \frac{X_1}{\pi_1}\left[\frac{\partial \pi_1}{\partial p_1} + X_1 \frac{\partial \pi_1}{\partial Y}\right].$$

From (3.22),

$$\frac{\partial X_1}{\partial y} = N\frac{\partial X_1}{\partial Y} \quad \text{and} \quad \frac{\partial \pi_1}{\partial y} = N\frac{\partial \pi_1}{\partial Y}.$$

Using these facts (3.25e) can be written

$$(3.25f) \quad \frac{\partial X_1^c}{\partial p_1} = \frac{\partial X_1}{\partial p_1} + \frac{X_1}{\pi_1}\frac{\partial X_1}{\partial Y} - \frac{X_1^2}{\pi_1^2}\frac{\partial \pi_1}{\partial Y} + \frac{X_1^2}{\pi_1}\frac{\partial \pi_1}{\partial Y}.$$

Rearranging (3.25f) yields (3.26) of the text. *Q.E.D.*

equal to $X_1\,dp_1$ *to that fraction* π_1 *of the population which consumes good* 1. But this would have the same effect on consumption as raising all incomes equally, by an aggregate amount $(X_1/\pi_1)\,dp_1$, since the extra payments to nonconsumers would have no impact (so long as $\partial\pi_1/\partial Y = 0$) on aggregate consumption X_1. Thus, the compensation would cause consumption of good 1 to rise by $(\partial X_1/\partial Y)(X_1/\pi_1)\,dp_1$, which gives the first income effect term in (3.26). The last term arises from the effects of income compensation on the choice probability itself.

Excess Burden

Following the procedure of Section 2, we approximate the demand curves by straight lines with slopes

$$\partial X_1/\partial p_1 = \alpha,$$

$$\partial X_1^c/\partial p_1 = \alpha^c,$$

$$\partial X_1/\partial Y = \omega.$$

We also let

$$\partial\pi_1/\partial Y = \nu.$$

As before, the compensating variation is given by equation (2.16); note, however, that this is no longer equivalent to (2.17).

We next proceed to calculate tax revenues under (a) partial and (b) general equilibrium assumptions, making use of the "Slutsky-like" equation (3.26), which can be written

$$(3.27) \quad \alpha^c = \alpha + (X_1/\pi_1)\omega - (1-\pi_1)(X_1/\pi_1)^2\nu.$$

(a) For the partial equilibrium case, tax revenue is simply given by (2.18), and excess burden by (2.22). Substituting (3.27) into (2.22) yields

$$(3.28) \quad \chi^P = \tfrac{1}{2}[-\alpha^c + 2(X_1/\pi_1)\omega - 2(1-\pi_1)(X_1/\pi_1)^2\nu](\Delta p_1)^2$$

$$(3.29) \quad = \tfrac{1}{2}[-\alpha + (X_1/\pi_1)\omega - (1-\pi_1)(X_1/\pi_1)^2\nu](\Delta p_1)^2.$$

Note that, in the absence of income effects on the choice probability ($\nu = 0$), the excess burden (3.29) is *larger* than would be calculated from (2.23), if discreteness were ignored.

(b) If tax revenues are returned uniformly in lump sum transfers, equation (2.20) remains valid, but is no longer equivalent to (2.21), corresponding to the fact that the compensation no longer returns each consumer, even approximately, to his initial utility level. Under this regime, the fraction π_1 who paid the tax have their real incomes reduced, while the others have them raised. In neither group

are individuals moving along their compensated demand curves. Subtracting (2.20) from (2.16) and applying (3.27) gives the excess burden:

$$\chi^G = \tfrac{1}{2}[\alpha^c - 2\alpha - 2X_1\omega](\Delta p_1)^2$$

$$(3.30) \qquad = \tfrac{1}{2}\left[-\alpha^c + 2\left(\frac{1}{\pi_1} - 1\right)X_1\omega - 2(1-\pi_1)\left(\frac{X_1}{\pi_1}\right)^2 \nu\right](\Delta p_1)^2$$

$$(3.31) \qquad = \tfrac{1}{2}\left[-\alpha + \left(\frac{1}{\pi_1} - 2\right)X_1\omega - (1-\pi_1)\left(\frac{X_1}{\pi_1}\right)^2 \nu\right](\Delta p_1)^2.$$

Comparing (3.31) with (2.25), we see again that, in the absence of income effects on the choice probability, the excess burden is larger than that computed by ignoring discreteness, though it is still smaller than in the partial equilibrium case (3.29).

So in general, if the choice probability is independent of income but the conditional demands have positive income elasticities, the usual approximation procedure which ignores discreteness underestimates the compensating variation and the excess burden of a price rise. There are two reinforcing tendencies causing the result. First, the usual Slutsky correction results in an overestimate of the magnitude of the compensated price elasticity, and hence an underestimate of the compensating variation. Second, the ameliorating effects of tax revenue redistribution on misallocation are less, because some of the distributions are "wasted" on nonconsumers of the good in question. We suspect that this result is a special case of a more general proposition that standard procedures tend to underestimate excess burden (i.e., the compensating variation less tax revenues) for groups of nonidentical consumers.

We conclude this section by compiling, in Table I, the results for computing excess burdens.[21] The formulae have been converted to elasticity form, using the definitions

$$\varepsilon = \frac{p_1}{X_1}\frac{\partial X_1}{\partial p_1},$$

$$\eta = \frac{Y}{X_1}\frac{\partial X_1}{\partial Y},$$

$$\psi = \frac{Y}{\pi}\frac{\partial \pi}{\partial Y},$$

$$\theta = \frac{X_1 p_1}{Y},$$

$$\pi = \pi_1.$$

[21] A sample caculation based upon an actual econometric study is available upon request to the authors.

The expressions given are those which should replace $[-\varepsilon^c]$ in the Harberger formula,

$$\chi^G = \tfrac{1}{2} X_1 p_1 [-\varepsilon^c](\Delta p_1/p_1)^2,$$

derived from (2.24).

<div align="center">

TABLE I

FACTORS FOR COMPUTING EXCESS BURDENS

</div>

	Continuous	Discrete
Partial equilibrium	$-\varepsilon + \theta\eta$	$-\varepsilon + \dfrac{1}{\pi}\theta\eta - \left(\dfrac{1}{\pi} - 1\right)\theta\psi$
General equilibrium	$-\varepsilon - \theta\eta$	$-\varepsilon + \left(\dfrac{1}{\pi} - 2\right)\theta\eta - \left(\dfrac{1}{\pi} - 1\right)\theta\psi$

Observe that when $\pi = 1$, the expressions in the second column equal their counterparts in the first. In both relative and absolute terms, the error introduced by ignoring discreteness is small when practically everybody consumes the good.[22] When π tends toward zero, on the other hand, terms involving θ also tend toward zero, whereas those involving θ/π tend to $\tilde{x}_1 p_1/y$.[23] Thus, the error introduced by ignoring discreteness, i.e., the difference between terms in Table I multiplied by $\tfrac{1}{2} X_1 p_1 (\Delta p_1/p_1)^2$, tends to zero absolutely (though not as a proportion of excess burden) as π approaches zero.

A final issue concerns the realism of the assumed pattern of redistribution of tax revenues, namely that they are returned to the population at large. Equity considerations or interest-group politics frequently lead instead to attempts to target revenue uses to the groups most affected by the tax. If it is possible to target the revenues only to the consumers of the taxed discrete good, the usual formulae would apply without modification. If some targeting occurs but it is incomplete, the excess burden would be somewhere between the "continuous" and "discrete" calculations.

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[22] For another consistency check, consider the special case in which π is independent of price and income. In that case, only a fraction π of the population experiences any effective price change. Calling this subpopulation "group A," we note that its price slope is $\alpha^A = \alpha$, and its income slope $\omega^A = \omega/\pi$. Substituting these values into equation (2.17) gives this group's compensating variation:

$$\Delta E^A = X_1^0 \Delta p_1 + \tfrac{1}{2}[\alpha + X_1^0 \omega/\pi](\Delta p_1)^2.$$

Subtracting total tax collections (2.18) or (2.20) yields (3.29) or (3.31), respectively, with $\nu = 0$. Thus, the results in the table are consistent, when π is constant, with the result obtained by computing excess burdens for the two groups separately.

[23]
$$\theta/\pi = \left(\frac{N\pi p_1 \tilde{x}_1}{Ny}\right)/\pi = \tilde{x}_1 p_1/y.$$

4. QUALITY CHANGES

Thus far, our focus has been the measurement of price-induced welfare changes. However, in many discrete choice problems the evaluation of quality changes is of primary interest. For example, we may wish to evaluate changes in frequency of bus services in contexts where consumers have a choice between bus and auto; or changes in noise levels when individuals have the options of staying in the affected locale or of moving.

One possibility for dealing with a quality change is to convert it to an equivalent price change, and then to apply the technique discussed in Section 3. Unfortunately, this is possible only in the very special case in which all consumers have constant and identical marginal rates of substitution between the relevant quality variable and the numeraire. For example, this assumption is often made in transportation studies by assuming a single constant value of time for everyone. Demand for a particular transportation service can then be viewed as a function of money plus time price, and consumer surplus techniques are applicable. However, this example illustrates the tenuousness of such an assumption, since both theoretical and empirical considerations suggest that value of time depends upon a variety of factors such as the wage rate.

An alternative tack is to extend the approach taken in Section 3, by adding a quality variable to the model. Suppose that good 1 in the model of the previous section has associated with it a quality variable described by the scalar q_1, which is considered exogenous by consumers. (The variable q_1 could also be interpreted as the level of provision of a public good which is complementary to good 1.) Thus

$$(4.1) \qquad u = u(x_n, x_1, x_2, q_1)$$

where, as before, u is assumed to be finite when either $x_1 = 0$ or $x_2 = 0$. We assume further that quality of good 1 does not matter unless good 1 is being consumed:

$$(4.2) \qquad \partial u(x_n, 0, x_2, q_1)/\partial q_1 = 0.$$

As in Section 3, the indirect utility function and the expenditure function are well defined, continuous, right and left differentiable in all arguments, and satisfy

$$(4.3) \qquad u = v(p_1, p_2, q_1, e(p_1, p_2, q_1, u)).$$

Two alternative formulations of the problem are obtained by implicitly differentiating (4.3) with respect to q_1 or p_1, respectively, keeping in mind that the right and left derivatives may differ at the crossover points.

Taking the *quality* derivative of (4.3) yields

$$(4.4) \qquad \partial e/\partial q_1 = -(1/\lambda)\partial v/\partial q_1,$$

where $\lambda \equiv \partial v/\partial y \equiv \partial u/\partial x_n$ is the marginal utility of income. This intuitive formula simply states that the marginal willingness-to-pay for a quality change is given by the marginal utility of quality, converted to monetary units via the marginal utility of income. For problems in which the indirect utility function is explicitly specified, as in Section 5 below, equation (4.4) is extremely useful, particularly if

it is reasonable to approximate λ (an empirical quantity, given the normalization implicit in the econometric procedure) by a constant. However, the constancy of λ is in no sense necessary for (4.4) to be valid.

Taking the *price* derivative of (4.3), on the other hand, and applying Roy's Identity, yields Shephard's Lemma, which was the starting point of deriving Theorem 1 of Section 3. Thus, by simply adding q_1 to the list of arguments of all the functions, Theorem 1 is seen to remain valid in the present context. This fact enables us to evaluate quality changes in terms of changes in consumer's surplus, along the lines explored by Bradford and Hildebrandt [7] and Willig [36] for continuous goods.

To see this, assume, following Bradford and Hildebrandt, that when the price of good 1 becomes high enough, the consumer does not care about its quality, presumably because none of the good is being consumed. More precisely, we assume that the quality derivative of the expenditure function becomes arbitrarily small as the price of good 1 rises:

$$(4.5) \qquad \lim_{p_1^f \to \infty} \partial e(p_1^f, p_2, q_1, u^0)/\partial q_1 = 0.$$

Thus, differentiating our previous result, (3.5), with respect to q_1 and letting $p_1^f \to \infty$ yields the following result:

THEOREM 2: *Let the utility function (4.1) satisfy (4.2) and all the assumptions of Theorem 1. Let $e(p_1, p_2, q_1, u)$, the minimum expenditure required to achieve utility u subject to constraints (3.2)–(3.4), satisfy (4.5). Then*

$$(4.6) \qquad \frac{\partial}{\partial q_1} e(p_1^0, p_2, q_1, u^0) = -\int_{p_1^0}^{\infty} \frac{\partial}{\partial q_1} x_1^c(p_1, p_2, q_1, u^0) \, dp_1,$$

where the derivatives are to be interpreted as left and right derivatives at points of nondifferentiability.

Note that (4.6) is the change in the area to the left of the compensated demand curve as it shifts in response to the change in q_1. This is the result derived for the case of continuous goods by Bradford and Hildebrandt [7], using a slightly different assumption on v. Thus, we have demonstrated that the Bradford–Hildebrandt formula remains valid in the discrete goods case.

Equation (4.6) is easily integrated over q_1 to obtain the compensating variation for a quality change from q_1^0 to q_1^f:

$$(4.7) \qquad \Delta e = -\int_{p_1^0}^{\infty} [x_1^c(p_1, p_2, q_1^f, u^0) - x_1^c(p_1, p_2, q_1^0, u^0)] \, dp_1.$$

Furthermore, direct aggregation of (4.7) is possible to obtain a formula in terms of the aggregate compensated demand function:

$$(4.8) \qquad \Delta E = -\int_{p_1^0}^{\infty} [X_1^c(p_1, p_2, q_1^f, \{u^i\}) - X_1^c(p_1, p_2, q_1^0, \{u^i\})] \, dp_1.$$

5. APPLICATION TO ECONOMETRIC MODELS

McFadden [21, 22] has shown that a broad class of discrete choice models can be derived by assuming that each consumer maximizes a utility function with a stochastic term accounting for differences in tastes that are unobservable to the investigator. Since such models underlie the bulk of current empirical work on discrete choice, it is useful to examine the form which the results of the previous two sections take when applied to stochastic utility models, in particular to the popular probit and logit models. Certain especially useful results are obtained in the special case when the marginal utility of income is approximately constant.

In our notation, the assumption behind all these models is that the *conditional* indirect utility function of consumer i for good j, $\tilde{v}_j^i$, is additively separable into three components as follows:[24]

$$(5.1) \qquad \tilde{v}_j^i(p_j, q_j, y^i) = V^i(y^i) + W_j(p_j, q_j, y^i; \underline{S}^i) + \varepsilon_j^i$$

where $W_j(\cdot)$ is a "universal" utility function whose form is identical for all consumers, and where ε_j^i is the realization for consumer i of a random variable ε_j which is independent of the arguments of W_j. The vector $\underline{S}^i$ contains observable characteristics of the consumer, and attributes of the discrete goods (e.g., locational attributes) which may vary across consumers.

By assuming a joint probability distribution on $\{\varepsilon_j\}$, we obtain the probability, conditional on prices, quality levels, and observable consumer characteristics, that utility is maximized by choosing good 1:

$$(5.2) \qquad \pi_1^i = \text{prob}\left[\varepsilon_j - \varepsilon_1 < W_1^i - W_j^i \text{ for all } j \neq i\right],$$

where W_j^i denotes the second term in (5.1). Given specific forms for the functions W_j, their parameters (up to a normalization convention) can be estimated from a sample of observed choices by maximizing the likelihood function associated with (5.2).

In order to forecast the aggregate demands, we need to measure the incomes y^i and other characteristics $\underline{S}^i$ of the individuals in the population under consideration. One way to do this is to sample the population, and assume that each sample member i is representative of a group of N^i individuals who are identical with respect to their incomes and *measured* characteristics.[25] The probability π_1^i calculated from (5.2) can therefore be interpreted as a prediction of the fraction of this particular class of consumers who will choose good 1. Assume further that the conditional (uncompensated) demand function $\tilde{x}_1^i$ either is nonstochastic, or contains a random term which is uncorrelated with ε_j of (5.1). The latter will be

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[24] The first term in (5.1) does not enter the econometric specification of the discrete choice model, since it is independent of j; it has usually been ignored in the literature. The remaining terms have generally been described as the "utility function" though it is clear from the inclusion of price as an argument that they are part of an *indirect* utility function. A true indirect utility function would have *all* prices as arguments and would not depend on j; hence the qualification "*conditional.*" See Small [31] for a fuller discussion of this point.

[25] Domenich and McFadden [12, p. 81] discuss an elaboration of this procedure, and derive equation (5.3) for the case of $\tilde{x}_1^i \equiv 1$ (see their equation 4.69).

the case, for example, if the values of $\tilde{x}_1^i$ used in the welfare computations are predicted values from the last stage of Heckman's estimation procedure described earlier. The expected aggregate demand from class i is therefore

(5.3) $\qquad X_1^i(p_1, p_2, q_1, y^i; \underline{S}^i) = N^i \pi_1^i \tilde{x}_1^i.$

Note the parallel with equation (3.20). Furthermore, applying Roy's Identity to (5.1) gives $\tilde{x}_1^i$ in terms of the price derivative of W_1 and the marginal utility of income,

$$\lambda^i \equiv \frac{\partial V^i}{\partial y^i} + \frac{\partial W_1^i}{\partial y^i}.$$

Substituting into (5.3) yields

(5.4) $\qquad X_1^i(p_1, p_2, q_1, y^i; \underline{S}^i) = -(N^i/\lambda^i)\pi_1^i \partial W_1^i/\partial p_1.$

We now proceed to apply Theorem 2 of the previous section in order to compute the expected compensating variation for members of class i arising from a change in the quality of good 1. A similar analysis could be performed, using Theorem 1, for price changes. We make use of the fact that the expectation operator is interchangeable with either differentiation or summation.[26] In the following discussion, we suppress the superscript i and the explicit dependence on S^i, which is assumed to be fixed.

Having specified functional forms for the components W_j of the indirect utility function, and having implicitly normalized by choosing a specific joint probability distribution function for the stochastic term ε_j, we are free to make assumptions about W_j on the basis of purely empirical considerations. We now make three such assumptions, which are likely to be valid in many applications: (a) that the marginal utility of income, λ, is approximately independent of the price and quality of good 1; (b) that the discrete goods are sufficiently unimportant to the consumer so that income effects from quality changes are negligible, i.e. that the compensated demand function in (4.6) is adequately approximated by the Marshallian demand function; and (c) that $\partial W_1/\partial q_1 \to 0$ as $p_1 \to \infty$. Equation (4.6) can therefore be aggregated over members of this consumer group to yield

$$\frac{1}{N}\frac{\partial E}{\partial q_1} = \frac{1}{\lambda}\frac{\partial}{\partial q_1}\int_{p_1^0}^{\infty} \pi_1(p_1, p_2, q_1, y)\frac{\partial W_1(p_1, q_1, y)}{\partial p_1}\, dp_1.$$

But (5.2) implies that π_1 depends on its arguments only through the functions W_1 and W_2. We may therefore make a change of variable, $\omega_1 = W_1(p_1, q_1, y)$, to

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[26] Domencich and McFadden [12, pp. 94–99], using this same stochastic utility model, compute the expected utility directly from the stochastic specification. They then demonstrate, using our restriction (a) below, that the change in expected utility due to a price change is equal to the conventional consumer's surplus. One virtue of our Theorems 1 and 2 is that they do not depend on any such restriction, but yield the Domencich-McFadden result as a special case. Since our welfare measure is not stochastic at the individual level, we have no need to compute expected utilities. The stochastic element enters instead as unobservable preferences affecting the demand functions used in Theorems 1 and 2, and we therefore need only to take expectations of demands, as in equation (5.3).

write

$$\frac{1}{N}\frac{\partial E}{\partial q_1} = \frac{1}{\lambda}\frac{\partial}{\partial q_1}\int_{\omega_1^0}^{\omega_1^\infty} \pi_1(\omega_1, W_2)\, d\omega_1$$

where $\omega_1^0 = W_1(p_1^0, q_1, y)$, $\omega_1^\infty = W_1(\infty, q_1, y)$. Now using a standard theorem for differentiating a definite integral with respect to its limits (Thomas [32, p. 720]) we find

$$\frac{1}{N}\frac{\partial E}{\partial q_1} = \frac{1}{\lambda}\left(\pi_1(\omega_1^\infty, W_2)\frac{\partial \omega_1^\infty}{\partial q_1} - \pi_1(\omega_1^0, W_2)\frac{\partial \omega_1^0}{\partial q_1}\right).$$

But according to assumption (c) above, $\partial\omega_1^\infty/\partial q_1 = 0$. Therefore,

$$\frac{1}{N}\frac{\partial E}{\partial q_1} = -\frac{1}{\lambda}\pi_1(\omega_1^0, W_2)\frac{\partial \omega_1^0}{\partial q_1}.$$

Integrating this expression over q_1 from initial to final values yields

$$(5.5) \qquad (\Delta E)/N = -(1/\lambda)\int_{W_1^0}^{W_1^f} \pi_1(W_1, W_2)\, dW_1$$

where $W_1^0 = W_1(p_1^0, q_1^0, y)$ and $W_1^f = W_1(p_1^0, q_1^f, y)$.[27]

Equation (5.5) is extremely useful for computing welfare effects using discrete choice models estimated from micro data, since it avoids the explicit use of the aggregate demand functions, which are not normally obtained in closed form from such studies. In fact, though equation (5.5) is derived above using the previous results on aggregate demand curves, it can instead be derived directly from the definition (4.3) of the expenditure function.

To see this, recall that the previous results were based on Shepherd's Lemma, obtained by differentiating (4.3) with respect to price. As mentioned in Section 4, one may instead differentiate (4.3) with respect to quality, leading to (4.4). The indirect utility v of (4.4) is equal to $\tilde{v}_1$ for that fraction π_1 of the consumer group which chooses good 1, and is equal to $\tilde{v}_2$ for the rest. Since only $\tilde{v}_1$ depends on q_1, aggregating (4.4) and calculating $\partial\tilde{v}_1/\partial q_1$ from (5.1) yields

$$(5.6) \qquad \partial E/\partial q_1 = -N(\pi_1/\lambda)\partial W_1/\partial q_1.$$

Equation (5.6) is perfectly general, but one must use caution in integrating it, since the income variable which appears as an argument in all the functions is compensated, i.e., is that value which maintains the original utility level. With the assumptions of constant λ and negligible income effects, however, integrating (5.6) yields (5.5) directly.

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[27] Equation (5.5) is easily generalized to many simultaneous price and quality changes, provided the Jacobian matrix $(\partial\pi_i/\partial W_j)$ is symmetric. In that case, the following line integral is unique, i.e., path independent (see Silberberg, [29]):

$$(5.5a) \qquad (\Delta E)/N = -(1/\lambda)\int_{\underline{W}^0}^{\underline{W}^f} \sum_j \pi_j(\underline{W})\, dW_j$$

where $\underline{W}$ is a vector of components W_j, and takes initial and final values $\underline{W}^0$ and $\underline{W}^f$ respectively.

A final point to notice about equation (5.5) is that it applies equally well to a price change, if W_1^0 and W_1^f are defined as the values taken by W_1 at the initial and final prices, respectively. This may be seen simply by replacing q_1 by p_1 in the above paragraph. Alternatively, the price-change version of (5.5) can be derived directly by integrating (5.4).

In the course of the econometric estimation of the choice model, specific functional forms for the W_j's are assumed, and estimates of the coefficients are obtained. Hence, one can compute W_1 at the initial and final quality levels, and can therefore compute the integral in (5.5). Furthermore, if price is included as a variable in the specification, one has an estimate of $\partial W_1/\partial p_1$. Using this information along with the conditional demand $\tilde{x}_1$, which is often known, λ can be computed by using Roy's Identity.[28] Equation (5.5) therefore represents an operational procedure for calculating welfare effects. We illustrate its use below for the binary probit and logit models.

Probit

In the binary probit model it is assumed that $(\varepsilon_1 - \varepsilon_2)$ is a standard normal variate. The probabilities are therefore given by

$$\pi_1 = \Phi(W_1 - W_2),$$

$$\pi_2 = 1 - \pi_1,$$

where Φ is the cumulative normal distribution function. Formula (5.5) for the compensating variation becomes

$$(5.7) \qquad (\Delta E)/N = -(1/\lambda) \int_{W_1^0}^{W_1^f} \Phi(W_1 - W_2)\, dW_1$$

$$= -(1/\lambda) \int_{\omega^0}^{\omega^f} \Phi(\omega)\, d\omega$$

where $\omega^0 = W_1^0 - W_2$ and $\omega^f = W_1^f - W_2$. Given a probit estimation of the functions W_1 and W_2, this quantity can be computed for any price or quality change.[29]

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[28] More specifically, suppose that α_p is the coefficient of the price variable in a probit or logit regression. Then $\lambda = -\alpha_p/\tilde{x}_1$, which, as noted above, is an empirical quantity given the normalization implicit in econometric procedure.

[29] The generalization to simultaneous changes in two price and two quality variables is straightforward, since the Jacobian condition $\partial\pi_1/\partial W_2 = \partial\pi_2/\partial W_1$ is satisfied. Equation (5.5a) in footnote 27 thus becomes

$$(5.7a) \qquad (\Delta E)/N = -\frac{1}{\lambda} \int_{W^0}^{W^f} \{\Phi(W_1 - W_2)\, dW_1 + [1 - \Phi(W_1 - W_2)]\, dW_2\}$$

$$= -\frac{1}{\lambda}\left[\Delta W_2 + \int_{\omega^0}^{\omega^f} \Phi(\omega)\, d\omega\right]$$

where $\Delta W_2 = W_2^f - W_2^0$. Recalling that W_j is an indirect utility function, the first term in (5.7a) is the negative of the willingness to pay for a uniform improvement in both goods of magnitude ΔW_2, whereas the second term gives the appropriate consumer surplus for any improvement in good 1 relative to good 2.

Logit

The logit model arises from the assumption that the stochastic utility components ε_j are independently and identically distributed with the extreme value or double-exponential distribution (McFadden [21, 22]). This leads to a formula for the probabilities:

$$(5.8) \qquad \pi_l = \frac{\exp\,(W_l)}{\sum_{j=1}^{2}\exp\,(W_j)}, \quad l = 1, 2.$$

In this case, the integral in (5.5) can be evaluated explicitly, leading to:

$$(5.9) \qquad (\Delta E)/N = -(1/\lambda)\left[\ln \sum_j \exp\,(W_j)\right]_{W_1^0}^{W_1^f}$$

where the square brackets indicate the difference in the expression inside when evaluated at the initial and final points.[30]

6. CONCLUSION

The analysis and estimation of discrete choice models is currently enjoying a popularity which is unlikely to abate in future years. Ultimately, both researchers and policy makers will be interested in ascertaining the welfare implications of these models. The purpose of this paper has been to set forth rigorous procedures for doing so. It has been shown how, under quite general conditions as well as for the special cases of probit and logit analysis, the usual techniques for measuring welfare effects must be modified for discrete choice problems. Particular attention has been focused on how the standard corrections for income effects might be altered. The results apply to the study of both price and quality changes.

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APPENDIX

A.1 *Proof of Theorem 1 for Different Types of Discreteness*

1. The good may be purchased only in discrete units which are so large that most consumers choose only one or two units.

The utility function is $u(x_n, x_1)$ where $x_n > 0$, and x_1 can take on values of only unity or zero. Utility is maximized subject to the budget constraint $x_n + x_1 p_1 = y$. If $x_1 = 0$, the optimal amount of x_n is trivially equal to y. This subproblem yields the expenditure function conditional on $x_1 = 0$, $\tilde{e}_n(u)$. Similarly, if

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[30] Like the Probit model, the Logit model satisfies the Jacobian symmetry conditions, and calculation of (5.5a) for many price and quality changes yields equation (5.9) altered only in that the limits for evaluating the term in the square brackets are the values $\underline{W}^0$ and $\underline{W}^f$ taken by the vector of *all* W_i at initial and final points. The term in square brackets has been identified in the transportation literature as the "inclusive price" or "composite cost" of an alternative composed of several sub-alternatives (Ben-Akiva [3]; Williams [34]).

$x_1 = 1$, the conditional expenditure function $\tilde{e}_1(p_1, u)$ is found trivially. As in the case analyzed in the text, the point of maximum utility is associated with the minimum of these two expenditure functions:

$$e(p_1, \bar{u}) = \min \{\tilde{e}_n(\bar{u}), \tilde{e}_1(p_1 \bar{u})\}.$$

Since $\tilde{e}_n$ and $\tilde{e}_1$ are both continuously differentiable in price (with $\partial \tilde{e}_n / \partial p_1 \equiv 0$) so is e, except at that price at which $\tilde{e}_1 = \tilde{e}_2$, and at this point e is continuous and right and left differentiable. The rest of the argument follows directly that used in the text.

2. Demand functions are continuous, but particular prices and incomes lead to corner solutions at which a good is not consumed at all.

The utility function $u(x_n, x_1)$, where $x_n > 0$, is maximized subject to $x_n + p_1 x_1 = y$ and $x_1 \geq 0$. For given utility $\bar{u}$, we have two price ranges divided by $p_1^*(\bar{u})$. For $p_1 < p_1^*(\bar{u})$, an interior solution in good 1 exists, and we denote its compensated demand and expenditure functions by $\tilde{x}_n^c$, $\tilde{x}_1^c$, and $\tilde{e}$. For $p_1 \geq p_1^*(\bar{u})$, a corner solution holds and good 1 is not consumed; we denote this solution by $\tilde{\tilde{x}}_n^c$ and $\tilde{\tilde{e}}$.

The budget constraint requires that these solutions satisfy

(A.1) $\tilde{x}_n^c(p_1, \bar{u}) + p_1 \tilde{x}_1^c(p_1, \bar{u}) = \tilde{e}(p_1, \bar{u}), \quad p_1 < p_1^*(\bar{u}),$

and

(A.2) $\tilde{\tilde{x}}_n^c(\bar{u}) = \tilde{\tilde{e}}(\bar{u}), \quad p_1 \geq p_1^*(\bar{u}).$

The price $p_1^*(\bar{u})$ is defined as the limiting price for the first case, i.e., that price for which

$$\lim_{p_1 \uparrow p_1^*} \tilde{x}_1^c(p_1, \bar{u}) = 0.$$

By assumption, the compensated demand function defined by

$$x_n^c(p_1, \bar{u}) = \begin{cases} \tilde{x}_n^c(p_1, \bar{u}), & p_1 < p_1^*(\bar{u}), \\ \tilde{\tilde{x}}_n^c(\bar{u}), & p_1 \geq p_1^*(\bar{u}), \end{cases}$$

is continuous at p_1^*. Therefore, taking limits of equations (A.1) and (A.2) yields

$$\lim_{p_1 \uparrow p_1^*} \tilde{e}(p_1, \bar{u}) = \tilde{\tilde{e}}(\bar{u}).$$

The overall expenditure function is therefore continuous, and the proof of Theorem 1 applies.

3. A good may be purchased in discrete units because nonconcavities in the utility function lead the consumer to choose among alternative corner solutions.

For cases in which discreteness arises from nonconcavity of the utility function, no general theory is available for use in deriving properties of the resulting demand functions. Indeed, so far as we know, no attempt has yet been made to derive formally statistical discrete choice models for such cases, and to do so would be beyond the scope of this paper. Nevertheless, it is not difficult to see that well-behaved examples can be treated within the framework presented in the text. Suppose, for example, that in a three-good world, all the indifference curves between x_1 and x_2 are concave to the origin, but indifference curves between either x_1 or x_2' and the numeraire are convex. Then at every price vector, consumption of either x_1 or x_2 will be zero; hence the constraint $x_1 x_2 = 0$ can be added without changing the problem. Furthermore, the assumptions guarantee that $u(x_n, 0, x_2)$ and $u(x_n, x_1, 0)$, viewed as functions of two variables, are well behaved. Thus the proof of Section 3 applies.

A.2 Proof of Theorem 1 for Arbitrary Numbers of Goods

1. More than one continuous good. The interpretation of x_n as a vector of goods presents no problems provided that the discussion of the "standard" case in Section 2 generalizes. That this is true is shown in Varian [33, pp. 209–211].

2. More than two discrete goods, all mutually exclusive. Utility is given by $u = u(x_n, x_1, x_2 \ldots, x_l)$, where $x_j \geq 0 (j = n, 1, \ldots l)$, and where $x_k > 0 (k = 1, \ldots l)$ implies that $x_r = 0$ for $r \neq k$. The budget constraint is $x_n + \sum_{k=1}^{l} p_k x_k = y$. If $x_k > 0$, then utility is found by maximizing $u(x_n, 0, \ldots x_k, \ldots, 0)$ subject to $x_n + p_k x_k = y$. The function u is well behaved when viewed as a function only of x_n and x_k, and its maximization leads to a (conditional) continuously differentiable expenditure function $\tilde{e}_k(p_k, u)$. The only difference between this case and that discussed in the text is that there are now l

such subproblems, rather than two. But the idea is the same: utility maximization leads to the expenditure function $e(p_1, \ldots, p_l, \bar{u}) = \min_k \tilde{e}_k(p_k, \bar{u})$. Since each of the $\tilde{e}_k$ is continuously differentiable in price, so is e except at those prices at which $\tilde{e}_l = \tilde{e}_r$, and at such cross-over points e is continuous and right and left differentiable. The rest of the argument follows directly from the proof of Theorem 1.

3. More than one set of mutually exclusive discrete goods. Utility is given by $u(x_n, x_{11}, x_{12}, x_{21}, x_{22}, \ldots x_{l1}, x_{l2})$, where all the arguments in the utility function are nonnegative. The constraints are $x_{k1}x_{k2} = 0$ for all $k = 1, \ldots, l$. The budget constraint is $x_n + \Sigma_{k=1}^{l} (p_{k1}x_{k1} + p_{k2}x_{k2}) = y$. In this case, we can imagine the individual solving 2^l subproblems of the sort discussed in Theorem 1. Each subproblem arises from the maximizing utility with one argument in each of the pairs equal to zero. This leads to well behaved conditional expenditure functions, the minimum of which is associated with the consumer's optimum. At each vector of "cross-over" prices, the minimum of these functions is continuous and right and left differentiable in each price. The rest of the argument follows as in the text.

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