Geometric Analysis@ UCI

Math 240BC

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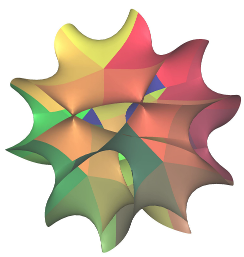

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Chapter 1 Preliminaries

1.1 Integration on manifolds

Let M be an n-dimensional manifold and let ω be an n-form on M with compact support. The integration of ω over M can be defined in the following way.

Let $\{U_j\}$ be a locally finite cover of M and let $\{\rho_j\}$ be the partition of unity subordinating to the cover. That is

1. $\rho_j \geq 0$;
2. $\operatorname{supp} \rho_i \subset U_j$ for all j;
3. $\sum_j \rho_j = 1$.

Then we can define

$$\int_M \omega = \sum_j \int_{U_j} \rho_j \omega,$$

where the right hand side is a sum of integrations on Euclidean space, hence well defined.

Exercise 1 *Prove that the definition of integral is independent to the choices of cover and the partition of unity.*

The definition of integration can be generalized to manifold with boundary. A manifold with boundary is a topological space such that

1. It can be covered by $\{U_j\}$, where U_j is either the unit ball of $\mathbb{R}^n$, or the upper half of the unit ball of $\mathbb{R}^n$ defined by
$$\{(x_1, \cdots, x_n) \mid \sum x_i^2 < 1, x_n \geq 0\}.$$
2. The transition functions are smooth (up to the boundary).

Let M be a manifold with boundary. Let ∂M be the boundary of M. Then we have

1. ∂M is a smooth manifold;
2. ∂M is a manifold without boundary. That is $\partial\partial M = 0$, or $\partial^2 = 0$.

The theorem of partition of unity can be generalized into the following setting.

Theorem 1.1

Let $\{U_j\}_I$ be a locally finite cover of M. Then there exists a partition of unity subordinating to the cover $\{\rho_j\}$. Moreover, let

$$J = \{j \in I \mid U_j \cap \partial M = \emptyset\}.$$

Then

$$\{U_j \cap \partial M\}_{j \in J}, \qquad \rho_j|_{\partial M \cap U_j}$$

is a partition of unity subordinating to the corresponding cover. ♡

Exercise 2 *Prove the above theorem.*

A manifold is called orientable, if there is a cover $\{U_j\}$ on M such that if $x_j^1, \cdots, x_j^n$ are the coordinates on U_j, then we have

$$\det \frac{\partial(x_j^1, \cdots, x_j^n)}{\partial(x_i^1, \cdots, x_i^n)} > 0$$

for any $U_i \cap U_j \neq \emptyset$. Such a set of coordinate charts is called an orientation of the manifold.

Exercise 3 *If M is an orientable manifold with boundary, prove that ∂M is also an orientable manifold.*

Theorem 1.2 (Stokes' Theorem)

Let M be an orientable manifold with the given orientation. Then there is a natural orientation on ∂M such that

$$\int_M d\omega = \int_{\partial M} \omega$$

for any $(n-1)$ forms on M. ♡

Proof. Let $\{U_i\}$ be a locally finite cover and let $\{\phi_i\}$ be the partition of unity subordinating to the cover. Since

$$d \sum_j \rho_j = 0,$$

we have

$$\int_M d\omega = \sum_{j \in I} \int_{U_j} \rho_j d\omega = \sum_{j \in I} \int_{U_j} d(\rho_j \omega), \qquad and$$

$$\int_{\partial M} \omega = \sum_{j \in J} \int_{\partial M \cap U_j} \rho_j \omega.$$

The Stokes' theorem follows from the following statements:

1. $$\int_{U_j} d(\rho_j \omega) = 0, \qquad j \notin J;$$

2. $$\int_{U_j} d(\rho_j \omega) = \int_{U_j \cap \{x_n = 0\}} \rho_j \omega, \qquad j \in J.$$

In the first case, we may assume that U_j is the unit ball of $\mathbb{R}^n$, and $\operatorname{supp}(\rho_j \omega) \subset U_j$. Write

$$\rho_j \omega = \sum_i (-1)^i f_i dx_1 \wedge \cdots \hat{dx_i} \wedge \cdots \wedge dx_n,$$

where $\hat{}$ means omit the term. Then

$$\int_{U_j} d(\rho_j \omega) = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n.$$

Since f_j vanishes on the boundary of ∂U_j, we have

$$\int_{U_j} d(\rho_j \omega) = 0$$

by the fundamental theorem of Calculus.

The proof of the second assertion is similar. We assume that

$$U_j = \{(x_1, \cdots, x_n) \mid \sum x_j^2 < 1, x_n \geq 0\}.$$

Then we have

$$d(\rho_j \omega) = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n.$$

If $i \neq n$, then

$$\int_{U_j} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n = 0.$$

With the appropriate orientation, we have

$$\int_{U_j} \frac{\partial f_n}{\partial x_n} dx_1 \wedge \cdots \wedge dx_n = \int_{U_j \cap \{x_n = 0\}} f_n dx_1 \wedge \cdots \wedge dx_{n-1}.$$

The theorem follows from the above two equations. ■

Exercise 4 *Provided the details in the last part of the proof of Stokes' Theorem.*

Lemma 1.1

On an orientable Riemannian manifold, the n-form

$$\sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n$$

is globally defined. ♡

Definition 1.1

Let f be a smooth function with compact support in a Riemannian manifold. Then

$$\int_M f dV_M = \int_M f \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n.$$

♣

Exercise 5 *Let*

$$X = \sum a^i \frac{\partial}{\partial x_i}$$

be a vector field of M. The divergence of the vector field X is defined to be

$$\operatorname{div} X = \sum_i \frac{\partial a^i}{\partial x^i} + \frac{1}{2} a^i \frac{\partial}{\partial x_i} \log \det(g_{k\ell}).$$

Prove that

$$\int_M \operatorname{div} X dV_M = 0.$$

1.2 The extension of the Levi-Civita connection

Let M be a Riemannian manifold and let ∇ be the Levi-Civita connection. We extend ∇ to all tensor fields as follows.

Definition 1.2

Let X, Y be vector fields.

1. *If $f \in \mathcal{C}^\infty(M)$, then $\nabla_X f = Xf$;*
2. *$\nabla_X Y$ is defined by the Levi-Civita connection;*
3. *If ω is a one-form, we use*

$$(\nabla_X \omega) Y = X(\omega(Y)) - \omega(\nabla_X Y)$$

   to define the one form $\nabla_X \omega$;
4. *In general, let $\{e_1, \cdots, e_n\}$ be a local frame of M and let $\omega_1, \cdots, \omega_n$ be the dual frame. Let*

$$T = a_{i_1 \cdots i_p j_1 \cdots j_q} e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes \omega_{j_1} \otimes \cdots \otimes \omega_{j_q}.$$

We define

$$\nabla_X T = X(a_{i_1\cdots i_p j_1\cdots j_q}) e_{i_1} \otimes \cdots \otimes \omega_{j_q}$$
$$+ \sum_{s=1}^{p} a_{i_1\cdots i_p j_1\cdots j_q} e_{i_1} \otimes \cdots \otimes \nabla_X e_{i_s} \otimes \cdots \otimes \omega_{j_q}$$
$$+ \sum_{t=1}^{q} a_{i_1\cdots i_p j_1\cdots j_q} e_{i_1} \otimes \cdots \otimes \nabla_X \omega_{j_t} \otimes \cdots \otimes \omega_{j_q}.$$

♣

Using the above notations, we have

Theorem 1.3

$\nabla_X ds^2 = 0$, *where* ds^2 *is the Riemannian metric.*

♡

Proof 1. Let $(x_1, \cdots, x_n)$ be a local coordinate system, and let Γ_{ij}^k be the Christoffel symbols. We then have

$$\nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i} = \Gamma_{ki}^{\ell} \frac{\partial}{\partial x_\ell}$$

and

$$\nabla_{\frac{\partial}{\partial x_k}} dx_j = -\Gamma_{k\ell}^j dx_\ell$$

Thus we have

$$\nabla_{\frac{\partial}{\partial x_k}} g_{ij} dx_i dx_j = \frac{\partial g_{ij}}{\partial x_k} dx_i dx_j - g_{ij}\Gamma_{k\ell}^i dx_\ell dx_j - g_{ij}\Gamma_{k\ell}^j dx_i dx_\ell = 0.$$

■

Proof 2. Using the intrinsic characterization of the Riemannian metric

$$(\nabla_X ds^2)(Y, Z) = X(\langle Y, Z\rangle) - \langle \nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle = 0.$$

■

We also have the following result which well justifies our definition of the curvature operator.

Theorem 1.4

Let d_{∇^2} *be the operator defined by the composition*

$$TM \xrightarrow{\nabla} TM \otimes T^*M \xrightarrow{\nabla} TM \otimes T^*M \otimes T^*M \xrightarrow{\wedge} TM \otimes \Lambda^2(M).$$

Then it is the curvature operator.

♡

Proof. We have

$$\nabla \frac{\partial}{\partial x_i} = \Gamma_{ki}^j \frac{\partial}{\partial x_j} \otimes dx_k,$$

and

$$\nabla^2 \frac{\partial}{\partial x_i} = \frac{\partial \Gamma_{ki}^j}{\partial x_\ell} \cdot \frac{\partial}{\partial x_j} \otimes dx_k \otimes dx_\ell$$
$$+ \Gamma_{ki}^j \Gamma_{\ell j}^m \frac{\partial}{\partial x_m} \otimes dx_k \otimes dx_\ell - \Gamma_{ki}^j \Gamma_{\ell m}^k \frac{\partial}{\partial x_j} \otimes dx_m \otimes dx_\ell.$$

Thus we have

$$d_{\nabla^2}\frac{\partial}{\partial x_i} = \frac{\partial \Gamma^j_{ki}}{\partial x_\ell} \cdot \frac{\partial}{\partial x_j} \otimes dx_k \wedge dx_\ell$$

$$+ \Gamma^j_{ki}\Gamma^m_{\ell j}\frac{\partial}{\partial x_m} \otimes dx_k \wedge dx_\ell - \Gamma^j_{ki}\Gamma^k_{\ell m}\frac{\partial}{\partial x_j} \otimes dx_m \wedge dx_\ell.$$

Obviously, the last term of the above equation is zero. Thus after changing indices, we have

$$d_{\nabla^2}\frac{\partial}{\partial x_i} = \left(\frac{\partial \Gamma^j_{ki}}{\partial x_\ell} - \Gamma^m_{ki}\Gamma^j_{\ell m}\right) \cdot \frac{\partial}{\partial x_j} \otimes dx_k \wedge dx_\ell.$$

By a straightforward computation, we get

$$d_{\nabla^2}\frac{\partial}{\partial x_i} = -\frac{1}{2}R_{imk\ell}g^{mj}\frac{\partial}{\partial x_k} \otimes dx_k \wedge dx_\ell,$$

where

$$R_{ijk\ell} = -g_{js}\left(\frac{\partial \Gamma^s_{ki}}{\partial x_\ell} - \frac{\partial \Gamma^s_{\ell i}}{\partial x_k} - \Gamma^m_{ki}\Gamma^s_{\ell m} + \Gamma^m_{\ell i}\Gamma^s_{km}\right).$$

■

As showed above, it is quite tedious to use local coordinates to compute the curvature. The natural frame

$$\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$$

may not be our best choice. For the rest of the lecture notes, in the most cases, we use orthonormal frames.

Let $e_1, \cdots, e_n$ be local orthonormal frame and let $\omega_1, \cdots, \omega_n$ be the dual frame. Then the Riemannian metric can be written as

$$ds^2 = \omega_1 \otimes \omega_1 + \cdots + \omega_n \otimes \omega_n = \omega_1^2 + \cdots + \omega_n^2,$$

where the tensor product is understood as the symmetric tensor product.

We write

$$\nabla_X e_j = \omega_{ij}(X)e_i,$$

where X is a vector field, and

$$R_{ijk\ell} = \langle \nabla_{e_k}\nabla_{e_\ell}e_j - \nabla_{e_\ell}\nabla_{e_k}e_j - \nabla_{[e_k,e_\ell]}e_j, e_i\rangle.$$

The Cartan's formulas are

$$\begin{cases} d\omega_j + \omega_i \wedge \omega_{ij} = 0, \\ d\omega_{ij} + \omega_{is} \wedge \omega_{sj} = \frac{1}{2}R_{ijk\ell}\,\omega_k \wedge \omega_\ell. \end{cases}$$

Exercise 6 *Prove that* $\omega_{ij} = -\omega_{ji}$.

Exercise 7 *Verify the above formulas.*

Exercise 8 *Prove that given* ω_j*, there are unique* ω_{ij} *with* $\omega_{ij} = -\omega_{ji}$ *satisfying the first Cartan equation.*

Exercise 9 *Prove that at any point* p*, one can choose an orthonormal frame such that at* p*, the connection matrix* ω_{ij} *is zero.*

Exercise 10 *Prove that, if the curvature is zero on a neighborhood, then on that neighborhood, one can choose an orthonormal frame such that the connection matrix is identically zero on that neighborhood.*

In the following we give two examples of computing the curvature using the Cartan's formula.

Example 1.1 On $\mathbb{R}^n$, we give the Riemannian metric to be

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{(1 + \frac{K}{4}\sum x_k^2)^2}.$$

Let

$$A = 1 + \frac{K}{4}\sum x_k^2.$$

and let $A_j = \frac{\partial A}{\partial x_j}$. We take

$$\omega_i = \frac{dx_i}{A}.$$

Let

$$\omega_{ij} = \frac{1}{A}(A_i\, dx_j - A_j\, dx_i).$$

Then we have

$$d\omega_j + \omega_i \wedge \omega_{ij} = 0.$$

Moreover, we have

$$d\omega_{ij} + \omega_{is} \wedge \omega_{sj} = K\, \omega_i \wedge \omega_j.$$

Thus the curvature tensor is

$$R_{ijk\ell} = K\,(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}),$$

and the sectional curvature is K.

Example 1.2 Define

$$S^n = \{(x_1, \cdots, x_{n+1}) \mid \sum x_k^2 = 1\}$$

and the Riemannian metric to be the inherited metric

$$ds^2 = dx_1^2 + \cdots + dx_n^2 + \frac{x_i x_j}{x_{n+1}^2}\, dx_i\, dx_j.$$

Define

$$\omega_i = dx_i - \frac{x_i}{1 + x_{n+1}}\, dx_{n+1}.$$

Then

$$\omega_{ij} = \frac{1}{1 + x_{n+1}}(x_i\, dx_j - x_j\, dx_i).$$

Thus we have

$$d\omega_j + \omega_i \wedge \omega_{ij} = 0,$$

and

$$d\omega_{ij} + \omega_{is} \wedge \omega_{sj} = \omega_i \wedge \omega_j.$$

Thus the sectional curvature is 1.

1.3 Covariant derivatives

The notation of connection on tensors field can be applied to co-tensors and yield the following

Lemma 1.2

Let

$$\eta = a_{i_1\cdots i_p}\omega_{i_1}\otimes\cdots\otimes\omega_{i_p}.$$

Then

$$\nabla_{e_k}\eta = \left(e_k(a_{i_1\cdots i_p}) - \sum_s a_{i_1\cdots \underset{\underset{sth}{\uparrow}}{r} \cdots\cdots i_p}\omega_{r\,i_s}(e_k)\right)\omega_{i_1}\otimes\cdots\otimes\omega_{i_p}.$$

Proof. A straightforward computation. ∎

Because of the above result, we make the following definition:

Definition 1.3

We define

$$a_{i_1\cdots i_p,k} = e_k(a_{i_1\cdots,i_p}) - \sum_s a_{i_1\cdots \underset{\underset{sth}{\uparrow}}{r} \cdots i_p}\omega_{r\,i_s}(e_k)$$

and call it the covariant derivative of the coeficients $a_{i_1\cdots i_p}$.

By the above definition, we have

$$\nabla\eta = a_{i_1\cdots i_p,k}\,\omega_k\wedge\omega_{i_1}\otimes\cdots\otimes\omega_{i_p}.$$

We have the following

Theorem 1.5

We have

$$a_{i_1\cdots i_p,k,\ell} - a_{i_1\cdots i_p,\ell,k} = a_{i_1\cdots \underset{\underset{sth}{\uparrow}}{r} \cdots i_p}R_{ri_sk\ell}.$$

Proof. To prove the above result, we introduce some notations. We let $I = (i_1,\cdots,i_p)$ and let $\omega_I = \omega_{i_1}\otimes\cdots\otimes\omega_{i_p}$. Then we have

$$\begin{aligned}&\frac{1}{2}(a_{I,k,\ell} - a_{I,\ell,k})\,\omega_k\wedge\omega_\ell\otimes\omega_I\\ &= a_{I,k,\ell}\,\omega_k\wedge\omega_\ell\otimes\omega_I\\ &= d_{\nabla^2}\eta.\end{aligned}$$

Using the above notations, we can re-write the definition of covariant derivatives as following

$$\nabla\eta = da_I - a_{I_r}\,\omega_{ri_s},$$

where

$$I_r = (i_1\cdots \underset{\underset{sth}{\uparrow}}{r} \cdots i_p).$$

We compute

$$\begin{aligned}d_{\nabla^2}\eta &= d_\nabla\left((\nabla\eta)_{I,k}\omega_k\otimes\omega_I\right)\\ &= -\left(d(\nabla\eta)_{I,k} - (\nabla\eta)_{I_r,k}\omega_{ri_s} - (\nabla\eta)_{I,r}\omega_{rk}\right)\wedge\omega_k\otimes\omega_I.\end{aligned} \tag{1.1}$$

The first and the third term of the above can be consolidated because

$$d(\nabla\eta)_{I,k}\wedge\omega_k - (\nabla\eta)_{I,r}\omega_{rk}\wedge\omega_k = d((\nabla\eta)_{I,k}\omega_k)$$

by the first Cartan's formula. Thus we have

$$d(\nabla\eta)_{I,k}\wedge\omega_k-(\nabla\eta)_{I,r}\omega_{rk}\wedge\omega_k=d(da_{I_r}-a_{I_r}\omega_{ri_s})=-da_{I_r}\wedge\omega_{ri_s}-a_{I_r}d\omega_{ri_s}.$$

Combining the above equation with (1.1), we have

$$\begin{aligned}d_{\nabla^2}\eta&=-(-da_{I_r}\wedge\omega_{ri_s}-a_{I_r}d\omega_{ri_s})\otimes\omega_I-(\nabla\eta)_{I_r,k}\omega_k\wedge\omega_{ri_s}\otimes\omega_I\\&=a_{I_r}(d\omega_{ri_s}+\omega_{r\ell}\wedge\omega_{\ell i_s})\otimes\omega_I+\sum_{s\neq t}a_{i_1\cdots\underset{tth}{\underset{\uparrow}{\mu}}\cdots\underset{sth}{\underset{\uparrow}{r}}\cdots i_p}\omega_{\mu i_t}\wedge\omega_{ri_s}\otimes\omega_I.\end{aligned}$$

For fixed $s\neq t$, we have

$$a_{i_1\cdots\underset{tth}{\underset{\uparrow}{\mu}}\cdots\underset{sth}{\underset{\uparrow}{r}}\cdots i_p}\omega_{\mu i_t}\wedge\omega_{ri_s}\otimes\omega_I+a_{i_1\cdots\underset{tth}{\underset{\uparrow}{\mu}}\cdots\underset{sth}{\underset{\uparrow}{r}}\cdots i_p}\omega_{\mu i_s}\wedge\omega_{ri_t}\otimes\omega_I=0.$$

Thus we have

$$d_{\nabla^2}\eta=\frac{1}{2}a_{I_r}R_{ri_sk\ell}\,\omega_k\wedge\omega_\ell\otimes\omega_I,$$

and the lemma is proved.

Let f be a smooth function. The following notations are often used

$$\begin{aligned}f_i&=e_i(f);\\f_{ij}&=f_{i,j};\\f_{ijk}&=f_{i,j,k}.\end{aligned}$$

We have

Corollary 1.1 (Ricci identity)

Using the above notations, we have

$$\begin{aligned}f_{ij}&=f_{ji};\\f_{ijk}-f_{ikj}&=f_rR_{rijk}.\end{aligned}$$

In particular, we have the following useful Ricci identity:

$$\sum_i f_{iji}-\sum_i f_{iij}=(Ric)_{rj}f_r.$$

Proof. We only need to prove $f_{ij}=f_{ji}$. To see this, we compute

$$f_{ij}\,\omega_j\wedge\omega_i=(df_i-f_r\omega_{ri})\wedge\omega_i=df_i\wedge\omega_i-f_r\,\omega_{ri}\wedge\omega_i.$$

By the Cartan's formula, the right hand side of the above is equal to

$$d(f_i\omega_i)=ddf=0.$$

Chapter 2 The Laplacian

2.1 The Laplace operator

Let M be a compact orientable Riemannian manifold and let $\Lambda^p(M)$ be the vector space of p forms. As in the last section, we use $\omega_1, \cdots, \omega_n$ to be the local orthonormal frame, and $\omega_1 \wedge \cdots \wedge \omega_n$ to be the volume form.

We write a p-form in the following expression

$$\eta = a_{i_1 \cdots i_p} \omega_{i_1} \wedge \cdots \wedge \omega_{i_p},$$

where we assume the coefficients are skew-symmetric. That is, we assume that

$$a_{\sigma(i_1)\cdots\sigma(i_p)} = (-1)^{sgn\,\sigma} a_{i_1\cdots i_p}$$

for any permutation σ of $\{1, \cdots, p\}$. The L^2 inner product is defined as

$$(\omega, \eta) = p! \int_M a_{i_1\cdots i_p} b_{i_1\cdots i_p} \omega_1 \wedge \cdots \wedge \omega_n. \tag{2.1}$$

We remark that in the above definition, we only require one of the coefficients to be skew-symmetric, which is useful in the following computation.

Exercise 11 *Prove the following: if*

$$\eta = b_{i_1\cdots i_p} \omega_{i_1} \wedge \cdots \wedge \omega_{i_p},$$

where $\eta = b_{i_1\cdots i_p}$ are not assumed to be skew-symmetric. Then we still have

$$(\omega, \eta) = p! \int_M a_{i_1\cdots i_p} b_{i_1\cdots i_p} \omega_1 \wedge \cdots \wedge \omega_n.$$

Lemma 2.1

We have

$$d\omega = (-1)^p a_{i_1\cdots i_p,k}\, \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge \omega_k.$$

Since the coefficients above are not skew symmetric, we may also rewrite the expression as

$$d\omega = \frac{(-1)^p}{p+1} \left(a_{i_1\cdots i_p,k} - \sum_{s=1}^{p} a_{i_1\cdots \underset{\underset{sth}{\uparrow}}{k} \cdots i_p, i_s} \right) \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge \omega_k.$$

♡

Proof. The proof is through a straightforward computation:

$$d\omega = da_{i_1\cdots i_p} \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} + \sum_{s=1}^{p} (-1)^{s-1} a_{i_1\cdots i_p} \wedge \omega_{i_1} \wedge \cdots \wedge d\omega_{i_s} \wedge \cdots \wedge \omega_{i_p}.$$

Using Cartan's formula $d\omega_{i_s} = -\omega_i \wedge \omega_{ii_s}$, we get the first formula. The second formula follows from the skew-symmetrization.

■

Lemma 2.2

Let δ be the dual operator of d with respect to the inner product (2.1). *We then have*

$$\delta\omega = (-1)^p p\, a_{i_1\cdots i_p, i_p} \omega_{i_1} \wedge \cdots \wedge \omega_{i_{p-1}}.$$

♡

Proof. Let

$$\eta = b_{i_1\cdots i_{p-1}}\omega_{i_1}\wedge\cdots\wedge\omega_{i_{p-1}}$$

be a smooth $(p-1)$-form with compact support. We verify that

$$(\delta\omega,\eta)-(\omega,d\eta)=0.$$

We have

$$(\delta\omega,\eta)=(-1)^p p!\int_M a_{i_1\cdots i_{p-1}i_p,i_p}b_{i_1\cdots i_{p-1}},$$

and

$$(\omega,d\eta)=(-1)^{p-1}p!\int_M a_{i_1\cdots i_p}b_{i_1\cdots i_{p-1},i_p}.$$

define an $(n-1)$ form α such that

$$\alpha=\sum_{k=1}^{n}(-1)^{k-1}a_{i_1\cdots i_{p-1}k}b_{i_1\cdots i_{p-1}}\,\omega_1\wedge\cdots\hat{\omega}_k\cdots\wedge\omega_n.$$

Then we have

$$d\alpha=(a_{i_1\cdots i_{p-1}i_p,i_p}b_{i_1\cdots i_{p-1}}+a_{i_1\cdots i_p}b_{i_1\cdots i_{p-1},i_p})\,\omega_1\wedge\cdots\wedge\omega_n.$$

The lemma follows from the Stokes' theorem

$$\int_M d\alpha=0.$$

■

Theorem 2.1 (Weitzenböck formula)

We have

$$\Delta\,\omega=\left(-a_{i_1\cdots i_p,k,k}+p\sum_{s=1}^{p-1}a_{i_1\cdots \underset{sth}{\underset{\uparrow}{r}}\cdots i_{p-1}k}R_{ri_si_pk}+p\,a_{i_1\cdots i_{p-1}r}(Ric)_{ri_p}\right)\omega_{i_1}\wedge\cdots\wedge\omega_{i_p}.$$

♡

Proof. By the above two lemmas we have

$$\delta d\omega=\left(-a_{i_1\cdots i_p,k,k}+\sum_{s=1}^{p}a_{i_1\cdots \underset{sth}{\underset{\uparrow}{k}}\cdots i_p,i_s,k}\right)\omega_{i_1}\wedge\cdots\wedge\omega_{i_p},$$

and

$$d\delta\omega=-p\,a_{i_1\cdots i_p,i_p,k}\omega_{i_1}\wedge\cdots\wedge\omega_{i_{p-1}}\wedge\omega_k.$$

By changing the indices and using the skew-symmetry of $a_{i_1\cdots i_p}$ in the above, we get

$$\delta d\omega=\left(-a_{i_1\cdots i_p,k,k}+pa_{i_1\cdots i_{p-1}k,i_p,k}\right)\omega_{i_1}\wedge\cdots\wedge\omega_{i_p},$$

and

$$d\delta\omega=-p\,a_{i_1\cdots k,k,i_p}\,\omega_{i_1}\wedge\cdots\wedge\omega_{i_{p-1}}\wedge\omega_{i_p}.$$

The theorem follows by applying Theorem 1.5.

■

Define the raw Laplacian $\nabla^*\nabla$ on a p-form to be

$$\nabla^*\nabla\omega=-\nabla_k\nabla_k\omega.$$

Then the Weitzenböck formula can be written as

$$\Delta\omega = \nabla^*\nabla\omega + E(\omega),$$

where E is a 0-th order differential operator depending on the curvature.

We would like to list the following special cases of the above theorem as exercises.

Exercise 12 *Using the above notations, we have*

$$\delta = (-1)^{n(p+1)+1} * d*$$

on $\Lambda^p(M)$.

Exercise 13 *Let* $\{e_1, \cdots, e_n\}$ *be an orthonormal frame. Then*

$$\delta = -\sum_{j=1}^{n} \iota(e_j)\nabla_{e_j}.$$

Exercise 14 *If f is a smooth function, then*

$$\Delta f = -f_{,k,k} = -f_{kk}.$$

Exercise 15 *If ω is a one-form, then*

$$\Delta\omega = \nabla^*\nabla\omega + a_r(Ric)_{rk}\omega_k.$$

2.2 Self-adjoint extension of the Laplace operator

In this section, we assume that M is Riemannian manifold, not necessarily compact nor complete.

Let $\Lambda^p(M)$ be the space of smooth p-forms and let $L^2(\Lambda^p(M))$ be the metric space completion of $\Lambda^p(M)$ under the inner product product defined in (2.1).

Exercise 16 *Prove that we can identify $L^2(\Lambda^p(M))$ to be the space of p-forms ω, where*

$$\omega = a_{i_1\cdots i_p}\omega_{i_1} \wedge \cdots \wedge \omega_{i_p},$$

such that $a_{i_1\cdots i_p}$ are locally L^2 integrable functions.

The reason we would like to use $L^2(\Lambda^p(M))$ in stead of $\Lambda^p(M)$ is, of course, that $L^2(\Lambda^p(M))$ is complete, allowing many applications in analysis.

Unfortunately, it is not possible to extend Δ to $L^2(\Lambda^p(M))$ as a symmetric linear operator. In what follows, we briefly explain the reason.

First, the Laplace operator is not a bounded operator on $\Lambda^p(M)$. We can give counterexamples even in one-dimensional case: let f be a smooth function on $[0, 1]$. Then there doesn't exist a constant C such that

$$\int_0^1 (f''(t))^2 dt \le C \int_0^1 (f(t))^2 dt.$$

Second, like most differential operators, the Laplace operator is a closed-graph operator. That is, if $\eta_j \to \eta$ and $\Delta\eta_j \to \eta'$ in L^2 and $\eta \in \Lambda^p(M)$, then we must have $\eta' = \Delta\eta$. To see this, we consider any smooth p form ω: We have

$$\int \langle \eta' - \Delta\eta, \omega\rangle = \lim_{j\to\infty} \int \langle \Delta\eta_j, \omega\rangle - \lim_{j\to\infty} \int \langle \eta_j, \Delta\omega\rangle = 0.$$

Thus we have $\eta' = \Delta\eta$.

If Δ could be extended to a linear operator of $L^2(\Lambda^p(M))$, then by the *Closed Graph Theorem*, Δ could have been bounded, which is a contradiction.

Because of the above result, the best we can do is to extend the Laplacian operator into a *densely defined* self-adjoint operator. Of course, $\Lambda^p(M)$ is dense in $L^2(\Lambda^p(M))$. But for such an operator (i.e, Δ and its domain $\mathcal{D}om(\Delta)$), we don't have the so-called spectral theorem. So in order to get meaningful results, we have to extend the Laplace operator first.

Definition 2.1

Let H be a Hilbert space, Given a densely defined linear operator A on H, its adjoint A^* is defined as follows:

1. *The domain $\mathcal{D}om(A^*)$ of A^* consists of vectors x in H such that*
$$y \mapsto \langle x, Ay \rangle$$
is a bounded linear functional, where $y \in \mathcal{D}om(A)$;
2. *By the Riesz Representation Theorem for linear functionals, if x is in the domain of A^*, there is a unique vector z in H such that*
$$\langle x, Ay \rangle = \langle z, y \rangle$$
for any $y \in \mathcal{D}om(A)$. This vector z is defined to be A^*x. It can be shown that the dependence of z on x is linear.

If $A^* = A$ (which implies that $\mathcal{D}om\,(A^*) = \mathcal{D}om\,(A)$), then A is called self-adjoint. ♣

For a (densely-defined) self-adjoint operator A, we have the *Spectral Theorem*. That is, there is a spectral measure such that
$$A = \int_{-\infty}^{+\infty} \lambda dE.$$

A densely defined self-adjoint operator $\bar{\Delta}$ is called an extension of Δ, if $\mathcal{D}om\,(\Delta) \subset \mathcal{D}om\,(\bar{\Delta})$, and
$$\Delta = \bar{\Delta}\,|_{\mathcal{D}om\,(\Delta)}.$$

For the rest of the section, we shall study the extensions of Δ. We shall prove that the extension of Δ always exists, but in general, they are not unique in general.

Define $H^1(M)$ to be the Sobolev space of the completion of $\Lambda^p(M)$ under the norm
$$||\eta||_1 = \sqrt{\int_M |\eta|^2 dV_M} + \sqrt{\int_M |\nabla\eta|^2\, dV_M}.$$
Similarly, let $\Lambda_0^p(M)$ be the space of smooth p-forms with compact support and define $H_0^1(M)$ to be the Sobolev space of the completion of $\Lambda_0^p(M)$ under the above norm. Then we have
$$H_0^1(M) \subset H^1(M).$$

In general, the above two spaces are not equal. However, we have

Theorem 2.2

If M is a complete Riemannian manifold, then
$$H_0^1(M) = H^1(M).$$
♡

Proof. Let $\phi \in H^1(M)$. Then there exists a sequence $\{\phi_j\} \in \Lambda^p(M)$ such that

$$\phi_j \to \phi, j \to \infty$$

in the $\| \quad \|_1$ norm.

Let $p \in M$ be a fixed point and let ρ be a smooth function on $\mathbb{R}$ with compact support such that in a neighborhood of 0, ρ is the constant 1. Let $d(x, y)$ be the distance function. Define

$$\rho_j(x) = \rho(j^{-1} d(p, x)).$$

Then we can prove that

$$\rho_j \phi_j \to \phi$$

in the $\| \quad \|_1$ norm. ■

Exercise 17 *Provide the detailed proof of the above theorem.*

We define the quadratic form Q by

$$Q(\omega, \eta) = \int_M (\langle d\omega, d\eta \rangle + \langle \delta\omega, \delta\eta \rangle) \, dV_M$$

for any $\omega, \eta \in H_0^1(M)$. By the Weitzenböck formula, we have

Proposition 2.1

If the curvature is bounded, then $\sqrt{Q(\phi, \phi) + \|\phi\|_{L^2}^2}$ *is equivalent to the* $\|\phi\|_1$.

The *Friedrichs extension* $\bar{\Delta}$ of Δ is defined by the following. Let

$$\begin{aligned} \mathcal{D}om(\bar{\Delta}) = \{ & \phi \in H_0^1(M) \mid \\ & \forall \psi \in \Lambda^p(M), \exists f \in L^2(\Lambda^p(M)), s.t. \, Q(\phi, \psi) = (f, \psi) \} . \end{aligned}$$

Lemma 2.3

Using the above notations, we have

$$\mathcal{D}om(\bar{\Delta}) = \mathcal{D}om(\bar{\Delta}^*).$$

Proof. For any $\phi, \psi \in \mathcal{D}om(\bar{\Delta})$, $(\bar{\Delta}\phi, \psi) = Q(\psi, \phi) = (\phi, \bar{\Delta}\psi)$ is a bounded functional. Thus $\phi \in \mathcal{D}om(\bar{\Delta}^*)$. On the other hand, if $\phi \in \mathcal{D}om(\Delta^*)$, then the functional $\psi \mapsto (\Delta\psi, \phi)$ is bounded. By the Riesz representation theorem, there is a unique $f \in L^2(\Lambda^p(M))$ such that $(\bar{\Delta}\psi, \phi) = (f, \psi)$. Thus we must have $Q(\phi, \psi) = (\bar{\Delta}\psi, \phi) = (f, \psi)$, and $\phi \in \mathcal{D}om(\bar{\Delta})$. ■

Theorem 2.3

The Laplace operator has a self-adjoint extension.

Theorem 2.4

Let M be a complete Riemannian manifold. Then the extension of the Laplace operator is unique.

Proof. Let Δ_2 be another self-adjoint extension of Δ. Then for $\phi \in \mathcal{D}om(\bar{\Delta})$ the functional

$$\psi \mapsto (\phi, \Delta\psi) + (\phi, \psi) = (\Delta_2\phi, \psi) + (\phi, \psi)$$

for $\psi \in \Lambda_0^p(M)$ is bounded under the norm $\| \quad \|_1$. By Proposition 2.1, it is also bounded under the norm

$$\sqrt{Q(\cdot, \cdot) + (\cdot, \cdot)}.$$

By the Riesz representation theorem, there exists an element $\mu \in H^1(M)$ such that

$$(\phi, \Delta\psi) + (\phi, \psi) = Q(\mu, \psi) + (\mu, \psi).$$

Therefore $\phi = \psi \in H^1(M)$. Since M is complete, By Theorem 2.2, $\phi \in H_0^1(M)$. For any $\psi \in \Lambda_0^p(M)$, we have

$$(\Delta_2\phi, \psi) = (\phi, \Delta\psi) = Q(\phi, \psi)$$

and by taking limit, the above equality is also true for any $\psi \in H_0^1(M)$. Thus we prove

$$\mathcal{D}om\,(\Delta_2) \subset \mathcal{D}om\,(\bar{\Delta}).$$

On the other hands, since both $\bar{\Delta}$ and Δ_2 are self-adjoint, we have

$$\mathcal{D}om\,(\bar{\Delta}) = \mathcal{D}om\,(\bar{\Delta}^*) \subset \mathcal{D}om\,(\Delta_2^*) = \mathcal{D}om\,(\Delta_2),$$

and thus we prove

$$\mathcal{D}om\,(\bar{\Delta}) = \mathcal{D}om\,(\Delta_2).$$

Thus $\Delta_2 = \bar{\Delta}$, and the extension is unique. ■

In general, the self-adjoint extension is not unique.

Example 2.1 Let M be a compact orientable manifold with smooth boundary. Let $\mathcal{C}_0^\infty(M)$ be the space of smooth functions whose supports are within the interior of M, and let $\mathcal{C}^\infty(M)$ be the space of smooth functions (smooth up to the boundary). We can define two kinds of Laplacians: the first one Δ_D is called the Dirichlet Laplacian, which is the same as the Friedrichs extension $\bar{\Delta}$. The second one Δ_N is called the Neumann Laplacian, defined by

$$\mathcal{D}om(\Delta_N) = \Big\{\phi \in H^1(M) \mid \forall \psi \in \mathcal{C}^\infty(M)\ s.t. \frac{\partial \psi}{\partial n} = 0, \exists f \in L^2(M), s.t. Q(\phi, \psi) = (f, \psi)\Big\}.$$

For $\phi, \psi \in \mathcal{C}^\infty(M)$, we have the Green's formula

$$(\Delta\phi, \psi) = \int_{\partial M} \frac{\partial \phi}{\partial n}\psi - \int_M \phi\,\frac{\partial \psi}{\partial n} + (\phi, \Delta\psi).$$

Therefore, if $\phi \in \mathcal{D}om(\Delta_D) \cap \mathcal{C}^\infty(M)$, we have $\phi|_{\partial M} = 0$, and if $\phi \in \mathcal{D}om(\Delta_N) \cap \mathcal{C}^\infty(M)$, we have $\frac{\partial \phi}{\partial n} = 0$. These justify the names.

Many results related to the Laplace operator are true even on noncomplete manifolds.

Theorem 2.5

Let $f \in \ker(\bar{\Delta})$. Assume that $f \in L^2(M)$. Then f is a constant. ♡

This is a theorem of Yau in the case of complete manifold. But we shall prove that it is also true for any noncompete manifold.

Proof. Let ρ_j be the smooth function defined in Theorem 2.2. Then we have

$$0 = (\bar{\Delta} f, \rho_j^2 f) = Q(f, \rho_j^2 f).$$

Thus we have

$$\int_M \rho_j^2 |\nabla f|^2 = -2 \int_M \rho_j f \nabla \rho_j \nabla f \leq 2 \sqrt{\int_M |\nabla \rho_j|^2 f^2 \, dV_M} \cdot \sqrt{\int_M \rho_j^2 |\nabla f|^2 \, dV_M}.$$

It follows that

$$\int_M \rho_j^2 |\nabla f|^2 \, dV_M \leq \frac{C}{j^2} \int_M f^2 \, dV_M \to 0$$

and the theorem is proved. ■

Example 2.2 Let $M = \mathbb{R}^n - \{0\}$. Then any L^2 harmonic function extends to the origin. Thus by the theorem of Yau, we know that it has to be a constant. Let M be a bounded domain. Then the fact that f is in the domain of the Laplace operator implies that f vanishes on the boundary. Hence f is zero.

2.3 Variational characterization of spectrum

Let Δ be the Laplace operator on functions of a manifold M. As in the last section, we make the convention that the operator is a negative operator.

Let λ be a complex number, $(\lambda I - \Delta)^{-1}$ is called the resolvent of Δ. If for some λ, $(\lambda I - \Delta)^{-1}$ doesn't exist, or it does exist but is an unbounded operator, then we call λ spectrum point of Δ. We use $\sigma(\Delta)$ or $\operatorname{Spec}(\Delta)$ to denote the set.

Theorem 2.6

Using the above notations, we have

$$\operatorname{Spec}(\Delta) \neq \emptyset.$$

♡

Proof. This is a general fact about the spectrum of any linear operator on a Banach space. We assume that Δ is densely defined in a Banach space $\mathcal{B}$. Assume that $\operatorname{Spec}(\Delta) = \emptyset$. Let $f \in \mathcal{B}$ and let ℓ be abounded linear functional on $\mathcal{B}$, then

$$\ell((\lambda I - \Delta)^{-1} f)$$

is a bounded holomorphic function of $\mathbb{C}$, which has to be a constant. More over, it must be the zero function. Since ℓ and f are arbitrary, we conclude that $(\lambda I - \Delta)^{-1} = 0$ which is not possible for all λ. This is a contradiction. ■

When M is a compact manifold, the best possible things happen: by the Hodge theorem, the spectrum of Δ is made from eigenvalues. More precisely, we have

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_k \to +\infty,$$

such that for each λ_j, the space

$$E_j = \{f \mid \Delta f = -\lambda_j f\}$$

is not trivial and is of finite dimensional. By the above result, we can prove the following variational (or min-max) principal.

Theorem 2.7

We have

$$\lambda_k = \inf\left\{ \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2} \;\middle|\; f|_{\partial\Omega} = 0,\ f \not\equiv 0 = \int_\Omega f\phi_j,\ \forall j < k \right\}, \tag{2.2}$$

where φ_j are the eigenfunctions of λ_j.

Proof. Let $\{f_\alpha\}$ be a sequence such that

$$\int_M f_\alpha \phi_j = 0, \qquad \forall j < k,$$

and

$$\frac{\int_M |\nabla f_\alpha|^2}{\int_M f_\alpha^2} \to \lambda_k.$$

If we normalize f_α such that $\int_M f_\alpha^2 = 1$, then the sequence $\{f_\alpha\}$ is bounded in $H_1(M)$. Therefore, there is an $f \in H_1(M)$ such that

$$f_\alpha \to f$$

in the weak sense. In particular, for any ϕ smooth, we have

$$\int_M \nabla f_\alpha \nabla \phi \to \int_M \nabla f \nabla \phi.$$

We have

$$\left| \int_M \nabla f \nabla \phi \right| \le \int_M \nabla f_\alpha \nabla \phi \le ||\nabla f_\alpha||_{L^2} ||\nabla \phi||_{L^2} \to \sqrt{\lambda_k} ||\nabla \phi||_{L^2}.$$

Since smooth functions are dense in $H_1(M)$, we have

$$\int_M |\nabla f|^2 \le \liminf_{k\to\infty} \int_M |\nabla f_\alpha|^2,$$

which is known as the Fatou Lemma.

On the other hand, by the Rellich Lemma, $f_\alpha \to f$ strongly in $L^2(M)$. Thus we have

$$\int_M f^2 = 1,$$

hence

$$\lambda_k = \frac{\int_M |\nabla f|^2}{\int_M f^2}$$

Let ϕ be a smooth function such that

$$\int_M \phi\phi_j = 0$$

Then for any ε, we must have

$$\frac{\int_M |\nabla f + \varepsilon \nabla \phi|^2}{\int_M (f + \varepsilon\phi)^2} \ge \frac{\int_M |\nabla f|^2}{\int_M f^2}$$

Since this is true for any ε, we must have

$$\Delta f = -\lambda_k f$$

in the weak sense. By the elliptic regularity, f has to be smooth. ∎

Unlike in the case of compact manifold, in general, a complete non-compact manifold doesn't admit any pure point spectrum. For example, there are no L^2-eigenvalues on $\mathbb{R}^n$. That is, for any $\lambda \in \mathbb{R}$, if $\Delta f + \lambda f = 0$

and $f \in L^2(\mathbb{R}^n)$, then we have $f \equiv 0$.

Let Δ be the Laplace operator on a complete non-compact manifold M. We extend Δ naturally to a self-adjoint densely defined operator on $L^2(M)$, which we still denote as Δ for the sake of simplicity.

The pure point spectrum of Δ are those $\lambda \in \mathbb{R}$ such that

1. there exists an L^2 function $f \neq 0$ such that
$$\Delta f + \lambda f = 0.$$
2. the multiplicity of λ is finite;
3. in a neighborhood of λ, it is the only spectrum point.

From the above discussion, $\sigma(\Delta)$ decomposes as the union of pure point spectrum, and the so-called essential spectrum, which is, by definition, the complement of the pure point spectrum.

The set of the essential spectrum is denoted as $\sigma_{ess}(\Delta)$. Using the above definition, $\lambda \in \sigma_{ess}(\Delta)$, if either

1. λ is an eigenvalue of infinite multiplicity, or
2. λ is the limiting point of $\sigma(\Delta)$.

The following theorems in functional analysis are well-known.

Theorem 2.8

A necessary and sufficient condition for the interval $(-\infty, \lambda)$ to intersect the essential spectrum of an self-adjoint densely defined operator A is that, for all $\varepsilon > 0$, there exists an infinite dimensional subspace $G_\varepsilon \subset \mathcal{D}om(A)$, for which $(Af - \lambda f - \varepsilon f, f) < 0$.

♡

Theorem 2.9

A necessary and sufficient condition for the interval $(\lambda - a, \lambda + a)$ to intersect the essential spectrum of A is that there exists an infinite dimensional subspace $G_\varepsilon \subset \mathcal{D}om(A)$ for which $||(A - \lambda I)f|| \leq a||f||$ for all $f \in G_\varepsilon$.

♡

Using the above result, we give the following variational characterization of the lower bound of spectrum and the lower bound of essential spectrum.

Theorem 2.10

Using the above notations, define
$$\lambda_0 = \inf_{f \in C_0^\infty(M)} \frac{\int_M |\nabla f|^2}{\int_M f^2},$$
and
$$\lambda_{ess} = \sup_K \inf_{f \in C_0^\infty(M \setminus K)} \frac{\int_M |\nabla f|^2}{\int_M f^2},$$
where K is a compact set running through an exhaustion of the manifold. Then λ_0 and λ_{ess} are the least lower bound of $\sigma(\Delta)$ and $\sigma_{ess}(\Delta)$, respectively.

♡

Proof. We prove the formula for λ_{ess}. The formula for λ_0 is similar. Let
$$\lambda'_{ess} = \inf \sigma_{ess}(\Delta).$$
If $\sigma_{ess}(\Delta) = \emptyset$, we define $\lambda'_{ess} = +\infty$. When $\lambda'_{ess} = +\infty$, we prove that λ_{ess} is also infinity.

Assume not, then there is a constant C, such that for any compact set K, there is a function

$f \in \mathcal{C}_0^\infty(M \backslash K)$, we have

$$\frac{\int_M |\nabla f|^2}{\int_M f^2} < C.$$

By inductively choosing f, we can make sure that the support of these functions are disjoint. Thus the functions span an infinite dimensional space G with

$$(-\Delta f - Cf, f) < 0,$$

which contradicts to the fact that the essential spectrum is an empty set.

The proof of the general case is similar. We fist prove that $\lambda_{ess} \leq \lambda'_{ess}$ By definition, for $\varepsilon > 0$,

$$(-\infty, \lambda'_{ess} + \varepsilon) \cap \sigma_{ess}(\Delta) \neq \emptyset.$$

Therefore we can find infinite dimensional space V such that for any $f \in V$

$$(-\Delta f - (\lambda'_{ess} + \varepsilon) f, f) < 0.$$

In what follows, we shall prove that, using the cut-off functions, there exist infinitely many elements in V such that their supports are disjoint.

We assume that K is a compact set. Let K' be a large ball containing K. Let ρ be the cut-off function such that $\rho = 1$ on K but $\rho = 0$ outside K'. We claim that for any $\varepsilon > 0$, there is an $f \in V$ with $\int f^2 = 1$ but

$$\int_M \rho^2 f^2 < \varepsilon.$$

If the above is not true, then there is an $\varepsilon_0 > 0$ such that for any $f \in V$,

$$\int_M \rho^2 f^2 \geq \varepsilon_0,$$

Since the set $f \in V$ is of infinite dimensional, the set ρf is of infinite dimensional as well, otherwise the above inequality is not valid. Thus we can find an orthogonal basis

$$\int_M \rho^2 f_i f_j = 0,$$

if $i \neq j$, while we still keeping $\int_M f_i^2 = 1$. We consider

$$\int_M |\nabla(\rho f_i)|^2 \leq 2 \int_M |\nabla \rho|^2 f_i^2 + 2 \int_M |\nabla f_i|^2.$$

By $\int_M |\nabla f_i|^2 \leq \lambda'_{ess} + \varepsilon$, we have Thus we have

$$\frac{\int_M |\nabla(\rho f_i)|^2}{\int_M (\rho f_i)^2} \leq \frac{2C + 2(\lambda'_{ess} + \varepsilon)}{\varepsilon_0}$$

for any i. This is a contradiction because on the compact set K', the are only finitely many eigenvalues (counting multiplicity) below a fixed number.

Now we can prove our theorem. For all $\varepsilon > 0$. We can find an f with $\int_M f^2 = 1$ but

$$\int_M \rho^2 f^2 < \varepsilon.$$

We take $K = B(r)$ and $K' = B(R+1)$ of balls of radius $r, R+1$, respectively. Both r, R are large numbers. Let $\tilde{\rho}$ be a cut-off function such that

$$\{\tilde{\rho} = 1\} \subset B(R).$$

We also assume that $|\nabla\tilde{\rho}| \le C$. Let $\rho_1 = 1 - \tilde{\rho}$. Then we have

$$\operatorname{supp}\rho_1 \cap K = \emptyset.$$

We have

$$\int_M |\nabla(\rho_1 f)|^2 = \int_M \rho_1^2 |\nabla f|^2 + 2\int_M \rho_1 f \nabla\rho_1 \nabla f + \int_M f^2 |\nabla\rho_1|^2.$$

From

$$\int_M f^2|\nabla\rho_1|^2 \le C\varepsilon,$$

and

$$2\int_M \rho_1 f\nabla\rho_1\nabla f \le 2\sqrt{\int_M |\nabla(\rho_1 f)|^2}\cdot\sqrt{\int_M |\nabla\rho_1|^2 f^2} \le C\sqrt{\varepsilon}\sqrt{\int_M |\nabla(\rho_1 f)|^2},$$

we get

$$\int_M |\nabla(\rho_1 f)|^2 \le \lambda'_{ess} + C\varepsilon.$$

Since

$$\int_M \rho_1^2 f^2 \ge 1 - C\varepsilon,$$

we get

$$\lambda_{ess} \le \frac{\lambda'_{ess} + C\varepsilon}{1 - C\varepsilon}.$$

and thus $\lambda_{ess} \le \lambda'_{ess}$. The other direction is easier to prove.

∎

It is particularly interesting to get the lower bound estimate for the essential spectrum because of the following theorem.

Theorem 2.11 (Variational principal)

Suppose $\lambda_0 < \lambda_{ess}$, then λ_0 is en eigenvalue of M with finite dimensional eigenspace. ♡

That is, there exists an L^2 function $f \neq 0$, such that

$$\Delta f = -\lambda_0 f,$$

which is a very strong result.

Example 2.3 Let $H = -\Delta + |x|^2$ on $\mathbb{R}^n$. Then the set of essential spectrum is empty. As a result, all spectrum of H are eingevalues of finite multiplicity.

2.4 The new Weyl criteria

Let $\mathfrak{X}$ be Hilbert space. Let $\mathscr{B}(\mathbb{C})$ and let $\mathcal{P}(X)$ be the space of self-adjoint projection operators. A spectral measure is a function

$$E : (\mathbb{C}) \to (X)$$

satisfying the following properties

1. $E(\emptyset) = 0, E(\mathbb{C}) = I$;
2. $E(\bigcup B_n) = \sum E(B_n)$ for disjoint Borel sets $\{B_n\}$.

Let f be a measurable function and let $x, y \in X$. Define the operator

$$A = \int_{-\infty}^{\infty} f \, dE$$

such that

$$(Ax, y) = \int_{-\infty}^{\infty} f \, dE_{(x,y)}.$$

Let H be a self-adjoint operator on a Hilbert space X. The norm and inner product in X are respectively denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Let $\sigma(H), \sigma_{\text{ess}}(H)$ be the spectrum and the essential spectrum of H, respectively. Let $\mathrm{Dom}\,(H)$ be the domain of H. The Classical Weyl criterion states that

Theorem 2.12 (Classical Weyl's criterion)

A nonnegative real number λ belongs to $\sigma(H)$ if, and only if, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}(H)$ such that

1. $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1,$
2. $(H-\lambda)\psi_n \to 0,$ *as* $n \to \infty$ *in* $\mathcal{H}$.

Moreover, λ belongs to $\sigma_{\text{ess}}(H)$ of H if, and only if, in addition to the above properties

3. $\psi_n \to 0$ *weakly as* $n \to \infty$ *in* $\mathcal{H}$.

♡

We have the following functional analytic result, which generalizes the weak Weyl criterion. To the authors' knowledge, this result seems to be new.

Theorem 2.13 (Charalambous-Lu)

Let f be a bounded positive continuous function over $[0,\infty)$. A nonnegative real number λ belongs to the spectrum $\sigma(H)$ if, and only if, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}(H)$ such that

1. $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1,$
2. $(f(H)(H-\lambda)\psi_n, (H-\lambda)\psi_n) \to 0,$ *as* $n \to \infty$ *and*
3. $(\psi_n, (H-\lambda)\psi_n) \to 0,$ *as* $n \to \infty$.

Moreover, λ belongs to $\sigma_{\text{ess}}(H)$ of H if, and only if, in addition to the above properties

4. $\psi_n \to 0,$ *weakly as* $n \to \infty$ *in* $\mathcal{H}$.

♡

Proof. Since H is a densely defined self-adjoint operator, the spectral measure E exists and we can write

$$H = \int_0^{\infty} \lambda \, dE. \tag{2.3}$$

Assume that $\lambda \in \sigma(H)$. Then by Weyl's criterion, there exists a sequence $\{\psi_n\}$ such that

$$\|(H-\lambda)\psi_n\| \to 0, \quad \|\psi_n\| = 1$$

as $n \to \infty$.

We write

$$\psi_n = \int_0^{\infty} dE(t)\psi_n$$

as its spectral decomposition. Then

$$(f(H)(H-\lambda)\psi_n, (H-\lambda)\psi_n) = \int_0^{\infty} f(t)(t-\lambda)^2 d\|E(t)\psi_n\|^2.$$

Since f is a bounded positive function, we have

$$(f(H)(H-\lambda)\psi_n,(H-\lambda)\psi_n)\leq C\int_0^\infty(t-\lambda)^2d\|E(t)\psi_n\|^2=C\|(H-\lambda)\psi_n\|^2.$$

Moreover,

$$(\psi_n,(H-\lambda)\psi_n)\leq C\,\|\psi_n\|\cdot\|(H-\lambda)\psi_n\|.$$

Thus the necessary part of the theorem is proved.

Now assume that $\lambda>0$ and $\lambda\notin\sigma(H)$. Then there is a $\lambda>\varepsilon>0$ such that $E(\lambda+\varepsilon)-E(\lambda-\varepsilon)=0$. We write

$$\psi_n=\psi_n^1+\psi_n^2, \tag{2.4}$$

where

$$\psi_n^1=\int_0^{\lambda-\varepsilon}dE(t)\psi_n,$$

and $\psi_n^2=\psi_n-\psi_n^1$.

Then

$$\begin{aligned}&(f(H)(H-\lambda)\psi_n,(H-\lambda)\psi_n)\\&=(f(H)(H-\lambda)\psi_n^1,(H-\lambda)\psi_n^1)+(f(H)(H-\lambda)\psi_n^2,(H-\lambda)\psi_n^2)\\&\geq c_1\|\psi_n^1\|^2+(f(H)(H-\lambda)\psi_n^2,(H-\lambda)\psi_n^2)\geq c_1\|\psi_n^1\|^2,\end{aligned}$$

where the positive number c_1 is the infimum of the function $f(t)(t-\lambda)^2$ on $[0,\lambda-\varepsilon]$. Therefore

$$\|\psi_n^1\|\to 0$$

by *(2)*. On the other hand, we similarly get

$$(\psi_n,(H-\lambda)\psi_n)\geq\varepsilon\|\psi_n^2\|^2-\lambda\|\psi_n^1\|^2.$$

If the criteria *(2), (3)* are satisfied, then, by the two estimates above, we conclude that both ψ_n^1,ψ_n^2 go to zero. This contradicts $\|\psi_n\|=1$, and the theorem is proved.

Note that for $\lambda=0$, ψ_n^1 is automatically zero, and the second half of the proof would give the contradiction.

Theorem 2.14 (Charalambous-Lu)

Let $\sigma_p(M)$ be the spectrum of the Laplacian on p-forms. If $\lambda\in\sigma_p(\Delta)$ and $\lambda\neq 0$, then $\lambda\in\sigma_{p-1}(\Delta)$ or $\lambda\in\sigma_{p+1}(\Delta)$.

Remark If M is a compact manifold, then the result is trivially true: if $f\neq 0$ is the eigenform of the eigenvalue λ.

$$\Delta f-\lambda f=0.$$

Since $d\Delta=\Delta d$, $\delta\Delta=\Delta\delta$, we have

$$\Delta df-\lambda df=0,\quad \Delta\delta f-\lambda\delta f=0.$$

Since $\lambda\neq 0$, we have either $df\neq 0$, or $\delta f\neq 0$. The result follows.

Proof. If $0\neq\lambda\in\sigma_p(\Delta)$, then for any $\varepsilon>0$, there exists $f\neq 0$ smooth with compact support, such that

$$\|\Delta f - \lambda f\|_{L^2} \le \varepsilon \|f\|_{L^2}.$$

We then have

$$\left| \|df\|_{L^2}^2 + \|\delta f\|_{L^2}^2 - \lambda \|f\|_{L^2}^2 \right| = |(f, \Delta f - \lambda f)| \le \varepsilon \|f\|_{L^2}^2.$$

In particular, if ε is small enough, we must have either

$$\|df\|_{L^2}^2 \ge \frac{\lambda}{4} \|f\|_{L^2}^2,$$

or

$$\|\delta f\|_{L^2}^2 \ge \frac{\lambda}{4} \|f\|_{L^2}^2.$$

For the rest of the proof, we assume that the former is correct. Using our new Weyl Criterion, we need to prove

$$(df, (\Delta - \lambda) df)| \le \varepsilon \|f\|_{L^2}^2.$$

But this follows from

$$|(\delta df, (\Delta - \lambda) f)| \le \|(\Delta - \lambda) f\|_{L^2} \cdot \|\delta df\|_{L^2},$$

and the fact that

$$\int_M |\Delta f|^2 = \int_M |df|^2 + \int_M |\delta f|^2 \ge \int_M |df|^2.$$

Similarly, we can prove that for function $g(t) = (t+1)^{-1}$, we have

$$|(g(\Delta)(\Delta - \lambda) df, (\Delta - \lambda) df)| \le \varepsilon \|df\|_{L^2}^2.$$

■

Corollary 2.1

Let X be a Kähler manifold and let $L \to X$ be a positive line bundle. Let Δ_0, Δ_1 be the Laplacians of L^m-valued sections and $(0,1)$-forms, respectively. We assume that both of them are self-adjoint. Then

$$\sigma_0(\Delta_0) \subset \{0\} \cup \sigma_1(\Delta_1).$$

Moreover, if $0 \in \sigma_0(\Delta_0)$, then holomorphic sections exists.

♡

2.5 Poincaré inequality and Sobolev inequality

Let M be a compact manifold without boundary (we call such a manifold *closed*). Then by the Hodge theorem, for any function f such that

$$\int_M f = 0,$$

we have

$$\int_M |\nabla f|^2 \ge C \int_M f^2$$

for a positive constant C. For manifold with boundary, we have similar versions of Poincaré inequalities with respect to the boundary conditions.

The other fundamental inequality is the Sobolev inequality

Sobolev inequality. Assume that M is a compact manifold with boundary, then there is a constant C such that

$$C\left(\int_M f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le \int_M |\nabla f| \tag{2.5}$$

for any smooth functions satisfies the Dirichlet or Neumann boundary conditions.

The Soboleve inequality is equivalent to the so-called isoperimetric inequality:

Isoperimetric inequality. Let Ω be a domain of M which is relatively compact. Then there is a constant C such that

$$C(\mathrm{Vol}\,(\Omega))^{\frac{n-1}{n}} \le \mathrm{Vol}\,(\partial\Omega). \tag{2.6}$$

To prove the equivalence, we first assume the Sobolev inequality (2.5). We take the function

$$f_\varepsilon(x) = \begin{cases} 1, & x \in \Omega, d(x, \partial\Omega) \ge \varepsilon, \\ \frac{d(x,\partial\Omega)}{\varepsilon}, & x \in \Omega, d(x, \partial\Omega) \le \varepsilon, \\ 0 & otherwise \end{cases}$$

Using the Soboleve inequality on f_ε and lettting $\varepsilon \to 0$, we get the isoperimetric inequality. On the other hand, we have the following co-area formula

Theorem 2.15 (Co-area formula)

Let M be a compact manifold with boundary. $f \in H^1(M)$. Then for any nonnegative function g on M, we have

$$\int_M g = \int_{-\infty}^{\infty} \left(\int_{\{f=\sigma\}} \frac{g}{|\nabla f|}\right) d\sigma.$$

♡

Exercise 18 *Proof the above co-area formula.*

For the sake of simplicity, we assume that $f \ge 0$ and we assume that the isoperimetric inequality (2.6) is valid. By the co-area formula, we have

$$\int_M |\nabla f| = \int_0^\infty \mathrm{Area}\,(f=\sigma)d\sigma.$$

At the same time we have

$$\int_M |f|^{\frac{n}{n-1}} = \int_0^\infty \mathrm{Vol}\,(f^{\frac{n}{n-1}} > \lambda)d\lambda = \frac{n}{n-1}\int_0^\infty \mathrm{Vol}\,(f>\sigma)\sigma^{\frac{n}{n-1}}d\sigma.$$

Using the isoperimetric inequality, we have

$$\int_M |\nabla f| \ge C\int_0^\infty \mathrm{Vol}\,(f>\sigma)^{\frac{n-1}{n}}d\sigma.$$

Therefore, we just need to prove that

$$\int_0^\infty \mathrm{Vol}\,(f>\sigma)^{\frac{n-1}{n}}d\sigma \ge C\left(\int_0^\infty \mathrm{Vol}\,(f>\sigma)\sigma^{\frac{1}{n-1}}d\sigma\right)^{\frac{n-1}{n}}.$$

Let

$$\begin{aligned} F(\sigma) &= \mathrm{Vol}\,(f>\sigma), \\ \phi(t) &= \int_0^t F(\sigma)^{\frac{n-1}{n}}d\sigma, \\ \psi(t) &= \left(\int_0^t F(\sigma)\sigma^{\frac{1}{n-1}}d\sigma\right)^{\frac{n-1}{n}}. \end{aligned}$$

Then $\phi(0) = \psi(0)$. It is not hard to see that $\phi'(t) \ge \frac{n}{n-1}\psi'(t)$. Thus $\phi(\infty) \ge \frac{n}{n-1}\psi(\infty)$.

QED

Chapter 3 Gradient estimates

3.1 Comparison Theorems

Let X be a Riemannian manifold with the Riemannian metric ds^2. Let $\omega_1, \dots, \omega_n$ be one forms such that

$$ds^2 = \omega_1^2 + \cdots + \omega_n^2.$$

Then by the theorem of Cartan, there are one forms $\{\omega_{ij}\}$ with $\omega_{ij} = -\omega_{ji}$, called the connection forms, such that

$$\begin{cases} d\omega_i = -\omega_{ij} \wedge \omega_j \\ d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj} + \frac{1}{2} R_{ijkl}\, \omega_k \wedge \omega_l \end{cases}.$$

The tensor R_{ijkl} is called the curvature tensor of the Riemannian metric ds^2.

Let $U \subset \mathbb{R}^{n-1}$ be an open set. Let $\eta_2, \dots, \eta_n$ be smooth one forms on U such that

$$(ds^1)^2 = \eta_2^2 + \cdots + \eta_n^2$$

defines a Riemannian metric on U. Let $\theta_2, \dots, \theta_n$ be local coordinates of U. Let $f(r, \theta)$ be a positive function on $\mathbb{R}^+ \times U$. Define a Riemannian metric

$$ds^2 = dr^2 + f^2(r, \theta) \sum_{i=2}^{n} \eta_i^2$$

on $\mathbb{R}^+ \times U$.

Example of the above setting appears in the Euclidean metric under polar coordinates. For example, in $\mathbb{R}^2$, the Euclidean metric can be written as

$$ds^2 = dr^2 + r^2 d\theta^2.$$

For this particular example, of course, the curvature of the above metric is zero. We shall see that, even in the more general setting, certainly components of the curvature tensor appear to be quite simple.

We let

$$\begin{aligned} \omega_1 &= dr \\ \omega_k &= f(t, \theta)\eta_k \end{aligned}$$

for $k > 1$. Then

$$ds^2 = \omega_1^2 + \cdots + \omega_n^2.$$

Let $e_1, \dots, e_n$ be the dual basis of $\omega_1, \dots, \omega_n$, and let η_{kl} be the connection form for $(ds^1)^2$. Define

$$\begin{aligned} \omega_{1l} &= -e_1(\log f)\omega_l; \\ \omega_{kl} &= \eta_{kl} - e_k(\log f)\omega_l + e_l(\log f)\omega_k \end{aligned}$$

for $k, l > 1$. A straightforward computation gives

$$d\omega_i = -\omega_{ij} \wedge \omega_j.$$

Note that $e_1 = \frac{\partial}{\partial r}$. We claim that

$$d\omega_{1l} + \omega_{1k} \wedge \omega_{kl} = -\frac{e_k\left(\frac{\partial f}{\partial r}\right)}{f} \omega_k \wedge \omega_l.$$

Verification: Since

$$\omega_{1l} = -\frac{\partial}{\partial r}(lgf)\omega_l,$$

we have

$$d\omega_{1l} = -e_k\left(\frac{\partial}{\partial r}(\log f)\right)\omega_k\wedge\omega_l - \frac{\partial}{\partial r}(\log f)d\omega_l.$$

On the other hand, we have

$$\begin{aligned}&\omega_{1k}\wedge\omega_{kl}\\ &\quad= -\frac{\partial}{\partial r}(\log f)\omega_k\wedge(\eta_{kl} - e_k(\log f)\omega_l + e_l(\log f)\omega_k)\\ &\quad= \frac{\partial}{\partial r}d\eta_l + \frac{\partial}{\partial r}(\log f)e_k(\log f)\omega_k\wedge\omega_l.\end{aligned}$$

Since

$$e_m\left(\frac{\partial}{\partial r}\log f\right) = f^{-1}e_m\frac{\partial f}{\partial r}(\log f) - e_m(\log f)\frac{\partial}{\partial r}(\log f),$$

we have

$$d\omega_{1l} + \omega_{1k}\wedge\omega_{kl} = -\frac{e_m\left(\frac{\partial f}{\partial r}\right)}{f}\omega_m\wedge\omega_l.$$

The claim is proved. By the Cartan's formula, we have

$$R_{1lmn} = -\delta_{ln}\frac{e_m\left(\frac{\partial f}{\partial r}\right)}{f} + \delta_{lm}\frac{e_n\left(\frac{\partial f}{\partial r}\right)}{f}.$$

The Ricci curvature at $\frac{\partial}{\partial r}$ direction is

$$R_{11} = R_{1l1l} = -(n-1)\frac{1}{f}\frac{\partial^2 f}{\partial r^2}.$$

In general, A Riemannian metric under the polar coordinates can be written as

$$ds^2 = (dr)^2 + h_{ij}(r,\theta)d\theta_i d\theta_j.$$

In order to compute/estimate the curvature, we use Uhlenbeck's trick which we introduce now.

Let $\omega_1 = dr$ and $\omega_2,\ldots,\omega_n$ be defined such that

$$h_{ij}d\theta_i d\theta_j = \sum_{j>1}\omega_j^2.$$

We write

$$\frac{\partial\omega_j}{\partial r} = \sum_{j>1}a_{ij}\omega_j$$

for $i>1$ for some matrix valued function (a_{ij}). The Uhlenbeck's trick is that, by an orthogonal change of the co-frame, we can make (a_{ij}) a symmetric matrix.

To see this, we write the above equations into matrix form. Let

$$\omega = \begin{pmatrix}\omega_2\\ \vdots\\ \omega_n\end{pmatrix},\quad A = (a_{ij}).$$

Then

$$\frac{\partial\omega}{\partial r} = A\omega.$$

Let $A = B + C$ be the decomposition of the matrix A into symmetric and skew-symmetric parts. Let Q be an orthogonal matrix-valued function. We solve the equation

$$\frac{\partial Q}{\partial r} = -QC$$

with the initial value $Q(0) = I$. It is not hard to verify that Q are orthogonal matrices: we have

$$\frac{\partial(Q^T Q)}{\partial r} = -\frac{\partial Q^T}{\partial r}Q + Q^T\frac{\partial Q}{\partial r} = 0$$

using the differential equation and the fact that C is skew-symmetric.

Changing the co-frame from ω to $Q\omega$, we have

$$\frac{\partial}{\partial r}(Q\omega) = QBQ^T(Q\omega).$$

Since QBQ^T is symmetric, up to an orthogonal change of the co-frame, we may assume thst the matrix A is symmetric. This proves the Uhlenbeck's trick.

Now back to the computation of the curvature. Let $(\eta_{ij})(i, j > 1)$ be the connection forms of the Riemannian metric

$$h_{ij}d\theta_i d\theta_j.$$

Define

$$\begin{aligned} \omega_{i1} &= \frac{\partial \omega_i}{\partial r}; \\ \omega_{ij} &= \eta_{ij} \end{aligned}$$

for $i, j > 1$. Then we have

$$d\omega_i = -\sum_{j=1}^{n}\omega_{ij}\wedge\omega_j$$

for $i > 1$. On the other hand, since A is symmetric, we have

$$\omega_{i1}\wedge\omega_i = 0.$$

Therefore, $d\omega_1 = 0 = -\omega_{1j}\wedge\omega_j$. Thus (ω_{ij}) is connection forms of $(ds)^2$ with respect to the co-frame $(\omega_1, \cdots, \omega_n)$.

We compute

$$d\omega_{i1} + \omega_{ik}\wedge\omega_{k1} = \frac{1}{2}R_{i1kl}\omega_k\wedge\omega_l.$$

If we only count the terms of the form $\{dr\wedge\omega_i\}$ in the above equation, we have

$$dr\wedge\frac{\partial w_{i1}}{\partial r} = R_{i11j}\omega_1\wedge\omega_j.$$

Thus we have

$$\mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = R_{11} = \frac{\partial}{\partial r}\left(\sum_{i>1}a_{ii}\right) + \sum_{i,j>1}a_{ij}^2.$$

On the other hand, we have

$$\frac{\partial}{\partial r}(\omega_1\wedge\cdots\wedge\omega_n) = (\sum_{i>1}a_{ii})\omega_1\wedge\cdots\wedge\omega_n.$$

Therefore we have

$$\frac{\partial}{\partial r}\sqrt{\det h_{ij}} = \sum a_{ii}\sqrt{\det h_{ij}}.$$

Using the Cauchy inequality, we have

$$\mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \geq \frac{\partial^2}{\partial r^2}\sqrt{\det h_{ij}} + \frac{1}{n-1}\left(\frac{\partial}{\partial r}\sqrt{\det h_{ij}}\right)^2. \tag{3.1}$$

Using the above computation, we get the following comparison theorem.

Theorem 3.1 (Laplacian comparison theorem)

Let X be a complete Riemannian manifold of dimension n. Let

$$\mathrm{Ric}(X) \geq -(n-1)K.$$

Let N be an n-dimensional simply connected space form of constant sectional curvature $-K$. Let ρ_M, ρ_N be the distance functions to fixed reference points, respectively. If $x \in X$ and $y \in N$ such that

$$\rho_M(x) = \rho_N(y).$$

Then in the sense of distribution, we have

$$\Delta \rho_M(x) \leq \Delta \rho_N(y).$$

♡

Proof: Outside the cut-locus and the reference points, the function ρ_M is smooth. Since ρ_M is the distance function,

$$|\nabla \rho_M| = 1.$$

The Laplacian can be written as

$$\Delta_M = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right).$$

Under the assumption that

$$ds^2 = d\rho^2 + \sum_{i,j} h_{ij}\, d\theta_i d\theta_j,$$

we have

$$\Delta \rho = \frac{\partial f}{\partial r},$$

where $f = \sqrt{\det h_{ij}}$. By (3.1), we have

$$\frac{\partial^2 f}{\partial r^2} + \left(\frac{\partial f}{\partial r} \right)^2 \geq -\mathrm{Ric} \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right).$$

The corresponding function on N satisfies the *equality*

$$\frac{\partial^2 f_0}{\partial r^2} + \left(\frac{\partial f_0}{\partial r} \right)^2 = (n-1)K. \tag{3.2}$$

If $r \to 0$, the asympototics f and f_0 are

$$\frac{\partial f}{\partial r} \sim \frac{n-1}{r}, \quad \frac{\partial f_0}{\partial r} \sim \frac{n-1}{\partial r}.$$

Since

$$-\mathrm{Ric} \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) \leq (n-1)K,$$

By the maximum principle

$$\frac{\partial f}{\partial r} \leq \frac{\partial f_0}{\partial r_0}.$$

This proves the comparison theorem at the smooth points of f.

Solving the equation (3.2), we get

$$f_0(r) = \frac{1}{\cosh \sqrt{K}} \cdot \sinh \sqrt{K} r.$$

Thus we have

$$\Delta_N \rho = \frac{n-1}{\rho} k\rho \coth k\rho \leq \frac{n-1}{\rho}(1 + k\rho).$$

Thus the Laplacian comparison theorem can be written in the following more useful form:

$$\Delta_M \rho \le \frac{n-1}{\rho}(1+k\rho).$$

We shall prove that the above inequality is true even at singular points of ρ, in the sense of distribution. To see this, we let Ω be the domain in X such that ρ is smooth on Ω. Ω is a star-like domain in X. Apparently

$$X = \Omega \cup Cut(p)$$

where $Cut(p)$ is the cut-locus of X. Since ρ is at least continuous, and since the measure of $Cut(p)$ is zero, we must have

$$\int_X \rho\Delta\varphi = \int_\Omega \rho\Delta\varphi$$

for any smooth function φ. Let Ω_ε be an exhaustion of Ω in the sense that $\Omega_\varepsilon \subset \Omega$. $\Omega_\varepsilon \subset \Omega_{\varepsilon'}$ if $\varepsilon > \varepsilon'$, and

$$\lim_{\varepsilon\to 0} \Omega_\varepsilon = \Omega.$$

Then we have

$$\begin{aligned} \int_X \rho\Delta\varphi &= -\int_X \nabla\rho\nabla\rho \\ &= \lim_{\varepsilon\to 0}(-1)\int_{\Omega\varepsilon} \nabla\rho\nabla\varphi. \end{aligned}$$

Using the Green's formula, we have

$$-\int_\Omega \nabla\rho\nabla\varphi = \int_\Omega \nabla u\cdot\varphi - \int_{\partial\Omega\varepsilon} \varphi\frac{\partial\rho}{\partial r}.$$

On the boundary $\partial\Omega\varepsilon$, we have

$$\frac{\partial\rho}{\partial r} \ge 0.$$

Thus we have

$$-\int_\Omega \nabla\rho\nabla\varphi \le \int_{\Omega\varepsilon} \Delta\rho\cdot\varphi \le \int_{\Omega\varepsilon} \frac{n-1}{\rho}(1+k\rho)\varphi$$

Finally, we get

$$\begin{aligned} \int_X \rho\nabla\varphi &\le \lim_{\varepsilon\to 0}\int_{\Omega\varepsilon} \frac{n-1}{\rho}(1+k\rho)\varphi \\ &= \int_X \frac{n-1}{\rho}(1+k\rho)\varphi \end{aligned}$$

which finishes the proof of the theorem. QED

Using exactly the same method, we have the following volume comparison theorem of Bishop.

Theorem 3.2

Let X be an n-dimensional complete Riemannian manifold. Let

$$\mathrm{Ric}(X) \ge -(n-1)k^2.$$

Then

$$\frac{\mathrm{vol}\,\partial B(R)}{\mathrm{vol}_k\,(\partial B(R))}\downarrow$$

is a decreasing function. ♡

Corollary 3.1 (Bishop)

Let X be an n-dimensional Riemannian manifold. If $\mathrm{Ric} \ge (n-1)k$. Then for any $R > 0$ ♡

$$\frac{\mathrm{vol}\,(B_X(R))}{\mathrm{vol}(k,R)}$$

is a decreasing function. In particular

$$\mathrm{vol}B_X(R) \leq V(k,R)$$

where $V(k,R)$ is the volume of ball of radius R in space form of curvature k.

Remark The Bochner formula gives another proof of (3.1). Let f be a smooth function. Then we have

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \frac{\partial}{\partial r}\Delta f + \mathrm{Ric}(\nabla f, \nabla f).$$

If we specialize f to be r, then $|\nabla r|^2 = 1$ and $\Delta|\nabla r|^2 \equiv 0$. On the other hand

$$|\nabla^2 f|^2 \geq \sum_i f_{ii}^2 \geq \frac{1}{n-1}\left(\sum f_{ii}\right)^2 = \frac{1}{n-1}(\Delta f)^2,$$

and the inequality follows.

Remark The proof of the laplacian comparison theorem in this section is not that far away from the usual proof using the second variational formula of the geodesic distance. The fact that we can write the Riemannian metric in the form

$$dr^2 + h_{ij}d\theta_i d\theta_j$$

is ensured by the Gauss Lemma, which is proved by the first variational formula of the geodesic distance function.

3.2 Gradient estimates and maximal principle

If X is a compact manifold, then any smooth real function on X reaches its maximum point. At the maximum point, the first derivatives are zero, and the Hessian matrix at the point is non-positive.

For non-compact manifold, the above statement is not true in general. In order to estimate a function we sometimes need a differential inequality. Usually such a differential inequality is obtained by doing the so-called gradient estimate.

We start with the following generalized maximum principle:

Theorem 3.3

Let f be a positive smooth function on a complex non-compact Riemannian manifold X. We assume that $\mathrm{Ric}(X) \geq -(n-1)k^2$ for some number k. Let φ_1, φ_2 be two smooth functions on X with φ_1 bounded. Assume that there are constants $\alpha > 0, C_1, C_2$ such that

♡

$$\nabla f \geq C_1 f^{l+\alpha} + \varphi_0 \nabla\varphi_1 \nabla f - C_2. \qquad \star$$

Then f is bounded. Furthermore, there is a constant

$$C = C(\alpha, C_1, C_2\|\varphi_0\|C_0\nabla\varphi_1)$$

such that

$$f \leq C.$$

Proof: We first assume that f is bounded. That is

$$\sup f < +\infty.$$

We claim that there is a sequence $\{x_k\}$ in X such that for any $\varepsilon > 0$, if k is large enough, we have

$$f(x_k) > \sup f - \varepsilon$$
$$|\nabla f|(x_k) < \varepsilon$$
$$\Delta f(x_k) < \varepsilon.$$

To prove the claim, we first take a sequence $\{y_k\}$ such that

$$\lim_{k\to\infty} f(y_k) = \sup f.$$

Define the cut-off function ρ as follows

$$\varphi : \mathbb{R} \to \mathbb{R}$$

smooth, $0 \le \rho \le 1, \rho(t) = 1$ for $0 \le t \le 1$. $\rho(t) = 0$ for $t > 2$ and $\rho'(t) \le 0$ for all $t \in \mathbb{R}$. Let R be a large number to be determined later. Let $d(x) = \mathrm{dist}(x, y_k)$ be the distance function. In general, the function $d(x)$ is only a continuous function. But let

$$g(x) = \rho\left(\frac{d^2(x)}{R^2}\right) f(x).$$

If the maximum point of $g(x)$ happens to be not smooth, we can always perturb the reference point by a little. So without loss of generality, we assume that $g(x)$ is smooth. Let x_k be the maximum point of $g(x)$. By the definition of x_k, we have

$$g(x_k) \ge g(y_k) = f(y_k) \ge \sup f - \varepsilon.$$

Using the definition, we also have

$$1 - \rho\left(\frac{d^2(x_k)}{R^2}\right) < \frac{\varepsilon}{\sup f}.$$

Since

$$0 = \nabla g(x_k) = \rho' \frac{2}{k^2} d\nabla d f(x_k) + \rho \nabla f(x_k)$$

we have

$$|\nabla f|(x_k) = -\frac{\rho'}{\rho}\frac{2}{k^2} d|\nabla d| f(x_k).$$

Because $|\nabla d| = 1$, if R is big enough, we have

$$|\nabla f|(x_k) < \varepsilon.$$

Finally, we have

$$0 \ge \Delta g(x_k) = \rho f + 2\nabla\rho\nabla f + f\Delta\rho.$$

In order to prove the claim, we just need to prove that

$$\Delta\rho\left(\frac{d^2(x, y_k)}{R^2}\right) < \varepsilon$$

for k large enough. A straight forward computation gives that

$$\Delta\rho = \frac{2}{R^2}\rho' + \frac{4d^2}{R^4}\rho'' + \frac{2}{R^2}\rho' d\Delta d.$$

Since the Ricci curvature of X is bounded from below, by Laplacian comparison theorem, we have

$$d\Delta d \ge -C$$

for some constant C depending only on the lower bound of the Ricci curvature. Since $\rho' \leq 0$, for R large enough we have

$$\Delta\rho < \varepsilon$$

for any ε. The claim is proved.

Using the differentiable inequality ($\star$), we have

$$\varepsilon \geq C_1 f(x_k)^{1+\alpha} = \|\varphi_0\| \, \|\nabla\varphi_1\|\varepsilon - C_2.$$

Thus

$$f(x_k) \leq \sqrt[1+\alpha]{\frac{C_2 + \|\varphi_0\| \, \|\nabla\varphi_1\|}{C_1}}.$$

Taking $k \to \infty$, we get the effective bound for f.

Now we assume that $\sup f = +\infty$. Let

$$v = 1 - \frac{1}{(1+f)^\beta}$$

for $\beta > 0$ to be determined later. Then by the previous result, there is a sequence $\{x_k\}$ such that for any $\varepsilon > 0$, if k is large enough, we have

$$\begin{aligned} v(x_k) &> 1 - \varepsilon \\ |\nabla v|(x_k) &< \varepsilon \\ \Delta v(x_k) &< \varepsilon. \end{aligned}$$

By the definition of v, we have

$$\Delta v = \beta \frac{\Delta f}{(1+\delta)^{1+\beta}} - \frac{1+\beta}{\beta} |\nabla v|^2 (1+f)^\beta$$

Applying the above inequality to ($\star$), we get

$$\varepsilon > \beta \frac{C_1 f^{1+\alpha} - \|\varphi_0\| \, \|\nabla\varphi_1\| \cdot \varepsilon - C_2}{(1+f)^{1+\beta}} - \frac{1+\beta}{\beta} \cdot \varepsilon (1+f)^\beta.$$

Letting $k \to \infty$, we get a uniform bound of $\sup f$ from the above inequality.

Let f be a smooth function on a complete non-compact manifold. We say that f is a harmonic function, if

$$\Delta f = 0.$$

Then we have the following:

Theorem 3.4 (Yau)

Let $p > 1$, if $f \in L^p, \Delta f = 0$. Then f is a constant. ♡

For the sake of simplicity, we only prove the theorem for $p = 2$.

Let f be an L^2 harmonic function. Take a fixed point x_0. Let

$$d(x) = \mathrm{dist}(x, x_0).$$

Let ρ be a cut-off function such that $\mathrm{supp}\rho$ is compact. We consider

$$g(x) = \rho\left(\frac{d(x)}{R}\right) f(x).$$

We have

$$\int_M \rho^2 |\nabla f(x)|^2 = -2 \int \rho f \nabla\rho \nabla f \frac{1}{R}.$$

Using the Cauchy inequality, we get

$$\int_M \rho^2|\nabla f|^2 \leq \frac{2}{R}\left(\int_M \rho^2|\nabla f|^2\right)^{\frac{1}{2}} \int_M |\nabla\rho|^2 f.$$

Since $|\nabla\rho| \leq C$. We get

$$\int_M \rho^2|\nabla f|^2 \leq C\frac{1}{R}\int_M f^2.$$

Letting $R \to \infty$, we get $\nabla f \equiv 0$ so f is a constant.

By the above theorem, for a complete manifold, the only interesting harmonic functions may be bounded harmonic functions.

In order to study the bounded harmonic functions, we first introduce the Ricci identity.

Let f be a smooth function of M. Define the derivative of f using the following formula

$$f_i\omega_i = df.$$

Using the same idea, we define

$$f_{ij}\omega_j = df_i - f_s\omega_{sj}.$$

The matrix (f_{ij}) is called the Hessian matrix. We have

$$f_{ij}\omega_j \wedge \omega_i = df_i \wedge \omega_i - f_s\omega_{sj} \wedge \omega_i = 0.$$

Thus the Hessian matrix is always symmetric.

The 3^{rd} order co-variant derivatives are defined as

$$f_{ijk}\omega_k = df_{ij} - f_{is} \wedge \omega_{sj} - f_{sj} \wedge \omega_{si}.$$

A careful computation gives

$$f_{ijk}\omega_k \wedge \omega_j = -\frac{1}{2}f_s R_{sikj}\omega_k \wedge \omega_j.$$

Thus we have

$$f_{ijk} - f_{ikj} = +f_s R_{sikk}$$

In particular, we have the following Ricci identity

$$f_{iik} - f_{kii} = -f_s \mathrm{Ric}_{sk}$$

Remark We have $\Delta f = f_{ii}$.

With the preparation above, we prove the following.

Theorem 3.5

Let M be a complete Riemannian manifold, $\dim M = n \geq 2$. $\mathrm{Ric}(M) \geq -(n-1)k, k \geq 0$. *Let u be a positive harmonic function. Then on any geodesic ball $B_a(x)$, we have*

$$\frac{|\nabla u|}{u} \leq C_n\left(\frac{1+ak^{\frac{1}{2}}}{a}\right).$$

where C_n is a constant only depends on n. We first prove that

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \sum_{ij} u_{ij}^2 - (n-1)k|\nabla u|^2. \tag{$\triangle$}$$

To see this, we do the following

$$\left(\sum u_j^2\right)_i = 2u_j u_{ji}.$$

Thus,

$$\frac{1}{2}\Delta|\nabla u|^2 + (u_j u_{ji})_i = u_{ji}^2 + u_j u_{jii}.$$

Using the Ricci identity, we have

$$\frac{1}{2}\Delta|\nabla u|^2 = u_{ji}^2 + u_j(\Delta u)_j + \mathrm{Ric}(\nabla u, \nabla u).$$

Thus $(\triangle)$ follows from the assumption on the Ricci curvature and harmonicity of u.

We now consider the points such that $\nabla u \neq 0$. By changing a frame, we may assume that

$$u_i \neq 0,\ u_j = 0 \text{ for } j < 1.$$

From $(\triangle)$, we conclude that

$$\Delta|\nabla u| = \frac{\Delta|\nabla u|^2}{2|\nabla u|} - \frac{1}{4}\frac{|\nabla|\nabla u|^2|^2}{|\nabla u|^3}.$$

Using the above information, we get

$$\Delta|\nabla u| \geq \frac{1}{|\nabla u|}\left(\sum_{j>1} u_{ij}^2 + \sum_{j>1} u_{jj}^2\right) - (n-1)k|u_1|.$$

Since

$$\sum_{j>1} u_{jj}^2 \geq \frac{1}{n-1}\left(\sum_{j>1} u_{jj}\right)^2 = \frac{1}{n-1}u_{11}^2.$$

we have

$$\Delta|\nabla u| \geq \frac{1}{n-1}\frac{1}{|\nabla u|}\sum_j u_{ij}^2 - (n-1)k|u_1|.$$

Now we assume that $\varphi = |\nabla u|/u$. Then we have

$$\Delta\varphi = \frac{\Delta|\nabla u|}{u} + 2\nabla|\nabla u|\nabla\frac{1}{u} + |\nabla u|\Delta\left(\frac{1}{u}\right).$$

Since

$$2\nabla|\nabla u|\nabla\frac{1}{u} = 2\nabla(\varphi u)\nabla\left(\frac{1}{u}\right) = -2\frac{\nabla\varphi\nabla u}{u} - 2\varphi^3.$$

We have

$$\nabla\varphi = \frac{\Delta|\nabla u|}{u} - 2\frac{\nabla\varphi\nabla u}{u}.$$

We have

$$\frac{\nabla\varphi\nabla u}{u} = -\varphi^3 - 2\nabla|\nabla u|\nabla\frac{1}{u}.$$

Thus if $\varepsilon > 0$ is small enough

$$\frac{\Delta|\nabla u|}{u} - \varepsilon\frac{\nabla\varphi\nabla u}{u} \geq \varepsilon\varphi^3.$$

Thus we get

$$\Delta\varphi \geq -(n-1)k\varphi - (2-\varepsilon)\frac{\nabla\varphi\nabla u}{u} + \varepsilon\varphi^3.$$

Using the maximum principle, we have φ is bounded.

Corollary 3.2

Let M be a complete Riemannian manifold, if $\mathrm{Ric}(M) \geq 0$. Then any positive harmonic function is a constant. ♡

Proof: If Ric ≥ 0 then $k = 0$. We have

$$\Delta\varphi \geq -(2-\varepsilon)\frac{\nabla\varphi\nabla u}{u} + \varepsilon\varphi^3.$$

For $\varepsilon = \frac{2}{n-1}$. Thus using the generalized maximal principal, $\varphi \equiv 0$. QED

Corollary 3.3 (Harnack **)

Let M be n-dimensional Riemannian manifold, $\mathrm{Ric}(M) \geq -(n-1)k$. *If u is a positive harmonic function in Ba. Then*

$$\sup_{Ba/2} u \leq C(n,a,k) \inf_{Ba/2} u$$

where $C(n,a,k)$ are constants depending only on n, a, k. ♡

Proof: By the above theorem, we have

$$\sup_{Ba} \frac{|\nabla u|}{u} \leq C(n,a,k).$$

Thus

$$|\nabla lgu| \leq C(n,a,k)$$

the conclusion follows.

Corollary 3.4

Suppose $\mathrm{Ric}(M) \geq 0$. *Then any positive harmonic function must be constant.* ♡

Let the Riemann metric be written as

$$(dr)^2 + h_{ij}(r,\theta)\, d\theta_i d\theta_j.$$

What is the asymptotic behavior of $h_{ij}(r,\theta)$ when $r \to 0$.

Note that the Riemannian metric at the reference point is regular. Thus we can define $(\theta_2, \ldots, \theta_n)$ as

$$\theta_j = \frac{x_j}{r}, \quad r = \sqrt{\sum x_j^2}.$$

Suppose

$$ds^2 = g_{ij} dx_i dx_j.$$

Let's compare

$$g_{ij} dx_i dx_j \sim (dr)^2 + h_{ij} d\theta_i d\theta_j.$$

We have

$$\begin{aligned} g_{ij} dx_i dx_j &= g_{ij}(dr\theta_i + r d\theta_i)(dr\theta_j + r d\theta_j) \\ &= g_{ij}\theta_i\theta_j (dr)^2 + 2g_{ij} x_i dr d\theta_j + g_{ij} r^2 d\theta_i d\theta_j. \end{aligned}$$

By comparison, we have

$$g_{ij}\theta_i\theta_j = 1$$
$$\sum_i g_{ij} x_i = 0$$

and

$$h_{ij} = r^2 \left(g_{ij} - g_{ij}\frac{\theta_i}{\theta_1} - g_{i1}\frac{\theta_j}{\theta_1} + g_{11}\frac{\theta_i\theta_j}{\theta_1^2} \right).$$

Thus
$$\sqrt{\det(h_{ij})} = \gamma^{n-1} \cdot \mu$$
for μ being a regular function (at least for fixed θ_i). Thus
$$\Delta\rho = \frac{\partial f}{\partial r} \sim \frac{n-1}{r} + \text{ small terms.}$$
If we choose (g_{ij}) to be normal, we can actually compute
$$\det\left(\delta_{ij} + \frac{\theta_i\theta_j}{\theta_1^2}\right) = \frac{1}{\theta_1^2}.$$
Thus
$$\sqrt{\det h_{ij}}$$
can be extended as a regular function near 0.

3.3 Harmonic functions revisited

We assume that u is a positive harmonic function defined on $B(a) \subset M$, where M is a complete Riemannian manifold with $\mathrm{Ric}(M) \geq -(n-1)k$.

In this section, we re-prove the Harnack inequality using the de Giorgi-Nash-Moser estimates. Note that our **result is weaker** than the differential Li-Yau Harnark inequality. However, the methods we use here are useful in non-linear case, even if we only use a linear problem as the example.

The reference book of this section is

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We first prove the following estimate.

Theorem 3.6

Under the above assumptions, then for any $p > 0, 0 < \theta < 1$, there is a constant
$$C = C(n, p, \theta, k) > 0$$
such that
$$\sup_{B(a\theta)} u \leq C(f_{B(a)} u^p)^{\frac{1}{p}}.$$

Proof: We first assume that $p \geq 2$. Let $\varphi \geq 0, \varphi \in C_0^\infty(B(a))$. Then since $\Delta u = 0$, we have
$$\int_{B(a)} \nabla u \nabla \varphi = 0.$$
Let ρ be a smooth function with compact support in $B(a)$. Then we have
$$\int_{B(a)} \nabla u \nabla(\rho^2 u^{p-1}) = 0.$$
Here p is a real number to be specialized later.

Expanding the above inequality, we get
$$(p-1)\int_{B(a)} \rho^2 u^{p-2}|\nabla u|^2 \leq -2\int_{B(a)} u^{p-1}\rho\nabla u\nabla\rho.$$
Using the Cauchy inequality, we get
$$(p-1)^2\int_{B(a)} \rho^2 u^{p-2}|\nabla u|^2 \leq C\int_{B(a)} |\nabla\rho| u^p.$$

Note that

$$u^{p-2}|\nabla u|^2 = \frac{4}{p^2}\left|\nabla u^{\frac{p}{2}}\right|^2$$

we have

$$\int_{B(a)} \rho^2 |\nabla u^{\frac{p}{2}}\Big|^2 \le C\int_{B(a)} |\nabla \rho|^2 u^p$$

if we allow C to be a little bigger, we shall get

$$\int_{B(a)} |\nabla(\rho u^{\frac{p}{2}})\Big|^2 \le C\int_{B(a)} |\nabla \rho|^2 u^p.$$

We let $2^* = \frac{2n}{n-2} > 2$. Using the Sobolev inequality we have

$$\left(\int_{B(a)} (\rho u^{\frac{p}{2}})^{2^*}\right)^{\frac{1}{2^*}} \le C\int_{B(a)} |\nabla \zeta|^2 u^p. \tag{$\star$}$$

We let

$$R_k = a\left(\theta + \frac{1-\theta}{2^k}\right).$$

Let $\rho_k \in C_0^\infty(B(R_k)), 0 \le \rho_k \le 1, \rho_k \equiv 1$ on $B(R_{k+1})$. We further assume that

$$|\nabla \rho_k| \le \frac{2}{R_k - R_{k+1}} = \frac{2^{k+1}}{(1-\theta)a}.$$

From ($\star$), we have

$$\left(\int_{B(R_{k+1})} U^{\frac{np}{n-p}}\right)^{\frac{n-2}{n}} \le \frac{C\cdot 4^k}{(1-\theta)^2 a^2}\int_{B(R_k)} u^p.$$

Specializing $p = p_k$, where $p_k = p\left(\frac{n}{n-2}\right)^k$, we have

$$\begin{aligned}\|u\|_{L^{p_{k+1}}(B(R_{k+1}))} \le \left(\frac{C\cdot 4^k}{(1-\theta)^2a^2}\right)^{\frac{1}{p_k}} \\ \times \|u\|_{L^p(B(R_k))}.\end{aligned}$$

Iterating, we have

$$\|u\|_{L^{p_{k+1}}(B(R_{k+1}))} \le \Pi\left(\frac{C\cdot 4^k}{(1-\theta)^2a^2}\right)^{\frac{1}{p_k}} \|u\|_{L^p(B(a))}.$$

We need to prove that

$$\Pi\left(\frac{C\cdot 4^k}{(1-\theta)^2a^2}\right)^{\frac{1}{p_k}} \le \frac{C}{((1-\theta)^2a^2)^{\frac{n}{2p}}} \tag{Exercises}$$

Since the right-hand side is independent of k, we let $k \to \infty$ and get

$$\|u\|_{L^\infty(B(\theta a))} \le \frac{C}{((1-\theta)a)^{\frac{n}{p}}}\|u\|_{L^p(B(a))}.$$

This proves the theorem for $p \ge 2$.

Now we assume that $0 < p < 2$. Using the result for $p = 2$, we have

$$\begin{aligned}\|u\|_{L^{p\infty}(B(\theta a))} \quad &\le \quad \frac{C}{((1-\theta)a)^{\frac{n}{2}}}\|u\|_{L^\infty(B(a))^{1-\frac{p}{2}}} \\ &\times \quad \left(\int_{B(a)} u^p\right)^{\frac{1}{2}}.\end{aligned}$$

Using the Young inequality, we get

$$\begin{aligned}\|u\|_{L^\infty(B(\theta a))} &\le \frac{1}{2}\|u\|_{L^p(B(a))}\\ &+\frac{C}{((1-\theta)a)^{\frac{n}{2}}}\|u\|_{L^p(B(a))}.\end{aligned}$$

Let $\varphi(s)=\|u\|_{L^\infty(B(a))}$, we get

$$\varphi(s)\le\frac{1}{2}\varphi(t)+\frac{C}{(t-s)^{\frac{n}{p}}}\|u\|_{L^p(B(a))}$$

$\forall\, 0<s<t\le a$. Iterating again, we get

$$\varphi(s)\le\frac{C}{(1-s)a)^{\frac{n}{p}}}\|u\|_{L^p(B(a))} \tag{Ex}$$

Theorem 3.7 (Weak Harnack inequality)

There is a constant $C>0, p_0>0$ such that

$$\inf_{B(a\theta)} u \ge \frac{1}{C}\left(⨍_{B(a)} u^{p_0}\right)^{\frac{1}{p_0}}.$$

Here p_0, C only depends on a, $(1-\theta)^{-1}$ and Sobolev constants. ♡

Proof. Without loss of generality, we assume that $u\ge\varepsilon>0$. Otherwise we can use $u+\varepsilon$ in place of u. We also assume that $a=1$. By a straight forward computation we get that

$$\Delta u^{-1}=\frac{2|\nabla u|^2}{u^3}\ge 0.$$

Thus u^{-1} is a subsolution. Using the above lemma, for any p, we have

$$\sup_{B(\theta)} u^{-p}\le C\int_{B(1)} u^{-p}.$$

Thus we must have

$$\begin{aligned}\inf_{B(\theta)} u &\ge C^{-\frac{1}{p}}\left(\int_{B_1} u^{-p}dx\right)^{-\frac{1}{p}}\\ &= C^{-\frac{1}{p}}\left[\int_{B_1} u^{-p}\cdot\int_{B_1} u^p\right]^{\frac{1}{p}}\\ &\times\left(\int_{B_1} u^p\right)^{\frac{1}{p}}.\end{aligned}$$

In order to prove the theorem, we just need to prove that for $p>0$ small enough

$$\int_{B(1)} u^{-p}\int_{B_1} u^p\le C.$$

We let

$$\omega=\log u-\beta$$

where

$$\beta=⨍\log u.$$

We shall establish

$$\int_{B(1)} e^{p|\omega|}\le C \tag{$\star$}$$

for $p>0$ small enough. We first prove that

$$\int_{B(\sigma)}|\nabla\omega|^2\le C$$

for some $\sigma > 1$. To see this, let ρ be the cut-off function whose support is within $B(\bar{\sigma})$ for some $\bar{\sigma} > \sigma$. Since u is harmonic, we have

$$\int_{B(\bar{\sigma})} \nabla u \nabla (u^{-1}\rho^2) = 0.$$

It follows that

$$\begin{aligned}\int_{B(\bar{\sigma})} \rho^2 |\nabla \omega|^2 &\leq \int_{B(\bar{\sigma})} \nabla \omega \nabla \rho^2 \\ &\leq 2\sqrt{\int_{B(\bar{\sigma})} \rho^2 |\nabla \omega|^2}\sqrt{\int_{B(\bar{\sigma})} |\nabla \rho|^2}.\end{aligned}$$

Since ρ only depends on $\sigma, \bar{\sigma}$, we get the desired inequality.

By the Poincaré inequality, we have

$$\int_{B(\bar{\sigma})} |\omega|^2 \leq C.$$

To get the estimate $(\star)$, we still use the Moser iteration. First observe that

$$\Delta \omega = -|\nabla \omega|^2.$$

Let ρ be a cut-off function to be determined later. Then we have

$$-\rho^2 |\omega|^{2q} \Delta \omega = \rho^2 |\omega|^{2q} |\nabla \omega|^2.$$

It follows that

$$\begin{aligned}\int \rho^2 |\omega|^{2q} |\nabla \omega|^2 &= \int \nabla \omega \nabla (\rho^2 |\omega|^{2q}) \\ &= 2\int \rho \nabla \rho \omega |\omega|^{2q} + 2q \int \rho^2 |\omega|^{2q-1} |\nabla \omega|^2.\end{aligned}$$

We use the Young inequality to get

$$2q|\omega|^{2q-1} \leq \frac{2q-1}{2q}|\omega|^{2q} + (2q)^{2q-1}. \qquad (\triangle)$$

Inserting the above inequality into the equation, we get

$$\begin{aligned}\int \rho^2 |\omega|^{2q} |\nabla \omega|^2 &\leq (2q)^{2q} + 2\sqrt{\int \rho^2 |\omega|^{2q} |\nabla \omega|^2} \\ &\times \sqrt{\int |\nabla \rho|^2 |\omega|^{2q}}.\end{aligned}$$

Using the Cauchy inequality $ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$, we get

$$\int \rho^2 |\omega|^{2q} |\nabla \omega|^2 \leq C((2g)^{2q} + \int |\omega|^{2q} |\nabla \rho|^2).$$

With a slightly larger constant C, we have

$$\int \rho^2 |\nabla |\omega|^{q+1}|^2 \leq Cq^2((2q)^{2q} + \int |\omega|^{2q} |\nabla \rho|^2).$$

Thus we have

$$\int (\nabla(\rho^2 |\omega|^{q+1}))^2 \leq 2Cq^2((2q)^{2q} + \int |\omega|^{2q+2} |\nabla \rho|^2).$$

Thus replacing $q+1$ by q and choosing the suitable cut-off function, we get

$$\left(\int_{B(\bar{\sigma})} |\omega|2^{qk}\right)^{\frac{1}{k}} \leq C\left((2q)^{2q} + \tau^{-2} q^2 \int_{B(q+\tau)} |\omega|^{2q}\right).$$

Let $\kappa = \frac{n}{n-1}$, $q_i = \kappa^{i-1}$, $\delta_0 = \bar{\sigma} > 1$.

$$\delta_i = \delta_{i-1} - \frac{\bar{\sigma} - 1}{2^i}.$$

Then we have

$$\begin{aligned} \left(\int_{B(\delta i)} |\omega|^{2+1} \right)^{\frac{1}{k}} &\leq C\kappa^{2(i-1)}\kappa^{i-1} \\ &+ C(4k)^i \int_{B(\delta_{i-1})} |\omega|^{2\kappa^{i-1}} \end{aligned}$$

We let

$$I_j = \left(\int_{B\partial_1} |\omega|^{2k^j} \right)^{\frac{1}{2k^3}}.$$

Then we get

$$I_1 \leq C^{\frac{1}{k^{i-1}}} k^{i-1} + C^{\frac{1}{k^{i-1}}} (4k)^{\frac{1}{k^{i-1}}} I_{i-1}.$$

Using the standard iteration we get

$$I_i \leq C + \frac{1}{2k^i}.$$

For any $q \geq 2$ let j be such that

$$2\kappa^{i-1} \leq q \leq 2\kappa^j.$$

Using Hölder and $||\omega||_{L^2} \leq C$ we get

$$\|\omega\|_{L^q} C I_j \leq \tilde{C} q \ \geq 2.$$

Since $q^q \leq e^q q!$, we have

$$\int_{B_1} |\omega|^q \leq \tilde{C}^q q^q \leq (\tilde{C} e^q) q!.$$

For ε enough

$$\int_{B_1} e^{\varepsilon|\omega|} \leq \sum_{q=2}^{\infty} (C\varepsilon e)^q \leq C.$$

This proved the weak Harnack inequality.

Theorem 3.8 (Harnack inequality)

Let u be a positive harmonic function on $B(a)$. Then for any $0 < \theta < 1$ we have

$$\sup_{B(a\theta)} u \leq C \inf_{B(a\theta)} u$$

where C only depends on n, $(1-\theta)^{-1}$, and the Sobolev constants.

♡

Chapter 4 Heat kernel and Green's functions on complete manifold

4.1 Heat kernel

Let M be a Riemannian manifold. The Laplace operator Δ can be extended as a densely defined self-adjoint operator. Thus by the spectrum theorem, we can write

$$\Delta = \int \lambda dE$$

where E is the corresponding spectrum measure. Using functional analysis, we define

$$e^{-\Delta t} = \int_0^\infty e^{-\lambda t}\, dE$$

to be the heat operator, and

$$\int_0^\infty e^{-\Delta t}\, dt$$

to be the Green's operator.

The heat semi-group, or the heat operator, has a kernel.

Theorem 4.1

Let M be a complete Riemannian, then there is a heat kernel $H(x,y,t) \in C^\infty(M \times M \times \mathbb{R}^+)$ such that

$$(e^{-\Delta t} f)(x) = \int_M H(x,y,t) f(y)$$

$\forall f \in L^2(M)$ such that

(1) $H(x,y,t) = H(y,x,t)$

(2) $\lim_{t\to 0^+} H(x,y,t) = \delta_x(y)$

(3) $\left(\Delta - \frac{\partial}{\partial t}\right) H = 0$

(4) $H(x,y,t) = \int H(x,z,t-s)H(z,y,s)\, dz.$

♡

Remark Let M be a compact manifold and let $\{f_i\}$ be an orthonormal basis of eigenfunctions. Let λ_i be the corresponding eigenvalues. Then

$$H(x,y,t) = \sum_{k=1}^\infty e^{-\lambda_k t} f_k(x) f_k(y).$$

As in the case of harmonic functions, for positive solutions of the heat equations

$$\left(\Delta - \frac{\partial}{\partial t}\right) u = 0$$

we also have the differentiable Harnack inequality. The theorem is as follows:

Theorem 4.2

Let M be a compact Riemannian manifold with boundary, Ric$(M) \geq 0$. If $\partial M \neq \emptyset$, we assume that ∂M is convex. In this case, we assume that $u(x,t)$ satisfies the Neumann boundary.

$$\frac{\partial u}{\partial n} = 0$$

on $\partial M x(0,\infty)$, where $\frac{\partial}{\partial n}$ is the outer normal direction. Then on $Mx(0,\infty)$, we have

$$\frac{|\nabla u|^2}{u^2} - \frac{ut}{u} \le \frac{n}{2t}.$$

♡

For manifold with $\mathrm{Ric}(M) \ge -k$, the estimates are more complicated but the same principle applies. The Harnack inequalities follows from the gradient estimates:

Using the gradient estimates and the Harnack inequality we obtain:

Theorem 4.3

Let M be a complete manifold without boundary, Ric$(M) \ge -k, k \ge 0$. Let $H(x,y,t)$ be the fundamental solution of the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x,t) = 0.$$

Then for any $\delta \in (0,1)$, we have

$$H(x,y,t) \le C(\delta,n) V_x^{-\frac{1}{2}}(\sqrt{t}) V_y^{-\frac{1}{2}}(\sqrt{t}) \exp\left(-\frac{r^2(x,y)}{(4+\delta)t} + C_1 \delta k t\right).$$

♡

For the proof, see the book of Yau and Schoen. We give some applications of the above theorem.

Theorem 4.4 (Gromov)

Let M be a compact manifold without boundary, let Ric$(M) \ge 0$ and d be the diameter of M. Then

$$\lambda_k \ge \frac{C(n)}{d^2}(k+1)^{\frac{2}{n}}$$

where λ_k is the kth eigenvalue of the manifold.

♡

Proof We set $x = y$ and $k = 0$ in the above theorem. Then we have

$$H(x,x,t) \le C(n) V_x^{-1}(\sqrt{t}).$$

Since

$$H(x,y,t) = \sum_{k=1}^{\infty} e^{-\lambda k t} f_k(x) f_k(y)$$

we have

$$H(x,x,t) = \sum_{k=1}^{\infty} e^{-\lambda_k t}.$$

As a result we have

$$\sum_{k=1}^{\infty} e^{-\lambda k t} \le C(n) \int_M V_x^{-1}(\sqrt{t})\, dx.$$

If $\sqrt{t} \ge d$, then $V_x^{-1}(\sqrt{x}) = \mathrm{vol}(M)$, if $\sqrt{t} \le d$, then by the Bishop volume comparison theorem

$$\frac{V_x(\sqrt{t}}{V_x(d)} \ge \left(\frac{\sqrt{t}}{d}\right)^n.$$

Thus

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \le C(n) \begin{cases} \left(\frac{d}{\sqrt{t}}\right)^n & t \le d^2 \\ 1 & t > d^2 \end{cases}. \tag{$\star$}$$

For fixed k, by the monotonicity, we have

$$ke^{-\lambda_k t} \le C(n) \begin{cases} \left(\frac{d}{\sqrt{t}}\right)^n & t \le d^2 \\ 1 & t > d^2 \end{cases}.$$

We let

$$\sqrt{t} = k^{-\frac{1}{n}} d$$

and the result follows.

We can compare the above eigenvalue estimate with the result of Cheng-Li.

From $(\star)$, we can get

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \le C t^{-\frac{n}{2}} \quad t \le d^2$$

where C is the absolute d constant. On the other hand, if we consider

$$\begin{aligned} \frac{\partial}{\partial t} \int_M H(x,y,t)^2 dy &= 2 \int_M H(x,y,t) \Delta H(x,y,t)\, dy \\ &= -2 \int |\nabla y H(x,y,t)|^2 \, dy \\ &\le -2C \left(\int_M |H(x,y,t)|^{\frac{2n}{n-2}} \, dy \right)^{\frac{n-2}{n}} \end{aligned}$$

where C is the Sobolev constant. Since for any $t > 0$

$$\int H(x,z,t)\, dz \le 1.$$

So we have

$$\left| \int_M |H|(x,y,t)|^{\frac{2n}{n-2}} \, dy \right|^{\frac{n-2}{n}} \ge \left(\int_M |H(x,y,t)|^2 \, dy \right)^{\frac{2+n}{n}}$$

QED

(**Proof.** Let $f = H(x,y,t)$. Then

$$\int f^2 = \int f^{\frac{2n}{n+2} + \frac{4}{n+2}} \le \left(\left(\int f^{\frac{2n}{n-2}} \right)^{\frac{n+2}{2n}} \left(\int f \right)^{\frac{4}{n+2}} \right).$$

Thus we have

$$\frac{\partial}{\partial t} \int_M H(x,y,t)^2 \, dy \le -C \left(\int_M |H(x,y,t)^2| dy \right)^{\frac{2+n}{n}}.$$

Since

$$\lim_{t \to 0} H(x,x,t) = \infty.$$

Thus

$$H(x,x,2t) \le \left(\frac{4}{n} C t \right)^{-\frac{n}{2}}.$$

Using the same method as before, we have

$$\lambda_k \ge C \left(\frac{k}{\mathrm{vol}(M)} \right)^{\frac{2}{n}}.$$

The second application of the heat kernel estimate is the following resolvent estimate.

Theorem 4.5

For any $\beta > 0$ there is $n \in N, \alpha < 0, C \subset \infty$ such that the integral kernel $g_\alpha^{(\frac{n}{2})}(x,y)$ of $(\Delta - \alpha)^{-\frac{n}{2}}$ satisfies.

$$g_\alpha^{(\frac{n}{2})}(x,y) \le C\varphi(x)^2 e^{-\beta d(x,y)}$$

where $\varphi(x) = (v(B_1(x))^{-\frac{1}{2}}$.

Proof. First, using the heat kernel $H(x,t)$ we can write

$$g_\alpha^{(\frac{n}{2})}(x,y) = C_2 \int_0^\infty H(x,y,t) t^{\frac{n}{2}-1} e^{\alpha t}\, dt.$$

By the volume comparison theorem, we have

$$\begin{aligned} v^{\frac{n}{2}}(B_{\sqrt{t}}(x)) v^{\frac{-1}{2}}(B_{\sqrt{t}}(y)) &\le C_2 \varphi(x)^2 \sup(1, t^{-\frac{n}{2}}) \\ &\times e^{\beta_2 d(x,y)}. \end{aligned}$$

By an element calculation, for any $\beta_3 > 0$, we have

$$\exp\left(-\frac{d^2(x,y)}{C_1 4t}\right) \le \exp(\beta_3 d(x,y)) \exp(C_1 \beta_3^2 t).$$

Thus for any $\beta > 0$, there exists $\alpha < 0, C_3 < \infty$ such that

$$H(x,y,t) \le C_3 \varphi^2(x) e^{-\beta_2 d(x,y)} \sup\{t^{-\frac{n}{2}}, 1\} t^{-(\alpha+1)t}$$

which implies

$$g_\alpha^{(\frac{n}{2})}(x,y) \le C_t \phi^2(x) e^{-\beta d(x,y)}.$$

Cheeger-Yau's heat kernel comparison theorem. ■

Theorem 4.6

Let M be a complete Riemannian manifold such that $Ric(M) \ge 0$. Fixing $x \in M, r_0 > 0$. The heat kernel $H(x,y,t)$ in $B(x,r_0)$ and the heat kernel $\varepsilon(r(x,y),t)$ in $V(k,r_0)$ satisfies the following

$$\varepsilon(r(x,y),t) \le H(x,y,t).$$

(For both Dirichlet and Neumann conditions).

Proof. Using the property of the heat kernel, we have

$$\begin{aligned} &H(x,y,t) - \varepsilon(x,y,t) \\ &\quad = \int_0^t \int_{B(x_1 r_0)} \frac{d}{ds}(\varepsilon(x,z,t-s) H(z,y,s))\, dzds \\ &\quad = -\int_0^t \int_{B(x_1 r_0)} \left(\frac{d}{ds}\varepsilon(r(x,z),t-s)\right) H(z,y,s)\, dzds \\ &\quad + \int_0^t \int_{B(x_1 r_0)} \varepsilon(r(x,z),t-s) \frac{d}{ds} H(z,y,s)\, dzds \\ &\quad = -\int_0^t \int_{B(x_1 r_0)} \tilde{\Delta}\varepsilon(r(x,z),t-s) H(z,y,s)\, dzds \\ &\quad + \int_0^t \int_{B(x_1 r_0)} \varepsilon(r(x,z),t-s) \Delta H(z,y,s)\, dzds. \end{aligned}$$

Using the Green's formula, under either the Dirichlet or Neumann boundary condition, we have

$$\int_{B(x_1 r_0)} \varepsilon(r(x,z), t-s)\Delta H(z,y,s)\, dz = \int_{B(x_1 r_0)} \Delta\varepsilon(r(x,z), t-s) H(z,y,s)\, dz.$$

Since $H(\xi, y, s) > 0$, we just need to prove that

$$\tilde{\Delta}\varepsilon(r(x,z), t-s) \leq \Delta\varepsilon(r(x,z), t-s).$$

This essentially follows from the Laplacian comparison theorem: Let $x = (r, \zeta), \zeta \in S^{n-1}$. Then

$$\begin{aligned} \tilde{\Delta} &= \frac{\partial^2}{\partial r^2} + m(r)\frac{\partial}{\partial r}, m(r)\frac{\partial}{\partial r}\log\det\sqrt{\tilde{g}} \\ \Delta &= \frac{\partial^2}{\partial r^2} + m(r,\zeta)\frac{\partial}{\partial r}, m(r,\zeta)\frac{\partial}{\partial r}\log\det\sqrt{g}. \end{aligned}$$

Since $\mathrm{Ric}(M) \geq (n-1)_k$, using the volume comparison theorem, we have

$$m(r,\zeta) \leq m(r).$$

Since

$$\frac{\partial\varepsilon}{\partial r} < 0$$

we have

$$\tilde{\Delta}\varepsilon(r, t-s) \leq \Delta\varepsilon(r, t-s).$$

QED

In the above proof, we didn't take the cut-locus into account. Using some kind of limiting process, we can overcome the difficulty.

Theorem 4.7

Let M be a complete Riemannian manifold such that $Ricci(M) \geq (n-1)k, n = \dim M$. We use $B(x_0, r)$ to denote the ball centered at x_0 with radius r. Let $V(k,r)$ be the ball of radius r in a simply connected space form. Then with the Dirichlet boundary condition, we have

$$\lambda_1(B(x_0, r)) \leq \lambda_1(V(k,r)).$$

Proof Let $H(x,y,t)$ and $\varepsilon(x,y,t)$ be the corresponding heat kernel. Then we have

$$\begin{aligned} H(x,y,t) &= \sum e^{-\lambda_1 t}\varphi_1^2(x) \\ \varepsilon(x,y,t) &= \sum e^{-\tilde{\lambda}_1 t}\tilde{\varphi}_1^2(x). \end{aligned}$$

If we let $t \to \infty$, we get $\lambda_1 \leq \tilde{\lambda}_1$. S.Y. Cheng concretely computed the upper bounds of the eigenvalue

1. if $\mathrm{Ric}(M) \geq 0$, then $\lambda_1 \leq \frac{2n(n+4)}{d^2}$
2. if $\mathrm{Ric}(M) \geq n-1$, then $\lambda_1 \leq n\frac{\pi^2}{d^2}$
3. if $\mathrm{Ric}(M) \geq (n-1)(-k)$, then

$$\lambda_1 \leq \frac{1}{4}k + \frac{C_n}{d^2} \quad C_k = 2n(n+4).$$

Unfortunately, for Neumann boundary condition, we don't have the comparison theorem for the 1st eigenvalues **directly**. Under, the Neumann boundary condition, the **first** eigenvalues are always 0. The comparison theorem for that is trivial.

It is thus interesting to have the following result.

Theorem 4.8

If M is a compact manifold without boundary, then

$$\lambda_1(M) \le \lambda_1\left(V\left(k, \frac{d}{2}\right)\right)$$

where d is the diameter of M.

Proof of a theorem of Brascamp and Lieb.

Theorem 4.9

Let Ω be a bounded convex domain of $\mathbb{R}^n$. Let u be the first Dirichlet eigenfunction. Let

$$\Delta u = -\lambda, u \quad u > 0.$$

Then $\log u$ must be concave.

Proof. We choose any function $u_0 > 0, u_0|_{\partial\Omega} = 0$ such that $-\log u_0$ is concave. Such a function always exists. For example, we can take the convex hull of the graph of $-\log u$.

Consider the flow

$$\frac{\partial u}{\partial t} = \Delta u + \lambda_i u \quad \frac{u}{\partial\Omega} = 0.$$

We **assume** that $u_t \to u$, the first eigenfunction. We are going to use the maximum principle. Let T be the biggest number such that $\det(-\Delta^2 \log u)$ is degenerated. Thus there is an $x_0 \in M$ and a direction i such that

$$-(\log u)_{ii}(x_0) = 0$$

and for other j, $(-\log u)_{jj}(x_0) \ge 0$. Let $\varphi = -\log u$. Then the envolution of φ is

$$\frac{\partial \varphi}{\partial t} = \Delta\varphi - |\nabla\varphi|^2 - \lambda_1.$$

By the maximum principle, $\varphi_{iik} = 0, \frac{\partial\varphi_{ii}}{\partial t} \le 0 \Delta\varphi_{ii} \ge 0$. Thus

$$\begin{aligned} 0 \ge \frac{\partial \varphi_{ii}}{\partial t} &= \Delta\varphi_{ii} - 2\varphi_k\varphi_{kii} - 2\varphi_{ki}^2 \\ &\ge -2\varphi_{ki}^2. \end{aligned}$$

However, by convexity, $\varphi_{ki}^2 \le \varphi_{ii} \cdot \varphi_{kk} = 0, \varphi_{ki} \equiv 0$. The theorem follows from strong maximum principle.

Let Ω be a bounded domain with smooth boundary. Let u be the first Dirichlet eigenfunction with eigenvalue λ_1. Then ■

Lemma 4.1

$u \ge 0$.

Proof. Otherwise, we may use $|u|$ in place of u. From the Kato's inequality

$$|\nabla|u|| \le |\nabla u|$$

we have

$$\frac{\int |\nabla|u||^2}{\int u^2} \le \frac{\int |\nabla u|^2}{\int u^2}.$$

Since $\frac{\int |\nabla u|^2}{\int u^2}$ is minimal, we must have

$$\frac{\int |\nabla |u||^2}{\int u^2} = \frac{\int |\nabla u|^2}{\int u^2}$$

and thus u is an eigenfunction. QED

Lemma 4.2

$u > 0$.

If not, assume that $u(x_0) = 0$. Let

$$u(x) = p^N(x) + 0(x^{N+\varepsilon}).$$

By the equation $\Delta u = -\lambda_1 u$, we must have

$$\Delta p^N(x) = 0.$$

Since $u(x) \geq 0$, we have $p^N(x) \geq 0$. This contradicts to the maximum principle.

Lemma 4.3

$\frac{\partial u}{\partial n} \neq 0$.

Proof. Using Taylor's expansion

$$u(x) = p^N(x) + 0(x^{N+\epsilon}).$$

Again, $\Delta p^N(x) = 0, p^N(x) > 0$ in Ω. By the Hopf lemma

$$\frac{\partial u}{\partial n} = \frac{\partial p^N}{\partial n} \neq 0.$$

Consider the flow

$$\frac{\partial u}{\partial t} = \Delta u = \lambda_1 u, \frac{u}{\partial \Omega} = 0.$$

■

We have

Lemma 4.4

For any smooth initial $u_0 > 0$, the flow exists and converges to the first eigenfunction.

Proof. Let

$$u_0 = \sum a_i f_i(x).$$

Then

$$u(t, x) = \sum a_i e^{(-a_i - \lambda_1)t} f_i(x)$$

solves the flow equation. Obviously,

$$\lim_{t \to \infty} u(t, x) = a_i f_1(x).$$

If we choose the u_0 such that $a_i \neq 0$. Then the flow converges to the first eigenfunction. ■

Lemma 4.5

If $u_0 > 0$, then $u(t, x) > 0$.

Proof. Maximum principle. ∎

Lemma 4.6

If Ω is convex, then near $\partial\Omega(-\log u)$ is convex. ♡

Proof. We solve $u(x_1 \dots, x_n) = 0$ to get

$$x_n = x_n(x_1, \dots, x_{n-1})$$

Ω is convex

$$\frac{\partial^2 x_n}{\partial x_i \partial x_j} > 0 \quad \text{(positive definite matrix)}.$$

Using implicitly function theorem, we have

$$-\frac{u_{ij}}{u_n} + \frac{u_{in}u_j}{u_n^2} + \frac{u_{jn}u_i}{u_n^2} - \frac{u_i u_j u_{nn}}{u_n^3} > 0.$$

Thus for any $(a_1, \cdots, a_n)$ if $\sum a_i u_i = 0$, we have

$$-u_{ij}a_i a_j \geq \varepsilon |a|^2.$$

For any $a + \mu b, b = (u_1 \dots, u_n)$, we have

$$\begin{aligned} - \quad & u_{ij}(a_i + \mu b_i)(a_j + \mu b_j) \\ & + \frac{1}{u}|\mu|^2 |b|^4 > 0. \end{aligned}$$

Thus $-\log u$ is convex.

For the maximal principle, see page 84–A.

For initial function, take any $\frac{u}{\partial\Omega} = 0 \; ue^{-c\sum x_i^2}$. ∎

4.2 Green's function and parobolicity

Let M be a complete non-compact Riemannian manifold. The Green's function is a smooth function on

$$M \times M \backslash \text{diag}(M)$$

such that

1. $G(x, y) = G(y, x)$ and fixing y, we have $\Delta_x G(x, y) = 0 \, \forall x \neq y$.
2. $G(x, y) \geq 0$.
3. Fixing y, when $x \to y$, we have the following
$$G(x, y) = \begin{cases} \rho_y(x)^{2-n}(1 + 0(1)) & n > 2 \\ -\log \rho_y(x)(1 + 0(1)) & n = 2 \end{cases}.$$

This last asymptotical expansion of the Green's function also implies that $\Delta_x G(x, y) = -\delta_{x \cdot y}$.

From the asymptotical behavior we can find that $n = 2$ and $n > 2$, the Green's functions are very different. We make the following.

Definition 4.1

A complete manifold is said to be parabolic, if and only if it doesn't admit a positive Green's function. Otherwise it is said to be non-parabolic. ♣

Definition 4.2

An End, E, with respect to a compact subset $\Omega \subset M$ is an unbounded connected component of $M \backslash \Omega$. The number of ends with respect to Ω, denoted by $N_\Omega(M)$, is the number of unbounded connected component of $M \backslash \Omega$.

♣

Definition 4.3

An End E is said to be parabolic, if it doesn't admit a positive harmonic function f satisfying

$$f \equiv \text{ on } \partial E$$

and

$$\frac{\lim}{n \to E(\infty)} f(y) < 1$$

where $E(\infty)$ denotes the infinity of E. Otherwise, E is said to be non-parabolic and the function f is said to be a barrier function of E.
We prove the following result:

♣

Theorem 4.10

Let E be an parabolic end. Let $A(R) = E \cap \partial B(R)$ where $B(R)$ is the ball of radius R with respect to some reference point. Let f be a harmonic function on E such that

$$f|_{\partial E} = 1, f|_{A(R)} = 0.$$

Then

$$\lim_{R \to \infty} \int_E |\nabla f|^2 \to 0.$$

♡

Proof. Using the Green's formula, we have

$$\int_E |\nabla f|^2 = -\int_E \frac{\partial f}{\partial r}$$

where $\frac{\partial}{\partial r}$ is the outer normal direction. We claim that $\frac{\partial f}{\partial r} \to 0$. To see this, we take a sequence $R_1 < R_2 < \dots R_k \to \infty$. The corresponding harmonic function $f_i = f_{R_i}$. By the maximal principle, f_i are increasing sequence on any compact set of E. Let

$$\lim_{i \to \infty} f_i = f.$$

Then f must be a positive harmonic function. By the parabolicity, $f \equiv 1$. By the maximal principle again

$$\frac{\partial f_i}{\partial r} \to 0$$

as $i \to \infty$. ■

Examples: $\mathbb{R}^2$ and $\mathbb{R}^n$.

Let $R > 0$ be a big number. Let

$$F(R) = \{f \in C_0^\infty(\mathbb{R}^n) | f = 1 \text{ for } |x| < R, f \text{ rotational symmetric}\}.$$

If $n > 2$, then for any $C > 0$, there exists an R_0 such that for any $R > R_0$

$$\int_{\mathbb{R}^n} |\nabla f|^2 > C$$

for any $f \in F(R)$. If $n = 2$, then for any $\varepsilon > 0$ there exists $R_0 > 0$ such that for any $R > R_0$, we can find an

$f_R \in F(R)$ for which

$$\int_{\mathbb{R}^2} |\nabla f|^2 < \varepsilon.$$

Proof. If $n > 2$, then

$$\int_{\mathbb{R}^2}^{\infty} \frac{1}{r^{n-1}}\, dr = \frac{1}{n-2} \frac{1}{R^{n-2}}.$$

Thus we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla f|^2 &\geq (n-2) \subset R^{n-2} \int_R^{\infty} r^{n-1} \left(\frac{\partial f}{\partial r}\right)^2 dr \int_0^{\infty} \frac{1}{r^{n-1}} dr \\ &\geq (n-1) \subset R^{n-2} \to +\infty. \end{aligned}$$

However, for $n = 2$, we define $f_R = \sigma_R(|x|($ such that

$$\sigma_R(t) = \begin{cases} 1 & t \leq R \\ \left(1 - \frac{\log R}{R}\right)^{-1} \left(\frac{\log R}{\log t} - \frac{\log R}{R}\right) & R < t < e^R \\ 0 & t \geq e^R \end{cases}.$$

Then a straightforward computation gives

$$\int_0^{\infty} +|\sigma_R'(t)|^2 dt \leq \frac{4}{3} \frac{1}{\log R} \quad \text{for } R >> 0.$$

The main result of this section is the following characterization of parabolicity. ■

Theorem 4.11

Let M be a complete manifold. If M is non-parabolic, then for any point $p \in M$, we have

$$\int_1^{\infty} \frac{dt}{Ap(t)} < +\infty$$

where $A_p(r)$ denotes the area of $\partial B_p(r)$. ♡

Proof. For $p \in M$, let $G(p, y)$ be the Green's function (assuming that it exists). Let $G_i(p, y)$ be the Green's function on $B_p(R_i)$ with the Dirichlet boundary condition, where $R_i \to \infty$. ■

For any $| < R < R_i$, let's denote that

$$\begin{aligned} S_i(1) &= \sup_{y \in \partial B_p(1)} G_i(p, y) \\ i_i(R) &= \inf_{y \in \partial B_p(R)} G_i(p, y). \end{aligned}$$

Let f be the harmonic function defined on $B_p(R) \backslash B_p(1)$ satisfying the boundary conditions

$$f(y) = S_i(1) \text{ on } \partial B_p(1)$$
$$f(y) = G_i(p, y) \text{ on } \partial B_p(R).$$

The maximum principle implies that

$$f(y) \geq G_i(p, y) \text{ on } B_p(R) \backslash B_p(1).$$

In particular, we have

$$\frac{\partial f}{\partial r} \leq \frac{\partial G_i}{\partial r} \text{ on } \partial B_p(R).$$

On the other hand, since $f(y)$ is harmonic, Stokes theorem implies

$$0 = \int_{B_p(R)|B_p(1)} \Delta f = \int_{\partial B_p(R)} \frac{\partial f}{\partial r} - \int_{\partial B_p(1)} \frac{\partial f}{\partial r}.$$

Also, we observe that

$$\int_{\partial B_p(R)} \frac{\partial G_i}{\partial r} = \int_{B_p(R)} \Delta G_i = -1.$$

Thus we get

$$\int_{\partial B_p(1)} \frac{\partial f}{\partial r} \le -1.$$

consider h to be the harmonic function defined on $B_p(R)\backslash B_p(1)$ satisfying the boundary conditions

$$\begin{aligned} h(y) &= S_i(1) \text{ on } \partial B_p(1) \\ h(y) &= i_i(R) \text{ on } \partial B_p(R). \end{aligned}$$

Again, the maximum principle implies that

$$h(y) \le f(y) \text{ on } B_p(R)\backslash B_p(1)$$

and

$$\frac{\partial h}{\partial r} \le \frac{\partial h}{\partial r} \text{ on } B_p(1).$$

Thus we have

$$\int_{\partial B_p(R)} \frac{\partial h}{\partial r} = \int_{\partial B_p(1)} \frac{\partial h}{\partial r} \le -1.$$

Define the function

$$g(r) = (s_i(1) - i_i(R)) \left(\int_1^R \frac{dt}{A_p(t)} \right)^{-1} \int_r^R \frac{dt}{A_p(t)} + i_i(R).$$

Then $g(r(y))$ will have the same boundary conditions as $h(y)$. The Dirichlet integral minimizing property for harmonic functions implies that

$$\begin{aligned} &\int_{B_p(R)\backslash B_p(1)} |\nabla h|^2 \le \int_{B_p(R)\backslash B_p(1)} |\nabla g|^2 \\ &= \int_1^R \left((s_i(1) - i_i(R)) \left(\int_1^R \frac{dt}{A_p(t)} \right)^{-1} \frac{1}{A_p(r)} \right)^2 A_p(r)\, dr \\ &= (s_i(1) - i_i(R)) \left(\int_1^R \frac{dt}{A_p(t)} \right)^{-1}. \end{aligned}$$

On the other hand, integration by parts yields

$$\begin{aligned} \int_{B_p(R)\backslash B_p(1)} |\nabla h|^2 &= i_i(R) \int_{\partial B_p(R)} \frac{\partial h}{\partial r} - s_i(1) \int_{\partial B_p(R)} \frac{\partial h}{\partial r} \\ &\le s_i(1) - i_i(R)). \end{aligned}$$

By taking the limit

$$\int_1^k \frac{dt}{A_p(t)} \le \sup_{y \in \partial B_p(1)} G(p, y) - \inf_{y \in \partial B_p(R)} G(p, y).$$

Corollary 4.1

Let M be a complete manifold such that

$$\mathrm{vol}(B(R)) \le CR^2$$

then M has to be parabolic. ♡

Corollary 4.2

Let M be a Riemann surface such that

$$\int_M |k| < +\infty$$

Then M is parabolic. ♡

Proof of a theorem of Brascamp and Lieb.

Theorem 4.12

Let Ω be a bounded convex domain of $\mathbb{R}^n$. Let u be the first Dirichlet eigenfunction. Let

$$\Delta u = -\lambda_1 u \quad u > 0.$$

Then $\log u$ must be concave. ♡

Proof We choose any function $u_0 > 0, u_0|_{\partial\Omega} = 0$ such that $-\log u_0$ is concave. Such a function always exists. For example, we can take the convex hull of the graph of $-\log u$.

Consider the flow

$$\frac{\partial u}{\partial t} = \Delta u + \lambda_1 u \quad \frac{u}{\partial \Omega} = 0.$$

We **assume** that $u_t \to u$, the first eigenfunction. We are going to use the maximum principle. Let T be the biggest number such that $\det(-\nabla^2 \log u)$ is degenerated. Thus there is an $x_0 \in M$ and a direction i such that

$$-(\log u)_{ii}(x_0) = 0$$

and for other j, $(-\log u)_{jj}(x_0) \geq 0$. Let $\varphi = -\log u$. Then the envolution of φ is

$$\frac{\partial \varphi}{\partial t} = \Delta\varphi - |\nabla\varphi|^2 - \lambda_1.$$

By the maximum principle, $\varphi_{ii}k = 0$, $\frac{\partial \varphi_{ii}}{\partial t} \leq 0$ $\Delta\varphi_{ii} \geq 0$. Thus

$$\begin{aligned} 0 \quad &\geq \quad \frac{\partial \varphi_{ii}}{\partial t} = \Delta\varphi_{ii} - 2\varphi_k\varphi_{kii} - 2\varphi_{kl}^2 \\ &\geq \quad -2\varphi_{ki}^2. \end{aligned}$$

However by convexity, $\varphi_{ki}^2 - \varphi_{ii}\varphi_{kk} = 0, \varphi_{ki} \equiv 0$. The theorem follows from the strong maximum principle.

Chapter 5 Eigenvalue problems

5.1 Eigenvalue

We first make different notations of the Laplace operator.

We assume that M is a Riemannian manifold with the Riemannian metric

$$ds^2 = \sum g_{ij} dx_i dx_j.$$

Let (g^{ij}) be the inverse matrix of (g_{ij}) and let $g = \det(g_{ij})$. Then under the local coordinates $(x_1, \ldots, x_n)$, we define

$$\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right).$$

Apparently, we can write

$$\Delta = g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \right) \frac{\partial}{\partial x_j}.$$

As before, we have the following computation

$$\begin{aligned}
&\frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} (g^{ij} \sqrt{g}) \\
&= \frac{1}{2} g^{ij} \frac{\partial}{\partial x_i} \log g + \frac{\partial}{\partial x_i} g^{ij} \\
&= \frac{1}{2} g^{ij} g^{kl} \frac{\partial g_{kl}}{\partial x_i} - g^{in} g^{jm} \frac{\partial g_{mn}}{\partial x_i} \\
&= -g^{kl} \Gamma^j_{kl}.
\end{aligned}$$

Thus we also have the following formula

$$\Delta = g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} - g^{kl} \Gamma^j_{kl} \frac{\partial}{\partial x_j}.$$

From the above representation, we have the following third formula for the Laplace operator.

Let $ds^2 = \sum_{j=1}^n \omega_j^2$. With respect to the frame, the connection (ω_{ij}) is well-defined. We are above the define the covariant derivatives of a function like the following

$$\begin{aligned}
df &= f_i \omega_i \\
f_{ij} \omega_j &= df_i - f_s \omega_{si}.
\end{aligned}$$

Using the above notations, we can define

$$\Delta f = \sum_{i=1}^n f_{ii} = \sum_i \nabla_i \nabla_i f.$$

Finally, we can define the Laplace operator on p-forms as follows:

Let $d : \wedge^p(M) \to \wedge^{p+1}(M)$ be the ordinary differential operator, where $\wedge^p(M)$ be the space of smooth p-forms. With respect to the Riemannian metric, $\wedge^p(M)$ becomes an infinite dimensional inner product space.

Let δ be the formal dual operator with respect to d. Then

$$\delta : \wedge^p(M) \to \wedge^{p-1}(M).$$

We can **prove** that δ is also a differential operator of first order. The Laplace operator can be defined as

$$\Delta = d\delta + \delta d.$$

In particular, on the space of functions, or 0-forms,

$$\Delta = \delta d.$$

We have the following:

Theorem 5.1 (Weitzenböck formula)

For function f, we have

$$\Delta f = \delta d f = -\sum \nabla_i \nabla_i f.$$

In general, we have the following

$$\Delta = -\sum \nabla_i \nabla_i + \textit{ curvature terms}$$

which is also called Weizenböck formula.

We go back to the Laplacian on functions. We know that $\Delta = \sum \nabla_i \nabla_i$ is well-defined on $C^\infty(M)$. Unfortunately, with the following L^2-inner product

$$\langle f \cdot g \rangle = \int_M f g dV$$

C^∞ (M) is not a complete metric space. The complete metric space is $L^2(M)$. However, there is no way that we can extend Δ on $L^2(M)$. ♡

Proof: The key point is that any differential operator is a closed-graph operator. Thus if Δ is extendable, then by ***, Δ has to be a bounded operator, as by the example of Heaviside function. QED

Figure 5.1: Heaviside Function

Thus we can only extend the operator into a densely defined self-adjoint operator.

Recall that an operator Δ is self-adjoint if

$$\mathrm{Dom}(\Delta) = \mathrm{Dom}(\Delta^*)$$

and

$$\langle \Delta f, g \rangle = \langle f, \Delta g \rangle$$

for any $f, g \in \mathrm{Dom}(\Delta)$.

In functional analysis, we have the following theorem. Let

$$Q(\varphi, \psi) = \int \nabla \varphi \nabla \psi.$$

Then Q is a non-negative quadratic form defined on $H_0^1(M)$. Then there is a unique densely defined operator A such that

$$Q(\varphi, \psi) = -(A\varphi, \psi).$$

Such an operator A is in fact called the Dirichlet Laplacian operator.

As an exercise, we prove that on an only manifold L^2 harmonic function must be constant. QED

Theorem 5.2

Let A be the Dirichlet extension of the Laplacian Δ. A function f is called A-harmonic, if $f \in \mathrm{Dom}(A)$ and $Af = 0$. If $f \in L^2(M)$, then f is a constant. ♡

Proof: The key point is that

$$Q(\rho^2 f, f) = Q\langle \rho^2 f, \Delta f\rangle = 0.$$

Thus using the same method as before, f is a constant. If M is a compact manifold with no boundary, we still use Δ to denote the Dirichlet extension of the Laplace operator. By the elliptic regularity, the spectrum of Δ are discrete. That is, there is a sequence

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

such that for any λ_i, there $\exists f_i$, $f_i \in L^2$

$$\Delta f_i = -\lambda_i f_i.$$

We have similar results for manifolds with boundary conditions. To be more precise, the Laplacians acting on functions with the following boundary conditions.

(A) Dirichlet boundary condition: Δ acting on f vanishing on the boundary.

(B) Neumann boundary condition: Δ acting on functions such that $\frac{\partial f}{\partial n} = 0$.

For eigenvalues, we have the following minimax principle. Assume that M is a closed manifold, then

$$\lambda_1 = \inf_{\int f = 0} \frac{\int |\nabla f|^2}{\int f^2}.$$

To prove the above result, we let φ be any smooth function such that

$$\int_M \varphi = 0.$$

Then by the definition of λ_1, we have

$$\frac{\int |\nabla (f + \varepsilon\varphi)|^2}{\int (f + \varepsilon\varphi)^2} \geq \lambda_1.$$

However, if we take the first order term, we get

$$\int (\delta f + \lambda_1 f)\varphi = 0.$$

Note that

$$\int (\Delta f + \lambda_1 f) = 0.$$

Then

$$\Delta f + \lambda_1 f \equiv 0$$

and λ_1 is the first eigenvalue.

By elliptic regularity, $\lambda_1 > 0$. Thus we have the following Poincaré-inequality: there exists a constant C, such that

$$\int |\nabla f|^2 \geq C \int f^2$$

for any function with

$$\int_M f = 0.$$

Before going further, let's prove the following co-area formula. QED

Theorem 5.3

Let M be a compact Riemannian manifold with boundary. Let $f \in H'(M)$. Then

$$\int_M g = \int_{-\infty}^{+\infty} \int_{\{f=0\}} \frac{g}{|\nabla f|} d\sigma.$$

Proof: Without loss of generality, we assume that $|\nabla f| \neq 0$. Thus by the implicit function theorem $\{f = \sigma\}$ is a smooth manifold.

Using the cut-off function, we may assume that $\operatorname{supp} g$ is contained in a coordinate chart. Thus we may assume that the Riemannian metric is given under the global coordinates $(x_1, \ldots, x_n)$ as follows

$$ds^2 = \sum g_{ig} dx_i ds_j$$

by definition

$$\int_M g = \int_M g\sqrt{\det(g_{ij})}\, dx_1, \ldots, dx_n.$$

Since $\nabla f \neq 0$, we can solve the equation

$$f = \sigma$$

by

$$x_1 = x_1(\sigma, x_2, \ldots, x_n)$$

or in other words, by the implicit function theory $(\sigma, x_2, \ldots, x_n)$ is a local coordinate system as well. The Jacobian of the transformation is

$$dx_1 \wedge \ldots \wedge dx_n = \frac{\partial x_1}{\partial \sigma} d\sigma \wedge dx_2 \wedge \ldots \wedge dx_n.$$

On the other side, restricting to $f = \sigma$, the Riemann metric can be written as

$$\left(g_{11} \frac{\partial x_1}{\partial x_k} \cdot \frac{\partial x_1}{\partial x_l} + g_{1l} \frac{\partial x_1}{\partial x_l} + g_{k1} \frac{\partial x_1}{\partial x_k} + g_{kl} \right) dx_k dx_l.$$

If we choose local coordinates such that $g_{kl} = \delta_{kl}$. Then we have

$$\left(\frac{\partial x_1}{\partial x_k} \cdot \frac{\partial x_1}{\partial x_l} + \delta_{kl} \right) dx_k dx_l.$$

The volume form of the above is

$$\begin{aligned} \det\left(\delta_{kl} + \frac{\partial x_1}{\partial x_k} \cdot \frac{\partial x_1}{\partial x_l} \right) &= 1 + \sum \left| \frac{\partial x_1}{\partial x_k} \right|^2 \\ &= \frac{1}{|\frac{\partial f}{\partial x_1}|^2} \sum \left| \frac{\partial f}{\partial x_k} \right|^2 = \frac{|\nabla f|^2}{|\frac{\partial f}{\partial x_1}|^2}. \end{aligned}$$

Thus we have

$$\begin{aligned} \int_M g &= \int_M g\sqrt{\det(g_{ij})} dx_1 \ldots dx_n \\ &= \int_M \sqrt{\det(g_{ij})} \frac{\partial x_1}{\partial \sigma} d\sigma \wedge dx_2 \ldots \wedge dx_n. \end{aligned}$$

By ($\triangle$), we must have

$$dV ds^2 = dV_{f=\sigma} \cdot \frac{\left|\frac{\partial f}{\partial x_i}\right|}{|\nabla f|}.$$

Thus

$$\int_M g = \int_{-\infty}^{+\infty} \left(\int_{f=\sigma} \frac{g}{|\nabla f|} \right) d\sigma.$$

Of course, in general, there are points such that $\nabla f = 0$. But by a theorem of Sand, the set

$$\{yf(y)|\nabla f(y) = 0\}$$

is of zero measure. Using the standard covering technique, we can prove the same result.

As an application of the above co-area formula, we prove the following result of Sobolev inequality.

Sobolev inequality. Let M be a compact manifold with boundary. Then there is a constant $C > 0$ such that

$$C\left(\int_M |f|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq \int_M |\nabla f|$$

for any smooth function $f/2M = 0$ (D-condition) or $\int f = 0$ (Neumann condition).

Isopermetric inequality. Let Ω be a domain in $M, \Omega \subset\subset M$. Then there is a constant independent to Ω such that

$$C\text{vol}(\Omega)^{\frac{n-1}{n}} \leq \text{vol}(\partial\Omega).$$

We want to prove that the Isopermetric inequality is equivalent to the Sobolev inequality.

At least one-side of the implication was clear: assuming the Sobolev inequality, if we let

$$f_\varepsilon(x) = \left\{\begin{array}{cl} 1 & x \in \Omega,\ d(x, \partial\Omega) \geq \varepsilon \\ \frac{d(x,\partial\Omega}{\varepsilon} & x \in \Omega,\ d(x, \partial\Omega) \leq \varepsilon \\ 0 & \text{otherwise} \end{array}\right\}.$$

Then using the Sobolev inequality, the isopermetric inequality follows by letting $\varepsilon \to 0$.

In order to prove that the isopermetric inequality implies the Sobolev inequality, we use the co-area formula. We assume that $f \geq 0$. Then

$$\int_M |\nabla f| = \int_0^\infty \text{ Area } (f = \sigma)\, d\sigma.$$

We also have

$$\begin{aligned} \int_M |f|^{\frac{n}{n-1}} &= \int_0^\infty \text{vol}(f^{\frac{n}{n-1}} > \lambda)\, d\lambda \\ &= \frac{n}{n-1}\int_0^\infty \text{vol}(f > \sigma)\sigma^{\frac{n}{n-1}}\, d\sigma. \end{aligned}$$

Using the isopermetric inequality, we have

$$\int_M |\nabla f| = \int_0^\infty \text{Area}(f = \sigma)\, d\sigma \geq C\int_0^\infty \text{vol}(f > \sigma)^{\frac{n}{n-1}}\, d\sigma.$$

Thus in order to prove the Sobolev inequality, we just need to prove that

$$\int_0^\infty \text{vol}(f > \sigma)^{\frac{n}{n-1}}\, d\sigma \geq C\left(\int_0^\infty \text{vol}(f > \sigma)\sigma^{\frac{n}{n-1}}\, d\sigma\right)^{\frac{n}{n-1}}.$$

We let

$$\begin{aligned} F(\sigma) &= \text{vol}(f > \sigma) \\ \varphi(t) &= \int_0^t F(\sigma)^{\frac{n}{n-1}}\, d\sigma \\ \psi(t) &= \left(\int_0^t F(sigma)\sigma^{\frac{n}{n-1}}\, d\sigma\right)^{\frac{n-1}{n}}. \end{aligned}$$

Then $\varphi(0) = \psi(0)$. Using the monotonicity of $F(\sigma)$ we can prove that

$$\varphi'(t) \geq \frac{n}{n-1}\psi'(t).$$

Thus

$$\varphi(\infty) \geq \frac{n}{n-1}\psi(\infty).$$

Corollary 5.1

$$\left(\int f^{\frac{n-p}{np}}\right)^{\frac{np}{n-p}} \leq C\left(\int |\nabla f|^p\right)^{\frac{1}{p}}$$

for any $p > 1$.

Definition 5.1

Let M *be a compact Riemannian manifold if* $\partial M \neq \emptyset$.

$$h_D(M) = \inf\left\{\frac{\mathrm{vol}(\partial\Omega)}{\mathrm{vol}(\Omega)}\bigg|\, \Omega \subset\subset M\right\}$$

if $\partial M = \emptyset$.

$$h_N = \inf\left\{\frac{\mathrm{vol}(H)}{\min(\mathrm{vol}(M_1), \mathrm{vol}(M_2))}\bigg|\, H \textit{ is a hypersurface}\right\}.$$

Theorem 5.4 (Cheeger)

For Dirichlet condition, we have

$$\lambda_1 \geq \frac{1}{4}h_D^2.$$

For Neumann condition

$$\lambda_1 \geq \frac{1}{4}h_N^2(M).$$

Proof: We only prove the case for Dirichlet condition. We first observe that, if there is a constant μ such that

$$\int_M |\nabla\varphi| \geq \mu \int_M |\varphi|$$

for any φ with $\varphi|_{\partial M} = 0$. Then $\lambda_1 \geq \frac{1}{4}\mu^2$. To see this, we consider $\varphi = f^2$

$$\mu \int_M f^2 \leq 2\int_M |f||\nabla f| \leq 2\left(\int_M f^2\right)^{\frac{1}{2}}\left(\int |\nabla f|^2\right)^{\frac{1}{2}}.$$

Thus

$$\frac{1}{4}\mu^2 \int_M f^2 \leq \int_M |\nabla f|^2.$$

Since the above is true for any function f, we must have

$$\lambda_1 \geq \frac{1}{4}\mu^2.$$

Finally, we prove a result which is well known but can't readily be found in the literature. QED

Theorem 5.5

Let M *be a compact manifold with smooth boundary. Let*

$$M_1 = \inf_{\int f = 0} \frac{\int |\nabla f|^2}{\int f^2}.$$

Then we have the following result: let f *be a minimizer, and let* f *be smooth. Then*

$$\frac{\partial f}{\partial n} = 0.$$

Proof: Let φ be a smooth function with compact support such that

$$\int \varphi = 0.$$

Then we have

$$\int |\nabla(f+\varepsilon\varphi)|^2 \geq \mu_1 \int (f+\varepsilon\varphi)^2.$$

Thus we have

$$\int \nabla f \nabla\varphi = -\mu_1 \int_M f\varphi.$$

by Green's formula

$$\int_M \nabla f\nabla\varphi - \int_{\partial M} \varphi\frac{\partial f}{\partial n} - \int_M \Delta f\varphi.$$

Thus since $\Delta f = -\mu_1 f$, we have

$$\int_{\partial M} \varphi\frac{\partial f}{\partial n} = 0$$

and we must have $\frac{\partial f}{\partial n} = 0$.

QED

5.2 Eigenvalue problems(II)

By the variational characterizing of the eigenvalues, we know that it is usually more difficult to get the lower bound estimate of eigenvalues. Among all the eigenvalues, the lower bound of the first eigenvalue is particularly important.

The Cheeger's result did give a lower bound estimate of the first eigenvalues. But the bounds are not "computable." In geometry, "computable" bounds provide effective versions of Poincaré and Sobolev inequalities.

The following Lichnerowicz theorem gives a good lower bound of the first eigenvalue for closed manifold.

Theorem 5.6 (Lichnerowicz)

Let M be a closed n-dimensional Riemannian manifold. Assume that

$$\mathrm{Ric}(M) \geq (n-1)k > 0$$

Then $\lambda_1 \geq nk$

Proof: One line proof: let u be the first eigenfunction. Then using the Ricci identity we have

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \sum u_{ij}^2 + \nabla u\Delta u + \mathrm{Ric}(\nabla u, \nabla u).$$

We have

$$\begin{aligned} \sum u_{ij}^2 &\geq \sum u_{ii}^2 \geq \frac{1}{2}\left(\sum u_{ii}\right)^2 = \frac{\lambda_1^2}{n} uU^2 \\ \mathrm{Ric}(\nabla u, \nabla u) &\geq (n-1)k|\nabla u|^2 \end{aligned}$$

QED

Thus we have

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{\lambda_1^2}{n}u^2 - \lambda_1|\nabla uU|^2 + (n-1)k|\nabla u|^2.$$

Taking integration on both sides, we get

$$\frac{\lambda_1^2}{n} - \lambda_1^2 + (n-1)k\lambda_1 \leq 0.$$

The theorem follows.

In 1962, Obata proved, if $\lambda_1 = nk$, then M has to be the standard sphere.

For the rest of this section, we use the gradient estimates to find "computable" lower bounds of the first eigenvalue.

We prove the following theorem.

Theorem 5.7 (Li-Yau)

Let M be a closed manifold and

$$\mathrm{Ric}(M) \geq 0.$$

Then $\lambda_1 \geq \pi^2/2d^2$, d is the diameter.

Proof: Let u be the first eigenfunction. After normalization we may assume that

$$1 = \sup u > \inf u = -k \geq -1$$

for some $1 \geq k > 0$. Let

$$\tilde{u} = \frac{u - \frac{1-k}{2}}{\frac{1+k}{2}}.$$

Then after this linear change of u. We have

$$\begin{cases} \Delta \tilde{u} = -\lambda_1(\tilde{u}+1) \\ \sup \tilde{u} = a \\ \inf \tilde{u} = -1 \end{cases}$$

for $a = \frac{1-k}{1+k}$. $1 > a \geq 0$.

Let $g = \frac{1}{2}(|\nabla \tilde{u}|^2 + (\lambda_1 + \varepsilon)\tilde{u}^2)$ for some $\varepsilon > 0$ to be determined later. Assume that at x_0

$$g(x_0) = \mathrm{Max}\, g.$$

Using the maximum principle, at x_0, we have

$$\tilde{u}_j \tilde{u}_j + (x_1 + \varepsilon)\tilde{u} u_i = 0$$

and

$$\begin{aligned} 0 \geq \Delta g \quad &= \quad \hat{u}_{ij}^2 \mathrm{Ric}(\nabla \tilde{u}, \nabla \tilde{u}) + \nabla \tilde{u} \nabla \Delta \tilde{u} \\ &\quad +(\lambda_1 + \varepsilon)|\nabla \tilde{u}|^2 + (\lambda_1 + \varepsilon)\tilde{u}\Delta\tilde{u} \\ &\quad \geq \tilde{u}_{ij}^2 - \lambda_1 |\nabla \tilde{u}|^2 + (\lambda_1 + \varepsilon)|\nabla \tilde{u}|^2 \\ &\quad -\lambda_1(\lambda_1 + \varepsilon)\tilde{u}(\tilde{u} + a \end{aligned}$$

if at x_0, $\nabla \tilde{U} = 0$. Then we have

$$|\nabla \tilde{u}|^2 + (\lambda_1 + \varepsilon)\tilde{u}^2 \leq (\lambda_1 + \varepsilon).$$

In particular, we have

$$|\nabla \tilde{u}|^2 + \lambda_1(1+a)\tilde{u}^2 \leq (1+a)$$

if $\nabla\tilde{u}(x_0) \neq 0$. Then using the Cauchy inequality

$$\tilde{u}_{ij}^2 \geq \frac{(\tilde{u}_i\tilde{u}_j\tilde{u}_{ij})^2}{|\nabla\tilde{u}|^4} = (\lambda_1+\varepsilon)^2\tilde{u}^2.$$

Thus we have

$$\begin{aligned} 0 \geq \Delta g(x_0) \quad \geq \quad & (\lambda_1+\varepsilon)^2\tilde{u}^2 + \varepsilon|\nabla\tilde{u}|^2 \\ & -\lambda_1(\lambda_1+\varepsilon)\tilde{u}^2 - \lambda_1(\lambda_1+\varepsilon) \\ & \geq 2\varepsilon g - \lambda_1(\lambda_1+\varepsilon)a. \end{aligned}$$

For any $\varepsilon > \lambda_1 a$, the above gives

$$|\nabla\tilde{u}|^2 + \lambda_1(1+a)\tilde{u}^2 \leq \lambda_1(1+a).$$

Let

$$f(t) = \arcsin\tilde{U}(\sigma(t))$$

where $\sigma(t)$ is the arc-length curve connecting the minimal point and the maximum point of $\tilde{u}$. Then by the above argument, we have

$$|f'(t)| \leq \sqrt{\lambda_1(1+a)}.$$

Let d be the diameter of the manifold, then we have

$$\begin{aligned} d\sqrt{\lambda_1(1+a)} \quad \geq \quad & \int_0^d (f'(t))dt \geq \arcsin 1 \\ & -\arcsin(-1) = \pi. \end{aligned}$$

Thus

$$\lambda + 1 \geq \frac{1}{1+a}\frac{\pi^2}{d^2}.$$

Since $a < 1$, this gives

$$\lambda_1 \geq \frac{\pi^2}{2d^2}$$

We let $\theta = \arcsin\tilde{u}$. Then Zhong-Yau proved $\oint$ the following surprising theorem.

Theorem 5.8 (***)

Let

$$\psi(\theta) = \left(\frac{4}{\pi}\right)\theta + \cos\theta\sin\theta) - 2\sin\theta.$$

Then

$$\frac{|\nabla\tilde{u}|^2}{1-\tilde{u}^2} \leq \lambda_1(1+a\psi(\theta)).$$

The method is maximal principle, very surprising and mysterious.

Using the above sharpened inequality, observed that $\psi(\theta)$ is an odd function, we can prove that

$$\lambda_1 \geq \frac{\pi^2}{d^2}.$$

Assume that $\mathrm{Ric}(M) \geq -(n-1)k$ *for* $k > 0$*. Then by estimating* $|\nabla u|^2 + \lambda_1(1-u)^2$*, Li-Yau was able to prove that*

$$\lambda_1 \geq \frac{C}{d^2}\exp(-C_1\sqrt{kd^2}).$$

**** was able to modify the above and proved that*

$$\lambda_1 \geq \frac{\pi^2}{d^2}\exp(-C_1\sqrt{kd^2}).$$

When $\mathrm{Ric}(M) > 0$*, or* $\mathrm{Ric}(M) \geq (n-1)k > 0$ *the above inequality is not optimal. In fact, it is far from being optimal. Let* $M = s^n$*. Then by Lichnerowicz theorem,* $\lambda_1 \geq n$ *(in fact,* $\lambda_1 = n$*).* $d(s^n) = \pi$*. Thus ** gives*

$$\lambda_1 \geq 1.$$

In this direction, we have the following Peter Li Conjecture.

conjecture 1 (P. Li) *If* $\mathrm{Ric}(M) \geq (n-1)k$*. Then*

$$\lambda_1 \geq \frac{\pi^2}{d^2} + (n-1)k.$$

Such a conjecture, if true, will sharpen both the result of Zhong-Yang and Lichnerowicz because by Myer's theorem

$$\frac{\pi^2}{d^2} \geq k.$$

Not much was known to the proof of the conjecture. D. Yang proved that

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{1}{4}(n-1)k$$

Ling Jun proved a bigger number

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \alpha(n-1)k\alpha > \frac{1}{4}.$$

On the other end, if $\mathrm{Ric}(M) \geq (n-1)k, k > 0$. Then

$$\lambda_1 \geq \frac{\pi^2}{d^2} - (n-1)k.$$

Recently, *** was able to prove that

$$\lambda_1 > \frac{\pi^2}{d^2}.$$

Very interesting result.

We end this section by citing a result of Li and Croke.

Theorem 5.9 *Let* M *be a* $\oint$ *manifold, with boundary. Then there is a constant* $C = (n, d, V, k) > 0$ *such that the Sobolev constant is* $> c > 0$.

5.3 first Eigenvalue

Let M be an n-dimensional Riemannian manifold with or without boundary. Let the metric ds^2 be represented by

$$ds^2 = \sum g_{ij}dx_i dx_j,$$

where $(x_1, \cdots, x_n)$ are local coordinates. Let

$$\Delta = \frac{1}{\sqrt{g}}\sum \frac{\partial}{\partial x_i}(g^{ij}\sqrt{g}\frac{\partial}{\partial x_j})$$

be the Laplace operator, where $(g^{ij}) = g_{ij}^{-1}$, $g = \det(g_{ij})$.

The operator Δ acts on smooth functions. If $\partial M \neq \emptyset$, then we may define one of the following two boundary conditions:

①. Dirichlet condition: $f|_{\partial M} = 0$.

②. Neumann condition: $\frac{\partial f}{\partial n}|_{\partial M} = 0$, where n is the outward normal vector of the manifold ∂M.

By the elliptic regularity, if M is compact, then the spectrum of Δ consists of eigenvalues

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \to +\infty$$

of finite multiplicity.

By the variational principal, we have the following Poincaré inequality

$$\int |\nabla f|^2 \geqslant \lambda_1 \int f^2$$

To our special interests, we would like to give "computable" lower bound estimates of the first eigenvalue. Here by "computable" we mean the geometric quantities like the diameter, the bounds of the curvature, *etc*, that are readily available.

Li-Yau [**li-yau**] discovered the method of gradient estimates to give "computable" lower bounds of the first eigenvalue. The prototype of the estimates is as follows:

Theorem 5.10 (Li-Yau)

Let M be a compact manifold without boundary. Let d be the diameter of M. Assume that the Ricci curvature of M is non-negative. Then we have the following estimate

$$\lambda_1 \geqslant \frac{\pi^2}{4d^2}.$$

♡

Proof. Let u be the first eigenfunction such that

$$\max u^2 = 1.$$

Let

$$g(x) = \frac{1}{2}(|\nabla u|^2 + (\lambda_1 + \varepsilon)u^2),$$

where $\varepsilon > 0$.

The function g is a smooth function. Let x_0 be the maximal point of g. Then at x_0 we have

$$u_j u_{ji} + (\lambda_1 + \varepsilon)uu_i = 0 \tag{5.1}$$

and

$$0 \geqslant u_{ji}^2 + u_j u_{jii} + (\lambda_1 + \varepsilon)|\nabla u|^2 + (\lambda_1 + \varepsilon)u\Delta u.$$

Using the Ricci identity, we have

$$u_j u_{jii} = u_j(\Delta u)_j + Ric(\nabla u) \geqslant u_j(\Delta u)_j.$$

Thus we have

$$0 \geqslant u_{ji}^2 + u_j(\Delta u)_j + (\lambda_1 + \varepsilon)|\nabla u|^2 + (\lambda_1 + \varepsilon)u\Delta u. \tag{5.2}$$

Suppose that at the maximum point of $g(x)$, $\nabla u \neq 0$. Then we have

$$u_{ji}^2 \geqslant |\nabla u|^{-4}(\sum_{i,j} u_j u_i u_{ij})^2$$

by the Cauchy inequality. Using the first order condition, we conclude

$$u_{ji}^2 \geqslant (\lambda_1+\varepsilon)^2 u^2.$$

Putting the above inequality into ⊛, we get

$$0 \geqslant \varepsilon|\nabla u|^2 + \varepsilon(\lambda_1+\varepsilon)u^2$$

which is not possible. Thus at the maximum point, we must have $\nabla u = 0$. Therefore we have

$$g(x) \leqslant \frac{1}{2}(\lambda_1+\varepsilon)\max u^2 = \frac{1}{2}(\lambda_1+\varepsilon).$$

From the above estimate, we get

$$\frac{|\nabla u|^2}{1-u^2} \leqslant \lambda_1+\varepsilon.$$

Since ε is arbitrary, we let it go to zero and obtain

$$\frac{|\nabla u|^2}{1-u^2} \leqslant \lambda_1.$$

By changing the sign of u, we may assume that $\max u = 1$. Let $u(p) = 1$ for $p \in M$. Since $\int u = 0$, there is a point $q \in M$ such that $u(q) = 0$. Let $\sigma(t)$ be the minimal geodesic line connecting q and p. Consider the function

$$\arcsin u(\sigma(t))$$

By the above inequality, we get

$$|(\arcsin u(\sigma(t)))'| \leqslant \sqrt{\lambda_1}|\sigma'(t)|$$

Integrating the above inequality along the geodesic line, we get

$$\frac{\pi}{2} \leqslant \sqrt{\lambda_1}d$$

and the theorem is proved.

QED

Several extensions of the above method can be obtained when

① The manifold has boundary;

② The Ricci curvature has a lower bound.

We first address the Neumann boundary condition.

Lemma 5.1

If $\partial M \neq 0$ and ∂M is convex, then $g(x)$ doesn't attain its maximum on ∂M unless at the point $\nabla u = 0$. ♡

Proof. Let $x_0 \in \partial M$ such that $g(x)$ attains the maximum at x_0. Then

$$\frac{\partial g(x_0)}{\partial n} \leqslant 0,$$

where $\vec{n}$ is the outward unit normal vector. By the definition of $g(x)$ and the fact $\frac{\partial u}{\partial n} = 0$, we have

$$u_j \frac{\partial u_j}{\partial n} \leqslant 0.$$

Let h_{ij} be the second fundamental form. Then

$$0 \geqslant u_j \frac{\partial u_j}{\partial n} = h_{ij}u_i u_j \geqslant 0.$$

If the equality is true, then we must have $\nabla u(x_0) = 0$.

QED

The next question is to sharpen the Li-Yau estimates. Even for the unit circle, Li-Yau estimate is not sharp.

Let's consider the circle $x^2 + y^2 = R^2$. Let the parameter, or the coordinate, of the circle be

$$x = R\cos\theta, y = R\sin\theta$$

Then the Laplace operator is

$$\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}$$

As a result, $u = \cos\theta, \sin\theta$ are the two eigenfunctions with the first eigenvalue $1/R^2$. If $u = \cos\theta$, then with the induced Riemannian metric,

$$|\nabla u|^2 = \frac{1}{R^2}\sin^2\theta.$$

Thus we have

$$g(\theta) = \frac{1}{2R^2},$$

and thus

$$\frac{|\nabla u|^2}{1-u^2} \leqslant \lambda_1,$$

which is not sharp.

The problem is that in general, we don't know whether the first eigenfunction is always symmetric. More precisely, if we assume that

$$1 = \sup u > \inf u = -k \geqslant -1$$

we don't know whether $k = 1$. From Li-Yau's basic estimate, we can improve the estimate $\lambda_1 \geqslant \pi^2/4d^2$ to

$$\lambda_1 \geqslant (\frac{\pi}{2} + \arcsin k)^2 d^{-2}.$$

The above inequality is essentially useless because we know nothing about k. However, using a simple trick, we can double the estimate of Li-Yau:

Take

$$\widetilde{u} = \frac{u - \frac{1-k}{2}}{\frac{1+k}{2}}.$$

Then using the standard gradient estimate, we get

$$|\nabla\widetilde{u}|^2 \leqslant \lambda_1(1+a)(1-\widetilde{u}^2) \tag{5.3}$$

where

$$a = \frac{1-k}{1+k}.$$

Now the function $\widetilde{u}$ is symmetric: $\max\widetilde{u} = -\min\widetilde{u} = 1$. Using the same method, we get

$$\lambda_1 \geqslant \frac{\pi^2}{(1+a)d^2} \geqslant \frac{\pi^2}{2d^2}$$

Zhong-Yang [**zhong-yang**] took one more step and proved the following result.

Theorem 5.11 (Zhong-Yang)
Let M be a compact Riemannian manifold with non-negative Ricc curvature. Then

$$\lambda_1 \geqslant \frac{\pi^2}{d^2}.$$

QED

The estimate is called "optimal" in the sense that for 1 dimensional manifold, the lower bound is achieved. We shall soon see that the estimate, in general, is far from being optimal.

The basic idea of the proof is still the maximum principle. From the estimate (5.3), we suspect that there is an odd function $\varphi(\arcsin u)$ such that

$$|\nabla \tilde{u}|^2 \leqslant \lambda_1(1 + a\varphi(\arcsin u))(1 - \tilde{u}^2). \tag{5.4}$$

If such function φ exists, then we have

$$\sqrt{\lambda_1} d \geqslant \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 + a\varphi(\theta)}} \geqslant \pi$$

by the convexity of the function $\frac{1}{\sqrt{1+x}}$, which implies the optimal inequality.

To prove the inequality (5.4), we use the maximal principle. At the point x_0 such that the equality of (5.4) holds, we have

$$\varphi(\arcsin u) \leqslant \tilde{u} - \tilde{u}\sqrt{1 - \tilde{u}^2}\varphi'(\arcsin u) + \frac{1}{2}(1 - \tilde{u}^2)\varphi''(\arcsin u).$$

We define a function

$$\psi(\theta) = \begin{cases} (\frac{4}{\pi}(\theta + \cos\theta\sin\theta) - 2\sin\theta)\cos^{-2}\theta, & \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \psi(\frac{\pi}{2}) = 1, \psi(-\frac{\pi}{2}) = -1 \end{cases}.$$

Then a straightforward computation gives

$$\begin{cases} \psi'(\theta) \geqslant 0 \\ \psi - \sin\theta + \sin\theta\cos\theta\psi' - \frac{1}{2}\cos^2\theta\psi'' = 0 \end{cases}.$$

Using the maximum principle we get $\varphi(\arcsin u) \leqslant \psi(\arcsin u)$. Since ψ is an odd function, the theorem is proved.

Remark Recently, Hang-Wang [**hang-wang**] proved that, in fact,

$$\lambda_1 > \frac{\pi^2}{d^2}$$

unless the manifold is of one dimensional.

The Li-Yau-Zhong-Yang estimate is still effective when the Ricci curvature is not "too negative". Namely, let

$$Ric(M) \geqslant -(n-1)K$$

for some constant $K > 0$. Then

$$\lambda_1 \geqslant \frac{\pi^2}{d^2} - (n-1)K.$$

Thus as long as the right hand side of the above is positive, the estimate is effective.

When K is very negative, we need to modify the basic gradient estimate. The following theorem belongs to Li-Yau.

Theorem 5.12

Let M be a compact Riemannian manifold without boundary. Assume that

$$Ric(M) \geqslant -(n-1)K$$

for $K > 0$. Then

$$\lambda_1 \geqslant \frac{1}{(n-1)d^2}\exp(-[1+(1+4(n-1)^2d^2K)^{1/2}]),$$

where d is the diameter of M.

Proof. Let u be the normalized first eigenfunction. That is

$$1 = \sup u > \inf u \geqslant -1.$$

Let $\beta > 1$. Consider

$$G(x) = \frac{|\nabla u|^2}{(\beta-u)^2}.$$

Let x_0 be the maximum point of $G(x)$. Then

$$\nabla G(x_0) = 0, \Delta G(x_0) \leqslant 0.$$

Since

$$G(x)(\beta-u)^2 = |\nabla u|^2,$$

we have

$$\Delta G(\beta-u)^2 + 2\nabla G\nabla(\beta-u)^2 + G\Delta(\beta-u)^2 = \Delta|\nabla u|^2.$$

Thus at x_0, we have

$$\begin{aligned} 0 &\geqslant \Delta|\nabla u|^2 - G\Delta(\beta-u)^2 \\ &= 2\sum u_{ij}^2 + 2\sum u_i u_{ijj} - 2G[(\beta-u)(-\Delta u) + |\nabla u|^2] \\ &= 2u_{ij}^2 + 2u_i(\Delta u_i)_i \\ &+ 2Ric(\nabla u, \nabla u) - 2G[\lambda_1 u(\beta-u) + |\nabla u|^2]. \end{aligned}$$

That is

$$u_{ij}^2 - \lambda_1|\nabla u|^2 - (n-1)K|\nabla u|^2 - G(\lambda_1 u(\beta-u) + |\nabla u|^2) \leqslant 0.$$

We choose a local coordinate system at x_0 such that $u_j = 0$ $(j = 2, \cdots, n)$, $u_1 = |\nabla u|$. Then $u_1 \neq 0$ (or otherwise, $G(x_0) = 0$ which is not possible). From $\nabla G(x_0) = 0$, we have

$$\begin{cases} u_{11} = -|\nabla u|^2(\beta-u)^{-1} \\ u_{1i} = 0, i \neq 1 \end{cases}.$$

Using the following trick

$$\begin{aligned} \sum_{i\cdot j=2}^{n} u_{ij}^2 &\geqslant \sum_{i=2}^{n} u_{ii}^2 \geqslant \frac{1}{n-1}\left(\sum_{i=2}^{n} u_{ii}\right)^2 = \frac{1}{n-1}(\Delta u - u_{11})^2 \\ &= \frac{1}{n-1}(\lambda u + u_{11})^2 = \frac{1}{n-1}(\lambda^2u^2 + 2\lambda u u_{11} + u_{11}^2) \\ &\geqslant \frac{u_{11}^2}{2(n-1)} - \frac{1}{n-1}\lambda^2 u^2, \end{aligned}$$

we have

$$\frac{1}{2(n-1)}\frac{|\nabla u|^4}{(\beta-u)^2}-\frac{\lambda^2u^2}{n-1}-(\lambda_1+(n-1)K)|\nabla u|^2-\lambda_1\frac{|\nabla u|^2u}{\beta-u}\leqslant 0.$$

Let $\alpha=u(\beta-u)^{-1}$. Then

$$\alpha\leqslant\frac{1}{\beta-u}\leqslant\frac{1}{\beta-1}.$$

Thus

$$\frac{1}{2(n-1)}G^2-\frac{\lambda^2}{n-1}\alpha^2-(\lambda_1+(n-1)K)G-\lambda_1G\alpha\leqslant 0,$$

which gives

$$G(x)\leqslant G(x_0)\leqslant 4(n-1)\left(\frac{\lambda\beta}{\beta-1}+(n-1)K\right).$$

Let l be the geodesic line connecting x_1 and x_2, where $u(x_1)=0$, $u(x_2)=\sup u=1$. Then we have

$$\log\frac{\beta}{\beta-1}\leqslant\int_\gamma\frac{|\nabla u|}{\beta-u}\leqslant\left[4(n-1)\left(\frac{\beta\lambda_1}{\beta-1}+(n-1)K\right)\right]^{1/2}d,$$

or in other words

$$\lambda_1\geqslant\frac{\beta-1}{\beta}\left[\frac{1}{4(n-1)d^2}\left(\log\frac{\beta}{\beta-1}\right)^2-(n-1)K\right]$$

Choosing β_0 such that the right side above maximized, we proved the theorem.

QED

The optimal estimate, in this direction, was obtained by Yang:

Theorem 5.13

Let M be a compact Riemannian manifold.

$$Ric(M)\geqslant-(n-1)K,\ \ (K>0),\ \ d=diam(M)$$

Then

$$\lambda_1\geqslant\frac{\pi^2}{d^2}\exp(-C_nKd^2)$$

where $C_n=\sqrt{n-1}$ for $n>2$ and $C_n=\sqrt{2}$ for $n=2$. ♡

The case when the Ricci curvature is positive is also very interesting. The following theorem of Lichnerowicz is well known.

Theorem 5.14

Let M be a compact Riemannian manifold. Assume that d is the diameter of the manifold and

$$Ric(M)\geqslant-(n-1)K>0.$$

for $K>0$. Then

$$\lambda_1\geqslant nK.$$ ♡

QED

In seeking the common generalization of the above theorem and the Zhong-Yang estimate, Peter Li (see [**yangd**]) proposed the following conjecture.

conjecture 2 *For a compact manifold with $Ric(M)\geqslant(n-1)K>0$ the first eigenvalue λ_1, with respect to* *the closed, the Neumann, or the Dirichlet Laplacian satisfies*

$$\lambda_1 \geqslant \frac{\pi^2}{d^2} + (n-1)K.$$

Here is $\partial M \neq \emptyset$, we assume that ∂M is convex.

Note that by Myer's theorem, we always have $\pi^2/d^2 \geqslant K$. Thus the conjecture, if true, will give a common generalization of the result of Lichnerowicz's and the one obtained by the gradient estimate.

In this direction, D-G Yang [**yangd**] proved that the first Dirichlet eigenvalue of the Laplacian satisfies

$$\lambda_1 \geqslant \frac{\pi^2}{d^2} + \frac{1}{4}(n-1)K,$$

if the manifold has weakly convex boundary. He also proved that the first closed eigenvalue and the first Neumann eigenvalue of the Laplacian satisfies

$$\lambda_1 \geqslant \frac{\pi^2}{d^2} + \frac{1}{4}(n-1)K,$$

if the manifold has convex boundary.

Ling [**ling**] was able to improve the above estimate into

$$\lambda_1 \geqslant \frac{\pi^2}{d^2} + \frac{31}{100}(n-1)K$$

Further improvements are possible, see Ling-Lu [**ling-lu**] for example.

We end this lecture by making the following

conjecture 3 *Let M be a compact Ricci flat Riemannian manifold such that*

$$\lambda_1 - \frac{\pi^2}{d^2} < \varepsilon$$

for a sufficiently small $\varepsilon > 0$. Then $M = S' \times M_0$, where M_0 is a Ricci flat compact Riemannian manifold.

5.4 The lower bound of $\lambda_2 - \lambda_1$

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $\lambda_1\lambda_2$ be the first two eigenvalues with respect to the Dirichlet boundary condition. Then

$$\lambda_2 - \lambda_1 \geq \frac{\bar{\pi}^2}{4d^2}$$

where d is the diameter.

Proof. Let u_2, u_1 be the second and the first eigenfunctions, respectively. By the variational principle, we must have $u_1(x) > 0$. Furthermore, near the boundary, we must have $\nabla u_1(x) \neq 0$. Thus the function

$$v = \frac{u_2}{u_1}$$

is smooth up to the boundary. By a simple computation, we obtained

$$\Delta v = -\lambda v - 2(\nabla v \cdot \nabla \log u_1)$$

where $\lambda = \lambda_2 - \lambda_1 > 0$. Let G be the function $\bar{\Omega} \to \mathbb{R}$

$$G = |\nabla v|^2 + \lambda(u-v)^2 \; \mu > \sup v.$$

Then G is a smooth function. Let $x_0 \in \bar{\Omega}$ be the maximum point of G. Then we claim that

$$G \leq \sup_{\Omega} \lambda(u - v)^2.$$

In fact, if $x_0 \in \partial\Omega$. By choosing an orthonormal frame $\{\ln_1, \ldots, \ln_n\}$ such that $\ln_1$ be the out normal direction if we let $\frac{\ln_1}{\partial\Omega} = \frac{\partial}{\partial x_1}$. Then

$$\frac{\partial G}{\partial x_1}(x_0) = 2\sum_{i=1}^{n} v_i v_{i1} - 2\lambda v_1(\mu - v).$$

We claim that

$$\frac{\partial G}{\partial x_1}(x_0) \geq 0.$$

To see this, we first observed that $\frac{\partial v}{\partial x_1} = 0$. This can be proved using the variational principle. Thus

$$\frac{\partial G}{\partial x_1}(x_0) = 2\sum_{i=2}^{n} v_i v_{i1}.$$

Using the definition of the second fundamental form, we get

$$v_{i1} = -\sum h_{ij} v_j.$$

Thus we have

$$0 \leq \frac{\partial G}{\partial x_1}(x_0) = -2\sum h_{ij} v_j.$$

Since Ω is assumed to be convex, all $v_i = 0$. Thus

$$G(x) \leq \sup_{x \in \bar{\Omega}} \lambda(\mu - v)^2.$$

Next we assume that $x_0 \in \Omega$. Then at the maximum point, we have

$$\nabla G(x_0) = 0, \ \Delta G(x_0) \leq 0.$$

We apply the standard gradient estimate: assume that at $x_o, \nabla v \neq 0$. Then we have

$$\begin{aligned} 0 &= G_i(x_0) = 2v_j v_{ji} - 2\lambda(\mu - v)v_i \\ 0 \geq \Delta G &= 2\sum v_{ij}^2 + 2v_j v_{jii} + 2\lambda \sum v_i^2 - 2\lambda(\mu - v)\sum v_{ii}. \end{aligned}$$

Since $\nabla v \neq 0$, we can choose local orthonormal frame such that $v_1 \neq 0, v_i = 0 (i > 1)$. Thus at x_0, we have

$$v_{11}(x_0) = \lambda(\mu - v)$$
$$v_{i1}(x_0) = 0 \ 2 \leq i \leq n.$$

Thus

$$0 \geq \Delta G = 2\sum v_{ij}^2 + 2\lambda(\mu - v)v - 4v_1^2(\log u_1)_{11}.$$

Using a result of Brascamp and Lieb, $\log u_1$ is a concave function. Thus $(\log u)_{11}(x_0) \leq 0$. Thus

$$\sum v_{ij}^2 + \lambda^2(\mu - v)\nu \leq 0$$

or, in other words,

$$v_{11}^2 + \lambda^2 v(\mu - v)|_{x_0} \leq 0.$$

which is not possible. Thus we must have $\nabla v = 0$ at x_0 and thus

$$G(x) \leq \sup \lambda(\mu - r)^2.$$

We have

$$\sqrt{\lambda} \geq \frac{|\nabla v|^2}{\sqrt{(\sup v - \inf v)^2 - (\sup v - v)^2}}.$$

Using the same method as in the estimate of the first eigenvalue, we get

$$\lambda_1 \geq \frac{\pi^2}{4d^2}.$$

Remark Using the method of Zhong-Yang, Zhong-Yu was able to modify the above estimate to

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2}.$$

However, even the above estimate is not optimal. Van de Berg (see also Yau, Problem session) conjectured that

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{d^2}.$$

The estimate is asymptotically accurate for a very thin rectangular.

Not much progress was made in the direction of this conjecture. In JFA, 176, 368–399 (2000). Banuelos and Mendez-Hernandez proved that if Ω is a convex domain in $\mathbb{R}^2$ which is symmetric with respect to both x- and y-axises, then the Van der Berg conjecture is true.

For a triangle, Lu and Rowlett proved the following.

Theorem 5.15

Let $\triangle$ be a triangle and let $d = d(\triangle)$ be the diameter of $\triangle$. That is, d is the longest side of the triangle. Then $\forall C > 0$.

$$(\lambda_2(\triangle) - \lambda_1(\triangle))d^2(\triangle) \leq C$$

is a compact set. ♡

Proof. When a sequence of triangles doesn't converge, then the smallest angle must go to zero. We assume that the smallest angle is $\alpha\pi$ and we assume that the diameter $d = 1$. QED

Figure 5.2: Figure

The key part of the proof is the following cutting lemma. Let P_1, P_2 be two points on BC such that

$$|P_1C| = \alpha^3,\ |P_2C| = 2\alpha^\varepsilon$$

where $\varepsilon > \frac{2}{q}$. For the sake of simplicity, we assume that $< ACB = \pi/2$. We are going to prove that the eigenvalues of ABC and $A\theta_2P_2C$ are almost the same, thus cutting an acute angle won't make too much difference.

We let $U - A\theta_1P_1C$ and $U' = A\theta_2P_2C$.

Let f_i be the eigenfunctions for $\lambda_i = \lambda_i(ABC), i = 1, 2$. The height of $V = ABC\backslash U$ is at most

$$(1 - \alpha^\varepsilon)\tan\alpha\pi \approx (1 - a^\varepsilon)a.$$

Above we have dropped a constant factor of π on the right side for simplicity in the arguments to follow. By the one-dimensional Poincaré inequality

$$\frac{\int_u |\nabla f_i|^2}{\int_V f_i^2} \geq \frac{1}{(1-\alpha^\varepsilon)^2\alpha^2}\frac{\int_{V'} |\nabla f_i|^2}{\int_{V'} |f_i|^2} \geq \frac{1}{(1-2d^\varepsilon)^2\alpha^2}$$

where as before $V = ABC\backslash U, V' = ABC\backslash V'$.

On the other hand, we always have

$$\frac{\int_u |\nabla f_i|^2}{\int_u f_i^2} \geq \frac{1}{\alpha^2}, \ \frac{\int_{u'} |\nabla f_i|^2}{\int_{u'} f_i^2} \geq \frac{1}{\alpha^2}.$$

We normalize f_i to be

$$\int_{ABC} f_i^2 = 1.$$

Let

$$\int_V f_i^2 = \beta.$$

Then by the variational principle, we have

$$\frac{\beta}{(1-a^\varepsilon)^2\alpha^2} + \frac{1-\beta}{\alpha^2} \leq \int_V |\nabla f_i|^2 + \int_u |\nabla f_i|^2 = \lambda_i.$$

By the asymptotic estimate of the eigenvalues using the Bessel functions, we have

$$\lambda_i(ABC) \sim \frac{1}{\alpha^2} + \frac{C_2}{\alpha^{\frac{4}{3}}}.$$

Therefore we have

$$\beta \leq \frac{\alpha^{\frac{2}{3}} C_2 (1-\alpha^\varepsilon)^2}{\alpha^\varepsilon (2-\alpha^\varepsilon)} \leq \frac{C_2}{2} \alpha^{\frac{2}{3}-\varepsilon}.$$

For simplicity in the arguments to follow, we will replace the constant factor $C_2/2$ by a constant factor 1, since we are considering $\alpha \to 0$.

Let ρ be a smooth compactly supported function so that

$$\rho|_v \equiv 1, \ \rho|_{v'} \equiv 0.$$

We may also assume that

$$|\nabla \rho| \leq \frac{1}{d^\varepsilon}, |\Delta \rho|, |\Delta \rho^2| \leq \frac{1}{d^{2\varepsilon}}.$$

For the arguments to follow, we use the sign convention for Euclidean Laplacian so that $-\Delta$ has positive spectrum. Note that

$$-(\rho f_i)\Delta(\rho f_i) = \lambda_i \rho^2 f_i^2 - f_i^2 \rho \Delta \rho - 2 f_i \rho \nabla \rho \nabla f_i.$$

In the estimates to follow, we will absorb all constants multiplying factors of a^δ for $\delta > 0$ into a factor of one, since it is clear that as $\alpha \to 0$, no generality is lost by this assumption.

1. Estimate for $\lambda_i(v')$. We may use ρf_i as a test function for the Rayleigh quotient on v' to estimate $\lambda_1(v')$ for above. By $(\star)$, we have

$$\lambda_1(v') \leq \lambda_1(ABC) + \frac{\int_{v'} -\rho \Delta \rho f_1^2 - 2\rho \nabla \rho f_1 \nabla f_1}{\int_{v'} \rho^2 f_1^2}.$$

Since

$$\int_{v'} \rho \nabla \rho f_1 \nabla f_1 = \frac{1}{2} \int_{v'} \nabla \rho^2 \nabla f_1^2 = -\frac{1}{2} \int_{v'} \Delta \rho^2 f_1^2$$

and $\nabla \rho, \Delta \rho = 0$ on v, we must have

$$\int_{u'} -\rho \Delta \rho f_1^2 - 2\rho \nabla \rho f_1 \nabla f_1 \leq \frac{1}{d^{2\varepsilon}} \int_{v'-v} f_1^2 \leq \alpha^{\frac{2}{3}-3\varepsilon}.$$

Noting that

$$\int_{u'} \rho^2 f_1^2 \geq \int_u \rho^2 f_1^2 \geq 1 - \beta \geq 1 - \alpha^{\frac{2}{3}-\gamma}$$

we then have

$$\lambda_1(v') \leq \lambda_1(ABC) + \frac{\alpha^{\frac{2}{3}-3\varepsilon}}{1-\alpha_{\frac{2}{3}-\varepsilon}} \leq \lambda_1(ABC) + \alpha^{\frac{2}{3}-3\varepsilon}.$$

By modifying the above argument, we are able to prove that

$$\lambda_2(v') - \lambda_2(ABC) \leq \alpha^{\frac{2}{3}-3\varepsilon} + (\lambda_2 - \lambda_1)\alpha^{\frac{2}{3}-3\varepsilon}.$$

Solving the above inequality, we have

$$\lambda_2 - \lambda_1 \geq \lambda_2(v') - \lambda_1(v') - 0(\alpha^{\frac{2}{3}-3\varepsilon}).$$

By the gap theorem, and using the fact that the diameter of v' is at most $4\alpha^{\varepsilon}$, we get

$$\lambda_2(v') - \lambda_1(v') \geq \frac{\pi^2}{16\alpha^{2\varepsilon}}$$

the compactness theorem follows.

5.5 Spectrum gap of the first two eigenvalues

5.5.1 Heat flow proof of a theorem of Brascamp-Lieb

The following result was first proved by Brascamp-Lieb [**blieb**]. In Singer-Wong-Yau-Yau [**swyy**], a simplified proof was given. In this subsection, we give a heat flow proof.

Theorem 5.16

Let Ω be a bounded convex domain of $\mathbb{R}^n$. Let u be the first Dirichlet eigenfunction with the eigenvalue λ_1. Then (up to a sign), u is positive and $\log u$ is concave. ♡

We begin by the following lemmas.

Lemma 5.2

Up to a sign, $u \geqslant 0$. ♡

Proof. Otherwise, we may use $|u|$ in place of u, From Kato's inequality, we have

$$|\nabla |u|| \leqslant |\nabla u|.$$

Thus we have

$$\frac{\int |\nabla |u||^2}{\int u^2} \leqslant \frac{\int |\nabla u|^2}{\int u^2}.$$

By the variational characterizing of the first eigenvalue, we know that the right side of the above is minimized. Thus the equality must hold and $|u|$ is an eigenfunction of the first eigenvalue.

The multiplicity of the first eigenfunction must be one. Thus up to a sign, the eigenfunction must be non-negative. QED

Lemma 5.3

$u > 0$ inside Ω. ♡

Proof. If not, assume that $u(x_0) = 0$ at a point x_0 in the interior of Ω. Let

$$u(x) = p^N(x) + O(x^{N+\varepsilon})$$

be the Taylor's expansion of the eigenfunction at x_0, where $p^N(x)$ is the polynomial of degree N. From the equation $\Delta u = -\lambda_1 u$, we have $\Delta p^2(x) = 0$. Since $u(x) \geqslant 0$, we must have $p^2(x) \geqslant 0$, a contradiction to the maximal principal. QED

Lemma 5.4

Using the above notations, we have

$$\frac{\partial u}{\partial n} < 0$$

on $\partial\Omega$, the boundary of Ω.

Proof. This follows from the strong maximum principle. By the above lemma, we have

$$\frac{\partial u}{\partial n} \leqslant 0$$

If $\frac{\partial u}{\partial n} = 0$, then since the function u vanishes on the boundary, we have $\nabla u = 0$ at the point. Using the Taylor's expansion for the boundary point, we get $p^2 \geqslant 0$ and $\Delta p^2 = 0$. Since p^2 is harmonic, by the strong maximum principle, $\partial p^2/\partial n < 0$ unless $p^2 = 0$. But if $p^2 \equiv 0$ then $u \equiv 0$. This completes the proof. QED

QED

We now consider the heat flow

$$\frac{\partial u}{\partial t} = \Delta u + \lambda_1 u, \quad u|_{\partial\Omega} = 0.$$

we have

Lemma 5.5

For any smooth initial function $u_0 > 0$, the flow exists and converges to the first eigenfunction.

Proof. Let

$$u_1 = \sum a_j f_j(x),$$

where $f_j(x)$ is the eigenfunction of the j-th eigenvalue. Then the solution of the equation is

$$u(t, x) = \sum a_j e^{-(\lambda_j - \lambda_i)t} f_j(x)$$

Obviously, we have

$$\lim_{t\to\infty} u(t, x) = a_1 f_1(x).$$

If we choose u_0 such that $a_1 \neq 0$, then the flow converges to the first eigenfunction. $f_1(x) > 0$ by the above lemmas. By our choice of u_0

$$a_1 = \int_\Omega u_0 f_1(x) > 0.$$

The proof is complete. QED

Lemma 5.6

Let w be any smooth positive function on Ω such that

$$w|_{\partial\Omega} = 0 \text{ and } \nabla w|_{\partial\Omega} \neq 0$$

Then near the boundary of Ω, $\log w$ is concave.

Proof. Since w is a smooth function vanishes on the boundary, it can be viewed as the defining function of Ω. By the implicit function theorem, we solve the equation

$$w(x_1, \cdots, x_n) = 0$$

to get the function

$$x_n = x_n(x_1, \cdots, x_{n-1}).$$

If Ω is convex, then

$$\frac{\partial^2 x_n}{\partial x_i \partial x_j} > 0,$$

is a positive definite matrix. Using the chain rule, the above inequality is equivalent to

$$-\frac{w_{ij}}{w_n} + \frac{w_{in}w_j}{w_n^2} + \frac{w_{jn}w_i}{w_n^2} - \frac{w_i w_j w_{nn}}{w_n^3} > 0.$$

where $1 \leqslant i, j \leqslant n-1$. However, if we allow i or j to be n, then as the $n \times n$ matrix, we still have

$$-\frac{w_{ij}}{w_n} + \frac{w_{in}w_j}{w_n^2} + \frac{w_{jn}w_i}{w_n^2} - \frac{w_i w_j w_{nn}}{w_n^3} \geqslant 0.$$

Moreover, for any $(a_1, \cdots, a_n)$, if $\sum a_j w_j = 0$, we have

$$-w_{ij}a_i a_j \geqslant \varepsilon |a|^2$$

for some positive $\varepsilon > 0$. A generic vector has the form $a + \mu b$, where $b = (w_1, \cdots, w_n)$. For w small enough, we have

$$-w_{ij}(a_i + \mu b_i)(a_j + \mu b_j) + \frac{1}{w}|\mu|^2|b|^4 > 0.$$

Thus $\nabla^2 \log w$, whose matrix entries are

$$\frac{w_{ij}}{w} - \frac{w_i w_j}{w^2},$$

is negative definite for w small enough. QED

Remark The above proof is purely elementary. We can use differential geometry to give another proof. Assume $e_1, \cdots, e_n$ are the local frame fields at the boundary point such that e_n is normal to the boundary and the rest are tangent to the boundary. Then for $1 \leqslant i, j \leqslant n-1$, we have

$$\nabla^2(e_i, e_j)w = e_i(e_j(w)) - \nabla e_i e_j w = h_{ij}\frac{\partial w}{\partial n},$$

where h_{ij} is the second fundamental form, and $\frac{\partial}{\partial n}$ is the outward normal vector field. Thus $\frac{\partial w}{\partial n} \leqslant 0$. To prove

$$\frac{w_{ij}}{w} - \frac{w_i w_j}{w^2}$$

is negative definite, we write $V = V_1 + \mu e_n$, where V_1 is tangent to $\partial\Omega$. Then there is a constant C such that

$$\nabla^2 \log w(V, V) \leqslant w^{-1}\left|\frac{\partial w}{\partial n}\right|(\pi(V_1, V_1) + C\mu\|V_1\| + C\mu^2) - w^{-2}\mu^2\left|\frac{\partial w}{\partial n}\right|^4.$$

Since $\frac{\partial w}{\partial n} \neq 0$, for points sufficient close to the boundary, w is small enough. Using the Cauchy inequality, we can prove the above negativeness.

Now we begin to prove the theorem: We will choose a function u_0 such that $u_0 > 0$ and $\log u_0$ is concave. Then we shall prove that the log-concavity is preserved under the heat flow. The theorem thus follows from the above lemmas.

To construct the required u_0, we first pick up any smooth function $w > 0$ on Ω with $w|_{\partial\Omega} = 0$, $\nabla w|_{\partial\Omega} \neq 0$. Let

$$u_0 = we^{-C\sum x_j^2}$$

for a constant $C > 0$ sufficiently large. By the above lemma, the function is log-concave near the boundary. Away from the boundary, since $w > \delta > 0$ for some constant $\delta > 0$, we can choose C large enough so that u_0 is log-concave.

Using the matrix version of the maximum principle, we can prove that the flow keeps the log-concavity. Let $\varphi = \log u$, where u is the solution of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u + \lambda_1 u.$$

The flow of φ is

$$\frac{\partial \varphi}{\partial t} = \Delta \varphi - |\nabla \varphi|^2 - \lambda_1. \tag{5.5}$$

By the maximum principle, if T is the first time the matrix $\nabla^2\varphi$ is generated, then there is an $x_0 \in M$ and a direction i such that

$$-\varphi_{ii} = 0$$

and for other j's, $-\varphi_{jj} \geqslant 0$. Moreover, at x_0, $\varphi_{iik} = 0$, $\frac{\partial \varphi_{ii}}{\partial t} \leqslant 0$ and $\Delta\varphi_{ii} \geqslant 0$.

Differentiating (5.2) by i, i, we have

$$0 \geqslant \frac{\partial \varphi_{ii}}{\partial t} = \Delta\varphi_{ii} - 2\varphi_k\varphi_{kii} - 2\varphi_{ki}^2 \geqslant -2\varphi_{ki}^2.$$

By the convexity, $\varphi_{ki}^2 \leqslant \varphi_{ii}\varphi_{kk} = 0$. Thus $\varphi_{ki} = 0$ for $k \neq i$. The theorem follows from the strong maximum principle.

This completes the proof of the Brascamp-Lieb theorem.
QED

5.5.2 Gap of the first two eigenvalues

For the sake of simplicity, we only consider bounded smooth domain in $\mathbb{R}^n$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let λ_1, λ_2 be the Dirichlet first two eigenvalues. Since λ_1 must be simple (of multiplicity one) we have

$$\lambda_2 - \lambda_1 > 0.$$

The question we would like to answer is that, how to get the lower bound estimate of $\lambda_2 - \lambda_1$?

Let φ_1, φ_2 be the eigenfunctions with respect to λ_1, λ_2. We set

$$\varphi = \frac{\varphi_2}{\varphi_1}.$$

Then a straightforward computation gives

$$\Delta\varphi + \partial\nabla \log\varphi_1 \nabla\varphi = -\lambda\varphi,$$

where $\lambda = \lambda_2 - \lambda_1$. Moreover, since $\varphi_1|_{\partial\Omega} = 0$, we must have

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0.$$

We have a notion of Bakry-Émery Ricci tensor. Let

$$f = \varphi_1^2$$

Then the Bakry-Émery Laplacian is defined as

$$\Delta f = \Delta + \nabla \log f \nabla$$

which is a self-adjoint operator with respect to the volume form $\varphi_1^2 dV$.

The Bakry-Émery Ricci tensor is defined as

$$-\nabla^2 \log f$$

By the Brascamp-Lieb theorem, the Barkey-Emery Ricci curvature is non-negative.

We make the following conjecture:

conjecture 4 *The method of gradient estimates can be generalized to the Bakry-Émery case without additional difficulties.*

The above conjecture, if true, will give a unified proof between the estimation of the first eigenvalue and gap estimate.

We can go one more step further. Since the Barkey-Emery Ricci tensor comes from the setting $(M, ds^2, e^f dV)$, which can be considered as the limit of the wrap product

$$e^f dr^2 + ds^2$$

we make the following definition.

Let

$$\Omega_\varepsilon = \{(x, y) | x \in \Omega, 0 \leqslant y \leqslant \varepsilon\varphi_1^2(x)\}.$$

Let μ_ε be the first Neumann eigenvalue. Then

conjecture 5 *Using the above notations, we have*

$$\lambda_2 - \lambda_1 = \lim_{\varepsilon \to 0} \mu_\varepsilon.$$

Similarly, we make the following

conjecture 6 *Let $(\mu_\varepsilon)^k$ be the k-th Neumann eigenvalue of Ω_ε. Then*

$$\lim_{\varepsilon \to 0} (\mu_\varepsilon)^k = \lambda_{k+1} - \lambda_1.$$

We can prove the following

Theorem 5.17

$$\lambda_2 - \lambda_1 \geqslant \lim_{\varepsilon \to 0} \mu_\varepsilon.$$

♡

Proof. We first note that $\frac{\partial h}{\partial n}|_{\partial\Omega_\varepsilon} = 0$ on the part $y = \varepsilon\varphi_1(x)^2$ can be written as

$$\nabla h(\varepsilon\nabla\varphi_1^2, -1) = 0.$$

We define $\widetilde{U}_\varepsilon$ as follows

$$\widetilde{U}_\varepsilon = \varphi + y^2 \nabla \log \varphi_1 \nabla\varphi.$$

Then a straightforward computation gives

$$\begin{cases} \Delta\widetilde{U}_\varepsilon + \lambda\widetilde{U}_\varepsilon &= O(\varepsilon^2) \\ \frac{\partial \widetilde{U}_\varepsilon}{\partial n}\Big|_{\partial\Omega_\varepsilon} &= O(\varepsilon^2) \end{cases}.$$

From the above, we have

$$\int_{\Omega_\varepsilon} \widetilde{U}_\varepsilon = O(\varepsilon^2).$$

Let α be a number such that

$$\int_{\Omega_\varepsilon} (\widetilde{U}_\varepsilon - \alpha) = 0.$$

Then $\alpha = O(\varepsilon)$. By the variational principle, we have

$$\mu_\varepsilon \leqslant \frac{\int_{\Omega_\varepsilon} |\nabla \widetilde{U}_\varepsilon|^2}{\int_{\Omega_\varepsilon} (\widetilde{U}_\varepsilon - \alpha)^2}.$$

However, since

$$\int_{\Omega_\varepsilon} \widetilde{U}_\varepsilon^2 \geqq C\varepsilon.$$

We have

$$\mu_\varepsilon \leqslant \frac{\int_{\Omega_\varepsilon} \lambda \widetilde{U}_\varepsilon^2}{\int_{\Omega_\varepsilon} \widetilde{U}_\varepsilon^2} + O(\varepsilon).$$

and the theorem is proved. QED

QED

Remark We consider the wrap product

$$e^f dr^2 + d^2$$

over Ω_ε, we see the relation between two settings. This also gives the relation between the Barkey-Emery geometry with respect to the ordinary Riemannian geometry.

Some applications.

Theorem 5.18 (Singer-Wong-Yau-Yau, Yu-Zhong)

Let Ω be a convex domain in $\mathbb{R}^n$. Then

$$\lambda_2 - \lambda_1 \geqslant \frac{\pi^2}{d^2}.$$

♡

Proof. We consider the domain Ω_ε and let U_ε be the first Neumann eigenfunction. By what we proved in the last lecture, we have

$$\mu_\varepsilon \geqslant \frac{\pi^2}{d_\varepsilon^2}$$

where d_ε is the diameter of Ω_ε. Since $d_\varepsilon \to d$, the theorem is proved. QED

H

ow to recover the recent result of Yau?

We are going to use the following result of Chen-Li.

Theorem 5.19

Let M be an m-dimensional domain in $\mathbb{R}^m$. Let M be star-shaped. Let R be the radius of the largest ball centered at $p \in M$ contained in M and let R_0 be the smallest ball centered at p containing M. Then there is a constant C, depending only on m, such that

$$\eta_1 \geqslant C \frac{R^m}{R_0^{m+2}}$$

♡

Let U_ε be the first Neumann eigenfunction of Ω_ε. Then asymptotically we can write

$$U_\varepsilon \sim \varphi + y^2 \nabla \log \varphi_1 \nabla \varphi$$

In general, power series method can be used to prove the conjecture.
Remark Conjecture 5 was proved by Lu-Rowlett [**lu-rowlett**].

Appendix: Eigenvalues of collapsing domain

It is important to study the asymptotic behavior of eigenvalues when a domain collapses. In this appendix, we give some preliminary results in this direction.

We begin with the following observation. Consider the sector
What is the asymptotic behavior of the Dirichlet eigenvalues when $\alpha \to 0$?

As is well-known, the eigenfunctions of the sector are of the form

$$f(r) \sin \frac{\theta}{\alpha}$$

where $f(r)$ is the so-called Bessel function

$$f'' + \frac{1}{r} f' + \frac{1}{r^2}(r^2 - \frac{1}{\alpha^2}) f = -\lambda f$$

If we set

$$g(r) = \sqrt{r} f(r)$$

Then the corresponding equation becomes

$$-g''(r) + \frac{1}{r^2}(\frac{1}{\alpha^2} - \frac{1}{4}) g = \lambda g$$

When $\alpha \to 0$, we take the following renomorization: set $1 - r = \alpha^{2/3} x$. Then we have

$$\frac{1}{r^2} - 1 \sim 2x\alpha^{2/3}$$

Let $\lambda = \widetilde{\lambda} + \frac{1}{\alpha^2}$. Then we have

$$-g''(r) + \left(\frac{1}{r^2} \left(\frac{1}{\alpha^2} - \frac{1}{4} \right) - \frac{1}{\alpha^2} \right) g = \widetilde{\lambda} g$$

and if $\alpha \to 0$, we have

$$-g'' + 2xg = \lambda g$$

This is the Airy's function.

Friedlander and Solomyak was able to generalize the above result in the following setting:

Let $h(x) > 0$ be a piecewise linear function defined on $[-a, b]$, where $a, b > 0$. We assume that

$$h(x) = \begin{cases} M - C_+ x & x > 0 \\ M - C_- x & x < 0 \end{cases}$$

where the choice of M, C_+, C_- are so that $h(-a) = h(b) = 0$.

For any positive $\varepsilon > 0$, let

$$\Omega_\varepsilon = \{(x, y) | x \in I, 0 \leqslant y \leqslant \varepsilon h(x)\}$$

Theorem 5.20 (Friedlander-Solomyak)

Let $\alpha = 2/3$. Let $l_j(\varepsilon)$ be the Dirichlet eigenvalues of Ω_ε. Let

$$\mu_j = \lim_{\varepsilon\to 0} \varepsilon^{2\alpha}\left(l_j(\varepsilon) - \frac{\pi^2}{M^2\varepsilon^2}\right).$$

exists. Then $\{\mu_j\}$ are eigenvalues of the Schrödinger operator H on $L^2(\mathbb{R})$, where

$$H = -\frac{d^2}{dx^2} + q(x)$$

where

$$q(x) = \begin{cases} 2\pi^2 M^{-3} C_+ x & x > 0 \\ 2\pi^2 M^{-3} C_- x & x < 0 \end{cases}$$

Note that if $C_+ = C_-$, then H turns to the harmonic oscillator. If $C_+ = +\infty$, then it turns to the above discussed case.

Theorem 5.21 (Lu-Rowlett)

Let M be a triangle and let d be the diameter of M. Then

$$d^2(\lambda_2 - \lambda_1) \to +\infty$$

if the triangle collapses. In other words, the "gap" function is a proper function on the moduli space of triangles.

The original proof of the above result is independent to the work of Friedlander *et.la.*

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et $h(x) > 0$ be a piecewise smooth function on a bounded domain Ω of $\mathbb{R}^n$. Let

$$\Omega_\varepsilon = \{(x, y) | x \in \Omega, 0 \leqslant y \leqslant \varepsilon h(x)\}$$

Then what is the asymptotical behavior of the Dirichlet (Neumann) eigenvalues of Ω_ε.

The question is very important in answering the following conjecture of Van den Berg and Yau.

conjecture 7 *Let Ω be a convex bounded domain in $\mathbb{R}^n$. Then*

$$\lambda_2 - \lambda_1 \geqslant \frac{3\pi^2}{d^2}$$

The conjecture is asymptotically optimal for thin rectangles.

In some sense, the result of Friedlander gives the compactification of the Laplacians on the moduli space. Namely, if $\alpha \to 0$, then the Laplace operators tend to the Schrödinger operator defined above.

As an application, we relate the result to the following conjecture.

conjecture 8 *Does it exist a number N such that the first N Dirichlet eigenvalues determine to triangle.*

The result of Chang-Deturck gives partial answer to the above conjecture:

Theorem 5.22

There exists $N = N(\lambda_1, \lambda_2)$ such that $\lambda_1, \cdots, \lambda_N$ determines the triangle.

Unfortunately, if $\alpha \to 0$, $N = N(\lambda_1, \lambda_2) \to +\infty$.

In order to solve this "hearing the shape of a triangle" problem, we consider the following parametrization of the moduli space of a triangle when one of the angle is small
where $b \leqslant 1$, and we use (α, b) as coordinates.

conjecture 9 *Let $\xi = \alpha^{2/3}$. Define two functions*

$$\begin{aligned} P(\zeta, b) &= \lambda_2/\lambda_1 \\ Q(\zeta, b) &= (\lambda_3 - \lambda_2)/(\lambda_2 - \lambda_1) \end{aligned}$$

Then P, Q are analytic functions on $[0, \varepsilon) \times [\frac{1}{2}, 1]$.

Note that by the result of Friedlander-Solomyak, we know that

$$\begin{aligned} P(\zeta, b) &= 1 + a\zeta + O(\zeta) \\ Q(\zeta, b) &= 1 + o(1) \end{aligned}$$

The hearing problem is implied by proving

$$(\zeta, b) \mapsto (P, Q)$$

is invertible.

Since the limit of the Laplacian is the 1-d Schrödinger operator, we must first solve the problem of hearing the Schrödinger operator. Gelfand-Levitan theory doesn't apply directly here.

Chapter 6 Essential Spectrum on complete non-compact manifold

6.1 Spectrum on complete non-compact manifolds

Unlike in the case of compact manifold, in general, a complete manifold doesn't admit any eigenvalues. For example, there are no L^2-eigenvalue on $\mathbb{R}^n$. That is, $\forall \lambda \in \mathbb{R}$. If

$$\Delta f + \lambda f = 0$$

and $f \in L^2(\mathbb{R}^n)$, then $f \equiv 0$.

Escobar proved that if M has a rotational symmetric metric, then there is no L^2-eigenvalue. Let Δ be the Laplace operator on a complete non-compact manifold M. By the argument before, Δ naturally extends to a self-adjoint densely defined operator, which we still denote Δ for the sake of simplicity.

It is well-known that there is a spectrum measure such that

$$\Delta = \int_0^\infty \lambda dE$$

(Here we assume Δ is the geometric Laplacian, which is a positive operator).

Define $e^{-\Delta t}$ for any $t > 0$. Obviously, it is a bounded operator. Thus by the Hahn-Banach $e^{-\Delta t}$ is a bounded operator.

The heat kernel is defined as

$$e^{-\Delta t} f(x) = \int H(x, y, t) f(y)\, dy.$$

The Green's function is defined as

$$G(x, y) = \int_0^\infty H(x, y, t)\, dt.$$

Of course, we need to prove the existence of these functions when M is complete non-compact.

The pure point spectrum of Δ are those $\lambda \in \mathbb{R}$ such that

1. There exist a L^2 function $f \neq 0$ such that

$$\Delta f + \lambda f = 0$$

2. The multiplicity of λ is finite
3. In a neighborhood of λ, it is the only spectrum point.

We define

$$\rho(\Delta) = \left\{ y \in \mathbb{R} \;\middle|\; \begin{array}{r} (\Delta - y)^{-1} \text{ is a bounded} \\ \text{operator} \end{array} \right\}$$

$\sigma(\Delta) = \mathbb{R} - \rho(\Delta)$ is the spectrum set of Δ. From the above discussion, $\sigma(\Delta)$ decomposes $\emptyset$ as the union of pure point spectrum, and the so-called essential spectrum, which is by definition, the complement of pure point spectrum.

Using the above definition, $\lambda \in \sigma(\Delta)$ belongs to the set $\sigma_{ess}(\Delta)$, if either

1. λ is an eigenvalue of infinite multiplicity, or
2. λ is the limiting point of $\sigma(\Delta)$.

The following theorems in functional analysis characterizing the essential spectrum.

Theorem 6.1

A necessary and sufficient condition for the interval $(-\infty, \lambda)$ to intersect the essential spec of an self-adjoint densely defined operator A is that, for all $\varepsilon > 0$, there exists an infinite dimensional subspace $G_\varepsilon \subset \mathrm{Dom}(A)$, for which $(Af - \lambda f - \varepsilon f, f) < 0$.

Theorem 6.2

A necessary and sufficient condition for the interval $(\lambda - \sigma, \lambda + \sigma)$ to intersect the essential spectrum of A is that there exists an infinite dimensional subspace $G < \mathrm{Dom}(A)$ for which $\|A - \lambda I)f\| < \sigma\|f\|, f \in G$.

For reference, see Donnelly, Topology 20, 1–14, 1981.

Using the above result, we give the following variational characterization of the lower bound of spectrum and the lower bound of essential spectrum.

Theorem 6.3

Let

$$\lambda_0 = \inf_{f \in C_0^\infty(M)} \frac{\int_M |\nabla f|^2}{\int_M f^2}$$

and

$$\int_{ess} = \sup_k \inf_{f \in C_0^\infty(M \setminus k)} \frac{\int_M |\nabla f|^2}{\int_M f^2}$$

where k is a compact set running through an exhaustion of the manifold. Then λ_0 and λ_{ess} are the lower bound of $\sigma(\Delta)$ and $\sigma_{ess}(\Delta)$ respectively.

Proof We first prove the formula for λ_{ess}. Let

$$\lambda'_{ess} = \inf \sigma_{ess}(\Delta)$$

is $\sigma_{ess}(\Delta) = \emptyset$, we define, $\lambda'_{ess} = +\infty$. By the definition, $\forall \epsilon > 0, (-\infty, \lambda'_{ess} + \varepsilon) \cap \sigma_{ess}(\Delta) \neq \emptyset$. Then we can find infinite dimensional space V such that for any $f \in V$

$$\langle \Delta f - (\lambda'_{ess} + \varepsilon) f, f \rangle < 0. \tag{$\star$}$$

Without loss of generality, we may assume that

$$\int_M f^2 = 1.$$

Also, without loss of generality, we may assume that all elements in V are smooth.

We leave as an exercise, to prove that $(\star)$ implies

$$\int_M |\nabla f|^2 \leq (\lambda'_{ess} + \varepsilon).$$

Now we assume that k is a compact set. Let k' be a larger ball containing k. Let ρ be the cut-off function such that $\rho \equiv 1$ on k but $\rho \equiv 0$ outside k'. We claim (***) that $\forall \varepsilon > 0$, there is an $f \in V$ with $\int f^2 = 1$ but

$$\int_M \rho^2 f^2 < \varepsilon.$$

If the above is not true, then for any $f \in V$.

$$\int_M \rho^2 f^2 \geq \varepsilon_0, \text{ if } \int f^2 = 1.$$

Since the set $f \in V$ is of infinite dimensional, the set ρf is of infinite dimensional. Thus we can find an orthogonal basis

$$\int \rho^2 f_i f_j = 0 \;\; i \neq j$$

while we can still keep $\boxed{\int f_i^2 = 1}$. We consider

$$\begin{aligned} \int |\nabla(\varphi f_i)|^2 &\leq 2\int |\nabla\rho|^2 f_i^2 + 2\int \rho^2 |\nabla f_i|^2 \\ &\leq 2C + 2(\lambda'_{ess} + \varepsilon). \end{aligned}$$

Thus

$$\frac{\int \nabla(\rho f_i)^2}{\int(\rho f_i)^2} \leq \frac{2C + 2(\lambda'_{ess} + \varepsilon}{\varepsilon_0}$$

for infinitely dimensional space. This is a contradiction because on the compact set k' the eigenvalues go to infinity.

With the above preparation, we can prove our theorem $\forall \varepsilon > 0$. We find an f with $\int f^2 = 1$ but

$$\int \rho^2 f^2 < \varepsilon.$$

Consider $\rho_1 = 1 - \rho$

$$\begin{aligned} \int (\nabla(\rho_1 f)|^2 &= \int \rho_1^2 |\nabla f|^2 + 2\int \rho_1 f \nabla\rho_1 \nabla f \\ &\quad + \int f^2 |\nabla \rho_1|^2. \end{aligned}$$

Using the above inequality, we have

$$\int f^2 |\nabla\rho_1|^2 \leq C\varepsilon$$

$$2\int \rho_1 f \nabla\rho_1 \nabla f = -\frac{1}{2}\int \Delta\rho_1^2 \cdot f^2$$
$$\leq C\varepsilon$$

because the supp of $\nabla\rho_1, \Delta\rho_1^2$, are within k'. Thus

$$\begin{aligned} \int |\nabla(\rho_1 f)|^2 &\leq (\lambda'_{ess} + \varepsilon) + C\varepsilon \\ \int \rho_1^2 f^2 &\geq 1 - \varepsilon. \end{aligned}$$

Using the definition, we get

$$\lambda_{ess} \leq \lambda'_{ess} + \varepsilon$$

and thus $\lambda_{ess} \leq \lambda'_{ess}$. The other direction is easier to prove.

Using the same method, we can prove the case for λ_0, provided that we need a similar theorem like Theorem 1.

It is an interesting question to compute the set of essential. It is particularly interesting to get the lower bound estimate for the essential spectrum because of the following theorem.

Theorem 6.4

Suppose $\lambda_0 < \lambda_{ess}$, then λ_0 is an eigenvalue of M with finite dimensional eigenspace. That is, there exists an L^2 function $f \neq 0$, such that

$$\Delta f = -\lambda_0 f$$

which is a very strong result. ♡

In what follows we use $\mathbb{R}^n$ to elaborate our theorem.

First let's compute λ_{ess} for $\mathbb{R}^n$. We claim that $\lambda_{ess} = 0$. By definition, this is equivalent to say that $\forall \varepsilon > 0 \,\forall\, R > 0$, there is a function $f \in C_0^\infty(\mathbb{R}^n - B(R))$ such that

$$\int |\nabla f|^2 < \varepsilon \int f^2.$$

This is obvious: let $B(x_0, R')$ be a ball of radius R' with center x_0 such that $|x_0| > 2R + 1 + R'$. Then

$$B(x_0, R') \subset \mathbb{R}^n \backslash (R).$$

Let f be the first Dirichlet eigenfunction of $B(X_0, R')$. Then if $R' \to \infty$

$$\int |\nabla f|^2 / \int f^2 < \varepsilon$$

zero extending f to $\mathbb{R}^n \backslash B(R)$ we get the result.

Using Theorem 2, we can even prove that

$$\sigma_{ess}(\Delta) = [0, +\infty).$$

To prove the above, we make the following observation $\forall \lambda > 0, \forall m \in \mathbb{Z}$, we can find a square of size $m\bar{\pi}\sqrt{\frac{\lambda}{n}}$ such that

$$f = \sin\sqrt{\frac{\lambda}{n}} x_1 \cdot \sin\sqrt{\frac{\lambda}{n}} x_2 \ldots \sin\sqrt{\frac{\lambda}{n}} x_n$$

is an eigenfunction with Dirichlet condition: $\Delta f + \lambda f = 0$.

However, we can't use Theorem 2 directly. The reason is that f is **not** second differentiable near the boundary. Thus we need to use cut-off function. Without loss of generality, we may assume that the square is in the first quadratic. We denote such a square to be s.

Figure 6.1: S Square

Let s_1 be a square with the same center as s_1 but with smaller size. We assume that the distance of the boundary of s_1 to s is d, which is to be determined later.

We choose a cut-off function ρ such that $\rho \equiv 1$ on s_1 and $\rho \equiv 0$ outside s. By the same argument as before, we can prove that

$$\|\Delta(\rho f) + \lambda \rho f\|_{L_2} \subset \varepsilon \|\rho f\|_{L_2}.$$

Thus $ess(\Delta) = [0, +\infty)$.

Unfortunately, the above argument doesn't apply to the general case. Thus the following result of *** is very surprising and interesting.

Theorem 6.5 (J-P. Wang)

Let M be a complete manifold with non-negative Ricci curvature. Then

$$ess(\Delta) = [0, \infty)$$

♡

Note that λ_{ess} was known before, e.g., P.Li-Wang, Brooks.

In this lecture, we study the essential spectrum of a complete non-compact manifold with non-negative Ricci curvature. We are going to prove that the set of essential spectrum is $[0, \infty)$. The reference papers for this lecture are

1. Sturm, J. Funct. Anal. 118, 442–453, 1993
2. Wang, Math. Research Letters. 4, 473–479, 1997

Through this lecture, M is a complete Riemannian manifold with non-negative Ricci curvature. We say the volume (M, g) grows uniformly sub-exponentially, if for any $\varepsilon > 0$, there is a constant $C < \infty$ such that for all $r > 0$ and all $x \in M$

$$v(B_r(x)) \le Ce^{\varepsilon r} v(B, (x))$$

Theorem 6.6

If the volume of (M, g) grow uniformly sub exponentially, then the spectrum $\sigma(\Delta p)$ of Δp acting on $L^p(M)$ is independent of $p \in [1, \infty)$. In particular, it is a subset of the real line.
Note that by the Bishop volume comparison theorem, Ricci non-negative implies uniformly sub-exponentially volume growth.

The theorem can be proved using the resolvent estimates, which are based on the previous heat kernel estimate.

We begin with the following.

Lemma 6.1

If the volume of (M, g) grows uniformly subexponentially, then for any $\varepsilon > 0$

$$\sup_{x \in M} \int_M e^{-\varepsilon d(x,y)} (v(B_1(x)))^{-\frac{1}{2}} (v(B_1(y)))^{-\frac{1}{2}} d\nu(y) < \infty.$$

Proof: We take $r = d(x, y)$. Then since

$$B_1(y) \subset B_{r+1}(x)$$

we must have

$$\nu(B_1(y))^{-\frac{1}{2}} \ge \nu(B_{r+1}(x))^{-\frac{1}{2}} \ge Ce^{-\frac{1}{2}(r+1)} \nu(B_1(x))^{-\frac{1}{2}}$$

for any x. Thus the integration in the lemma if less than

$$C \int_M e^{-\varepsilon r} \cdot e^{\frac{1}{2}\varepsilon(r+1)} \nu(B_1(x))^{-1} dy.$$

We let $f(r) = \text{vol}(\partial B_r(x))$ and $F(r) = \int_0^r f(t)\, dt$. Then up to a constant, the above expression is less than

$$(\nu(B_1(x)))^{-1} \int_0^\infty e^{-\frac{1}{2}\varepsilon r} f(f)\, dr.$$

By volume comparison again $f(r) \le cr^{n-1}\nu(B_1(x))$. The lemma follows.

Lemma 6.2

For any $\beta > 0$, there is an $n \in N, \alpha < 0, c < \infty$ such that the integral kernel $g_\alpha^{(\frac{n}{2})}(x, y)$ of $(\Delta - \alpha)^{-\frac{n}{2}}$ exists and satisfies

$$g_\alpha^{\frac{n}{2}}(x, y) \le Ce^{-\beta d(x,y)} \varphi(x)^2$$

where $\varphi(x) = (\nu(B_1(x)))^{-\frac{1}{2}}$.
The lemma was proved on page 67, using the heat kernel estimates.

Before going further, let's make some remarks on the kernel of an operator. Let A be an operator on functions. If there is a function $g(x, y)$ such that

$$Af(x) = \int_M g(x, y)f(y)\, dy$$

then we call $g(x, y)$ the "kernel" of A. However, in general, the kernel doesn't exist.

To see why the kernel in general does not exist, we let $\xi \in \rho(\Delta)$. To be more specific, Δ is an operator on $L^2(M)$ so we assume that $\xi \in \rho(\Delta_2)$. The operator $(\Delta_2 - \xi)^{-1}$ is called the resolvent. It is a bounded operator form $L^2(M) \to L^2(M)$. (By Hahn-Banach theorem, it can be extended to whole $L^2(M)$. However, in general, the kernel doesn't exist: if not, we let $f_i \to \delta$ be a sequence converges to the δ-function in $L^1(M)$ then $\Delta_2 f_i$ could have been bounded. We need an estimate to extend Δ_2 from $L^2(M)$ to $L^1(M)$.

Lemma 2 told us that for $\alpha < 0$, $(\Delta - \alpha)^{-\frac{n}{2}}$ has a kernel. The operator can be extended to $L^1(M)$. Further more, the kernel exponentially decays.

The Laplacian we used here is the geometric Laplacian. That is, it is a positive operator.

For our purpose, we just need to prove $\sigma(\Delta_1) \subset \sigma(\Delta_2)$, which is also the major part of the paper of Sturm.

Recall that $x \in (A)$, if, $(A - xI)^{-1}$ is a bounded operator. We define the spectrum $\sigma(A) = \mathbb{R} - \rho(A)$. Note that $\sigma(\Delta_2) \subset [0, \infty)$. Then for $\alpha < 0$, $(\Delta_2 - \alpha I)^{-1}$ is bounded. Since

$$\Delta - \alpha I = (\Delta - \zeta I)(1 - (\alpha - \zeta)(\Delta - \zeta I)^{-1}).$$

we have

$$(\Delta - \zeta I)^{-1} = (1 - (\alpha - \zeta)(\Delta - \zeta)^{-1}(\Delta - \alpha I)^{-1}.$$

or for any n

$$(\Delta - \zeta I)^{-n} = (1 - (\alpha - \zeta)(\Delta - \zeta)^{-1})^n(\Delta - \alpha I)^{-n}.$$

From the above identity, the kernel for $(\Delta - \zeta I)^{-n}$ exists. Let

$$A = (1 - (\alpha - \zeta)(\Delta - \zeta)^{-1})^n.$$

Then

$$A_x(g_\alpha^{(\frac{n}{2})}(x, y)$$

is a kernel of $(\Delta - \zeta I)^{-n}$.

Lemma 6.3

Let $g(x, y)$ be the kernel of $(\Delta - \zeta I)^{-n}$. Then

$$|g(x, y)| \leq Ce^{-\varepsilon d(x,y)}\varphi(x)\varphi(y)$$

where $\varphi(x) = v(B_1(x))^{-\frac{1}{2}}$.

We omit the proof of the above lemma. By Lemma 1,

$$\sup_x \int |g(x, y), dy < +\infty.$$

Thus $(\Delta_2 - \zeta)^{-n}$ is an operator from $L^1 \to L^1$.

Since Δ_2 and Δ_1 are the same acting on C^∞ functions, $(\Delta_1 - \zeta)^{-n}$ is a bounded operator from $L^1(M) \to L^1(M)$.

♡

Lemma 6.4

If $(\Delta_1 - \zeta)^{-n}$ is a bounded operator, then $\zeta \in \rho(\Delta_1)$.

♡

Proof. Since $(\Delta_1 - \zeta)^{-n}$ is bounded, there is a neighborhood of ζ such that for any ζ' in the neighborhood, $(\Delta_1 - \zeta')^{-n}$ is also bounded. By

$$(\Delta_1 - \zeta)^{-1} = (\Delta_1 - \zeta')^{-1}(1 - (\zeta - \zeta')(\Delta_1 - \zeta')^{-1})^{-1}$$

and the fact that the latter can be expended to a convergent series, $(\Delta_1 - \zeta)^{-1}$ is bounded in $L'(M)$. Thus $\zeta \in \rho(\Delta_1)$.

From the above lemma, we have

$$\sigma(\Delta_1) \subset \sigma(\Delta_2) \subset [0, \infty).$$

The theorem thus follows from

Theorem 6.7 (Wang)

Let M be a complete Riemannian manifold with non-negative Ricci curvature. Then

$$\sigma(\Delta_1) = [0, \infty).$$

Before giving the proof, we use the following theorem. ♡

Theorem 6.8

Let M be a complete non-compact Riemannian manifold. Ric$(M) \geq 0$. Then

$$\text{vol}B_p(R) \geq C(n, \text{vol}B_p(1))R.$$

♡

Proof. Fixing $x_0 \in \partial B_p(R)$. Using the comparison theorem, we have

$$\Delta p^2 \leq 2n.$$

For any $\varphi \in C_0^\infty(M), \varphi \geq 0$, we have

$$\int_M \varphi \Delta \rho^2 \leq 2n \int_M \varphi.$$

We choose a standard cut-off function $\varphi = \psi(\rho(x))$, where

$$\psi(t) = \begin{cases} 1 & 0 \leq t \leq R-1 \\ \frac{1}{2}(R+1-t), & R-1 \leq t \leq R+1 \\ 0 & t \geq R+1 \end{cases}.$$

By Stokes theorem

$$\begin{aligned} \int_M \varphi \Delta \rho^2 &= -\int \nabla\varphi\nabla\rho^2 = -2\int \psi'\rho|\nabla\rho|^2 \\ &= \int_{Bx_0(R+1)\backslash Bx_0(R-1)} \rho \\ &\geq (R-1)\text{vol}(x_0(R+1)\backslash Bx_0(R-1)). \end{aligned}$$

Thus

$$(R-1)\text{vol}(Bx_0(R+1) - Bx_0(R-1)) \leq 2n\int \varphi \leq 2n\,\text{vol}Bx_0(R+1)$$

Obviously

$$B_p(1) \subset Bx_0(R+1)\backslash Bx_0(R-1).$$

Thus

$$2n\,\text{vol}Bx_0(R+1) \geq (R-1)\text{vol}B_p(1).$$

Since $B_p(2(R+1) \supset Bx_0(R+1)$, we have

$$\begin{aligned} 2n\,\mathrm{vol}B_p(2(R+1)) &\geq (R-1)\mathrm{vol}B_p(1) \\ \mathrm{vol}B_p(2(R+1)) &\geq \frac{R-1}{2n}\mathrm{vol}B_p(1). \end{aligned}$$

Wang modified the above argument and proved that

Lemma 6.5

There is a constant $C(n)$ such that for $a \leq r \leq R$

♡

$$V_q(r) \leq C\frac{r}{R}V_1(R). \tag{$\star$}$$

If we choose $r = \varepsilon R$ such that $C\varepsilon < \frac{1}{2}$, we have

$$V_q(\varepsilon) < \frac{1}{2}V_q(R)$$

Furthermore, we have

$$A_q(\varepsilon r) \leq \frac{C}{R}(V_q(2r) - V_q(\varepsilon r)). \tag{$\star\star$}$$

↑

This is inverse Laplacian comparison theorem!!!

We pick a cut-off function ψ such that $\psi(r) = 1$ for $2\varepsilon < r < 2, \psi(r) = 0$ for $r > 2, r < \varepsilon, 0 \leq \psi \leq 1, |\psi'| + |\psi'' < C(s)$. From now on, we fix $\varepsilon > 0$. Let

$$\varphi_k = \psi\left(\frac{r(x)}{k}\right)\sin\sqrt{\lambda}r.$$

Then $\{\varphi_k\}$ forms an infinite dimensional vector space.

A straightforward computation gives

$$|\Delta\varphi_k + \lambda\varphi_k| \leq \frac{C}{k} + C|\Delta r|.$$

We have known $\Delta r \leq \frac{n-1}{r}$. Thus

$$|\Delta r| \leqq \frac{n-1}{r} - \Delta r + \frac{n-1}{r} = \frac{2(n-1)}{r} - \Delta r.$$

Thus we have

$$\begin{aligned} |\Delta\varphi_k + \lambda\varphi_k| &\leq \frac{C}{k} - \frac{C}{k}\Delta r \\ |\Delta\varphi_k + \lambda\varphi_k|_{L'(M)} &\leq \frac{C}{k}(V(2k) - V(\varepsilon k)) - \frac{C}{k}\int_{B(2k)-B(\varepsilon k)} \Delta r \end{aligned}$$

On the other hand,

$$|\varphi_k|_{L'} \leq C(V(k) - V(\varepsilon k)).$$

By the volume comparison $(\star)$

$$\frac{C}{k}(V(2k) - V(2(\varepsilon k)) \leq \frac{C}{k}|\varphi_k|_{L'}$$

On the other hand

$$\begin{aligned} -\int_{B(2k)-B(\varepsilon k)} &= \textstyle\int_{\partial B(\varepsilon k)} 1 - \int_{\partial B(2k)} 1 \\ &\leq \textstyle\int_{\partial B(\varepsilon k)} 1 = A_q(\varepsilon k). \end{aligned}$$

Using $(\star)(\star)$, we get the desired estimate.

Further readings:

Griffiths-Harris: Principle of algebraic geometry.

6.2 The L^p-spectrum of the Laplacian

6.2.1 The Laplacian on L^p space

Definition 6.1

A one-parameter semi-group on a complex Banach space B is a family T_t of bounded linear operators, where $T_t : B \to B$ parameterized by real numbers $t \geqslant 0$ and satisfies the following relations:

①. $T_0 = 1$;

②. *If* $0 \leqslant s_1 t < +\infty$, *then*

$$T_s T_t = T_{s+t}$$

③. *The map*

$$t_1 f \mapsto T_t f$$

from $[0, +\infty) \times B$ *to* B *is jointly continuous.* ♣

The (infinitesimal) generator Z of a one-parameter semi-group T_t is defined by

$$Zf = \lim_{t \to 0^+} t^{-1}(T_t f - f)$$

The domain $Dom(Z)$ of Z being the set of f for which the limit exists. It is evident that $Dom(Z)$ is a linear space. Moreover, we have

Lemma 6.6

The subspace $Dom(Z)$ is dense in B, and is invariant under T_t in the sense that

$$T_t(Dom(Z)) \subset Dom(Z)$$

for all $t \geqslant 0$. Moreover

$$T_t Z f = Z T_t f$$

for all $f \in Dom(Z)$ and $t \geqslant 0$. ♡

Proof. If $f \in B$, we define

$$f_t = \int_0^t T_x f dx$$

The above integration exists in the following sense: since $T_x f$ is a continuous function of x, we can define the integration as the limit of the corresponding Riemann sums. In a Banach space, absolute convergence implies the conditional convergence. Thus in order to prove the convergence of the Riemann sums, we only need to verify that

$$\int_0^t \|T_x f\| dx$$

is convergent. But this follows easily from the joint continuity in the definition of semi-group. $\|T_x f\|$ must be uniformly bounded for small x.

We compute

$$
\begin{aligned}
& \lim_{h\to 0^+} h^{-1}(T_h f_t - f_t) \\
= & \lim_{h\to 0^+} \left\{ h^{-1}\int_h^{t+h} T_x f dx - h^{-1}\int_0^t T_x f dx \right\} \\
= & \lim_{h\to 0^+} \left\{ h^{-1}\int_t^{t+h} T_x f dx - h^{-1}\int_0^h T_x f dx \right\} \\
= & T_t f - f
\end{aligned}
$$

Therefore, $f_t \in Dom(Z)$ and

$$
Z(f_t) = T_t f - f
$$

Since $t^{-1} f_t \to f$ in norm as $t \to 0^+$, we see that $Dom(Z)$ is dense in B. QED

The generator Z, in general, is not a bounded operator. However, we can prove the following

Lemma 6.7

The generator Z is a closed operator. ♡

Proof. We first observe that

$$
T_t f - f = \int_0^t T_x Z f dx
$$

if $f \in Dom(Z)$. To see this, we consider the function $r(t) = T_t f - f - \int_0^t T_x Z f dx$. Obviously we have $r(0) = 0$, and $r'(t) \equiv 0$. Thus $r(t) \equiv 0$. QED

Using the above formula, we have

$$
T_f f - f = \lim_{n\to\infty}(T_t f_n - f_n) = \lim_{n\to\infty}\int_0^t T_x Z f_n dx
$$

By the Lebegue theorem, the above limit is equal to

$$
\int_0^t T_x g dx
$$

Thus we have

$$
\lim_{t\to 0^+} t^{-1}(T_t f - f) = g
$$

and therefore $f \in Dom(Z)$, $Zf = g$.

Lemma 6.8

If B is a Hilbert space, then Z must be densely defined and self-adjoint. ♡

Let M be a manifold of dimension n, not necessarily compact or complete. The semi-group can formally be defined as

$$
e^{\Delta t}
$$

More precisely, the following result is true

Theorem 6.9

Let M be a manifold, then there is a heat kernel

$$
H(x, y, t) \in C^\infty(M \times M \times \mathbb{R}^t)
$$

such that

$$(T_t f)(x) = \int_M H(x,y,t) f(y) dy$$

satisfying

①. $H(x,y,t) = H(y,x,t)$.
②. $\lim_{t\to 0^+} H(x,y,t) = \delta_x(y)$.
③. $(\Delta - \frac{\partial}{\partial t})H = 0$.
④. $H(x,y,t) = \int_M H(x,z,t-s)H(z,y,s)dz$.

♡

In [Getzler], the above theorem was proved. One of the feature of the above theorem is that the proof is independent to the fact that Δ can be extended as a densely defined self-adjoint operator on $L^2(M)$. In particular, we don't need to assume M to be complete. The infinitesimal generator on $L^2(M)$ is in fact the Dirichlet Laplacian.

We let Δ_p denote the Laplacian on L^p space. With this notation, for most of the theorems in linear differential geometry, the completeness assumption can be removed.

Example 6.1 Let $f \in Dom(\Delta_2)$ such that $f \in L^2(M)$ and $\Delta f = 0$. Then f has to be a constant.

When M is a complete manifold, the above is a theorem of Yau. However, it is interesting to see that even when M is incomplete, the above result is still true, and the proof is exactly the same as the original proof of Yau.

Examining some special cases of the above setting is interesting.

Ⓐ. If $\partial M \neq \emptyset$ and if ∂M is an $(n-1)$-dimensional manifold, then A_2 is the Dirichlet Laplacian.
Ⓑ. If $M = \mathbb{R}^n - \{0\}$. Then if $f \in L^2(M)$, $\Delta f = 0$ and $f \in Dom(\Delta_2)$. Then $f(0) = 0$ and f must be bounded near 0. By the removable singularity theorem, f extends to a harmonic function on $\mathbb{R}^n$, which must be a constant.
©. Δ_2 is particularly useful on moduli spaces, where it is very difficult to describe the boundary.

6.2.1.1 Variational characterization of spectrum

Unlike in the case of compact manifold, in general, a complete manifold doesn't admit any pure point spectrum. For example, there are no L^2-eigenvalues on $\mathbb{R}^n$. That is, for any $\lambda \in \mathbb{R}$, if $\Delta f + \lambda f = 0$ and $f \in L^2(\mathbb{R}^n)$, then we have $f \equiv 0$.

The above well-known result was generalized by Escobar, who proved that if M has a rotational symmetric metric, then there is no L^2-eigenvalue.

Let Δ be the Laplacian on a complete non-compact manifold M. By the argument in the previous section, Δ naturally extends to a self-adjoint densely defined operator, which we still denote as Δ for the sake of simplicity.

It is well-known that there is a spectrum measure E such that

$$-\Delta = \int_0^\infty \lambda dE$$

The heat kernel is defined as

$$e^{\Delta t} f(x) = \int H(x,y,t) f(y) dy$$

and the Green's function is defined as

$$G(x,y) = \int_0^\infty H(x,y,t)dt$$

The pure-point spectrum of Δ are these $\lambda \in \mathbb{R}$ such that

① There exists an L^2 function $f \neq 0$ such that

$$\Delta f + \lambda f = 0$$

② The multiplicity of λ is finite.

③ In a neighborhood of λ_1 it is the only spectrum point.

We define

$$\rho(\Delta) = \{y \in \mathbb{R} | (\Delta - y)^{-1} \text{ is a bounded operator}\}$$

and we define $\sigma(\Delta) = \mathbb{R} - \rho(\Delta)$ to be the spectrum of Δ. From the above discussion, $\sigma(\Delta)$ decomposes as the union of pure point spectrum, and the so-called essential spectrum, which is, by definition, the complement of the pure-point spectrum.

The set of the essential spectrum is denoted as $\sigma_{ess}(\Delta)$. Using the above definition, $\lambda \in \sigma_{ess}(\Delta)$, if either

① λ is an eigenvalue of infinite multiplicity, or

② λ is the limiting point of $\sigma(\Delta)$.

The following theorems in functional analysis are well-known. For reference, see Donnelly.

Theorem 6.10

A necessary and sufficient condition for the interval $(-\infty, \lambda)$ to intersect the essential spectrum of an self-adjoint densely defined operator A is that, for all $\varepsilon > 0$, there exists an infinite dimensional subspace $G_\varepsilon \subset Dom(A)$, for which

$$(Af - \lambda f - \varepsilon f, f) < 0$$

♡

Theorem 6.11

A necessary and sufficient condition for the interval $(\lambda - a, \lambda + a)$ to intersect the essential spectrum of A is that there exists an infinite dimensional subspace $G \subset Dom(A)$ for which $\|(A - \lambda I)f\| \leqslant a\|f\|$ for all $f \in G$.

♡

Using the above result, we give the following variational characterization of the lower bound of spectrum and the lower bound of essential spectrum.

Theorem 6.12

Using the above notations, define

$$\lambda_0 = \inf_{f \in C_0^\infty(M)} \frac{\int_M |\nabla f|^2}{\int_M f^2}$$

and

$$\lambda_{ess} = \sup_K \inf_{f \in C_0^\infty(M \setminus K)} \frac{\int_M |\nabla f|^2}{\int_M f^2}$$

where K is a compact set running through an exhaustion of the manifold. Then λ_0 and λ_{ess} are the least lower bound of $\sigma(\Delta)$ and $\sigma_{ess}(\Delta)$, respectively.

♡

Corollary 6.1

If $\lambda_0 < \lambda_{ess}$, then λ_0 is an eigenvalue of M with finite dimensional eigenspace. ♡

In this case, λ_0 is called the ground state.

In the following, we give a non-trivial application of the above principle.

Theorem 6.13 (Lin-Lu)

Let M be a complex complete surface embedded in $\mathbb{R}^3$. Assume that M is not totally geodesic, but asymptotically flat in the sense that the second fundamental form goes to zero at infinity. Define

$$\Omega = \{y \in \mathbb{R}^3 | d(y, M) \leqslant a\}$$

for a small positive number $a > 0$. Then Ω is a 3-d manifold with boundary. The Dirichlet Laplacian of Ω has a ground state. ♡

Sketch of the proof: Since M is asymptotically flat, at infinity

$$\Omega \approx M \times [-a, a]$$

As a result

$$\lambda_{ess} = \frac{\pi^2}{4a^2}$$

Thus the main difficulty in the proof of the above theorem is to prove

$$\lambda_0 < \frac{\pi^2}{4a^2}$$

which can be obtained by careful analysis of the Gauss and the mean curvatures.

Remark Exner *et al.* proved that under the condition

$$\int K \leqslant 0, \int |K| < \infty$$

and M being asymptotically flat, the ground state exists. Thus we make the following conjecture to give the complete picture.

conjecture 10 *Let M be a complete, no-totally geodesic, and asymptotically embedded surface in $\mathbb{R}^3$. Let Ω be defined as before. Let K be the Gauss curvature. If*

$$\int_M |K| < +\infty$$

then the ground state exists.

The difficulty of the above conjecture is that even the surface is asymptotically flat, we still don't known the long-range behavior of the surface.

6.2.2 On the theorem of Sturm

Let Mbe a complete Riemannian manifold. We say that the volume (M, g) grows uniformly sub-exponentially, if for any $\varepsilon > 0$, there is a constant $C < \infty$ such that for all $r > 0$ and all $x \in M$, we have

$$v(B_r(x)) \leqslant Ce^{\varepsilon r} v(B_1(X))$$

Theorem 6.14 (Sturm)

If the volume of (M,g) grows uniformly and sub-exponentially, then the spectrum $\sigma(\Delta_p)$ of Δ_p acting on $L^p(M)$ is independent of $p\in[1,\infty)$. In particular, it is a subset of the real line. ♡

One feature of the concept "uniformly and sub-exponentially" is that it is self-dual. Take the following example: A hyperbolic space is not "uniformly and sub-exponentially". On the other side, let Γ be a discrete group acting on the hyperbolic space H, such that $\Gamma\backslash H$ has finite volume. Since the infinity of $\Gamma\backslash H$ are cusps, it is still not "uniformly and sub-exponentially".

A manifold with non-negative Ricci curvature satisfies the assumption that the volume grows "uniformly and sub-exponentially". However, for such a manifold, its volume is infinite. It doesn't has the finite volume counterpart.

The proof of Sturm's theorem depends on the heat kernel estimates. We begin with the following

Lemma 6.9

If the volume of (M,g) grows uniformly and sub-exponentially, then for any $\varepsilon>0$

$$\sup_{x\in M}\int_M e^{-\varepsilon d(x,y)}(v(B_1(x))^{-\frac{1}{2}}v(B_1(y))^{-\frac{1}{2}}dv(y)<\infty$$

♡

Proof. We take $r=d(x,y)$. Then since

$$B_1(y)\subset B_{r+1}(x)$$

we must have

$$v(B_1(y))^{-\frac{1}{2}}\geqslant v(B_{r+1}(x))^{-\frac{1}{2}}\geqslant Ce^{-\frac{1}{2}(r+1)}v(B_1(x))^{-\frac{1}{2}}$$

for any x. Thus the integration in the lemma is less than

$$C\int_M e^{-\varepsilon r}e^{\frac{1}{2}\varepsilon(r+1)}v(B_1(x))^{-1}dy$$

We let $f(r)=v(\partial B_r(x))$ and $F(r)=\int_0^r f(t)dt$. Then up to a constant, the above expression is less than

$$(v(B_1(x)))^{-1}\int_0^\infty e^{-\frac{1}{2}\varepsilon r}f(r)dr$$

By the volume growth assumption $f(r)\leqslant Cr^{n-1}vB_1(x)$, the lemma follows. QED

In fact, the assertion is true if

① The Ricci curvature of M has a lower bound;

②

$$\sup_{x\in M}\int_M e^{-\varepsilon d(x,y)}(v(B_1(x))^{-\frac{1}{2}}v(B_1(y))^{-\frac{1}{2}}dv(y)<\infty$$

The hard part is to prove $\sigma(\Delta_p)\subset\sigma(\Delta_2)$ for all $p\in[1,\infty]$. If this is done, then it is easy to prove

$$\sigma(\Delta_2)\subset\sigma(\Delta_p)$$

as follows:

Let $\xi\in\rho(\Delta_p)$. Then $(\Delta_p-\xi)^{-1}$ is a bounded operator on $L^p(M)$. Let $\frac{1}{p}+\frac{1}{q}=1$. By dualization $(\Delta_q-\zeta)^{-1}$ is bounded in $L^q(M)$. By the interpolation theorem, $(\Delta_2-\zeta)^{-1}$ is bounded and this $\zeta\in\rho(\Delta_2)$.

In order to prove $\sigma(\Delta_p) \subset \sigma(\Delta_2)$, or $\rho(\Delta_2) \subset \rho(\Delta_p)$, we need some estimates. Let $\zeta \in \rho(\Delta_2)$. Then

$$(\Delta_2 - \zeta)^{-1}$$

is bounded from $L^2 \to L^2$. In order to prove that the operator is bounded on L^p, we need to prove that it has a kernel $g(x, y)$ such that

$$(\Delta_2 - \zeta)^{-n} f = \int g(x, y) f(y) dy$$

Lemma 6.10

If $g(x, y)$ satisfies

$$\sup_{x \in M} \int_M |g(x, y)| dy \leqslant C$$

Then $(\Delta_2 - \zeta)^{-n}$ is a bounded operator on $L^p(M)$. ♡

Proof. This is essentially Hölder inequality:

$$\begin{aligned}
& \int_M \left(\int_M g(x, y) f(y) dy \right)^p dx \\
\leqslant \; & \int_M \left(\int_M g^{\frac{1}{q}} g^{\frac{1}{p}} f dy \right)^p dx \\
\leqslant \; & \int_M \left(\int_M g \right)^{\frac{1}{q}} \cdot \int g f^p dx \\
\leqslant \; & C^{\frac{1}{q}} \int_M \int_M g(x, y) f^p(y) dy dx \\
\leqslant \; & C^{1 + \frac{1}{q}} \int_M f^p(y) dy
\end{aligned}$$

QED

If we assume that $\sigma(\Delta_p)$ is a no-where dense set in $\mathbb{C}$, then we have

Lemma 6.11

If $(\Delta_2 - \zeta)^{-n}$ is bounded, so is $(\Delta_2 - \zeta)^{-1}$. ♡

Proof. For any ε, let $\zeta' \in \rho(\Delta_\rho)$ and $|\zeta - \zeta'| < \varepsilon$. Then from

$$\|(\Delta_2 - \zeta)^{-n}\| \leqslant C$$

we get

$$\|(\Delta_2 - \zeta')^{-n}\| \leqslant C + 1$$

provided that ε is small enough. Let $dist(\zeta', \sigma(\Delta_p))$ be the distance to the spectrum of Δ_p, then we have

$$C + 1 \geqslant \lim_{m \to \infty} \|(\Delta_2 - \zeta)^{-nm}\|^{\frac{1}{m}} \geqslant dist(\zeta', \sigma(\Delta_p))^{-n}$$

Thus

$$dist(\zeta', \sigma(\Delta_p)) \geqslant \frac{1}{(C+1)^{\frac{1}{n}}}$$

Since ζ' is arbitrary, we have $dist(\zeta, \sigma(\Delta_p)) > \delta > 0$. QED

6.3 On the essential spectrum of complete non-compact manifold

Let M be a complete non-compact manifold. We assume that there exists a small constant $\delta(n) > 0$, depending only on n such that for some point $q \in M$, the Ricci curvature satisfies

$$Ric(M) \geqslant -\delta(n)\frac{1}{r^2}$$

where $r(x)$, the distance from x to q is sufficiently large. J-P. Wang (cite Wang) proved the following theorem:

Theorem 6.15

Let M be the complete non-compact Riemannian manifold defined above. Then the spectrum of the Laplacian Δ_p acting on the space $L^p(M)$ is $[0, \infty)$ for all $p \in [1, \infty)$. ♡

Corollary 6.2

Let M be a complete manifold with non-negative Ricci curvature, then the L^2 essential spectrum of the Laplacian is $[0, +\infty)$. ♡

By the Bishop volume comparison theorem, we know that for any complete non-compact manifold with non-negative Ricci curvature, the volume growth is at most polynomial. In general, it is not correct to have the lower bound estimate. However, we have the following:

Theorem 6.16

Let M be a complete non-compact Riemannian manifold, and let $Ric(M) \geqslant 0$. Then there is a constant $C = C(n, v(B_1(p)))$ such that

$$v(B_p(R)) \geqslant C(n, v(B_1(p)))R.$$

♡

Proof. Let $p \in M$ be a fixed point. Let ρ be the distance function with respect to p, Let $R > 0$ be a large number. Fixing $x_0 \in \partial B_R(p)$. By the Laplacian comparison theorem, we have

$$\Delta\rho^2 \leqslant 2n.$$

It follows that for any $\varphi \in C_0^\infty(M)$, $\varphi \geqslant 0$, we have

$$\int_M \varphi\Delta\rho^2 \leqslant 2n\int_M \varphi. \tag{6.1}$$

We choose a standard cut-off function $\varphi = \psi(\rho(x))$, where

$$\psi(t) = \begin{cases} 1 & 0 \leqslant t \leqslant R-1 \\ \frac{1}{2}(R+1-t) & R-1 \leqslant t \leqslant R+1 \\ 0 & t \geqslant R+1 \end{cases}.$$

By the Stoke's theorem

$$\int_M \varphi\Delta\rho^2 = -2\int_M \rho\nabla\varphi\nabla\rho = -2\int_M \psi'\rho.$$

By the definition of ψ, the right hand side of the above is equal to

$$\int_{B_{R+1}(x_0)\backslash B_{R-1}(x_0)} \rho \geqslant (R-1)v(B_{R+1}(x_0)\backslash B_{R-1}(x_0)).$$

Combining the above equation with ⊛, we have

$$(R-1)v(B_{R+1}(x_0) - B_{R-1}(x_0)) \leqslant 2n\int \varphi \leqslant 2nv_{R+1}(x_0).$$

Obviously, we have

$$B_1(p) \subset B_{R+1}(x_0)\backslash B_{R-1}(x_0).$$

Thus we have

$$2nv_{R+1}(x_0) \geqslant (R-1)v_1(p).$$

Since $B_{2(R+1)}(p) \supset B_{R+1}(x_0)$, we have

$$2nv_{2(R+1)}(p) \geqslant (R-1)v_1(p).$$

or in other word,

$$v_{2(R+1)}(p) \geqslant \frac{R-1}{2n} v_1(p).$$

QED

What Wang observed was the following inverse Laplacian comparison theorem: We don't have a lower bound for the Laplacian. However, we have the following:

$$\begin{aligned} \int_{B(R)\backslash K} |\Delta\rho| &\leqslant \int_{B(R)\backslash K} \partial n + \int_{B(R)\backslash K} |\partial n - \Delta\rho| \\ &\leqslant CR^n - \int_{\partial B(R)} \frac{\partial\rho}{\partial n} + \int_{\partial K} \frac{\partial\rho}{\partial n} \\ &\leqslant CR^n. \end{aligned}$$

Thus we can also estimate $\int \Delta\rho$ from below.

Using the above observation, Wang computed the L^1-spectrum. Using the theorem of Sturm, all L^p-spectrum, in particular the L^2-spectrum we are interested, are the same.

It is possible to compute the L^2-spectrum directly, but that would be more or less the same as repeating the proof of Sturm's theorem. In fact, we can get a little more information than Wang's theorem provided.

Lemma 6.12

Let M be a complete non-compact manifold with non-negative Ricci curvature. Let $B(R)$ be a very large ball of radius R. Let λ be a Dirichlet eigenvalue and let f be its eigenfunction of $B(R)$. Then there is a constant $C > 0$ such that

$$\int_{B(R)\cdot B(R-1)} f^2 \leqslant C \int_{B(R)} f^2.$$

♡

For the rest of this lecture we are seeking possible extensions of Wang's theorem. While we observe that Sturm's theorem is self-dual (M could be of infinite volume or finite volume), Wang's theorem is not. In what follows, we shall construct an example that all L^p-spectrum are the same of finite volume, L^1-spectrum computable, but doesn't satisfy the assumption of Wang.

The manifold we construct is of 2 dimensional rotational symmetric outside a compact set, and the Riemannian metric g can be written as

$$g = dr^2 + f(r)^2 d\theta^2, r > 1$$

where $f(r) = \frac{1}{r^\alpha}$ for some α large.

The Gauss curvature of g is $-f''/f = -\alpha(\alpha+1)\frac{1}{r^2}$. Thus the manifold doesn't satisfy Wang's assumption.

We prove that the volume of M grows uniformly and sub-exponentially. To see this, we observe that for any point (x)

$$v(B_1(x)) \geqslant \frac{C}{r(x)^\alpha}$$

for some constant $C > 0$. On the other hand, if $\alpha > 1$, the volume of manifold is finite. Thus

$$v(B_r(x)) \leqslant C \leqslant e^{\varepsilon r}\frac{C(\varepsilon)}{r^\alpha}$$

for any $r \gg 0$.

Thus the manifold satisfies the assumption of Sturm and as a result, all L^p-spectrum of M are the same.

We compute the L^1-spectrum concretely. Following Wang, we pick up a large number k. Let ψ be a cut-off function whose support is in $[1, 4]$, and is identically 1 on $[2, 3]$. Consider the function

$$g = \psi(\frac{r}{k})e^{i\sqrt{\lambda}r}.$$

We have

$$\Delta g = \Delta\psi e^{i\sqrt{\lambda}r} + 2\nabla\psi\nabla e^{i\sqrt{\lambda}r} + \psi\Delta e^{i\sqrt{\lambda}r}.$$

We have the following estimate

$$\|\nabla\psi\nabla e^{i\sqrt{\lambda}r}\|_{L^1} \leqslant \frac{C}{K}(V(4k) - V(k)).$$

where $V(r)$ is the volume of the manifold of radius r. A straightforward computation gives

$$\|\nabla\psi\nabla e^{i\sqrt{\lambda}r}\|_{L^1} \leqslant \frac{C}{k^\alpha}.$$

By the same reason

$$\|\Delta\psi e^{i\sqrt{\lambda}r}\|_{L^1} \leqslant \frac{C}{k^\alpha} + \frac{C}{k}\int_{B(4k)\setminus B(k)} |\Delta r|.$$

Since $dS^2 = dr^2 + f(r)^2 d\theta^2$, we have

$$|\Delta r| \leqslant \frac{\alpha}{r}.$$

Thus

$$\int_{B(4k)\setminus(k)} |\Delta r| = \int_k^{4k} \frac{\alpha}{r^{\alpha+1}}dr \leqslant \frac{C}{k^\alpha}.$$

Finally

$$\|\psi\Delta e^{i\sqrt{\lambda}r} + \lambda g\|_{L^1} = \|\sqrt{\lambda}\psi e^{i\sqrt{\lambda}r}\Delta r\|_{L^1} \leqslant \frac{C}{k^\alpha}.$$

On the other hand

$$\|g\|_{L^1} \geqslant \int_{2k}^{3k} 1 \geqslant V(3k) - V(2k) \geqslant \frac{C_1}{k^{\alpha-1}}.$$

Thus if k is sufficiently large

$$\|\Delta g + \lambda g\|_{L^1} \leqslant \varepsilon\|g\|_{L^1}$$

Thus there should be a finite volume version of Wang's theorem.

We end the lecture by some speculations of the essential spectrum.

Definition 6.2

A discrete group G is called amenable, if there is a measure such that

1. *The measure is a probability measure;*
2. *The measure is finitely additive;*
3. *The measure is left-invariant: given a subset A and an element g of G, the measure of A equals to the measure of gA.*

♣

In one sentence, G is amenable if it has finitely-additive left-invariant probability measure.

The following theorem of R. Brooks [**brooks**] is remarkable:

Theorem 6.17

([Brooks] Let M be a compact Riemannian manifold and let $\widetilde{M}$ be the universal cover of M. We assume that $\widetilde{M}$ is non-compact, then

$$\lambda_0(\widetilde{M}) = 0 \Leftrightarrow \pi_1(M) \text{ is amenable}.$$

♡

It would be interesting to ask

conjecture 11 *Using the same assumptions as above. Then*

$$\sigma_{ess}(\widetilde{M}) = [0, \infty).$$

In the case when $\pi_1(M) = \mathbb{Z}^n$, the above conjecture is true.

Lemma 6.13

Suppose $M = T^n$, $\widetilde{M} = \mathbb{R}^n$. Then

$$\sigma_{ess}(\widetilde{M}) = [0, \infty)$$

♡

Here the metric on M is an arbitrary metric. **Proof.** Let N be any finite cover of M. Let λ be an eigenvalue of N. Then

$$\lambda \in \sigma_{ess}(\widetilde{M})$$

In fact, let ρ be a cut-off function. Since

$$\Delta f + \lambda f = 0 \text{ on } N$$

Then on $\widetilde{M}$

$$\|\Delta(\rho f) + \lambda\rho f\|_{L^2} \leqq \|f\Delta\rho\|_{L^2} + \|\alpha\nabla\rho\nabla f\|_{L^2}$$

If $|\nabla\rho|$, $|\Delta\rho|$ are small, then

$$\|\Delta(f\rho) + \lambda\rho f\|_{L^2} \leqslant \varepsilon\|f\rho\|_{L^2}$$

If the result is not true, since $\sigma_{ess}(\Delta)$ is a closed set, there is an interval (a, b) such that for any N, there is no eigenvalues in (a, b).

We prove this by contradiction. Let λ, μ be two consecutive eigenvalues such that $\lambda < a$ and $\mu > b$. By the above argument, we can find a C_0^∞ function on $\widetilde{M}$ such that

$$\begin{aligned} \|\Delta f + \lambda f\|_{L^2} &< \varepsilon\|f\|_{L^2} \\ \|\Delta g + \mu g\|_{L^2} &< \varepsilon\|g\|_{L^2} \end{aligned}$$

Let k, l be integers such that

$$\frac{k\int|\nabla f|^2 + l\int|\nabla g|^2}{k\int f^2 + l\int g^2} \in (a, b)$$

Then by repeating f k-times and g l-times we are done. QED

Remark Recently, Lu-Zhou [**lu-zhou**] proved that the essential spectrum is $[0, +\infty)$ for any complete non-compact manifold with asymptotic nonnegative Ricci curvature, generalizing Wang's result.

Chapter 7 The Hodge Thoerem

7.1 The Poincaré Lemma

In this section, we prove the following

Theorem 7.1

Let U be the unit ball of the original point of $\mathbb{R}^n$. Let ω be a p-form on U with $p \geq 1$. Then there is a $p-1$ form η such that $\omega = d\eta$. ♡

Proof. If ω is a 1-form, the proof is straightforward. Let

$$f(x) = \int_\ell \omega,$$

where ℓ is a curve connecting the original point to x. Then

$$df = \omega.$$

In general, we use the math induction. We write

$$\omega = \omega_1 + dx_n \wedge \omega_2,$$

where ω_1, ω_2 are forms without the factor dx_n. Let

$$\omega = \sum a_{i_1 \cdots i_{p-1}}(x_1, \cdots, x_n) dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}}.$$

Define

$$\eta = \int a_{i_1 \cdots i_{p-1}}(x_1, \cdots, x_n) dx_n \, dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}}.$$

Then $d(\omega - d\eta) = 0$ and $\omega - d\eta$ has not dx_n factor. As a result, the coefficients of $\omega - d\eta$ are independent to x_n. Thus the theorem follows from the inductive assumption.

QED

The following theorem is a nonlinear generalization of the above result:

Theorem 7.2

Let U be the unit tube of the original point of $\mathbb{R}^n$. Let ω is a skew-symmetric matrix valued 1-form on U. Assume that

$$d\omega + \omega \wedge \omega = 0.$$

Then on U, the equation

$$dg + \omega g = 0$$

is solvable. ♡

Proof. We assume that $g(0) = 1$. Assume that for $r > 0$, we can define $g(x_1, \cdots, x_r, 0, \cdots, 0)$ such that $dg + \omega g = 0$ on $\{x_{r+1} = \cdots x_n = 0\}$. We define $g(x_1, \cdots, x_{r+1}, 0, \cdots, 0)$ by the following ODE:

$$\begin{cases} \frac{\partial g}{\partial x_{r+1}} + \omega_{r+1} g = 0 \\ g(x_1, \cdots, x_r, 0, \cdots, 0) \text{ were defined.} \end{cases}$$

To complete the proof, we need to prove that for any $k \leq r$, we have

$$\frac{\partial g}{\partial x_k} + \omega_k g = 0. \tag{7.1}$$

Taking the derivative with respect to x_{r+1}, we get

$$\frac{\partial^2 g}{\partial x_{r+1}\partial x_k} + \frac{\partial \omega_k}{\partial x_{r+1}} g + \omega_k \frac{\partial g}{\partial x_{r+1}}.$$

Using the ODE, we get

$$-\frac{\omega_{r+1}}{\partial x_k} g - \omega_{r+1} \frac{\partial g}{\partial x_k} + \frac{\partial \omega_k}{\partial x_{r+1}} g + \omega_k \frac{\partial g}{\partial x_{r+1}}.$$

Using the fact that $d\omega + \omega \wedge \omega = 0$, we can prove that the expression in (7.1) satisfies the ODE with the zero initial value, hence must be zero.

QED

7.2 de Rham Theorem

The differential operator $d : \Lambda^p(M) \to \Lambda^{p+1}(M)$ satisfies the relation $d^2 = 0$. Using this, we can define the de Rham cohomology groups as follows:

Definition 7.1

Let

$$Z^k(M) = \{\omega \in \Lambda^k(M) \mid d\omega = 0\},$$
$$B^k(M) = \{d\eta \mid \eta \in \Lambda^{k-1}(M)\}.$$

Then $B^k(M) \subset Z^k(M)$*, and the quotient spaces*

$$H^k_{DR}(M) = Z^k(M)/B^k(M),$$

are called the de Rham cohomology groups. ♣

We have the following result

Theorem 7.3 (de Rham)

For any $k \geq 0$*, we have*

$$H^k_{DR}(M) = H^k_{sing}(M),$$

where $H^k_{sing}(M)$ *is the singular cohomology groups.* ♡

Proof. There are many fancy proofs of the above result. The proof provided below is the best one (and can be made rigorous).

Let $S_k(M)$ be the vector spaces generated by all dimension k submanifolds of M with boundaries. The map

$$\partial : S_k(M) \to S_{k-1}(M)$$

sends those submanifolds into their boundaries. Obviously, $\partial^2 = 0$. That is, the boundary of the boundary of a submanifold has no boundary. We can define the homology groups as

$$H_k(M) = \frac{\{V \mid \partial V = 0, V \in S_k(M)\}}{\{\partial W \mid W \in S_{k+1}(M)\}}$$

for all $k \geq 0$. We *assume* that

$$H^k_{sing}(M) = (H_k(M))^*,$$

which is a combinatorial result and the proof can be found in any book in algebraic topology.

There is a natural non-degenerated coupling between $\Lambda^k(M)$ and $S_k(M)$ by the integration

$$\Lambda^k(M) \times S_k(M) \to \mathbb{R}, \qquad (\omega, V) \mapsto \int_V \omega.$$

By the Stokes theorem, the above coupling descends to the map

$$H^k_{DR}(M) \times H_k(M) \to \mathbb{R}.$$

To complete the proof, we need to prove that the above coupling is non-degenerated. What we need to prove is that, for any p-form ω such that $d\omega = 0$ and for any p-dimensional submanifold K without boundary $\int_K \omega = 0$, then there is a $p-1$ form η such that $\omega = d\eta$. Let U, V be two open sets and suppose that the theorem is proved for U, V. Let $\omega = d\eta_1$ on U and $\omega = d\eta_2$ on V. Then on $U \cap V$, $d(\eta_1 - \eta_2) = 0$. We need to prove that for any $p-1$ dimensional submanifold L in $U \cap V$, if $\int_L (\eta_1 - \eta_2) = 0$, then there is a ξ such that $\eta_1 - \eta_2 = d\xi$. Thus $\eta_1 - d\xi = \eta_2$ provides the required solution.

QED

Exercise 19 *Provide the details of the proof.*

7.3 Formal Hodge Theorem

The de Rham Theorem introduced in the last section is among the very fundamental results of compact manifolds. However, in terms of the concrete computation of the cohomology groups, they are not very efficient. The elements of the groups are classes of differential forms.

The Hodge Theorem provides the way to pick up the best possible candidates in the classes. It asserts that, among the class of the form $\omega + d\eta$, there is a unique representative, called the *harmonic form*, which is better than other representatives of the class in the following sense.

We consider the following variational problem for a fixed element $[a]$ of $\Lambda^p(M)$: finding an η_0 such that

$$||\eta_0||^2 = \inf_{\eta \in [a]} \int_X \langle \eta, \eta \rangle dV_g. \tag{7.2}$$

By definition, each $\eta \in [a]$ can be represented by

$$\eta = \xi_0 + d\xi_1.$$

If the space of ξ_1 were of finite dimensional, then since $||\eta||^2 \geq 0$, we could have found a ξ_1' such that

$$||\xi_0 + d\xi_1'||^2 = \inf_{\eta \in [a]} ||\eta||^2,$$

and the variational problem (7.2) would have been solved.

Now let's *assume* that there is a unique p form η_0 that solves problem (5.2). Let ξ_1 be an arbitrary $p-1$ form. Then we have the inequality

$$||\eta_0 + \varepsilon d\xi_1||^2 \geq ||\eta_0||^2.$$

Since ε is arbitrary, we have

$$(\eta_0, d\xi_1) = 0.$$

Then

$$\delta\eta_0 = 0. \tag{7.3}$$

Then from (7.3), η_0 is harmonic, i.e., $\Delta\eta_0 = 0$. Thus the solution of the problem (5.2) must be harmonic. For any element η in a fixed cohomologyical class $[a]$, we have the following Hodge decomposition

$$\eta = \eta_0 + d\eta_1,$$

where η_0 is the harmonic form, and η_1 is a $p-1$ form. In general, for arbitrary p form η, we have the following Hodge decomposition:

$$\eta = \eta_0 + d\xi_1 + \delta\xi_2,$$

where η_0, ξ_1, and ξ_2 are smooth p, $p-1$, and $p+1$ forms, respectively, and η_0 is harmonic.

7.4 The Hodge theorem

Unfortunately, it is far from trivial that the variational problem (5.2) can be solved. The PDE theory, especially the elliptic regularity theory kicks in here to make the above arguments rigid.

Theorem 7.4 (real Hodge Theorem)

Let M be a compact orientable Riemannian manifold. Let $p \geq 0$. Define

$$\mathcal{H}^p(M) = \{\phi \in \Lambda^p(M)) \mid \Delta\phi = 0\}.$$

Then

1. $\dim \mathcal{H}^p(M) < +\infty$;
2. *Let η be an arbitrary smooth p-form. Then we have*

$$\eta = \eta_0 + d\eta_1 + \delta\eta_2,$$

where η_0 is a harmonic p-form; η_1 and η_2 are $(p-1)$ and $(p+1)$ forms, respectively,

3. $H^p_{DR}(M) = \mathcal{H}^p(M)$ *for* $p \geq 0$.

♡

7.5 Proof of the Hodge Theorem

The proof of the (real) Hodge theorem heavily depends on the analysis of the Laplacian Δ. The three basic PDE tools we are going to use are the Sobolev Lemma, the Rellich Lemma, and the Gårding inequality. For the first two lemmas, we need to introduce the Sobolev s-norms. The proof of the Gårding inequality depends on the Weitzenböck formula.

Let M be a compact Riemannian manifold. $X = \bigcup U_\alpha$ be a finite cover such that each $(U_\alpha, \{x^1_\alpha, \cdots, x^n_\alpha\})$ is a real local coordinate system.

Let $\{\rho_\alpha\}$ be the partition of unity subordinating to the cover $\{U_\alpha\}$. Let S be a smooth function. For any nonnegative integer s, define the Sobolev norm of S to be

$$\|S\|_s^2 = \sum_\alpha \sum_{|K|\leq s} \int_M |D^K_\alpha(\rho_\alpha S)|^2 dV_M,$$

where $K = (k_1, \cdots, k_n)$ is the multiple index; $|K| = \sum k_i$; and

$$D^K_\alpha S = \frac{\partial^{k_1+\cdots+k_n}}{\partial(x^1_\alpha)^{k_1}\cdots\partial(x^n_\alpha)^{k_n}} S.$$

Obviously, the definition of the Sobolev norms depend on the choice of the cover $\{U_\alpha\}$ and the partition of unity $\{\rho_\alpha\}$.

Exercise 20 *Prove that the Sobolev norms $\|\ \|_s$, defined by different covers and partitions of unity, are equivalent. In particular, the norm $\|\ \|_0$ is equivalent to the L^2 inner product in* (2.1).

There is a way to generalize the notation of Sobolev norms $|| \quad ||_s$ from nonnegative integers s to any nonnegative numbers s, using pseudo-differential operators, or using the "elementary" definition as follows:

We first define the Sobolev s-norms for $\rho_\alpha S$ on each U_α. Since U_α is a coordinate patch, we can assume, without loss of generality, that U_α is an open set of a torus $T = T_n = \mathbb{R}^n/\mathbb{Z}^n$. Thus the section $\rho_\alpha S$ can be extended as a smooth $\mathbb{C}^r$-valued function on T. Let

$$\rho_\alpha S = \sum_{\xi\in\mathbb{Z}^n} S_\xi e^{i\langle\xi,x\rangle}$$

be the Fourier expansion of S_α. We define

$$||\rho_\alpha S||_s^2 = \sum_{\xi\in\mathbb{Z}^n}(1+||\xi||^2)^s|(\rho_\alpha S)_\xi|^2.$$

The Sobolev s-norm of S is defined as

$$||S||_s^2 = \sum_\alpha ||\rho_\alpha S||_s^2.$$

Define $\mathcal{H}_s^p(M)$ to be the completion of the smooth p-forms under the norm $|| \quad ||_s$. As before, it is independent of the choice of the cover and the partition of unity.

Theorem 7.5 (Sobolev Lemma)

Using the above notations, we have

$$\bigcap_s \mathcal{H}_s^p(M) = \Lambda^p(M).$$

♡

Proof. The theorem being local, we assume that S is a $\mathbb{C}^r$-valued smooth function of T. By definition, for any $s > 0$, we have

$$\sum_{\xi\in\mathbb{Z}^n}(1+||\xi||^2)^s|S_\xi|^2 \leq C < \infty.$$

Let $K = (k_1, \cdots, k_n)$ be a multiple index. We consider the Fourier expansion of

$$D^K S \sim \sum_{\xi\in\mathbb{Z}^n}(i)^{|K|}\xi_1^{k_1}\cdots\xi_n^{k_n} S_\xi e^{i\langle\xi,x\rangle}.$$

By definition, if $S \in \bigcap_s \mathcal{H}_s^p(M)$, then

$$\sum_{\xi\in\mathbb{Z}^n}|\xi_1^{k_1}\cdots\xi_n^{k_n}|^2|S_\xi|^2 \leq \sum_{\xi\in\mathbb{Z}^n}(1+||\xi||^2)^s|S_\xi|^2 < +\infty$$

for any K. Thus $D^K S$ is well-defined. Since the differential operators D^K are closed operators, we conclude that S is K differentiable and hence prove the result.

■

In order to prove the Rellich Lemma, we give the following definition of *compact operators*.

Definition 7.2

Let $\mathcal{B}_1, \mathcal{B}_2$ be two Banach spaces. Let $A : \mathcal{B}_1 \to \mathcal{B}_2$ be a linear operator from $\mathcal{B}_1$ to $\mathcal{B}_2$. A is a compact operator, if the image of the unit ball of $\mathcal{B}_1$ under T is sequential compact in $\mathcal{B}_2$. In other word, if $\{x_i\}$ is a bounded sequence of $\mathcal{B}_1$, then a subsequence of $\{Ax_i\}$ converges to some point of $\mathcal{B}_2$.

♣

The following is the basic fact about a self-adjoin compact operator.

Theorem 7.6

Let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{H} \to \mathcal{H}$ be a compact operator. Then the essential spectrum $\sigma_{\text{ess}}(A) \subset \{0\}$. Moreover, if $\sigma_{\text{ess}}(A) = \emptyset$, then $\mathcal{H}$ is the direct sum of eigenspaces of each non-zero eigenvalues, that is, if E_m is the eigenspaces of the eigenvalue λ_m. Then we have

$$\mathcal{H} = \bigoplus_m E_m.$$

♡

Proof. If $\lambda \in \sigma_{rmess}(A)$ and $\lambda \neq 0$. Then for any $\varepsilon > 0$, by the Weyl's criterion, there exists an infinite dimensional vector space G_ε such that for any $\phi \in G_\varepsilon$, we have

$$\|A\phi - \lambda\phi\| < \varepsilon.$$

We choose an orthonormal set $\{\phi_i\} \subset G_\varepsilon$. Since A is compact, we may assume that $A\phi_i$ is convergent. Assume that for i, j big enough but $i \neq j$, we have

$$\|A\phi_i - A\phi_j\| \leq \varepsilon.$$

Then we have

$$\lambda\|\phi_i - \phi_j\| \leq \|A(\phi_i - \phi_j)\| + \|A\phi_i - \lambda\phi_i\| + \|A\phi_j - \lambda\phi_j\| \leq 3\varepsilon.$$

Since $\phi \perp \phi_j$, we have $\|\phi_i - \phi_j\| = \sqrt{2}$. Thus we have a contradiction

$$\sqrt{2}\,\lambda \leq 3\varepsilon.$$

This proves the first part of the theorem. If $\sigma_{\text{ess}}(A) = \emptyset$, then is follows from the spectrum theorem that the whole space is the direct sum of the eigenspaces.

■

Theorem 7.7 (Rellich Lemma)

For $s > r$, the inclusion

$$H_s^p(M) \to \mathcal{H}_r^p(M)$$

is compact.

♡

Proof. Similar to the previous theorem, we can work on a torus T. Assume that $\{u_k\}$ is a bounded sequence of $\mathcal{H}_s^p(T)$. We wish to prove that a subsequence of $\{u_k\}$ will converge in the $|| \quad ||_r$ norm.

Let $\{(u_k)_\xi\}$ be the Fourier coefficients of u_k for $k \geq 1$. Then

$$\sum_{\xi \in \mathbb{Z}^n} (1 + ||\xi||^2)^s |(u_k)_\xi|^2 \leq C \tag{7.4}$$

for some constant C independent of k. For any fixed ξ, a subsequence of $(1+||\xi||^2)^{s/2}(u_k)_\xi$ converges. By the standard diagonalization, we can find a subsequence of $\{u_k\}$, which we still denote as $\{u_k\}$ for abusing of notations, such that for each ξ, $(1 + ||\xi||^2)^{s/2}(u_k)_\xi$ is convergent.

We prove that $\{u_k\}$ is a Cauchy sequence under the normal $|| \quad ||_r$. For any $\varepsilon > 0$, we choose R large so that

$$\frac{1}{(1 + R^2)^{s-r}} < \varepsilon. \tag{7.5}$$

We consider all ξ with $||\xi|| \leq R$. There are only finitely many such ξ's. By the assumption on the sequence $\{u_k\}$, there is an $N > 0$, such that

$$\sum_{||\xi|| \leq R} (1 + ||\xi||^2)^r |(u_k)_\xi - (u_l)_\xi|^2 < \varepsilon$$

for $k, l > N$. Thus we have

$$||u_k - u_l||_r^2 \leq \varepsilon + \sum_{||\xi|| > R} (1 + ||\xi||^2)^r |(u_k)_\xi - (u_l)_\xi|^2.$$

Using (7.4) and (7.5), we have

$$||u_k - u_l||_s^2 < (1 + 2C)\varepsilon,$$

and the theorem is proved. ∎

Exercise 21 *Let M be a compact manifold. Define two Banach spaces of M. Let $\mathcal{C}^0(M)$ be the space of continuous functions with the maximum norm, and let $\mathcal{C}^1(M)$ be the space of $\mathcal{C}^1(M)$ functions with the norm*

$$\max(|f| + |\nabla f|).$$

Prove the Arzelá-Ascoli Theorem: the inclusion map

$$\iota : \mathcal{C}^1(M) \to \mathcal{C}^0(M)$$

is a compact operator.

The standard method in proving the above theorem, like in that of the Rellich Lemma, is to use the diagonal subsequence method, which is widely used. In the following, we give a different proof, using the Tychonoff Theorem.

We consider $F(M)$, the space of all bounded functions with bound 1 on M. By the identification

$$F(M) = \prod_{x \in M} [-1, 1],$$

and by the Tychonoff Theorem, we know that $F(M)$ is compact under the product topology. On the other hand, on $\mathcal{C}^0(M)$, we can define the so-called compact-open topology, whose subbase composed of the set of the form

$$C(K, U) = \{f \mid F(K) \subset U\},$$

where K is a compact subset of M and U is an open set of $\mathbb{R}$. We can prove that when we restrict to the space $\mathcal{C}^1(M)$, the product topology is the same as the compact-open topology. On the other hand, on $\mathcal{C}^0(M)$, compact-open topology defines the same topology as in the Banach space $\mathcal{C}^0(M)$, and the theorem is proved.

Exercise 22 *Provide the details.*

Next we shall introduce the Gårding inequality. Let

$$\phi, \psi \in \Gamma(X, \Lambda^p(M)),$$

and let

$$\mathfrak{D}(\phi, \psi) = (\phi, \psi) + (d\phi, d\psi) + (\delta\phi, \delta\psi) = (\phi, (1 + \Delta)\psi).$$

Then $\mathfrak{D}(\phi, \psi)$ defines an inner product on $\Lambda^p(M)$. Apparently we have the inequality

$$\mathfrak{D}(\phi, \phi) \leq C||\phi||_1^2$$

for some constant C. The Gårding inequality states that the inverse inequality is also true so that the norms $\sqrt{\mathfrak{D}(\phi, \phi)}$ and $||\phi||_1$ are equivalent.

Theorem 7.8 (Gårding Inequality)

For $\phi \in \Lambda^p(M)$, we have

$$||\phi||_1^2 \leq C\,\mathfrak{D}(\phi,\phi), \tag{7.6}$$

for some constant $C > 0$.

Proof. The proof of the inequality depends on the Weitzenböck formula, Theorem 2.1. Let $\phi \in \Lambda^p(M)$. Then

$$\Delta\phi = -\nabla^*\nabla\phi + E(\phi), \tag{7.7}$$

where E is a zero-th differential operator. Using integration by parts, we get

$$(\Delta\phi,\phi) \geq \int |\nabla\phi|^2 - C(\phi,\phi).$$

It follows that

$$\mathfrak{D}(\phi,\phi) \geq (\phi,\phi) + \frac{1}{2C}(\Delta\phi,\phi) \geq \frac{1}{2C}\int |\nabla\phi|^2 + \frac{1}{2}(\phi,\phi),$$

and the inequality is proved.

Proof of the Hodge Theorem For $\phi \in \Lambda^p(M)$, we define a linear functional

$$l(\psi) = (\phi,\psi)$$

for $\psi \in \Lambda^p(M)$. With respect to the inner product $\mathfrak{D}(\phi,\psi)$, l is bounded:

$$|l(\psi)| \leq ||\phi||_0\sqrt{\mathfrak{D}(\psi,\psi)}.$$

Thus by the Riesz representation theorem, there is an η such that

1. $(\phi,\psi) = (\eta,(1+\Delta)\psi)$ for any $\psi \in \Gamma(X,\mathcal{A}^{p,q}(E))$;
2. $||\eta||_1^2 \leq C\mathfrak{D}(\eta,\eta) < +\infty$. (Gårding inequality)

Let $A : \Lambda^p(M) \to \Lambda^p(M)$ be the linear operator defined by sending $\phi \in \Lambda^p(M)$ to the unique η defined above. Since A is a bounded operator, it can be extended to an operator on $L^2(M)$. Then since η is actually in $\mathcal{H}^p(M)$, the operator A is a compact operator by the Rellich Lemma. On the other hand, it is not hard to see that A is a self-adjoint operator[a].

According to the spectral theorem for compact, self-adjoint operators, there is a Hilbert-space decomposition

$$\Lambda^p(M) = \bigoplus_m E(\rho_m),$$

where ρ_m are the eigenvalues of A and $E(\rho_m)$ are the finite-dimensional eigenspaces corresponding to the eigenvalue ρ_m. Since A is one-to-one, all $\rho_m \neq 0$.

We claim that

$$\mathcal{H}^p(M) = E(1),$$

where $\mathcal{H}^p(M)$ is the space of harmonic forms. To see this, we assume that $\phi \in \mathcal{H}^p(M)$, and $A\phi = \phi$. By the Weitzenböck formula, Δ is an elliptic operator. Using the Sobolev Lemma and Schauder estimate, we can prove that ϕ is smooth. Thus $(1+\Delta)\phi = \phi$ and ϕ is harmonic. Since $E(1)$ is of finite dimensional, we have

$$\dim \mathcal{H}^p(M) < +\infty.$$

This proves the first assertion of the theorem. To prove the second assertion of the theorem, we note that the biggest possible eigenvalue of A is 1. By the property of compact operators, there is a gap between the eigenvalue 1 and the rest of the eigenvalues. Thus if $\phi \in \mathcal{H}^p(M)^\perp$ and if ϕ is smooth, we have

$$||\Delta\phi||_0 \geq \varepsilon||\phi||_0$$

for some $\varepsilon > 0$. Thus we can define the inverse operator G on $\mathcal{H}^p(M)^\perp$, called the Green's operator, such that $G = \Delta^{-1}|_{\mathcal{H}^p(M)^\perp}$.

If we compare G to the operator A (they have the same eigenspaces), we shall see that G is also a compact operator because $||G|| \leq C||A||$ for some constant C. Furthermore, by the elliptic regularity, G maps smooth p-forms to smooth p forms. Using this fact, the Hodge decomposition is given by

$$\phi = \eta_0 + d\delta G\phi + \delta d G\phi.$$

This completes the proof of the Hodge Theorem.

■

[a] Since A is a bounded operator, in order to prove that A is self-adjoint, we just need to verify the equality $(A\phi, \psi) = (\phi, A\psi)$ for smooth ϕ and ψ, which follows easily from integration by parts.

7.6 More about elliptic regularity

Let M be a compact Riemannian manifold and let $\phi \in \mathcal{H}_1^p(M)$ be a p-form. We say ϕ is a weak solution of the equation

$$(\Delta + I)\phi = \psi$$

for ψ a C^∞ p-form, if

1. $\phi \in \mathcal{H}_1^p(M)$;
2. for any η a smooth p-form, we have

$$(\phi, (\Delta + I)\eta) = (\psi, \eta).$$

In this section, we prove the following result.

Theorem 7.9

Let ϕ be a weak solution of the above equation, then ϕ must be smooth. ♡

Proof.

Let ρ be a smooth function. Then by the Weitzenböck formula, we have

$$(\Delta + I)(\rho\eta) = \rho(\Delta + I)\eta + J(\eta), \tag{7.8}$$

where J is some first order differential operator (depending on ρ) on the space of p-forms. Thus we have

$$(\rho\phi, (\Delta + I)\eta) = (\psi, \rho\eta) - (\phi, J(\eta)) \tag{7.9}$$

for any smooth form $\eta \in \Lambda^p(M)$.

Now we specify our choice of ρ. Let $x \in M$ be a fixed point. We choose a local normal coordinate system $(U, (x_1, \cdots, x_n))$. We choose a smooth function ρ such that the support of ρ is within U, and is constant 1 in a smaller neighborhood V of x such that $\bar{V} \subset U$. Without loss of generality, we may assume that $\rho\phi, \eta$ are p-forms of a torus T^n.

Let $(\ ,\ )_0$ be the inner product on $\Lambda^p(T^n)$ induced by the flat Riemannian metric and let $||\quad||_0$ be the corresponding norm. Then there is a constant $C > 0$ such that

$$C^{-1}||\quad||_0 \leq ||\quad|| \leq C||\quad||_0.$$

In fact, since we choose the local normal coordinate system, we may assume that near x, the Riemannian metrics are very close.

Let A be the zero-th differential operator such that

$$(\rho\phi, (\Delta_0 + I)\eta)_0 = (\rho\phi, A(\Delta_0 + I)\eta).$$

Let P be the operator

$$P = A(\Delta_0 + I) - (\Delta + I),$$

where Δ_0 is the Laplacian with respect to the flat metric of T^n. Then we have

$$(\rho\phi, (\Delta_0 + I)\eta)_0 = (\rho\phi, P\eta) + (\psi, \rho\eta) - (\phi, J(\eta)).$$

Let

$$\rho\phi \sim \sum_{\xi \in \mathbb{Z}^n} a_\xi e^{i\langle x, \xi \rangle}$$

be the Fourier expansion of $\rho\phi$, where a_ξ are p-forms. Let R, s be large real numbers and let

$$\eta_R = \sum_{||\xi|| < R} a_\xi e^{i\langle x, \xi \rangle}(1 + ||\xi||^2)^s$$

Since

$$\Delta_0 = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2},$$

We have

$$\Delta_0 e^{i\langle x, \xi \rangle} = ||\xi||^2 e^{i\langle x, \xi \rangle}.$$

Then we have

$$\sum_{||\xi|| < R} |a_\xi|^2 (1 + ||\xi||^2)^{s+1} = (\eta_R, (\Delta_0 + I)\eta_R)_0. \tag{7.10}$$

Let $\tilde{\Delta}$ be the raw Laplacian. Let

$$\tilde{P} = A(\Delta_0 + I) - (\tilde{\Delta} + I).$$

Then we have

$$(\rho\phi, P\eta) = (\rho\phi, \tilde{P}\eta) + (\rho\phi, E\eta),$$

where E is the zero-th operator in the Weitzenböck formula.

We claim that

1. $|(\rho\phi, \tilde{P}\eta_R)| \leq \varepsilon \|\phi\|^2_{\mathcal{H}_{(s+1)/2}}$ for $0 < \varepsilon \ll 1$;
2. $|(\rho\phi, E\eta_R)| \leq C\|\phi\|^2_{\mathcal{H}_{s/2}}$;
3. $|(\psi, \rho\eta_R)| \leq C\|\phi\|_{\mathcal{H}_{s/2}}$;
4. $|(\phi, J(\eta_R))| \leq C\|\phi\|^2_{\mathcal{H}_{s/2+1/4}}$.

We will postpone the proof of the above claim in the following. Now let's assume it is correct.

Letting $R \to \infty$ in (7.10), we have

$$\|\phi\|^2_{\mathcal{H}_{(s+1)/2}} \le \varepsilon \|\phi\|^2_{\mathcal{H}_{(s+1)/2}} + C\|\phi\|^2_{\mathcal{H}_{s/2}} + C\|\phi\|_{\mathcal{H}_{s/2}} + C\|\phi\|^2_{\mathcal{H}_{s/2+1/4}}.$$

This by induction, $\phi \in \mathcal{H}_s(M)$ for any s, and hence by the Sobolev Lemma, ϕ has to be smooth.

It remains to prove the claim. We shall only prove (1), the others begin similar.

$\tilde{P}$ is a differential operator with small coefficients. Let

$$\tilde{P} = a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + b_i \frac{\partial}{\partial x_i} + c.$$

Then we have

$$\tilde{\eta}_R = \sum_{\|\xi\|<R} a_\xi e^{i\langle x,\xi\rangle}(1+\|\xi\|^2)^s \cdot F(x,\xi),$$

where

$$F(x,\xi) = -a_{ij}\xi_i\xi_j + \sqrt{-1}b_i\xi_i + c.$$

We write

$$\overline{(\rho\phi, \tilde{P}\eta_R)} = \sum_{\tilde{\xi}\in\mathbb{Z}^n} \sum_{\|\xi\|<R} \bar{a}_{\tilde{\xi}}\, a_\xi (1+\|\xi\|^2)^s \int_M F(x,\xi) e^{i\langle x,(\xi-\tilde{\xi})\rangle}.$$

By the Riemann-Lebesgue lemma, we have

$$\left| \int_M F(x,\xi) e^{i\langle x,(\xi-\tilde{\xi})\rangle} \right| \le C\frac{1+\|\xi\|^2}{\|\xi-\tilde{\xi}\|^K}$$

for any $K>0$. Below we shall take $K=(s+1)/2$ and use the inequality that

$$(1+\|\xi\|^2)\cdot\|\xi-\tilde{\xi}\| \le C\,(1+\|\tilde{\xi}\|^2).$$

Thus we have

$$\begin{aligned}
&\left| \sum_{\xi\ne\tilde{\xi}} \bar{a}_{\tilde{\xi}}\, a_\xi (1+\|\xi\|^2)^s \int_M F(x,\xi) e^{i\langle x,(\xi-\tilde{\xi})\rangle} \right| \\
&\le \sum_{\xi\ne\tilde{\xi}} \bar{a}_{\tilde{\xi}}\, a_\xi (1+\|\xi\|^2)^{(s+1)/2} \cdot (1+\|\tilde{\xi}\|^2)^{(s+1)/2},
\end{aligned}$$

and this completes the proof of the claim. ∎

Chapter 8 Semigroups and heat kernels

8.1 Semigroup and its generator

In this section, we define abstract semigroup and prove some of the basic properties.

Definition 8.1

A one-parameter semigroup on a complex Banach space $\mathcal{B}$ is a family T_t of bounded linear operators, where $T_t : \mathcal{B} \to \mathcal{B}$ parameterized by real numbers $t \geq 0$ and satisfies the following relations:

1. $T_0 = 1$;
2. *If* $0 \leq s, t < \infty$, *then*
$$T_s T_t = T_{s+t}.$$
3. *The map*
$$(t, f) \to T_t f$$
from $[0, \infty) \times \mathcal{B}$ *to* $\mathcal{B}$ *is jointly continuous.*

♣

The (infinitesimal) generator Z of a one-parmeter semigroup T_t is defined by

$$Zf = \lim_{t \to 0^+} t^{-1}(T_t f - f).$$

The domain $\mathcal{D}om(Z)$ of Z being the set of f for which the limit exists. It is evident that $\mathcal{D}om(Z)$ is a linear space. Moreover, we have

Lemma 8.1

The subspace $\mathcal{D}om(Z)$ is dense in $\mathcal{B}$, and is invariant under T_t in the sense that

$$T_t(\mathcal{D}om(Z)) \subset \mathcal{D}om(Z)$$

for all $t \geq 0$. *Moreover*

$$T_t Z f = Z T_t f$$

for all $f \in \mathcal{D}om(Z)$ *and* $t \geq 0$.

♡

Proof. If $f \in \mathcal{B}$, we define

$$f_t = \int_0^t T_x f \, dx.$$

The above integration exists in the following sense: since $T_x f$ is a continuous function of x, we can define the integration as the limit of the corresponding Riemann sums. In a Banach space, absolute convergence implies conditional convergence. Thus in order to prove the convergence of the Riemann sums, we only need to verify that

$$\int_0^t ||T_x f|| \, dx$$

is convergent. But this follows easily from the joint continuity in the definition of the semigroup: $||T_x f||$ must be uniformly bounded for small x.

We compute

$$
\begin{aligned}
&\lim_{h\to 0^+} h^{-1}(T_h f_t - f_t)\\
&= \lim_{h\to 0^+}\left\{h^{-1}\int_h^{t+h} T_x f dx - h^{-1}\int_0^t T_x f dx\right\}\\
&= \lim_{h\to 0^+}\left\{h^{-1}\int_t^{t+h} T_x f dx - h^{-1}\int_0^h T_x f dx\right\}\\
&= T_t f - f.
\end{aligned}
$$

Therefore, $f_t \in \mathcal{D}om(Z)$ and

$$
Z(f_t) = T_t f - f.
$$

Since $t^{-1} f_t \to f$ in norm as $t \to 0^+$, we see that $\mathcal{D}om(Z)$ is dense in $\mathcal{B}$.

QED

The generator Z, in general, is not a bounded operator. However, we can prove the following

Lemma 8.2

The generator Z is a closed operator. ♡

Proof. We first observe that

$$
T_t f - f = \int_0^t T_x Z f dx
$$

for $f \in \mathcal{D}om(Z)$. To see this, we consider the function $r(t) = T_t f - f - \int_0^t T_x Z f dx$. Obviously we have $r(0) = 0$ and $r'(t) \equiv 0$. Thus $r(t) \equiv 0$.

Let $\{f_n\}$ be a sequence in the domain of Z such that $f_n \to f$ and $Z f_n \to g$. Using the above formula, we have

$$
T_t f - f = \lim_{n\to\infty}(T_t f_n - f_n) = \lim_{n\to\infty}\int_0^t T_x Z f_n dx.
$$

By the Lebesgue dominated convergence theorem, the above limit is equal to

$$
\int_0^t T_x g dx.
$$

Thus we have

$$
\lim_{t\to 0^+} t^{-1}(T_t f - f) = g,
$$

and therefore, $f \in \mathcal{D}om(Z), Zf = g$.

QED

Remark If Z is not a bounded operator on $\mathcal{D}om(Z)$, it is not possible to extend Z to the whole banach space $\mathcal{B}$ because otherwise since Z is closed, Z has to be bounded by the closed graph theorem.

Theorem 8.1

Let $\mathfrak{B}$ be a Hilbert space. If the operators T_t are self-adjoint operators, then Z is a densely defined self-adjoint operator. ♡

8.2 Heat kernel

In this section, we construct the heat kernels. That is, we are going to find smooth function $H(x, y, t)$ such that

$$T_t(f)(x) = \int_M H(x, y, t) f(y) dy.$$

For the sake of simplicity, we shall only consider the Laplace operator on functions. Moreover, the sign convention is that on the Euclidean space, $\Delta = \sum \frac{\partial^2}{\partial x_j^2}$. The semigroup is formally defined as $T_t = e^{\Delta t}$.

The main result of this section is the following

Theorem 8.2

Let M be a Riemannian manifold, then there is a heat kernel

$$H(x, y, t) \in C^\infty(M \times M \times \mathbb{R}^+),$$

such that

$$(T_t f)(x) = \int_M H(x, y, t) f(y) dy$$

satisfying

1. $H(x, y, t) = H(y, x, t)$;
2. $\lim_{t \to 0^+} H(x, y, t) = \delta_x(y)$;
3. $(\Delta - \frac{\partial}{\partial t}) H = 0$;
4. $H(x, y, t) = \int_M H(x, z, t - s) H(z, y, s) dz$ *for any* $0 < s \leq t$.

Before proving the theorem, we first *formally* construct the heat kernel. This formal construction also outlines the proof of the theorem.

The k-simplex Δ_k is the following subset of $\mathbb{R}^k$

$$\{(t_1, \cdots, t_k) \mid 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}.$$

For $t > 0$, we write $t\Delta_k$ for the rescaled simplex

$$\{(t_1, \cdots, t_k) \mid 0 \leq t_1 \leq \cdots \leq t_k \leq t\}.$$

We assume that $U(x, y, t)$ be a function on $M \times M \times \mathbb{R}^+$ such that

$$\lim_{t \to 0^+} U(x, y, t) = \delta_x(y)$$

be the the Dirac function. Let

$$R(x, y, t) = \frac{dU(x, y, t)}{dt} - \Delta_x U(x, y, t),$$

where Δ_x means the Laplace operator for the variable x. Note that formally, any function $g(x, y, t)$ defines one-parameter family of operators G_t by the formula

$$G_t f(x) = \int_M g(x, y, t) f(y) dy.$$

We use the R_t, U_t to denote the corresponding families of operators with respect to the functions $R(x, y, t)$ and $U(x, y, t)$, respectively. For any $k \geq 1$, define the operator

$$Q_t^k = \int_{t\Delta_k} U_{t - t_k} R_{t_k - t_{k-1}} \cdots R_{t_2 - t_1} R_{t_1} dt_1 \cdots dt_k,$$

and $Q_t^0 = U_t$. Let

$$R^{(k)}(s) = \int_{s\Delta_{k-1}} R_{s - t_{k-1}} \cdots R_{t_2 - t_1} R_{t_1} dt_1 \cdots dt_{k-1},$$

and $R^{(0)}(s) = 0$. Since the derivative of the integral of the form

$$\int_0^t a(t-s)b(s)ds$$

is equal to

$$\int_0^t \frac{da}{dt}(t-s)b(s)ds + a(0)b(t),$$

we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q_t^k = R^{(k+1)}(t) + R^{(k)}(t).$$

As a result, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) \sum_{k=0}^{\infty} (-1)^k Q_t^k = 0.$$

For $\mathbb{R}^n$, the fundamental solution of the heat equation

$$(\Delta - \frac{\partial}{\partial t})u = 0$$

is

$$\frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}},$$

where $r = d(x,y)$ is the Euclidean distance of x and y.

We wish to find the the following form of the fundamental solution of the heat equation:

$$U(x,y,t) \sim (4\pi t)^{-\frac{n}{2}} e^{-d^2(x,y)/4t} \left\{ \sum_{i\geq 0} \phi_i(x,y) t^i \right\} \tag{8.1}$$

where $d(x,y)$ is the distance function on the Riemannian manifold. $U(x,y,t)$ should satisfy

1. $\lim_{t\to 0^+} U(x,y,t) = \delta_x(y)$, where $\delta_x(y)$ is the Dirac function at x;
2. For any N, $\lim_{t\to 0^+} (\Delta - \frac{\partial}{\partial t})U(x,y,t) = O(t^N)$.

The function $U(x,y,t)$ is called the *paramatrix* of the heat kernel.

We pick a normal coordinate system $(y_1, \cdots, y_n)$, and let $r = d(x,y)$ be the Riemannian distance. We identify a neighborhood of y to a small ball of $T_y(M)$ by the exponential map. Under this map, the coordinates of x can be written as $(x_1, \cdots, x_n)$. On the other hand, let $(\theta_1, \cdots, \theta_{n-1})$ be a coordinate system on S^{n-1}, then $(r, \theta_1, \cdots, \theta_{n-1})$ gives a coordinate system at y also, and this coordinate system is called the polar coordinates.

Exercise 23 *Let ds^2 be the Riemannian metric. Then we can write*

$$ds^2 = dr^2 + \sum_{i,j=1}^{n-1} r^2 s_{ij}(x) d\theta_i d\theta_j.$$

That is, prove that $\frac{\partial}{\partial r}$ is orthogonal to any $\frac{\partial}{\partial \theta_j}$.

Let $\psi(r)$ be a function of r, and let $g = \det(s_{ij})$. Then we have

$$\Delta\psi = \frac{d^2\psi}{dr^2} + \frac{n}{r}\cdot\frac{\partial\psi}{\partial r} + \left(\frac{d\log\sqrt{g}}{dr}\right)\frac{d\psi}{dr},$$

$$\Delta(\phi\psi) = \phi\Delta\psi + \psi\Delta\phi + 2\frac{d\phi}{dr}\frac{d\psi}{dr}.$$

We let

$$\psi = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}},$$
$$\phi = \phi_0 + \phi_1 t + \cdots + \phi_N t^N,$$

where $\phi_j = \phi_j(x, y)$ are smooth local functions on $M \times M$. We assume that under our coordinate system, y is the origin[1]. Then

$$u_N = \psi\phi = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} \sum_{i=0}^{N} \phi_i t^i.$$

Therefore

$$\left(\Delta - \frac{\partial}{\partial t}\right) u_N = \phi \left(\Delta\psi - \frac{\partial\psi}{\partial t}\right) + \psi \left(\Delta\phi - \frac{\partial\phi}{\partial t}\right) + 2\frac{\partial\phi}{\partial r}\frac{d\psi}{dr}.$$

Since

$$\Delta\psi - \frac{\partial\psi}{\partial t} = \frac{d\log\sqrt{g}}{dr}\frac{d\psi}{dr},$$
$$\frac{d\psi}{dr} = -\frac{r}{2t}\psi.$$

we have

$$\left(\Delta - \frac{\partial}{\partial t}\right) u_N = \frac{\psi}{t} \sum_{k=0}^{N} \left[\Delta\phi_{k-1} - \left(k + \frac{r}{2}\frac{d\log\sqrt{g}}{dr}\right)\phi_k - r\frac{d\phi_k}{dr}\right] t^k. \tag{8.2}$$

Thus in order to find the paramatrix, we set

$$r\frac{d\phi_k}{dr} + \left(k + \frac{r}{2}\frac{d\log\sqrt{g}}{dr}\right)\phi_k = \Delta\phi_{k-1}$$

for $k = 0, \cdots, N$, where we let $\phi_{-1} = 0$. The solutions of the above ordinary differential equations are

$$\phi_0(x, y) = g^{-\frac{1}{4}}(x);$$
$$\phi_k(x, y) = g^{-\frac{1}{4}}(x) r(x, y)^{-k} \int_0^{r(x,y)} r^{k-1}(\Delta\phi_{k-1}) \left(\frac{rx}{r(x, y)}\right) g\left(\frac{rx}{r(x, y)}\right)^{\frac{1}{4}} dr. \tag{8.3}$$

Thus we have

$$\left(\Delta - \frac{\partial}{\partial t}\right) u_N = \frac{\psi}{t}(\Delta\phi_N) t^N.$$

From the above, we prove that

Lemma 8.3

There is a unique formal solution $U(x, y, t)$ of the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) U(x, y, t) = 0$$

of the form (8.1) *such that $\phi_i(x, y)$ are defined in* (8.3).

♡

QED

Let η be a smooth function such that $\eta = 1$ for $t < 1$ and $\eta = 0$ for $t > 2$. Let

$$p(x, y) = \eta\left(\frac{2r(x, y)}{\delta}\right)$$

be the cut-off function, where δ is the injectivity radius at y. Then for any N, we consider the function $u_N(x, y, t) = p(x, y) u_N(x, y, t)$. We shall prove that

[1] We use the obvious fact that there exists a smooth family of normal coordinate systems parametrized by any point of M. So all the functions we define below are smooth not only with respect to x but y.

Lemma 8.4

For any N sufficiently large, we have

1. $\lim_{t\to 0^+} u_N(x,y,t) = \delta_x(y)$;
2. *The kernel* $R_N(x,y,t) = \left(\Delta_y - \frac{\partial}{\partial t}\right) u_N(x,y,t)$ *satisfies the estimate*
$$||R_N(x,y,t)||_{C^l} \le C(l)t^{N-l/2-1}.$$

QED

Using (8.2) and the solutions of ϕ_k, we know that for $N \gg 0$, we have

$$||R_N(x,y,t)||_{C^l} \le Ct^{\alpha}.$$

It follows that

$$||R^{(k)}(s)||_{C^l} \le \frac{s^{k\alpha}(\alpha!)^k}{(\alpha k)!}.$$

Since

$$\sum_k \frac{s^{k\alpha}(\alpha!)^k}{(\alpha k)!} < +\infty,$$

our formal construction is convergent to the heat kernel.

Theorem 8.3

The Δ_p on $L^p(M)$ is well defined as the infinitesimal generator of the heat semi-group.

QED

For any self-adjoint extension $\tilde{\Delta}$ of Δ, the corresponding heat kernel is

Theorem 8.4

Using the above notations, we have

$$\tilde{H}(x,y,t) = U(x,y,t) - \int_0^t e^{\tilde{\Delta}(t-s)} \left(\frac{\partial}{\partial t} - \Delta_x\right) U(x,y,s)ds.$$

Exercise 24 *Construct heat kernels of the Laplacian of bundle-valued p-forms.*

Chapter 9 The PDEs

9.1 De Giorgi-Nash-Moser Estimates

In classical Schauder estimates or $W^{2,p}$ estimates, the continuity of the coefficients is required. Therefore, the methods are not usable in nonlinear case. In 1957, De Giorgi obtained the Hölder estimate on elliptic equations of divergence form with *measurable* coefficients. In 1958, Nash independently got the similar estimates on parabolic equations. In 1960, Moser obtained a simplified proof and the Harnark inequality. These methods now becomes crucial in geometric analysis.

We consider the following elliptic equation of divergence form:

$$-D_j(a^{ij}D_iu) + b^iD_iu + cu = 0 \tag{9.1}$$

such that

$$\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \le \Lambda|\xi|^2, \qquad \forall x \in \Omega,\ \xi \in \mathbb{R}^n$$
$$\sum_{i,j}||a^{ij}||_{L^\infty} + \sum_i ||b^i||_{L^\infty} + ||c||_{L^\infty} \le \Lambda.$$

For the sake of simplicity, we only consider the following equation

$$-D_j(a^{ij}D_iu) = 0. \tag{9.2}$$

Lemma 9.1

Let $\phi(s) \in C^{0,1}_{loc}(\mathbb{R})$ be a convex function

1. *If u is (9.2)'s weak subsolution, $\phi'(s) \ge 0$. Then $v = \phi(u)$ is also a weak subsolution;*
2. *If u is (9.2)'s weak supersolution, $\phi'(s) \le 0$. Then $v = \phi(u)$ is also a weak subsolution.*

♡

QED

Lemma 9.2 (Local maximum principle)

Let $v \in H^{1,2}(B_R)$ be a weak subsolution. Then for any $p > 0$, $0 < \theta < 1$, we have

$$\operatorname*{ess\,sup}_{B_{\theta R}} v \le C\left(⨍_{B_R}(v^+)^p\right)^{1/p},$$

where C only depends on n, Λ/λ, p, and $(1-\theta)^{-1}$.

♡

Proof. The lemma is true even if v is not bounded. However, for the sake of simplicity, we assume that v is essentially bounded.

We first assume that $p \ge 2$. Since v is a weak subsolution, so is v^+. Thus we can assume that $v \ge 0$.

For any testing function $\phi \ge 0$, we have

$$\int_{B_R} a^{ij}D_ivD_j\phi \le 0.$$

We let $\xi \in C_0^\infty(B_R)$ and $\phi = \xi^2v^{p-1}$. Then we have

$$(p-1)\int_{B_R}(a^{ij}D_ivD_jv)v^{p-2}\xi^2dx \le -2\int_{B_R}a^{ij}v^{p-1}\xi D_ivD_j\xi dx.$$

Thus there is a constant C such that

$$\int_{B_R}|D(\xi v^{p/2})|^2dx \le C\int_{B_R}|D\xi|^2v^pdx.$$

Using the Sobolev inequality, we have

$$\left(\int_{B_R} (\xi v^{p/2})^{2^*} dx\right)^{2/2^*} \leq C \int_{B_R} |D\xi|^2 v^p dx,$$

where

$$2^* = \frac{2n}{n-2}.$$

Let

$$R_k = R\left(\theta + \frac{1-\theta}{2^k}\right).$$

Define $\xi_k \in C_0^\infty(B_{R_k})$ such that $\xi_k \equiv 1$ on $B_{R_{k+1}}$. Then

$$|D\xi_k| \leq \frac{2^{k+1}}{(1-\theta)R}.$$

Using this we have

$$\left(\int_{B_{R_{k+1}}} v^{\frac{np}{n-2}} dx\right)^{\frac{n-2}{n}} \leq \frac{C \cdot 4^k}{(1-\theta)^2 R^2} \int_{B_{R_k}} v^p dx.$$

Thus we have

$$||v||_{L^{\frac{np}{n-2}}}(B_{R_{k+1}}) \leq (C \cdot 4^k)^{1/p} ||v||_{L^p}(B_{R_k}).$$

We let

$$p_1 = \frac{np}{n-2}, \cdots, p_k = \left(\frac{n}{n-2}\right)^k p.$$

Then we have

$$||v||_{L^{p_{k+1}}}(B_{R_{k+1}}) \leq (C \cdot 4^k)^{1/p_k} ||v||_{L^{p_k}}(B_{R_k}).$$

Since

$$\log\left(\prod_{k=1}^{\infty} (C \cdot 4^k)^{1/p_k}\right) = \sum_{k=1}^{\infty} \frac{1}{p_k}(\log C + k \log 4) < C,$$

there is a constant C such that

$$||v||_{L^\infty}(B_{\theta R}) \leq \frac{C}{((1-\theta)R)^{n/p}} \frac{1}{R^n} ||v||_{L^p}(B_R).$$

It remains to prove that the lemma is true for $0 < p < 2$. To see this we first use the inequality for $p = 2$. That is

$$\sup_{B_{\theta R}} v \leq \frac{C}{(1-\theta)^{n/2}} \left(⨍_{B_R} u^2\right)^{1/2}.$$

We have

$$\left(\int_{B_R} v^2\right)^{1/2} \leq \sup_{B_R} v^{1-p/2} \cdot \left(\int_{B_R} v^p\right)^{1/2}.$$

Using Young's inequality, we have

$$\phi(\theta R) = (1 - \frac{p}{2})\phi(R) + \frac{C}{((1-\theta)R)^{n/p}} ||v||_{L^p},$$

where

$$\phi(s) = \sup_{B_s} v.$$

Thus we have the inequality

$$\phi(t) \leq (1 - \frac{p}{2})\phi(s) + \frac{A}{(t-s)^\alpha} ||v||_{L^p}$$

for any $t \geq s$. Let

$$t_0 = R_0 < R, \qquad t_{i+1} = t_i + (1-\tau)t^i(R - R_0)$$

and $0 < \tau < 1$ to be determined later. Then we have

$$\phi(t_i) \leq (1 - \frac{p}{2})\phi(t_{i+1}) + \frac{A}{((1-\tau)t^i(R-R_0))^\alpha}||u||_{L^2}.$$

By iteration, we know that

$$\phi(R_0) \leq A||u||_{L^2}\frac{1}{((1-\tau)t^i(R-R_0))^\alpha}\left((1-\frac{p}{2})\tau^\alpha + (1-\frac{p}{2})^2\tau^{-2\alpha} + \cdots\right).$$

We can choose τ close to 1 such that $(1-p/2)t^{-\alpha} < 1$. Then the above sequence is convergent and the theorem is proved.

Lemma 9.3 (Weak Harnarck inequality)

Let v be the weak supersolution of (9.2). *Then there is a $p_0 > 0$ such that*

$$\operatorname*{ess\ inf}_{B_{\theta R}} v \geq \frac{1}{C}\left(\fint_{B_R} v^{p_0}dx\right)^{1/p_0}$$

♡

Proof. We assume that $v > 0$ on B_R. Otherwise we can use $v + \varepsilon$ instead.

Since v is a supersolution, v^{-1} is a subsolution. By the above lemma, we have

$$\operatorname*{ess\ sup}_{B_{\theta R}} v^{-p} \leq C\fint_{B_R} v^{-p}dx,$$

which is equivalent to

$$\operatorname*{ess\ inf}_{B_{\theta R}} v \geq C\left(\fint_{B_R} v^{-p}dx\right)^{-\frac{1}{p}}$$

We write

$$\begin{aligned}\operatorname*{ess\ inf}_{B_{\theta R}} v &\geq C\left(\fint_{B_R} v^{-p}dx\right)^{-\frac{1}{p}}\\ &= C\left(\fint_{B_R} v^{-p}dx\cdot\fint_{B_R} v^{p}dx\right)^{-1/p}\cdot\left(\fint_{B_R} v^{p}dx\right)^{1/p}.\end{aligned}$$

We just need to prove that there exists a $p_0 > 0$ such that

$$\fint_{B_R} v^{-p}dx\cdot\fint_{B_R} v^{p}dx \leq C.$$

We let $w = \log v - \beta$, where β is a constant to be determined. We wish to prove that

$$\int_{B_R} e^{p|w|}dx \leq C \tag{9.3}$$

for some $p > 0$. If the above inequality is true then we have

$$\int_{B_R} e^{p\log v - p\beta}dx \leq C, \qquad \int_{B_R} e^{-p\log v + p\beta}dx \leq C,$$

and the lemma follows.

The proof of (9.3) is somewhat complicated but the basic idea is still Moser iteration. Let $v^{-1}\xi^2$ be a test function, where $\xi \in C_0^\infty(B_R)$, we have

$$0 \geq \int a^{ij}v_i(v^{-1}\xi^2)_j = -\int \xi^2 a^{ij}w_iw_j + 2\int a^{ij}w_i\xi\xi_j.$$

Without loss of generality, we assume that $R = 1$. Then we have

$$\int_{B_\sigma} |Dw|^2 \leq C$$

for any $0 < \sigma < 1$. By choosing suitable β, we have the Poincaré inequality

$$\int_{B_\sigma} |w|^2 dx \le C.$$

Let $\xi \in C_0^\infty(B_1)$ and let $\phi = \xi^2|w|^{2q}$. Then we have

$$\int_{B_\sigma} \xi^2|w|^{2q}|D_i w|^2 \le 2q \int_{B_\sigma} \xi^2|w|^{2q-1} D_i w D_i|w| + 2\int_{B_\sigma} \xi|w|^{2q} D_i w D_i \xi.$$

Using the Young's inequality, we have

$$2q|w|^{2q-1} \le \frac{2q-1}{2q}|w|^{2q} + (2q)^{2q-1}.$$

Thus we have

$$\frac{1}{2q}\int_{B_\sigma} \xi^2|w|^{2q}|D_i w|^2 \le (2q)^{2q-1}\int_{B_\sigma} |D_i w|^2 + 4q\int_{B_\sigma} |w|^{2q}|D_i\xi|^2.$$

Finally, we get

$$\int_{B_\sigma} |D(\xi^2|w|^{2q})|^2 dx \le C(2q)^{2q} + C\tau^{-2}q^2 \int_{B_{\sigma+\tau}} |w|^{2q} dx,$$

where C only depends on n, Λ/λ, and $(1-\sigma)^{-1}$. Let $\kappa = n/(n-1)$. By Sobolev inequality, we have

$$\left(\int_{B_\sigma} |w|^{2q\kappa} dx\right)^{1/\kappa} \le C(2q)^{2q} + C\tau^{-2}q^2 \int_{B_{\sigma+\tau}} |w|^{2q} dx.$$

We let

$$q_i = \kappa^{i-1}, \quad \delta_0 = \sigma, \quad \delta_i = \delta_{i-1} - \frac{\sigma-1}{2^i}$$

Then

$$\left(\int_{B_{\delta_i}} |w|^{2\kappa^i} dx\right)^{1/\kappa} \le C(\kappa)^{2(i-1)\kappa^{i-1}} + C(4\kappa)^i \int_{B_{\delta_{i-1}}} |w|^{2\kappa^{i-1}} dx$$

Let

$$I_j = \left(\int_{B_{\delta_j}} |w|^{2\kappa^j}\right)^{1/2\kappa^j}.$$

Then

$$I_i \le C^{1/\kappa^{i-1}}\kappa^{i-1} + C^{1/\kappa^{i-1}}(4\kappa)^{i/\kappa^{i-1}} I_{i-1}.$$

Running the iteration we get

$$I_j \le C\kappa^j + cI_0.$$

Thus we have

$$||w||_{L^q(B_1)} \le Cq.$$

If $pCe < 1$, then we must have

$$\int_{B_1} e^{p|w|} \le C.$$

QED

Theorem 9.1 (Harnack Inequality)

Let u be a nonnegative solution of (9.2). *Then we have*

$$\operatorname*{ess\ sup}_{B_{\theta R}} u \le C \operatorname*{ess\ inf}_{B_{\theta R}} u$$

where C only depends on n, λ/λ, $(1-\theta)^{-1}$.

Theorem 9.2 (Hölder estimate)

Using the above notations, u is Hölder continuous in $B_{\theta R}$.

Proof. Let

$$M(R) = \operatorname*{ess\ sup}_{B_{\theta R}} u, \quad m(R) = \operatorname*{ess\ inf}_{B_{\theta R}} u.$$

Let $w(R) = M(R) - m(R)$. Then

$$u - m(R) \geq 0$$

on $B_{\theta R}$. Thus

$$M(\theta R) - m(R) \leq C(m(\theta R) - m(R)).$$

Therefore

$$w(\theta R) \leq \left(1 - \frac{1}{C}\right) w(R)$$

hence

$$w(\theta^s R) \leq \left(1 - \frac{1}{C}\right)^s w(R).$$

Let $R' < R$. Let s be such that

$$\theta^{s+1} R \leq R' < \theta^s R.$$

Then

$$w(R') \leq w(\theta^s R) \leq \left(1 - \frac{1}{C}\right)^s w(R).$$

Since

$$s = \left[\frac{\log(R'/R)}{\log \theta}\right] - 1,$$

We have

$$w(R') \leq \left(\frac{R'}{R}\right)^r w(R),$$

where

$$r = \left(1 - \frac{1}{C}\right)^{(\log\theta)^{-1}} < 1$$

Let $x, y \in B(\theta R)$. Then we have

$$|u(x) - u(y)| \leq C|x - y|^r$$

and the Hölder continuity follows.

QED

9.2 Aleksandrov maximum principle

The contents of this section is essentially from (cite chen)

Definition 9.1

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let $y \in \Omega$. Define

$$\chi(y) = \{p \in \mathbb{R}^n \mid u(x) \leq u(y) + p(x - y),\ \forall x \in \Omega\}.$$

Then χ defines a map from Ω to the set of subsets of $\mathbb{R}^n$. We call it the normal map.

Here is the geometric interpretation of the normal map. Let the lower graph of u in $\mathbb{R}^{n+1}$ be

$$\{(x,z)\in\mathbb{R}^n\times\mathbb{R}\mid x\in\Omega,-\infty<z<u(x)\}.$$

If $p\in\chi(y)$, then the hyperplane

$$z=u(y)+p\cdot(x-y)$$

is a supporting plane of the graph of u. $\chi(y)$ is the set of p such that $u(y)+p\cdot(x-y)$ is a supporting plane, and $(-p,1)$ is the normal vector of the corresponding the plane.

Definition 9.2

Let $u\in\mathcal{C}(\Omega)$. Let

$$\Gamma_u=\{y\in\Omega\mid\chi(y)\neq\emptyset\}=\{y\in\Omega\mid\exists\,p\in\mathbb{R}^n,u(x)\le u(y)+p\cdot(x-y)\}.$$

We call Γ_u the contact set *of u.* ♣

We consider the convex hull of the lower-graph of u and let $\hat{u}$ be the corresponding function. Then Γ_u is the intersection of the graphs of u and $\hat{u}$ projecting to $\{z=0\}$, which justifies its name: contact set.

If $u\in\mathcal{C}^1(\Omega)$, $y\in\Gamma_u$, then $\chi(y)=\{Du(y)\}$. If $u\in\mathcal{C}^2(\Omega)$, then $-D^2u(y)\ge 0$. In fact, a little bit more is true:

Lemma 9.4

Let $u\in W^{2,1}_{\mathrm{loc}}(\Omega)\cap\mathcal{C}(\Omega)$. Then

$$\chi(y)=\{Du(y)\},\qquad -D^2u(y)\ge 0,\quad a.e.\quad y\in\Gamma_u.$$

♡

Definition 9.3

Let

$$\chi(\Omega)=\chi(\Gamma_u)=\bigcup_{y\in\Omega}\chi(y).$$

♣

Example 9.1 Let $\Omega=B_d(x_0)$, where $B_d(x_0)$ is the ball of radiou d centered at x_0. Consider the function

$$u(x)=\frac{\lambda}{d}(d-|x-x_0|).$$

The graph of u is a cone with (x_0,λ) as the vertex and with $B_d(x_0)$ as the base. We have $\Gamma_u=\Omega$ and

$$\chi(y)=\begin{cases}B_{\lambda/d}(0) & y=x_0;\\ -\frac{\lambda}{d}\cdot\frac{y-x_0}{|y-x_0|} & y\neq x_0.\end{cases}$$

Definition 9.4

Let $\Omega\subset\mathbb{R}^n$, $x_0\in\Omega$. Construct a cone in $\mathbb{R}^n$ with (x_0,λ) being the vertex and $\{x\in\Omega,z=0\}$ being the base. Let ω be the function of the graph. Define

$$\Omega[x_0,\lambda]=\chi_\omega(\Omega)$$

♣

Lemma 9.5

Let $u\in\mathcal{C}(\Omega)$. Then

1. *for any $y\in\Gamma_u$,*

$$|p|\le\frac{2\sup|u|}{\mathrm{dist}(y,\partial\Omega)},\qquad\forall p\in\chi(y),$$

2. *the normal map maps a compact subset of Ω to a closed set of $\mathbb{R}^n$.* ♡

Proof. For all $y \in \Gamma_u$, we have

$$u(y) + p \cdot (x - y) \geq u(x), \qquad \forall x \in \Omega.$$

We draw a ray starting from y with direction $-p$. Suppose the ray hits $\partial\Omega$ at x_0. Since $x_0 - y$ is proportional to p (and in the opposite direction), we have

$$u(x_0) \leq u(y) + p(x_0 - y) = u(y) - |x_0 - y| \cdot |p|.$$

Thus

$$|p| \leq \frac{2 \sup |u|}{|x_0 - y|} \leq \frac{2 \sup |u|}{\operatorname{dist}(y, \partial\Omega)}.$$

To prove (2), let F be a compact subset of Ω, let $\{p_n\} \subset \chi(F)$, and let $p_n \to p$ as $n \to \infty$. Since $p_n \in \chi(F)$, there is a sequence $y_n \in F$ such that $p_n \in \chi(y_n)$. By definition, we have

$$u(y_n) + p_n(x - y_n) \geq u(x), \qquad \forall x \in \Omega$$

By passing the subsequence if necessary, we may assume that $p_n \to p_0, y_n \to y_0$. Then we have $p_0 \in \chi(y_0)$. ∎

Lemma 9.6

(1). Let Ω, A be open domains of $\mathbb{R}^n$, $\Omega \subset A$. Then if $x_0 \in \Omega$,

$$\Omega[x_0, \lambda] \supset A[x_0, \lambda].$$

(2). Let d be the diameter of Ω. Then

$$|\Omega[x_0, \lambda]| \geq \left(\frac{\lambda}{d}\right)^n w_n,$$

where w_n is the volume of the unit ball in $\mathbb{R}^n$. ♡

Proof. (1) is obvious and (2) follows from the above example. ∎

Lemma 9.7

Let $u \in C^2(\bar{\Omega}), g \in C(\bar{\Omega}), g \geq 0$, and let E be a measurable subset of Γ_u. Then

$$\int_{Du(E)} g(x(p))dp \leq \int_E g(x) \det(-D^2 u) dx,$$

where $x(p) = (Du)^{-1}(p)$ is defined and is continuous outside a measure zero set. ♡

Proof. Let $J(x) = \det(-D^2 u)$, $S = \{x \in \Omega \mid J(x) = 0\}$. By the Sard's theorem, $|Du(S)| = 0$.

We first assume that E is an open set; $E \backslash S$ is also open. We can find a sequence of parallel, disjoint cubes $\{C_\ell\}_{\ell=1}^\infty$ such that

$$E \backslash S = \bigcup_{\ell=1}^{\infty} C_\ell.$$

Moreover, we assume that C_ℓ are so small such that

$$Du : C_\ell \to Du(C_\ell)$$

are differmorphisms. Therefore

$$\int_{Du(C_\ell)} g(x(p))dp = \int_{C_\ell} g(x)J(x)dx.$$

Thus we have

$$\int_{Du(E\backslash S)} g(x(p))dp \leq \sum_\ell \int_{Du(C_\ell)} g(x(p))dp$$
$$= \sum_\ell \int_{C_\ell} g(x)J(x)dx = \int_{E\backslash S} g(x)J(x)dx.$$

By the Sard's Theorem, $|Du(S)| = 0$. Thus the result is true when E is an open set.

If E is a measurable subset of Γ_u, then there is an open set $G \supset E\backslash S$ such that on G, $J(x) > 0$. On the other side, since $E\backslash S$ is measurable, we can find open sets $\{O_\ell\}$ such that $E\backslash S \subset O_\ell$, and $|O_\ell\backslash(E\backslash S))| \to 0$ as $\ell \to \infty$. Thus we have

$$\int_{Du(E)} g(x(p))dp \leq \int_{Du(G\cap O_\ell)} g(x(p))dp \leq \int_{G\cap O_\ell} g(x)J(x)dx.$$

The lemma follows by letting $\ell \to \infty$.

■

Lemma 9.8

Let $u \in \mathcal{C}(\bar{\Omega})$. Assume that on $\partial\Omega$, $u \leq 0$; and there is an $x_0 \in \Omega$ such that $u(x_0) > 0$. Then

$$\Omega[x_0, u(x_0)] \subset \chi(\Gamma_u^+),$$

where $\Gamma_u^+ = \Gamma_u \cap \{u \geq 0\}$.

♡

Proof. Let $p \in \Omega[x_0, u(x_0)]$. Then by definition

$$u(x_0) + p \cdot (x - x_0) \geq \hat{u}(x) \geq 0.$$

Define

$$\lambda_0 = \inf\{\lambda \mid \lambda + p \cdot (x - x_0) \geq u(x), \forall x \in \bar{\Omega}\}.$$

By the continuity of u, we have

$$\lambda_0 + p \cdot (x - x_0) \geq u(x), \qquad \forall x \in \bar{\Omega},$$

and there is a ξ such that

$$\lambda_0 + p \cdot (\xi - x_0) = u(\xi).$$

1. if $\lambda_0 = u(x_0)$, then $x_0 \in \Gamma_u^+, p \in \chi(\Gamma_u^+)$;
2. if $\lambda_0 > u(x_0)$, then $\xi \notin \partial\Omega$, otherwise

$$u(\xi) > u(x_0) + p \cdot (\xi - x_0) \geq 0$$

contradicting to $u \leq 0$ on $\partial\Omega$. Since we have

$$u(\xi) + p \cdot (x - \xi) \geq u(x), \forall x \in \bar{\Omega},$$

where $\xi \in \Gamma_u^+$ and $p \in \chi(\Gamma_u^+)$.

■

With the above lemma we immediately get the following

Lemma 9.9 (Aleksandrov type estimate)

Let $u \in \mathcal{C}^2(\bar{\Omega})$ *and* $u|_{\partial\Omega} \le 0$. *Then*

$$\sup_{\Omega} u \le \frac{d}{\sqrt[n]{w_n}} \left[\int_{\Gamma_u^+} \det(-D^2 u) dx \right]^{1/n},$$

where $d = \operatorname{diam} \Omega$.

Proof. By Lemma 9.6 and Lemma 9.8, we have

$$|\chi(\Gamma_u^+)| \ge w_n \left[\frac{u(x_0)}{d} \right]^n.$$

Thus

$$u(x_0) \le \frac{d}{\sqrt[n]{w_n}} |\chi(\Gamma_u^+)|^{1/n}.$$

Using Lemma 9.7, we have

$$|\chi(\Gamma_u^+)| \le \int_{\chi(\Gamma_u^+)} dp \le \int_{\Gamma_u^+} \det(-D^2 u) dx.$$

■

We can weaken the smoothness of the function u to get the following

Theorem 9.3

Let $u \in \mathcal{C}(\bar{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$. *Then*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u + \frac{d}{\sqrt[n]{w_n}} \left[\int_{\Gamma_v^+} \det(-D^2 u) dx \right]^{1/n},$$

where $v = u - \sup_{\partial\Omega} u$.

Proof. By assumption, there is a sequence $\{u_m\} \subset \mathcal{C}^2(\bar{\Omega})$ such that $u_m \to u$ in $W^{2,n}_{\mathrm{loc}}(\Omega)$. That is, for any $\Omega' \subset\subset \Omega$, if $m \to \infty$, $u_m \to u$ in $W^{2,n}(\Omega')$. For any $\varepsilon > 0$, let

$$\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

Let

$$v_\varepsilon = u - \sup_{\Omega \setminus \Omega_\varepsilon} u - \varepsilon$$

$$v_{m,\varepsilon} = u_m - \sup_{\Omega \setminus \Omega_\varepsilon} u_m - \varepsilon$$

By Theorem 9.9, we have

$$\sup_{\Omega} v_{m,\varepsilon} \le \frac{d}{\sqrt[n]{w_n}} \left(\int_{\Gamma^+_{v_{m,\varepsilon}}} \det(-D^2 u_m) dx \right)^{1/n}.$$

Since $\Gamma^+_{m,\varepsilon} \subset \Omega_\varepsilon$, we write

$$\begin{aligned}&\int_{\Gamma^+_{v_{m,\varepsilon}}} \det(-D^2 u_m)dx \\ &= \int_{\Gamma^+_{v_{m,\varepsilon}}} (\det(-D^2 u_m) - \det(-D^2 u))dx + \int_{\Gamma^+_{v_{m,\varepsilon}}} \det(-D^2 u)dx \\ &= I_{1,m} + I_{2,m}\end{aligned}$$

Obviously, we have $I_{1,m} \to 0$ as $m \to \infty$. On the other hand, we have

$$\overline{\lim_{m\to\infty}}\ \Gamma^+_{v_{m,\varepsilon}} \subset \Gamma^+_{v_\varepsilon}.$$

In fact, if

$$x_0 \in \overline{\lim_{m\to\infty}}\ \Gamma^+_{v_{m,\varepsilon}}.$$

Then there is a subsequence m_k such that $x_0 \in \Gamma_{v_{m_k,\varepsilon}}$. By Lemma 9.5,

$$|p_k| \le \frac{4 \sup |u_m|}{\operatorname{dist}(x, \partial\Omega)} \le \frac{4}{\varepsilon} \sup |u_m|$$

for any $p_k \in \chi_{v_{m_k,\varepsilon}}(x_0)$.

Assuming $p_k \to p_0$. Then $p_0 \in \chi_{v_\varepsilon}(x_0)$, hence $x_0 \in \Gamma^+_{v_\varepsilon}$. Thus

$$\overline{\lim_{m\to\infty}}\ I_{2,m} \le \int_{\Gamma^+_{v_\varepsilon}} \det(-D^2 u)dx.$$

The theorem is proved. ∎

Theorem 9.4

Let $u \in C(\bar\Omega) \cap W^{2,n}_{\text{loc}}(\Omega)$ and let

$$-a^{ij} D_{ij} u \le f.$$

Then

$$\sup_\Omega u(x) \le \sup_{\partial\Omega} u(x) + \frac{d}{n\sqrt[n]{w_n}} \left|\left| f^+/\mathcal{D}^* \right|\right|_{L^n(\Gamma^+_u)},$$

where $v = u - \sup_{\partial\Omega} u(x)$, and $\mathcal{D}^* = (\det a^{ij})^{1/n}$. ♡

Proof. We know that on Γ_v, $-D^2 u \ge 0$ almost everywhere. Thus we have

$$-a^{ij} D_{ij} u \ge n(\det A(-D^2 u))^{1/n} = n\mathcal{D}^* \det(-D^2 u)^{1/n}.$$

Thus we have

$$\det(-D^2 u) \le \left(\frac{f^+}{n\mathcal{D}^*}\right)^n,$$

and the theorem is proved. ∎

Theorem 9.5 (Aleksandrov maximum principle)

Let $u \in \mathcal{C}(\bar\Omega) \cap W^{2,n}_{\text{loc}}(\Omega)$ such that

$$Lu \le f,$$

where

$$Lu = -a^{ij}D_{ij}u + b^i D_i u + cu = f.$$

Assume that

$$\begin{aligned} &(a^{ij}) \geq 0 \qquad \text{on } \Omega \\ &\sum_i \left\| \frac{b^i}{\mathcal{D}^*} \right\|_{L^n(\Omega)} \leq \beta \\ &c \geq 0 \end{aligned}$$

Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \left\| \frac{b^i}{\mathcal{D}^*} \right\|_{L^n(\Gamma_u)},$$

where C only depends on n, β and $\operatorname{diam}\Omega$*.* ♡

9.3 Krylov-Safonov Estimates

For elliptic equations of non-divergence form, in 1980, Krylov-Safonov obtained the Hölder estimate, which was almost 20 years after de Giorgi-Nash-Moser's results on elliptic equations of divergence form.

We shall use similar steps as in the de Giorgi-Nash-Moser case: we start from local maximum principle, weak Harnark inequality, Harnark inequality, and the C^α-estimate. The contents of this section is essentially from (Cite chen).

For the sake of simplicity, we only consider the following equation

$$Lu = -a^{ij}D_{ij}u = f \tag{9.4}$$

on Ω, where (a^{ij}) satisfies the uniform ellipticity condition:

$$\lambda > 0, \Lambda/\lambda \leq \gamma \tag{9.5}$$

on Ω, where λ, Λ are the minimum and maximum eigenvalues of (a^{ij}) respectively, and γ is a positive number.

Theorem 9.6 (local maximum principle)

Assume equation (9.4)*. Let $u \in W^{2,n}(\Omega)$ and let $Lu \leq f$ almost everywhere on Ω. Let $f/\lambda \in L^n(\Omega)$. Then for any $p > 0$, $B_{2R} \subset \Omega$, we have*

$$\sup_{B_R(y)} u \leq C \left[\fint_{B_{2R}(y)} (u^+)^p dx \right]^{1/p} + R \left\| \frac{f}{\lambda} \right\|_{L^n(B_{2R}(y))},$$

where C only depends on n, γ, p. ♡

Proof. By using the transformation

$$x \mapsto \frac{x-y}{2R},$$

we can assume that $y = 0$ and $R = 1/2$. Taking

$$\eta(x) = (1 - |x|^2)^\beta$$

as the cut-off function, then we have

$$\begin{aligned} &D_i\eta = -2\beta x_i(1-|x|^2)^{\beta-1};\\ &D_{ij}\eta = -2\beta\delta_{ij}(1-|x|^2)^{\beta-1} + 4\beta(\beta-1)x_ix_j(1-|x|^2)^{\beta-2};\\ &L\eta = [2\beta\sum_{i=1}^n a^{ii}(1-|x|^2) - 4\beta(\beta-1)\sum_{i,j=1}^n a^{ij}x_ix_j](1-|x|^2)^{\beta-2}. \end{aligned} \tag{9.6}$$

Let $v = \eta u$. Then we have

$$Lv = uL\eta + \eta Lu - 2a^{ij}D_i\eta D_j u.$$

On the contact set Γ_v defined in Definition 9.2, we have

$$|Dv| \le \frac{v(x)}{\mathrm{dist}(x,\partial B_1)} \le \frac{v(x)}{1-|x|}$$

by Lemma 9.5. Then we have

$$|Du| \le \frac{1}{\eta}|Dv - uD\eta| \le \frac{1}{\eta}\left[\frac{v(x)}{1-|x|} + u|D\eta|\right] \le 2(1+\beta)\eta^{-1/\beta}u.$$

Thus we have

$$Lv \le \eta f + (16\beta^2 + 2n\beta)\Lambda\eta^{-2/\beta}v \le C\lambda\eta^{-2/\beta}v + f$$

almost everywhere on Γ_v, where C only depends on n, β, γ.

By Theroem 9.4, we have

$$\begin{aligned} \sup_{B_1} v &\le C\left(||\eta^{-2/\beta}v^+||_{L^n(B_1)} + \left|\left|\frac{f}{\lambda}\right|\right|_{L^n(B_1)}\right)\\ &\le C\left((\sup v^+)^{1-2/\beta}||(u^+)^{2/\beta}||_{L^n(B_1)} + \left|\left|\frac{f}{\lambda}\right|\right|_{L^n(B_1)}\right). \end{aligned}$$

Using the Young's inequality, we have

$$\sup_{B_1} v \le C\left|\left|\frac{f}{\lambda}\right|\right|_{L^n(B_1)} + \varepsilon\sup_{B_1} v^+ + C_\varepsilon||(u^+)^{2/\beta}||^{\beta/2}_{L^n(B_1)}.$$

We take $\varepsilon = 1/2$ and if $p < n$, we let $\beta = 2n/p > 2$. Then we have

$$\sup_{B_{1/2}} u \le C\left(||u^+||_{L^p(B_1)} + \left|\left|\frac{f}{\lambda}\right|\right|_{L^n(B_1)}\right).$$

If $p \ge n$, then the estimate follows from the Hölder inequality.

■

The most complicated part of the estimate is the weak Harnark inequality. Let $K_R(y)$ be a tube on $\mathbb{R}^n$ with center in y and side length $2R$. The following argument was introduced by Krylov-Safonov. We start with the following

Lemma 9.10

Let K_0 be a tube in $\mathbb{R}^n$ and let Γ be a measurable subset of K_0. Let $0 < \delta < 1$. Let

$$\Gamma_\delta = \bigcup\{K_{3R}(y)\cap K_0 \mid K_R(y)\subset K_0, |\Gamma\cap K_R(y)| \ge \delta|K_R(y)|\}.$$

If $\Gamma_\delta \ne K_0$, then

$$|\Gamma| \le \delta|\Gamma_\delta|.$$

♡

Proof. If $|K_0 \cap \Gamma| > \delta |K_0|$, then $\Gamma_\delta = K_0$. Thus if $\Gamma_\delta \neq K_0$, we must have

$$|K_0 \cap \Gamma| \leq \delta |K_0|.$$

Dividing K_0 into 2^n congruent sub-cubes and denoting them as $\{K(i_1)\}_{i_1=1}^{2^n}$. On each $K(i_1)$, we have either of the following

1. $|\Gamma \cap K(i_1)| \leq \delta |K(i_1)|$;
2. $|\Gamma \cap K(i_1)| > \delta |K(i_1)|$.

Let the collection of $K(i_1)$ satisfying (2) be $\mathcal{F}_1$. If $K(i_1)$ satisfies (1), continue the subdivision such that

$$K(i_1) = \bigcup_{i_2=1}^{2^n} K(i_1, i_2).$$

For each $K(i_1, i_2)$, we still have two cases. We let $\mathcal{F}_2$ be the collection of all $K(i_1, i_2)$ satisfying (2). On the other hand, if $K(i_1, i_2)$ satisfying (1), continue the subdivision. Thus we have the sets $\mathcal{F}_1, \cdots, \mathcal{F}_m, \cdots$. Let

$$\mathcal{F} = \{K(i_1, \cdots, i_{m-1}) \mid K(i_1, \cdots, i_{m-1}, i_m) \in \mathcal{F}_m\}.$$

Thus if $K(i_1, \cdots, i_m) \in \mathcal{F}_m$, then

$$|K(i_1, \cdots, i_m) \cap \Gamma| > \delta |K(i_1, \cdots, i_m)|;$$
$$|K(i_1, \cdots, i_{m-1}) \cap \Gamma| \leq \delta |K(i_1, \cdots, i_{m-1})|.$$

By the definition of Γ_δ, we have

$$K(i_1, \cdots, i_{m-1}) \subset \Gamma_\delta.$$

Let

$$\tilde{\Gamma}_\delta = \bigcup_{K \in \mathcal{F}} K \subset \Gamma_\delta.$$

Then we have

$$|\tilde{\Gamma}_\delta \cap \Gamma| = \sum_{K \in \mathcal{F}} |K \cap \Gamma| \leq \delta \sum_{K \in \mathcal{F}} |K| = \delta |\tilde{\Gamma}_\delta| \leq \delta |\Gamma_\delta|.$$

On the other hand, by the definition of $\tilde{\Gamma}_\delta$, since Γ is measurable, by the Lebesgue Theorem, we know that for almost every point of Γ is dense. Thus we have

$$|\Gamma| = |\Gamma \cap \tilde{\Gamma}_\delta| \leq \delta |\Gamma_\delta|.$$

■

Theorem 9.7 (Weak Harnark inequality)

Suppose L satisfies the uniform ellipticity conditions (9.5); $u \in W^{2,n}(\Omega)$; $Lu \geq f$; $f/\lambda \in L^n(\Omega)$; $u \geq 0$ on $B_{2R}(y) \subset \Omega$. Then there is a $p > 0$, $C > 1$ such that

$$\left(\fint_{B_R(y)} |u|^p dx \right)^{1/p} \leq C \left[\inf_{B_R(y)} u + R \left\| \frac{f}{\lambda} \right\|_{L^n(B_{2R}(y))} \right],$$

where p, C only depends on n, R.

♡

Proof. The proof of this theorem is quite long so we divide it into the following 5 steps.

Step 1. Using rescaling

$$x \mapsto \frac{x-y}{2R},$$

we assume that $y = 0$, $R = 1/2$. Let

$$\tilde{u} = u + \left|\left|\frac{f}{\lambda}\right|\right|_{L^n(B_1)}, \quad \Gamma = \{x \in B_1 \mid \tilde{u}(x) \geq 1\}.$$

We want to prove that there exists $C > 0$, $0 < \delta < 1$ such that whenever

$$|\Gamma \cap K_\alpha| \geq \delta |K_\alpha|, \tag{9.7}$$

we have

$$\inf_{K_{3\alpha}} \tilde{u} \geq C^{-1}, \tag{9.8}$$

where K_α is the tube centered at 0 with side length 2α, $\alpha = 1/(6\sqrt{n})$, C, δ only depends on n, γ.

For any $\varepsilon > 0$. Let

$$w = \log(\tilde{u} + \varepsilon)^{-1}, g = \frac{f}{\tilde{u} + \varepsilon}.$$

Then we have

$$-a^{ij} D_{ij} w = \frac{1}{\tilde{u} + \varepsilon} a^{ij} D_{ij} u - a^{ij} D_i w D_j w \leq -g - a^{ij} D_i w D_j w.$$

Let $\eta = (1 - |x|^2)^\beta$. Let $v = \eta w$. Then we have

$$\begin{aligned} Lv &= \eta L w + w L\eta - 2a^{ij} D_i w D_j \eta \\ &\leq -\eta g - \eta a^{ij} D_i w D_j w - 2a^{ij} D_i w D_j \eta + vL\eta/\eta \\ &\leq -\eta g + \frac{1}{\eta} a^{ij} D_i \eta D_j \eta + \frac{vL\eta}{\eta}. \end{aligned}$$

By (9.6), we know when

$$(2\beta - 1)\lambda |x|^2 \geq n\Lambda,$$

we have $L\eta \leq 0$. Let $\alpha \in (0, 1)$. If $|x| > \alpha$, we take

$$\beta - 1 \geq \frac{n\gamma}{2\alpha^2}.$$

Then we have $L\eta \leq 0$ for $|x| \geq \alpha$. Thus we have

$$Lv \leq |g| + 4\beta^2 \Lambda + \sup_{B_\alpha} \left(\frac{L\eta}{\eta}\right) \chi(B_\alpha) v,$$

where $\chi(B_\alpha)$ is the characteristic function of B_α. Note that

$$\left|\left|\frac{g}{\lambda}\right|\right|_{L^n(B_1)} \leq 1.$$

By the Aleksandrov maximum principle, we have

$$\sup_{B_1} v \leq C[1 + ||v^+||_{L^n(B_\alpha)}], \tag{9.9}$$

where C only depends on n, γ, α.

In order to use the measure theory, we change ball into cube. We have

$$\sup_{B_1} v \leq C[1 + ||v^+||_{L^n(B_\alpha)}] \leq C(1 + |K_\alpha^+|^{1/n} \sup_{B_1} v),$$

where

$$K_\alpha^+ = \{x \in K_\alpha \mid v > 0\} = \{x \in K_\alpha \mid \tilde{u} + \varepsilon < 1\}.$$

If
$$\frac{|K_\alpha^+|}{K_\alpha} \leq \theta \triangleq \frac{1}{(2C)^n|K_\alpha|} = \frac{1}{(4C\alpha)^n}.$$
Then we have
$$\sup_{B_1} v \leq 2C,$$
or equivalently
$$\inf_{B_{1/2}} (\tilde{u} + \varepsilon) \geq \frac{1}{C}.$$
Letting $\varepsilon \to 0$, we get (9.8). On the other hand, if $\frac{|K_\alpha^+|}{K_\alpha} > \theta$, we let $\delta = 1 - \theta$, $\alpha = \frac{1}{6\sqrt{n}}$. Then we have $K_{3\alpha} \subset B_{1/2}$. If $|\Gamma \cap K_\alpha| \geq \delta|K_\alpha|$, then by Lemma 9.10, we have
$$|K_\alpha^+| = |K_\alpha \backslash (\Gamma \cap K_\alpha)| \leq \theta |K_\alpha|$$
which contradicts to the assumption (9.7).

Step 2. For any positive integer m, if
$$|\Gamma \cap K_\alpha| \geq \delta^m |K_\alpha|,$$
then
$$\inf_{K_\alpha} \tilde{u} \geq C^{-m},$$
where C is the constant in Step 1.

If $m = 1$, the statement is proved by the above step. Assume that the claim is true for $m \geq 1$. Assume that
$$|\Gamma \cap K_\alpha| \geq \delta^{m+1}|K_\alpha|.$$
Let $\tilde{K}_0 = K_\alpha$,
$$\Gamma_\delta = \bigcup \{K_{3r}(x) \cap \tilde{K}_0 \mid K_r(x) \subset \tilde{K}_0, |\Gamma \cap K_r(x)| \geq \delta |K_r(x)|\}.$$
By Lemma 9.10, we have either
$$\Gamma_\delta = \tilde{K}_0, \quad \text{or} \quad |\Gamma \cap \tilde{K}_0| \leq \delta |\Gamma_\delta|.$$
By the definition of Γ_δ and Step 1, we have
$$\inf_{\Gamma_\delta} \tilde{u} \geq C^{-1}.$$
If $\Gamma_\delta = \tilde{K}_0 = K_\alpha$, then the above implies the claim. If $|\Gamma \cap \tilde{K}_0| \leq \delta |\Gamma_\delta|$. Let $v = Cu$. Then v satisfiws
$$-a^{ij}D_{ij}v \geq Cf.$$
Let
$$\tilde{v} = v + \left|\left|\frac{Cf}{\lambda}\right|\right|_{L^n(B_1)} = C\tilde{u}.$$
Let
$$\tilde{\Gamma} = \{x \in B_1 \mid \tilde{v} \geq 1\}.$$
Then $\Gamma_\delta \subset \tilde{\Gamma}$ and we have
$$|\tilde{\Gamma} \cap \tilde{K}_0| \geq |\Gamma_\delta| \geq \frac{1}{\delta}|\Gamma \cap \tilde{K}_0| = \frac{1}{\delta}|\Gamma \cap K_\alpha| \geq \delta^m |K_\alpha| = \delta^m |\tilde{K}_0|.$$

By the inductive assumption, we have

$$\inf_{K_\alpha} \tilde{v} \geq C^{-m},$$

which is equivalent to

$$\inf_{K_\alpha} \tilde{u} \geq C^{-(m+1)}.$$

Step 3. Let

$$\Gamma_t = \{x \in B_1 \mid \tilde{u}(x) > t\}.$$

Then there is a $C > 1, \mu > 0$ such that

$$|B_\alpha \cap \Gamma_t| \leq C|B_\alpha| \left(\frac{\inf_{B_\alpha} \tilde{u}}{t}\right)^\mu, \tag{9.10}$$

where C and μ depend only on n, γ.

Let $v = u/t$ and $\tilde{v} = \tilde{u}/t$.

$$\tilde{\Gamma} \triangleq \{x \in B_1 \mid \tilde{v}(x) > 1\} = \Gamma_t.$$

If $|B_\alpha \cap \Gamma_t| = 0$, then (9.10) is obvious. Now assume that $|B_\alpha \cap \Gamma_t| \neq 0$, then there is a positive number m such that

$$\delta^m |K_\alpha| \leq |\tilde{\Gamma} \cap K_\alpha| \leq \delta^{m-1}|K_\alpha|.$$

That it,

$$\log \frac{|\tilde{\Gamma} \cap K_\alpha|}{K_\alpha} \cdot (\log \delta)^{-1} \leq m \leq 1 + \log \frac{|\tilde{\Gamma} \cap K_\alpha|}{K_\alpha} \cdot (\log \delta)^{-1}.$$

By Step 2, we have

$$\inf_{K_\alpha} \tilde{v} \geq C^{-m} \geq C^{-1} \left[\frac{|\tilde{\Gamma} \cap K_\alpha|}{K_\alpha}\right]^{\frac{\log C}{\log \delta^{-1}}}.$$

Let $\mu = \log \delta^{-1} / \log C$. Then we have

$$|\Gamma_t \cap K_\alpha| \leq (C \inf_{K_\alpha} \tilde{v})^\mu |K_\alpha|.$$

Step 4. There is a $p > 0$ such that

$$\left(\fint_{B_\alpha} u^p dx\right)^{1/p} \leq C\left(\inf_{B_\alpha} u + \left|\left|\frac{f}{\lambda}\right|\right|_{L^n(B_1)}\right).$$

We have

$$\int_{B_\alpha} u^p dx = p\int_0^\infty t^{p-1}|B_\alpha \cap \Gamma_t| dt = p\int_0^b t^{p-1}|B_\alpha \cap \Gamma_t|dt + p\int_b^\infty t^{p-1}|B_\alpha \cap \Gamma_t|dt,$$

where b is a number to be determined. Therefore, we have

$$\int_{B_\alpha} u^p dx \leq p\int_0^b \int_0^b t^{p-1}|B_\alpha|dt + p\int_b^\infty C m_0^\mu |B_\alpha| t^{p-\mu-1}dt,$$

where $m_0 = \inf_{B_\alpha} \tilde{u}$. Let $p = \mu/2$. Then

$$\int_{B_\alpha} u^p dx \leq b^p|B_\alpha| + C m_0^{2p} b^{-p} |B_\alpha|.$$

Let $b = C^{1/2p} m_0$. Then we have

$$\int_{B_\alpha} u^p dx \le 2C^{1/2} m_0^p \le 2C^{1/2} \left(\inf_{B_\alpha} u + \left|\left|\frac{f}{\lambda}\right|\right|_{L^n(B_1)} \right)^p .$$

Step 5. Using covering technique to prove

$$\left(\int_{B_{1/2}} u^p dx \right)^{1/p} \le C \left(\inf_{B_{1/2}} u + \left|\left|\frac{f}{\lambda}\right|\right|_{L^n(B_1)} \right).$$

We note that $u \in W^{2,n}(B_{\frac{1}{2}})$, there is a $x_0 \in \bar{B}_{1/2}$ such that

$$u(x_0) = \inf_{B_{\frac{1}{2}}} u.$$

Consider the integration

$$\int_{B_{\frac{1}{2}+\frac{\alpha}{4}}} dy \cdot \int_{B_{\frac{\alpha}{4}}(y)} u^p dx \ge |B_{\frac{\alpha}{4}}| \int_{B_{\frac{1}{2}}} u^p(y) dy.$$

Using the mean value theorem, there is a y such that

$$y_0 \in B_{\frac{1}{2}+\frac{\alpha}{4}}$$

such that

$$\int_{B(y_0)} u^p dx \ge \int_{B_{\frac{1}{2}}} u^p dx.$$

The theorem is proved.

■

Theorem 9.8 (Harnark inequality)

Let the coefficients of L satisfy the uniform elliptic condition. $u \in W^{2,n}(\Omega)$ satisfies $Lu = f$. Let $f/\lambda \in L^n(\Omega)$. Suppose $u \ge 0$ on $B_{2R}(y) \subset \Omega$. Then

$$\sup_{B_{\frac{R}{2}}(y)} u \le C \left(\inf_{B_{\frac{R}{2}}(y)} u + R \left|\left|\frac{f}{\lambda}\right|\right|_{L^n(B_{2R}(y))} \right).$$

♡

Proof. This follows from combining the local maximum principle and the weak Harnark inequality.

■

Theorem 9.9

Using the above notations, we have

$$\operatorname*{osc}_{B_R(y)} u \le C \left(\frac{R}{R_0}\right)^\alpha \left(\operatorname*{osc}_{B_{R_0}(y)} u + R_0 \left|\left|\frac{f}{\lambda}\right|\right|_{L^n(\Omega)} \right).$$

In particular, u is Hölder continuous.

♡

Chapter 10 Special topics

10.1 The Reverse Hölder Inequality

In this section, we prove the classical Reverse Hölder inequality of Gehring (cite gehring).

Let Q be a tube in $\mathbb{R}^n$. Let $f(x)$ be a measurable, bounded, and nonnegative function. Define the *maximum function* by

$$M(f,x) = \sup_{r>0} \fint_{B(x,r)} f(x)dx.$$

In this section, we shall prove

Theorem 10.1 (Reverse Hölder Inequality)

Let $q > 1$ such that

$$(M(f^q,x))^{1/q} \le bM(f,x)$$

a.e. x, where b is a constant. Then there is an $\varepsilon = \varepsilon(n,b,q) > 0$ such that

$$\fint_Q f^{q+\varepsilon}(x)dx \le C\left(\fint_Q f(x)dx\right)^{q+\varepsilon}.$$

♡

By rescaling, we may assume that

$$\fint_Q f^q(x)dx = 1. \tag{10.1}$$

Define

$$E(t) = \{x \in Q \mid f(x) > t\}.$$

We first prove the following lemma

Lemma 10.1

For all $t > 1$, there is a constant $a = a(n,b,q) > 0$ such that

$$\int_{E(t)} f^q(x)dx \le at^{q-1}\int_{E(t)} f(x)dx.$$

♡

Proof. Define

$$s = \alpha t,$$

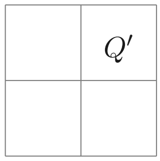
where $\alpha > 1$ is a constant to be determined. By assumption

$$\fint_Q f^q(x)dx = 1 \le s^q.$$

We use the function $f(x)$ and the number s^q to perform the *Calderón-Zygmund decomposition*. We then define a sequence of cubes $Q_j \subset Q$ such that

1. each side of Q_j is parallel to a side of Q; the interior of Q_j are disjoint;
2. $$s^q < \fint_{Q_j} f^q(x)dx \le 2^n s^q. \tag{10.2}$$
3. for *a.e.* $x \notin \bigcup_j Q_j$, we have $f(x) \le s$.

Briefly speaking, the sequences can be defined inductively in the following way: we first divide Q into 2^n sub-cubes like in Figure (ref fig2)



Let Q' be one of the sub-cubes. Then

$$\fint_{Q'} f^q(x)dx \le \frac{1}{m(Q')}\int_Q f^q(x)dx \le 2^n s^q.$$

If

$$\fint_{Q'} f^q(x)dx > s^q,$$

we say Q' is "good" and we assign Q' as one of the Q_j's. If Q' is not "good", then since

$$\fint_{Q'} f^q(x)dx \le s^q,$$

we can repeat the division as above. In this way, we get a sequence $\{Q_j\}$ such that they satisfy (10.2). Now for all $x \notin \bigcup_j Q_j$, there is a sequence Q_j'' shrinking to x such that

$$\fint_{Q_j''} f^q(x)dx \le s^q.$$

Thus by the Lebesgue Theorem, for *a.e.* x, $f(x) \le s$.

We define

$$G = \bigcup_j Q_j.$$

Then we have

$$\int_{E(s)} f^q(x)dx = \int_G f^q(x)dx = \sum_j \int_{Q_j} f^q(x)dx \le 2^n s^q m(G).$$

On the other hand, let $x \in Q_j$ and let $r = \operatorname{diam}(Q_j)$. Then

$$\frac{1}{m(B(x,r))}\int_{B(x,r)} f^q(x)dx \ge \frac{1}{m(B(x,r))} m(Q_j)s^q \ge c_1 s^q,$$

where $c_1 = c_1(n) > 0$ is a constant. By the definition of the maximum function, we have

$$M(f^q, x) \ge \fint_{B(x,r)} f^q(x)dx \ge c_1 s^q.$$

Thus by the assumption of the theorem, we have

$$M(f,x) \ge (c_1/b)^{1/q} s = (c_1/b)^{1/q}\alpha t > \beta t.$$

If we choose α large enough, we can assume that $\beta > 1$. By definition of the maximum function, for any x, there is an $r_x > 0$ such that

$$\fint_{B(x,r_x)} f(x)dx \ge \beta t.$$

Such a collection of balls $B(x, r_x)$ covers Q_j. Thus by the Vitali Covering Theorem, there is a countable subset of balls B_j which are disjoint and the union of $5B_j$ covers Q. We therefore have the inequality

$$m(G) \leq 5^n \sum_j m(B_j).$$

Thus we have

$$\beta t m(B_j) \leq \int_{B_j} f(x)dx \leq \int_{B_j \cap E(t)} f(x)dx + tm(B_j).$$

We therefore have

$$m(B_j) \leq \frac{1}{(\beta-1)t} \int_{B_j \cap E(t)} f(x)dx.$$

Thus we have

$$\begin{aligned}
&\int_{E(s)} f^q(x)dx \leq 2^n s^q m(G) \\
&\leq 10^n s^q \frac{1}{(\beta-1)t} \sum_j \int_{B_j \cap E(t)} f(x)dx \\
&\leq 10^n s^q \frac{1}{(\beta-1)t} \int_{E(t)} f(x)dx.
\end{aligned}$$

On the other hand, we have

$$\int_{E(t)\setminus E(s)} f^q(x)dx \leq s^{q-1} \int_{E(t)} f(x)dx.$$

Combining the above two inequalities completes the proof of the lemma.

■

Proof of the Reverse Hölder Inequality Define

$$h(t) = \int_{E(t)} f(x)dx.$$

Then it is easy to verify that

$$\int_{E(t)} f^r(x)dx = -\int_t^\infty s^{r-1} dh(s).$$

Thus the lemma can be written as

$$-\int_t^\infty s^{q-1} dh(s) \leq a t^{q-1} h(t). \tag{10.3}$$

Define

$$I(r) = -\int_1^\infty t^{r-1} dh(t).$$

and for $p > q > 0$, define

$$J = (p-q) \int_1^\infty t^{p-q-1} \left(-\int_t^\infty s^{q-1} dh(s)\right) dt.$$

Then by (10.3), we have

$$J \leq (p-q) \int_1^\infty a t^{p-2} h(t) dt = \frac{a(p-q)}{p-1} \int_1^\infty h(t) dt^{p-1} = \frac{a(p-q)}{p-1}(-I(1) + I(p)).$$

On the other hand, we use integration by parts to get

$$J = \int_1^\infty \left(-\int_t^\infty s^{q-1} dh(s)\right) dt^{p-q} = I(p) - I(q).$$

Thus we get

$$I(p) - I(q) \leq \frac{a(p-q)}{p-1}(-I(1) + I(p)).$$

Thus for p sufficiently close to q, we get

$$I(p) \leq C(I(q) + I(1)).$$

■

Second Proof of the theorem Since

$$\int_{E(t)} f^q(x)dx \leq at^{q-1} \int_{E(t)} f(x)dx,$$

for any $\varepsilon > 0$, we have

$$\int_1^\infty \left(t^{\varepsilon-1} \int_{E(t)} f^q(x)dx \right) dt \leq a \int_1^\infty \left(t^{q-2+\varepsilon} \int_{E(t)} f(x)dx \right).$$

We have

$$LHS = \int_1^\infty \int_X t^{\varepsilon-1} 1_{\{f(x)>t\}} f^q(x)dxdt = \int_X \int_1^\infty t^{\varepsilon-1} 1_{\{f(x)>t\}} f^q(x)dtdx$$
$$= \frac{1}{\varepsilon} \int_X (f^{q+\varepsilon}(x) - f^q(x))dx.$$

Similarly, we have

$$RHS = \frac{a}{q-1+\varepsilon} \int_X (f^{q+\varepsilon}(x) - f^{q-1+\varepsilon}(x))dx.$$

So the theorem is proved by assuming that $\varepsilon > 0$ is sufficiently small. ■

Remark Note that the last inequality is not homogeneous, which is fine, because we normalize the integral into (10.1).

10.2 The Grigor'yan-Netrusov-Yau cover

In this section, we give a simplified proof of (CITE GNY) Theorem 3.5.

Theorem 10.2

Let a pseudo-metric space (X, d) satisfy $(2, P)$-covering property. Let m be a Borel measure on X, and assume that there exist a positive real number v and ρ such that

$$\forall x \in X, \quad m(B(x, \rho/2)) \leq v, \quad \text{and} \quad \exists x_0 \in X, m(B(x_0, \rho)) > v. \tag{10.4}$$

Then, for any $\lambda > 1$, there exists a family $\mathfrak{A}$ of $[\varepsilon_0 m(X)/v]$ annuli in X satisfying the following properties

1. *$m(A) \geq v$ for any $A \in \mathfrak{A}$;*
2. *the annuli $\{\lambda A\}_{A \in \mathfrak{A}}$ are disjoint.*

Here ε_0 is a number independent to v. ♡

Let $N \geq 1$ be the maximum number such that there exists a family of annuli $\{A_i = B(x_i, r_i, R_i)\}$ with

1. $2C_3 v > m(A_i) > C_3 v$;
2. $\{\lambda A_i\}$ disjoint;
3. $$m(\bigcup_{i=1}^N B(x_i, 2\lambda^2 R_i)) < C_1 N v$$

for a constant $C_1 > 0$ independent to v.

Here C_3 is the constant such that when the radius is multiple by 8, then the volume is increased by C_3 times.

By (10.4), we know that the family of annuli exists at least for $N = 1$.

We shall get a contradiction by assuming that

$$N < \varepsilon_0 m(X)/v. \tag{10.5}$$

Definition 10.1

We say a subset

$$x_{i_1}, \cdots, x_{i_p}$$

of $\{x_1, \cdots, x_N\}$ *is admissible, if there are positive numbers* $T_{i_j} > 0$ *such that*

1. $$\bigcup_{j=1}^{p} B(x_{i_j}, \frac{1}{2\lambda} T_{i_j}) \supset \bigcup_{i=1}^{N} B(x_i, \lambda^2 R_i);$$

2. $$m(\bigcup_{j=1}^{p} B(x_{i_j}, T_{i_j})) \leq C_1 N v + C_2 v(N - p).$$

♣

Definition 10.2

Let

$$\sigma = \sup\{r \mid m(B(x,r)\backslash \bigcup_{j=1}^{p} B(x_{i_j}, T_{i_j}) \leq v, \quad \forall x \in X\}.$$

Then $\sigma < \infty$. *We call* σ *the Lu's number with respect to the balls*

$$B(x_{i_1}, T_{i_1}), \cdots, B(x_{i_p}, T_{i_p}).$$

♣

Let $x \in X$ be a point such that

$$m(B(x, 2\sigma)\backslash \bigcup_{j=1}^{p} B(x_{i_j}, T_{i_j}) > v.$$

Definition 10.3

A subset, say, $x_{i_s}, \cdots, x_{i_p}$, *of* $x_{i_1}, \cdots, x_{i_p}$ *is called selected, if*

$$B(x, 17\lambda^2\sigma) \cap B(x_{i_j}, \frac{1}{2} T_{i_j}) \neq \emptyset$$

for $j \geq s$. *In this case, we also call* x_{i_j} *selected if* $j \geq s$, *and* $B(x_{i_j}, T_{i_j})$ *selected if* x_{i_j} *is selected.*

♣

Note that there must be at least one selected ball, otherwise, we can add $B(x, 3/2\sigma)$ to $\mathfrak{A}$, a contradiction.

If there are more than one selected points, say $x_{i_{p-1}}, x_{i_p}$, then we remove x_{i_p}, and replace $B(x_{i_{p-1}}, T_{i_{p-1}})$ by

$$B(x_{i_{p-1}}, \max(300\lambda^3\sigma, T_{i_{p-1}})).$$

We claim that the new set of balls is still admissible.

First, we know that

$$m(\bigcup_{j=1}^{p-2} B(x_i, T_{i_j}) \bigcup B(x_{i_{p-1}}, 300\lambda^3\sigma)) \le C_1 N v + C_2(n-p+1)v.$$

We shall prove

$$\bigcup_{j=1}^{p-2} B(x_{i_j}, \frac{1}{2\lambda} T_{i_j}) \bigcup B(x_{i_{p-1}}, 150\lambda^2\sigma) \supset \bigcup_{i=1}^{N} B(x_i, \lambda^2 R_i).$$

To see this we first observe that

$$d(x, x_{i_j}) \le 17\lambda^2\sigma + \frac{1}{2} T_{i_j}$$

for $j = p-1, p$. On the other hand,

$$d(x, x_{i_j}) > T_{i_j} - 2\sigma$$

for any j. Therefore, we have

$$T_{i_j} \le (34\lambda^2 + 4)\sigma, \quad d(x, x_{i_j}) \le (34\lambda^2 + 2)\sigma$$

for $j = p-1, p$. Thus $B(x_{i_p}, T_{i_p}) \subset B(x_{i_{p-1}}, 150\lambda^2\sigma)$, the reduced set is still admissible.

Using the procedure, we can get a *minimal* admissible set

$$B(x_1, T_1), \cdots, B(x_k, T_k).$$

Without loss of generality, we can assume that

$$m(B(x_j, T_j)) > C_3 v. \tag{10.6}$$

Let $x \in X$ such that

$$m(B(x, 2\sigma') \backslash \bigcup_{j=1}^{k} B(x_j, T_j)) > v,$$

where σ' is the Lu's number for the minimal admissible set.

If $B(x, 2\lambda\sigma')$ doesn't touch any balls, then we can place one more ball $B(x, 2\sigma')$. If $B(x, 2\lambda\sigma')$ touch one of the balls, it has to touch only one of them, say, $B(x_1, T_1)$ otherwise the sequence is not minimal.

By the above same computation

$$T_1 \le 4(\lambda + 1)\sigma', \quad d(x, x_1) \le (4\lambda + 2)\sigma'.$$

We consider the annulus

$$B(x_1, \frac{1}{2} T_1, d(x, x_1) + 2\sigma').$$

We claim

$$B(x_1, \frac{1}{2} T_1, d(x, x_1) + 2\sigma') \supset B(x, 2\sigma').$$

To prove this, we first observe that $T_1 > 8\sigma'$, otherwise it contradicts to (10.6). Let $z \in B(x, 2\sigma') \backslash \bigcup_{j=1}^{k} B(x_j, T_j)$. Then for any $y \in B(x, 2\sigma')$, we have

$$d(y, x_1) \ge d(z, x_1) - 4\sigma' \ge T_1 - 4\sigma' \ge \frac{1}{2} T_1.$$

The inequality $d(y, x_1) \le d(x, x_1) + 2\sigma'$ is obvious. The claim is proved.

It remains to prove that the annulus

$$B(x_1, \frac{1}{2\lambda}T_1, \lambda(d(x, x_1) + 2\sigma'))$$

doesn't touch any other elements in λA.

To see this, we first observe that if $y \in B(x_2, \frac{1}{2}T_2)$, then

$$d(y, x_1) \geq d(x_2, x) - d(x, x_1) - \frac{1}{2}T_2 > 17\lambda^2\sigma' - d(x, x_1) > \lambda(d(x, x_1) + 2\sigma').$$

Remark[Final Remark] This is a lecture note for the Math 240BC at UCI. It is not a book or intended to be a book. The materials in this lecture note, I believe, are essential to differential geometers. Some of the materials are not standard so as to be easily found in any text books.