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Allocation of Network Error Correction Flow on Disjoint Paths

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Abstract: The diversity provided by disjoint paths can increase the survivability of communication networks. This paper considers the allocation of network error correction flow on a network that consists of disjoint paths from the source node to the destination node. Specifically, we propose an algorithm of allocating the path-flows to support the given rate with minimum cost. Our analysis shows that the asymptotic time complexity of this algorithm is linearithmic, and this algorithm is optimal in general.

Key words: network survivability; disjoint paths; maximum distance separable codes; network error correction

1 Introduction

Disjoint paths are commonly used to provide diversity and to improve the reliability and survivability of communication networks. Path protection mechanisms such as 1+1, 1:1, 1:M, and M:N protection are widely used in practical networks[1]. The 1+1 protection transmits identical information flow along both paths and uses a selector to choose a stream among received ones. Other protection mechanisms predefine backup paths, where information can be transmitted when some primary paths do not work[2]. Compared to 1:1, 1:M, and M:N, 1+1 protection, which can recover without switch at receiver side or feedback signaling to the transmitter side, can provide the fastest recovery, usually within 50 ms. Therefore, this mechanism is suitable for time-sensitive and high availability applications, especially in the cases that the delay of feedback is intolerable.

In 2002, Cai and Yeung[3] showed that network coding can be used to correct link errors in networks. The capacity of network error correction flow with unequal edge-flow to combat limited number of malicious links in networks was considered in Refs. [4–6]. Other researches on network error correction codes include Refs. [7, 8] and so on.

However, due to the high complexity of network coding and the operation limitation of routers in existing networks, almost all network coding systems in practical use can only encode and decode at source nodes and destination nodes. In these cases, network error correction flows are transmitted through disjoint paths.

There are a lot of existing results on finding disjoint paths. Optimal algorithms to find multiple edge-disjoint paths and vertex-disjoint paths were proposed in Ref. [9]. In these algorithms, an modified Dijkstra algorithm was used to find the shortest path in the graph with negative edges, which can be replaced by Bellman-Ford algorithm[10, 11] and its improved version (such as Ref. [12]).

In this paper, we propose an algorithm to allocate network error correction flow on disjoint paths. The algorithm firstly determines the number of paths to allocate the flow, then uniformly allocates path-flows on the paths with the least unit price. Further, Maximum Distance Separable (MDS) code is used on these path-flows as the network error correction code. The time complexity of this algorithm is $O\left(|\mathcal{P}| \log |\mathcal{P}|\right)$, where $|\mathcal{P}|$ is the number of disjoint paths. We also prove that this algorithm is optimal in general networks that consist of disjoint paths.

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The rest of paper is organized as follows: Section 2 formulates the model of network error correction. Section 3 characterizes the minimum cost network error correction flow. Section 4 presents the algorithm to allocate the network error correction flow. The time complexity of this algorithm is analyzed in Section 5, and the optimality of this algorithm is proved in Section 6. Section 7 draws the conclusion.

2 Problem Description

Consider a communication network, which can be denoted as a directed graph $\mathcal{G}$ with a source node v_s and a destination node v_t. Suppose that the graph $\mathcal{G}$ consists of $|\mathcal{P}|$ disjoint paths from v_s to v_t. Let $\mathcal{P}$ be the set of the disjoint paths. A session needs to transmit at the message rate R and to protect against fault on at most n_z paths.

Consider a flow $\boldsymbol{F}_{\mathcal{P}} = \left(f_p : p \in \mathcal{P}\right)$ on paths $\mathcal{P}$, where f_p is the flow on path p. Using the proof similar to Theorem 2 of Ref. [5], if $|\mathcal{P}| > n_z$ (where $n_z = 2z$), the maximum supported rate of this flow is the summation of minimal $|\mathcal{P}| - n_z$ elements in $\boldsymbol{F}_{\mathcal{P}}$; otherwise ($|\mathcal{P}| \leqslant n_z$), the maximum supported rate is zero. MDS codes can attain the rate. (Here is a brief explanation for $\boldsymbol{F}_{\mathcal{P}}$ such that the flow on every path is a multiple of $1/\alpha$ ($\alpha > 0$). Every flow on path p can be divided into $f_p\alpha$ streams, and there are $\sum\limits_{p\in\mathcal{P}} f_p\alpha$ streams in total. Path failings can only affect at most $\max\limits_{\mathcal{P}_1\subseteq\mathcal{P},|\mathcal{P}_1|\leqslant n_z} \sum\limits_{p\in\mathcal{P}_1} f_p\alpha$ streams. Thus,

$$\left(\sum_{p\in\mathcal{P}} f_p\alpha, \sum_{p\in\mathcal{P}} f_p\alpha - \max_{\mathcal{P}_1\subseteq\mathcal{P}:|\mathcal{P}_1|\leqslant n_z} \sum_{p\in\mathcal{P}_1} f_p\alpha\right)$$

MDS code can be used to support both n_z path redundancy and the rate

$$R = \sum_{p\in\mathcal{P}} f_p - \max_{\mathcal{P}_1\subseteq\mathcal{P},|\mathcal{P}_1|\leqslant n_z} \sum_{p\in\mathcal{P}_1} f_p$$

simultaneously.)

There are some forms of consumption (such as energy) for the flow, which is abstracted as "cost" in this paper. For simplicity, let the unit price of flow on path $p \in \mathcal{P}$ be a constant ψ_p. In this case, the overall expense of a flow $\boldsymbol{F}_{\mathcal{P}}$ is $\sum\limits_{p\in\mathcal{P}} \psi_p f_p$. Given the unit price vector $\boldsymbol{\Psi}_{\mathcal{P}} = (\psi_p : p \in \mathcal{P})$, the maximum failing path number n_z, and the rate R, the problem of allocating flow to minimize the cost can be formulated as follows:

$$\begin{aligned}
\text{min} \quad & \sum_{p\in\mathcal{P}} \psi_p f_p \\
\text{over} \quad & f_p, p \in \mathcal{P} \\
\text{s.t.} \quad & \sum_{p\in\mathcal{P}} f_p - \sum_{p\in\mathcal{P}_1} f_p \geqslant R, \mathcal{P}_1 \subseteq \mathcal{P} : |\mathcal{P}_1| \leqslant n_z, \\
& f_p \geqslant 0, \quad p \in \mathcal{P}.
\end{aligned}$$

The optimal flow $\boldsymbol{F}_{\mathcal{P}}^{\text{opt}}$ will be allocated on paths $\mathcal{P}$. The minimal cost c^{opt} will be the optimal overall expense. Moreover, MDS code will be selected to employ on such a flow.

Note that this problem is a linear programming with $O(\binom{|\mathcal{P}|}{n_z})$ constraints. Using a solver for linear programming, the optimal solution will be found in polynomial time. However, the polynomial order and the hidden constants are large, and the results can not be easily extended. Hereby, a better solution is proposed below.

3 Min-Cost Flow on Disjoint Paths

This section derives a characterization of minimum cost network error correction flow on disjoint paths.

Theorem 1 In a network that consists of $|\mathcal{P}|$ disjoint paths, the cost function is $\text{Cost}\,(\boldsymbol{F}_{\mathcal{P}}) = \sum\limits_{i=1}^{|\mathcal{P}|} \psi_i f_i$ where $\psi_1 \leqslant \psi_2 \leqslant \cdots \leqslant \psi_{|\mathcal{P}|}$. n_z has been determined. Let

$$n_r = \mathop{\arg\max}_{1\leqslant i\leqslant|\mathcal{P}|} \frac{\max\{i - n_z, 0\}}{\sum_{j=1}^{i} \psi_j} - n_z \tag{1}$$

When $n_r > 0$, the problem is feasible, and the flow

$$\boldsymbol{F}_{\mathcal{P}}^{\text{opt}} = \left\{\frac{R}{n_r}\mathbf{1}_{n_r+n_z}, \mathbf{0}_{|\mathcal{P}|-(n_r+n_z)}\right\}$$

is the minimum cost network error correction flow; otherwise, the problem is infeasible.

Proof Firstly, since $\psi_i < \psi_j$ ($1 \leqslant i < j \leqslant |\mathcal{P}|$), we can know that $f_i^{\text{opt}} \geqslant f_j^{\text{opt}}$ ($1 \leqslant i < j \leqslant |\mathcal{P}|$). Otherwise, suppose there exist i and j such that both $\psi_i < \psi_j$ and $f_i^{\text{opt}} < f_j^{\text{opt}}$. Let $\boldsymbol{F}' = (f_1^{\text{opt}}, \ldots, f_{i-1}^{\text{opt}}, f_j^{\text{opt}}, f_{i+1}^{\text{opt}}, \cdots, f_{j-1}^{\text{opt}}, f_i^{\text{opt}}, f_{j+1}^{\text{opt}}, \cdots, f_{|\mathcal{P}|}^{\text{opt}})$ be the flow that swapping the location of f_i^{opt} and f_j^{opt} in $\boldsymbol{F}^{\text{opt}}$. It is obvious that $\boldsymbol{F}'$ is feasible since the same session can be supported by swapping the location of code words on path p_i and p_j. Actually, $\boldsymbol{F}'$ is cheaper than $\boldsymbol{F}^{\text{opt}}$ since $\psi_i f_i^{\text{opt}} + \psi_j f_j^{\text{opt}} = \psi_i f_i^{\text{opt}} + \psi_j (f_j^{\text{opt}} - f_i^{\text{opt}}) + \psi_j f_i^{\text{opt}} >$

$$\psi_i f_i^{\rm opt} + \psi_i \left(f_j^{\rm opt} - f_i^{\rm opt} \right) + \psi_j f_i^{\rm opt} =$$
$$\psi_i f_j^{\rm opt} + \psi_j f_i^{\rm opt},$$

which contracts that $\boldsymbol{F}^{\rm opt}$ is the optimal flow.

Having proved $f_i^{\rm opt} \geqslant f_{i+1}^{\rm opt}$ $(1 \leqslant i < |\mathcal{P}|)$, we only consider the flow $\boldsymbol{F}$ such that $f_i \geqslant f_{i+1}$ $(1 \leqslant i < |\mathcal{P}|)$. Define $\Delta\boldsymbol{F} = (\Delta f_i : 1 \leqslant i \leqslant |\mathcal{P}|)$ as

$$\Delta f_i = \begin{cases} f_i - f_{i+1}, & i < |\mathcal{P}|; \\ f_{|\mathcal{P}|}, & i = |\mathcal{P}|. \end{cases}$$

And it's obvious to show that

$$f_j = \sum_{i=j}^{|\mathcal{P}|} \Delta f_i, \ \ 1 \leqslant j \leqslant |\mathcal{P}|.$$

On the one hand, the total expense is

$$\sum_{j=1}^{|\mathcal{P}|} \psi_j f_j = \sum_{j=1}^{|\mathcal{P}|} \psi_j \sum_{i=j}^{|\mathcal{P}|} \Delta f_i =$$
$$\sum_{i=1}^{|\mathcal{P}|} \sum_{j=1}^{i} \psi_j \Delta f_i = \sum_{i=1}^{|\mathcal{P}|} \phi_i \Delta f_i,$$

where $\phi_i = \sum_{j=1}^{i} \psi_j$. On the other hand, the summation of flow on $|\mathcal{P}| - n_z$ minimal paths is

$$\sum_{j=n_z+1}^{|\mathcal{P}|} f_j = \sum_{j=n_z+1}^{|\mathcal{P}|} \sum_{i=j}^{|\mathcal{P}|} \Delta f_i =$$
$$\sum_{i=n_z+1}^{|\mathcal{P}|} \sum_{j=1}^{i} \Delta f_i = \sum_{i=n_z+1}^{|\mathcal{P}|} i \Delta f_i = \sum_{i=1}^{|\mathcal{P}|} \varphi_i \Delta f_i,$$

where $\varphi_i = \max\{0, i - n_z\}$. Note that $\varphi_i = 0$ when $1 \leqslant i \leqslant n_z$, $1 \leqslant n_r \leqslant |\mathcal{P}| - n_z$, thus $\varphi_{n_z+n_r} = n_r$.

From the view of $\Delta\boldsymbol{F} = (\Delta f_i : 1 \leqslant i \leqslant |\mathcal{P}|)$, the "performance cost ratio" of Δf_i can be defined as $\rho_i = \varphi_i / \phi_i$. Let $-n_z \leqslant n_r \leqslant |\mathcal{P}| - n_z$ be an integer such that

$$\rho_{n_z+n_r} = \max_{1 \leqslant i \leqslant |\mathcal{P}|} \rho_i.$$

The best solution is to make full use of the $(n_r + n_z)$-th element in $\Delta\boldsymbol{F}^{\rm opt}$. Therefore, set

$$\Delta f_i^{\rm opt} = \begin{cases} \dfrac{R}{n_r}, & i = n_z + n_r; \\ 0, & i \neq n_z + n_r. \end{cases}$$

That is, the optimal solution is to allocate the flow equally on $n_z + n_r$ cheapest paths (and the flow on each path is R/n_r) and employ $(n_z + n_r, n_r)$ MDS code as the network error correction code. ■

This theorem tells us how to find the minimum cost network error correction flow on disjoint paths. First, we need to determine the value of n_r. Then we need to allocate the flow equally on $(n_r + n_z)$ path with minimum prices.

4 Algorithm

Algorithm 1 can determine the optimal flow allocation and corresponding network error correction code. The basic idea of this algorithm is as follows: At the beginning, the unit prices of each path are sorted in ascending order. Then, a loop is used to find the number of paths to allocate the flow. During the iterations, ψ is used to record the current unit price, and φ is used to record the summation of preceding unit prices. When the condition of an arbitrary if-block is met, the statement in the block breaks out of the loop. At this point, the value of integer n_r can indicate the number of paths to allocate the flow. Specifically, if $n_r > 0$, the problem is feasible, and the paths to allocate the flow should be the $n_r + n_z$ paths with minimum prices. Therefore, the flow is uniformly allocated in these $n_r + n_z$ paths, and MDS code is the network error correction code finally.

Example 1 Consider a network comprising of disjoint paths from $v_{\rm s}$ to v_t. The ordered unit price vector is $\boldsymbol{\Psi} = (1, 2, 2, 2, 3, 3, 3, 8, 9, 9)$. Let $n_z =$

Algorithm 1 Allocate Flow on Given Disjoint Paths

Input: $\boldsymbol{\Psi}_{\mathcal{P}} = (\psi_p : p \in \mathcal{P}), n_z, R$.
Output: $\boldsymbol{F}_{\mathcal{P}}^{\rm opt} = (f_p^{\rm opt} : p \in \mathcal{P}), c^{\rm opt}$.

Initialize: Sort $\psi_{\mathcal{P}}$ in ascending order to the vector $\boldsymbol{\Psi} = (\psi_1, \cdots, \psi_{|\mathcal{P}|})$. p_i is the path corresponding to i-th element in the vector $\boldsymbol{\Psi}$. Set $n_r \leftarrow (-n_z), \phi \leftarrow 0, i \leftarrow 1$.
loop
  if $i > |\mathcal{P}|$ **then**
    Break the loop.
  else
    $\psi \leftarrow \psi_i$.
  end if
  if $\phi \leqslant n_r \psi$ **then**
    Break the loop.
  else
    $n_r \leftarrow n_r + 1, \phi \leftarrow \phi + \psi, i \leftarrow i + 1$.
  end if
end loop
if $n_r > 0$ **then**
  $\boldsymbol{F}_{\mathcal{P}}^{\rm opt}$: Allocate $\frac{R}{n_r}$ on the path p_i $(1 \leqslant i \leqslant n_z + n_r)$. Other paths will not be used.
  $c^{\rm opt}$: $c^{\rm opt} \leftarrow \frac{R}{n_r}\psi$.
  Construct a $(n_z + n_r, n_r)$ MDS code for the flow.
else
  The problem is infeasible.
end if

2. The iterations are shown in Table 1. In the eighth iteration, $\phi \leqslant n_r \psi$, resulting in $n_r = 5$. Therefore, the optimal solution is to allocate flow on the cheapest $n_z + n_r = 7$ paths (the flow on each path is $R/n_r = R/5$) and to employ a (7, 5) MDS code.

5 Complexity

The complexity of ordering is $O(|\mathcal{P}|\log|\mathcal{P}|)$. The loop contains at most $|\mathcal{P}|$ iterations, and every iteration is $O(1)$. Therefore, the overall complexity of this algorithm is $O(|\mathcal{P}|\log|\mathcal{P}|)$.

6 Optimality

In this section, we will prove that this algorithm is optimal in general. In fact, the key step in this algorithm is to calculate the value of n_r. In order to justify the method to determine n_r, we need the following lemmas.

Lemma 1 Use the notations in previous context. Then

(1) For $1 < i \leqslant |\mathcal{P}| - n_z$, $\rho_{n_z+i-1} \leqslant \rho_{n_z+i}$ if and only if $i\psi_{n_z+i} \leqslant \phi_{n_z+i}$; $\rho_{n_z+i-1} < \rho_{n_z+i}$ if and only if $i\psi_{n_z+i} < \phi_{n_z+i}$. For $1 \leqslant i < |\mathcal{P}| - n_z$, $\rho_{n_z+i} \geqslant \rho_{n_z+i+1}$ if and only if $\phi_{n_z+i} \leqslant i\psi_{n_z+i+1}$; $\rho_{n_z+i} > \rho_{n_z+i+1}$ if and only if $\phi_{n_z+i} < i\psi_{n_z+i+1}$.

(2) $n_r\psi_{n_z+n_r} \leqslant \phi_{n_z+n_r} \leqslant n_r\psi_{n_z+n_r+1}$, $\psi_{n_z+n_r} \leqslant 1/\rho_{n_z+n_r} \leqslant \psi_{n_z+n_r+1}$.

(3) For $1 \leqslant i \leqslant n_r$, $i\psi_{n_z+i} \leqslant \phi_{n_z+i}$. For $n_r \leqslant i < |\mathcal{P}| - n_z$, $\phi_{n_z+i} \leqslant i\psi_{n_z+i+1}$.

(4) For any n such that $1 \leqslant n \leqslant |\mathcal{P}| - n_z$, $\rho_{n_z+n} = \max\limits_{1\leqslant i\leqslant|\mathcal{P}|-n_z} \rho_{n_z+i}$ if and only if $\rho_{n_z+1} \leqslant \rho_{n_z+2} \leqslant \cdots \leqslant \rho_{n_z+n-1} \leqslant \rho_{n_z+n} \geqslant \rho_{n_z+n+1} \geqslant \cdots \geqslant \rho_{|\mathcal{P}|}$.

Proof (1) On the one hand, for i such that $1 < i \leqslant |\mathcal{P}| - n_z$, $\rho_{n_z+i-1} \leqslant \rho_{n_z+i}$ is equivalent to

$$(i-1)/\phi_{n_z+i-1} \leqslant i/\phi_{n_z+i},$$

and is further equivalent to

$$(i-1)\phi_{n_z+i} \leqslant i\phi_{n_z+i-1}.$$

Table 1 Iterations of Example 1.

i	$\psi \leftarrow \psi_i$	Compare ϕ with $n_r\psi$	$n_r \leftarrow n_r + 1$	$\phi \leftarrow \phi + \psi$
0 (init)			−2	0
1	1	0 > −2	−1	1
2	2	1 > −2	0	3
3	2	3 > 0	1	5
4	2	5 > 2	2	7
5	3	7 > 6	3	10
6	3	10 > 9	4	13
7	3	13 > 12	5	16
8	8	16 ⩽ 40 (break)		

Considering that $\phi_{n_z+i-1} = \phi_{n_z+i} - l_{n_z+i}$, it is also equivalent to

$$(i-1)\phi_{n_z+i} \leqslant i(\phi_{n_z+i} - \psi_{n_z+i}),$$

and further equivalent to

$$i\psi_{n_z+i} \leqslant \phi_{n_z+i}.$$

On the other hand, for i such that $1 \leqslant i < |\mathcal{P}| - n_z$, $\rho_{n_z+i} \geqslant \rho_{n_z+i+1}$ is equivalent to

$$i/\phi_{n_z+i} \geqslant (i+1)/\phi_{n_z+i+1},$$

and is further equivalent to

$$i\phi_{n_z+i+1} \geqslant (i+1)\phi_{n_z+i}.$$

Considering that $\phi_{n_z+n_r+1} = \phi_{n_z+n_r} + \psi_{n_z+n_r+1}$, it is equivalent to

$$i(\phi_{n_z+i} + l_{n_z+i+1}) \geqslant (i+1)\phi_{n_z+i},$$

and is further equivalent to

$$\phi_{n_z+i} \leqslant i\psi_{n_z+i+1}.$$

Strict version of inequalities can be proved similarly.

(2) Since $\rho_{n_a+n_r} = \max\limits_{1\leqslant i\leqslant|\mathcal{P}|-n_z} \rho_{n_z+i}$, we have

$$\rho_{n_z+n_r-1} \leqslant \rho_{n_z+n_r} \geqslant \rho_{n_z+n_r+1},$$

which is equivalent to

$$n_r\psi_{n_z+n_r} \leqslant \phi_{n_z+n_r} \geqslant n_r\psi_{n_z+n_r+1}.$$

Consider $1/\rho_{n_z+n_r} = n_r/\phi_{n_z+n_r}$, $\psi_{n_z+n_r} \leqslant 1/\rho_{n_z+n_r} \leqslant \psi_{n_z+n_r+1}$.

(3) Mathematical induction is used here. For $1 < i \leqslant |\mathcal{P}| - n_z$ such that $i\psi_{n_z+i} \leqslant \phi_{n_z+i}$,

$$(i-1)\psi_{n_z+(i-1)} \leqslant (i-1)\psi_{n_z+i} = i\psi_{n_z+i} - \psi_{n_z+i} \leqslant \phi_{n_z+i} - \psi_{n_z+i} = \phi_{n_z+(i-1)}.$$

As in (2) $n_r\psi_{n_z+n_r} \leqslant \phi_{n_z+n_r}$, therefore, $i\psi_{n_z+i} \leqslant \phi_{n_z+i}$ is true for $1 \leqslant i \leqslant n_r$. Similarly, for $1 < i \leqslant |\mathcal{P}| - n_z$ such that $\phi_{n_z+i} \leqslant i l_{n_z+i+1}$,

$$\phi_{n_z+i+1} = \phi_{n_z+i} + \psi_{n_z+i+1} \leqslant i\psi_{n_z+i+1} + \psi_{n_z+i+1} = (i+1)\psi_{n_z+i+1} \leqslant (i+1)\psi_{n_z+i+2}.$$

As $\phi_{n_z+n_r} \leqslant n_r\psi_{n_z+n_r+1}$ in (2), $\phi_{n_z+i} \leqslant i\psi_{n_z+i+1}$ is true for $n_r \leqslant i < |\mathcal{P}| - n_z$.

(4) It is obvious that the condition is necessary so we have to prove only the sufficiency. For any $1 \leqslant n \leqslant |\mathcal{P}| - n_z$ such that $\rho_{n_z+n} = \max\limits_{1\leqslant i\leqslant|\mathcal{P}|-n_z} \rho_{n_z+i}$, due to Lemma 1(3), $i\psi_{n_z+i} \leqslant \phi_{n_z+i}$ when $i \leqslant n_r$, and $\phi_{n_z+i} \leqslant i\psi_{n_z+i+1}$ when $i \geqslant n_r$. According to Lemma 1(1), $\rho_{n_z+i} \leqslant \rho_{n_z+i+1}$ when $i \leqslant n_r$, and $\rho_{n_z+i} \geqslant \rho_{n_z+i+1}$ when $i \geqslant n_r$. That completes the proof. ■

Lemma 2 In the context of Theorem 1, let

$$\mathcal{I} = \{1 \leqslant i \leqslant |\mathcal{P}| : \phi_{i-1} \leqslant (i-1-n_z)\psi_i\}.$$

Then

$$n_r + n_z = \begin{cases} \min\limits_{i \in \mathcal{I}} i, & \mathcal{I} \neq \varnothing; \\ |\mathcal{P}|, & \mathcal{I} = \varnothing. \end{cases}$$

Proof Due to Eq. (1),

$$n_r = \mathop{\arg\max}_{-n_z < i \leqslant |\mathcal{P}| - n_z} \rho_{i+n_z}.$$

According to Lemma 1(4), n_r can be found by locating the minimal i ($-n_z \leqslant i < |\mathcal{P}| - n_z$) such that $\rho_{n_z+i} \geqslant \rho_{n_z+i+1}$. Lemma 1(1) tells that $\rho_{n_z+i} \geqslant \rho_{n_z+i+1}$ is equivalent to $\phi_{n_z+i} \leqslant i\,\psi_{n_z+i+1}$. Hence n_r equals the minimal i ($-n_z \leqslant i < |\mathcal{P}| - n_z$) such that $\phi_{n_z+i} \leqslant i\,\psi_{n_z+i+1}$, and $n_r + n_z$ equals the minimal i ($0 \leqslant i < |\mathcal{P}|$) such that $\phi_i \leqslant (i - n_z)\,\psi_{i+1}$. Actually, $(n_r + n_z)$ is also equivalent to the minimal i ($1 \leqslant i \leqslant |\mathcal{P}|$) such that $\phi_{i-1} \leqslant (i - n_z - 1)\,\psi_i$, which is exactly the minimum element in the set $\mathcal{I}$. When $\mathcal{I} = \varnothing$, there do not exist i such that $\rho_{n_z+i} \geqslant \rho_{n_z+i+1}$. In this case, $n_r + n_z = |\mathcal{P}|$. ■

What Algorithm 1 does follows Lemma 2 exactly. The proof of the optimality ends.

7 Conclusions and Further Work

This paper proposes an algorithm that uses MDS network error correction codes on disjoint paths to provide network diversity. We show that the complexity of this algorithm is low, and the resulting flow is the minimum cost flow. Actually, if combining the algorithm in this paper with the algorithm with multiple minimum weight disjoint paths algorithm in Ref. [9], we can further find the optimal flow allocation algorithm in general directed graph when recoding at intermediate nodes is not allowed. The overall time complexity of the combined algorithm is $O(|\mathcal{V}|^2\,|\mathcal{E}|)$, where $|\mathcal{V}|$ and $|\mathcal{E}|$ are the number of nodes and edges in the directed graph respectively. This result will be formally presented in our upcoming paper.

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