Preliminaries

Basic notions of time Petri nets and time labeled Petri nets

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Preliminaries

Let $\mathbb{N}$ and $\mathbb{Q}_0^+$ be the sets of non-negative integers and non-negative rational numbers, respectively. A Petri net is defined as a four-tuple $\mathcal{N} = (P, T, Pre, Post)$, where $P = \{p_1, p_2, ..., p_m\}$ is a finite set of places, $T = \{t_1, t_2, ..., t_n\}$ is a finite set of transitions with $P \cup T \neq \emptyset$ and $P \cap T = \emptyset$, and $Pre : P \times T \to \mathbb{N}$ and $Post : P \times T \to \mathbb{N}$ are the pre- and post-incidence functions, respectively, specifying the arcs directed from places to transitions, and transitions to places, respectively, in a net. Both functions can be alternatively represented as matrices in $\mathbb{N}^{m\times n}$, which facilitates algebraic operations in Petri net analysis. The incidence matrix of a Petri net is defined by $\mathbf{C} = Post - Pre$.

A marking is a mapping $M : P \to \mathbb{N}$ that assigns to each place a non-negative number of tokens. Given a marking M, we use $M(p)$ to denote the number of tokens in place p at the marking M. A Petri net system is denoted by $(\mathcal{N}, M_0)$, where M_0 denotes the initial marking of $\mathcal{N}$. From the initial marking M_0, net system $(\mathcal{N}, M_0)$ may reach other markings due to its dynamics. A transition t is logically enabled at a marking M if $M \geq Pre(\cdot, t)$ and may fire yielding a marking $M' = M + \mathbf{C}(\cdot, t)$, denoted as $M[t\rangle M'$. The set of transitions logically enabled at M is denoted as $En\,(M)$, i.e., $En\,(M) = \{t \in T \mid M \geq Pre\,(\cdot, t)\}$. If a transition t fires at marking M and yields a new marking M', the set of the newly enabled transitions at M' is denoted by $New(M, t)$, where $New(M, t) = En(M') \setminus \{t' \mid M[t'\rangle, M'[t'\rangle, t \neq t'\}$.

A transition sequence $\sigma = t_1 t_2 \cdots t_k \in T^*$ is enabled at marking M if there exist markings M_1, M_2, ..., M_k such that $M_1[t_1\rangle M_2[t_2\rangle \cdots M_{k-1}[t_k\rangle M_k$, denoted by $M[\sigma\rangle$ or simply $M[\sigma\rangle$ if M_k is of no interest. In such case, M_k is said to be reachable from marking M. The set of all markings reachable from the initial marking M_0 is denoted by $R(\mathcal{N}, M_0)$, where $R(\mathcal{N}, M_0) = \{M \in \mathbb{N}^m \mid \exists \sigma \in T^* : M_0[\sigma\rangle M\}$. A Petri net is said to be bounded if there exists an integer $k \in \mathbb{N}$ such that for all $p \in P$ and all $M \in R(\mathcal{N}, M_0)$, $M(p) \leq k$ holds.

A time Petri net (TPN) is a three-tuple $\mathcal{Z} = (\mathcal{N}, M_0, I)$, where $(\mathcal{N}, M_0)$ is a Petri net system and $I : T \to \mathbb{Q}_0^+ \times (\mathbb{Q}_0^+ \cup \{\infty\})$ is a time function such that each transition $t \in T$ is associated with a static closed time interval. Given a transition $t_k \in T$ with $I(t_k) = [a_k, b_k]$, where a_k and b_k are the lower and upper bounds of the time interval associated with t_k, respectively, it holds that $0 \leq a_k \leq b_k$ and $a_k \neq \infty$. Specifically, when a transition t_k becomes enabled, it cannot fire before a_k time units have elapsed, and it must fire before b_k time units, unless another transition t' fires before t_k such that t_k is disabled.

A state of a TPN is a pair $C_k = (M_k, \Phi_k)$, where M_k is a reachable marking from M_0, and $\Phi_k = \{a_i^k \leq \varphi_i \leq b_i^k \mid i = 1, 2, \ldots, |En(M_k)|\}$ is a set of inequalities with $|En(M_k)| \in \mathbb{N}$ representing the number of transitions logically enabled at M_k. An inequality $a_i^k \leq \varphi_i \leq b_i^k$ in the set Φ_k indicates that transition t_i is logically enabled at the marking M_k, and it can fire only after the a_i^k time units have elapsed and before the b_i^k time units have elapsed, unless another enabled transition $t' \in En(M_k)$ has fired such that t_i is disabled.

An evolution of a TPN is defined as a transition-time sequence (TTS) from the initial state and ending with a reachable state, i.e., $\sigma_t = (t_1, \tau_1)(t_2, \tau_2) \cdots (t_k, \tau_k) \in (T \times \mathbb{Q}_0^+)^*$, where τ_i represents the feasible firing time instant of transition t_i $(i = 1, 2, \ldots, k)$ with $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_k$. We write $(M_k, \Phi_k)[(t_i, \tau_i)\rangle(M_k', \Phi_k')$ to denote that a state (M_k', Φ_k') is generated from (M_k, Φ_k) by firing t_i at time τ_i. Analogously, we write $(M_k, \Phi_k)[\sigma_t\rangle(M'_k, \Phi'_k)$ to denote that (M'_k, Φ'_k) is reachable by firing σ_t at (M_k, Φ_k), or simply $(M_k, \Phi_k)[\sigma_t\rangle$ if (M'_k, Φ'_k) is of no interest. We denote by $log(\sigma_t)$ the logical transition sequence associated with σ_t, i.e., $log(\sigma_t) = t_1 t_2 \cdots t_k \in T^*$.

The set of all reachable states of a TPN $\mathcal{Z}$ from (M_0, Φ_0) is $R_t(\mathcal{Z}) = \{(M_k, \Phi_k) \mid \exists \sigma_t \in (T \times \mathbb{Q}_0^+)^* : (M_0, \Phi_0)[\sigma_t\rangle(M_k, \Phi_k)\}$. Let $C_k = (M_k, \Phi_k)$ be a state in $\mathcal{Z}$. A transition t_i is potentially enabled at M_n if there exists a TTS $\sigma_t = (t_1, \tau_1)(t_2, \tau_2) \cdots (t_k, \tau_k) \in (T \times \mathbb{Q}_0^+)^*$ such that $(M_n, \Phi_n)[\sigma_t\rangle$ holds and transition t_i along with its firing time τ_i, i.e., (t_i, τ_i), is included in σ_t. Let $\sigma_t = (t_1, \tau_1)(t_2, \tau_2) \cdots (t_k, \tau_k)$ be a TTS. A state-event sequence (M_0, Φ_0) $[(t_1, \tau_1)\rangle(M_1, \Phi_1)[(t_2, \tau_2)\rangle(M_2, \Phi_2) \cdots [(t_k, \tau_k)\rangle(M_k, \Phi_k)$ is called a path from (M_0, Φ_0) to (M_k, Φ_k) by firing σ_t. We use π to denote a general path in the case of no confusion. A TPN $\mathcal{Z} = (\mathcal{N}, M_0, I)$ is said to be bounded if $(\mathcal{N}, M_0)$ is bounded, i.e., there exists an integer $k \in \mathbb{N}$ such that for all $p \in P$ and all $M \in R(\mathcal{N}, M_0)$, $M(p) \leq k$ holds. This paper touches upon bounded TPNs only. Moreover, we assume that the considered TPN systems follow the single server semantics and the enabling memory policy.

A time labeled Petri net (TLPN) is a five-tuple $G = (\mathcal{N}, M_0, I, E, \ell)$, where $(\mathcal{N}, M_0, I)$ denotes a TPN system, E is a set of labels, and $\ell : T \to E \cup \{\varepsilon\}$ is a labeling function that assigns to a transition $t \in T$ either a label from E or the empty word ε. Accordingly, the set of transitions is divided into two disjoint sets $T = T_o \cup T_{uo}$ with $T_o \cap T_{uo} = \emptyset$, where $T_o = \{t \in T \mid \ell(t) \in E\}$ and $T_{uo} = \{t \in T \mid \ell(t) = \varepsilon\}$ represent the sets of observable and unobservable transitions, respectively. The transitions in T_o are said to be observable since their related labels are exposed when they fire, and those in T_{uo} are said to be unobservable. The labeling function can be extended to transition sequences, i.e., $\ell : T^* \to E^*$ by defining $\ell(\varepsilon) = \varepsilon$ and $\ell(\sigma t) = \ell(\sigma)\ell(t)$ with $\sigma \in T^*$ and $t \in T$. We use $\omega \in E^*$ to denote the string that is observed when a transition sequence $\sigma \in T^*$ fires, i.e., $\omega = \ell(\sigma)$. We write $M_i[\omega\rangle M_j$ to denote that there exists a sequence $\sigma \in T^*$ such that $\ell(\sigma) = \omega$ and the firing of σ at M_i yields M_j.

An observation in a TLPN G is defined as a time-labeled observation (TLO) $\rho = (e_1, \tau_1)(e_2, \tau_2) \cdots (e_k, \tau_k) \in (E \times \mathbb{Q}_0^+)^*$, where each $e_i \in E$ is an observable event and τ_i $(i = 1, 2, ..., k)$ denotes a time instant. That is to say, an observation is a sequence of pairs, which specifies the sequence of labels that have been observed and the corresponding time instants. We write $(M, \Phi)[(e, \tau)\rangle$ and $(M, \Phi)[\rho\rangle$ to denote that the time-label pair (e, τ) and the TLO ρ are enabled at the state (M, Φ), respectively. The feasible firing time of the final observable event of TLO ρ is denoted by $t_f(\rho)$. To avoid introducing too many different notations, we also use $t_f(\sigma_t)$ to denote the firing time of the final transition of a TTS σ_t.

Given a TTS $\sigma_t \in (T \times \mathbb{Q}_0^+)^*$, the natural projection of σ_t is defined by a projection function $P : (T \times \mathbb{Q}_0^+)^* \to (E \times \mathbb{Q}_0^+)^*$ such that

$$\begin{cases} P((t, \tau)) = \varepsilon, & \text{if } \ell(t) = \varepsilon; \\ P((t, \tau)) = (\ell(t), \tau), & \text{if } \ell(t) \in E; \\ P(\sigma_t(t, \tau)) = P(\sigma_t), & \text{if } \ell(t) = \varepsilon; \\ P(\sigma_t(t, \tau)) = P(\sigma_t)P(t, \tau), & \text{if } \ell(t) \in E. \end{cases} \tag{1}$$

We use $log(\rho) = e_1 e_2 \cdots e_k$ to denote the logical label sequence of ρ, i.e., the time instants associated with the observed label sequence are removed. Given a TLO $\rho \in (E \times \mathbb{Q}_0^+)^*$, $u \in (E \times \mathbb{Q}_0^+)^*$ is said to be a prefix of ρ if there exists $v \in (E \times \mathbb{Q}_0^+)^*$ such that $uv = \rho$, denoted by $u \preceq \rho$. The set of all prefixes of ρ naturally comes to be $[\rho] = \{u \in (E \times \mathbb{Q}_0^+)^* \mid u \preceq \rho\}$. The set of all reachable states of a TLPN G reachable from $C_0 = (M_0, \Phi_0)$ is $R_t(G) = \{C_k \mid \exists \sigma_t \in$

$(T \times \mathbb{Q}_0^+)^* : C_0[\sigma_t\rangle\ C_k\}$. The time reachability set $R_m(G)$ is the set of markings reachable from the initial marking. A TLPN G is said to be bounded if the set $R_m(G)$ is finite.