Petri Nets

To facilitate the reading, we briefly review the basics of Petri nets. For more details, a reader is referred to [1].

A Petri net is a four-tuple $N = (P, T, Pre, Post)$, where P is a finite set of h places, where $h \in \mathbb{N}$, T is a finite set of transitions with $P \cup T \neq \emptyset$ and $P \cap T = \emptyset$, where $\mathbb{N}$ is the set of non-negative integers. The *pre*-incidence functions of N is defined by $Pre : P \times T \to \mathbb{N}$, and the *post*-incidence functions is defined by $Post : P \times T \to \mathbb{N}$. Normally, we graphically represent a place with a circle and a transition by a box. Specifically, for a place p, a transition t, and $k \in \mathbb{N}$, $Pre(p, t) = k > 0$ means that there is an arc from p to t with weight k; $Post(p, t) = k > 0$ means that there is an arc from t to p with weight k. In the case of $k = 0$, there is no arc from p to t or t to p. $C = Post - Pre$ is defined as incidence matrix.

Given a node x in a Petri net, the pre-set of x is defined by ${}^\bullet x = \{y \in P \cup T \mid Pre(y, x) > 0\}$, and the post-set of x is defined by $x^\bullet = \{y \in P \cup T \mid Post(x, y) > 0\}$. Given a Petri net, let $P_o = \{p_{o1}, p_{o2}, \ldots, p_{oj}\}$ $(j \leq h)$ be a set of observable places. And $P_{uo} = P \setminus P_o$ is a set of unobservable places. A Kleene closure of the transitions T is defined as T^*, including all finite sequences composed of the transitions in T, and the empty transition sequence ε.

A marking is a mapping $M\colon P \to \mathbb{N}^h$, represented by a vector due to the finiteness of the place set for operation convenience. An entry $M(p)$ of a marking M indicates the number of tokens in place p at the marking M. A net system is represented as $\langle N, M_0 \rangle$, where M_0 is an initial marking.

A transition t is enabled at M if for all $p \in {}^\bullet t, M(p) \geq Pre(p, t)$, or denoted as $M \geq Pre(\cdot, t)$. The firing of an enabled transition t at marking M yields a marking M', denoted by $M[t\rangle M'$, and $M' = M + C(\cdot, t)$. A transition sequence $\sigma = t_1 t_2 \ldots t_n \in T^*$ is enabled at M if there exist markings M_1, M_2, $\ldots$, M_n such that $M_1[t_1\rangle M_2[t_2\rangle \ldots M_{n-1}[t_n\rangle M_n$, denoted by $M[\sigma\rangle M_n$ or simply $M[\sigma\rangle$ if M_n is of no interest. In this case, M_n is said to be reachable from M. The set of markings from the initial marking M_0 defines the reachability of net system $\langle N, M_0 \rangle$, denoted by $R(N, M_0) = \{M \in \mathbb{N}^h \mid \exists \sigma \in T^* : M_0[\sigma\rangle M\}$, called reachability set. Specially, if a transition sequence σ is an empty sequence, i.e., $\sigma = \varepsilon$ then $M[\sigma\rangle M$. The language of a net system $\langle N, M_0 \rangle$ is defined as

$$L(N, M_0) = \{\sigma \in T^* \mid M_0[\sigma\rangle\},$$

which is a set of transition sequences that are enabled from the initial marking. Write, by a slight abuse of notation, $t \in \sigma$ to represent that transition sequence σ contains transition t. The set of symbols forming σ, denoted by $||\sigma|| = \{t \in T \mid t \in \sigma\}$, is called the support of σ.

Given two markings M and M', $M = M'$ means that, for arbitrary place p, $M(p) = M'(p)$. Furthermore, write $M < M'$ if a marking M is less than a marking M'. It means that, for an arbitrary place p, $M(p) < M'(p)$. Similarly, if a marking M is less than or equal to a marking M', then it is denoted as $M \leq M'$. It means that for an arbitrary place p, $M(p) \leq M'(p)$. By analogy, we can get the meanings of $M > M'$ and $M \geq M'$.

Given a transition sequence $\sigma = t_1 t_2 ... t_n \in T^*$ and markings $M_1, M_2, \ldots, M_n$, a circuit in a Petri net is defined as $M_1[t_1\rangle M_2[t_2\rangle \ldots M_{n-1}[t_n\rangle M_n$, where $M_1 = M_n$. A self-loop in a Petri net is the simplest case of circuits, i.e., given a transition t, ${}^\bullet t = t^\bullet = {}^\bullet t \cup t^\bullet$ and $|{}^\bullet t| = |t^\bullet| = 1$. A Petri net is said to be self-loop free if it contains no self-loop. If there is no circuit in a net system, the net system is said to be *acyclic*.

A function $\pi : T^* \to \mathbb{N}^n$ that associates a sequence $\sigma \in T^*$ with a vector $y_\sigma = \pi(\sigma) \in \mathbb{N}^n$defines the Parikh vector of the transition sequence σ, where $n = |T|$ is the number of transitions in a net. Specifically, $y_\sigma(t) = k$ means that transition t appears k times in σ.

A reachability graph of a net system $\langle N, M_0 \rangle$, denoted by $RG(N, M_0)$, is a digraph starting from the initial marking M_0, whose nodes are markings in $R(N, M_0)$ and an edge form M to M' labeled with t if $M[t\rangle M'$ holds.

A net system $\langle N, M_0 \rangle$ is bounded if there is an integer $K > 0$ such that for all reachable markings $M \in R(N, M_0)$ and for all places $p \in P$, $M(p) \leq K$ holds. Otherwise is unbounded. For an unbounded net system, the number of tokens in an unbounded place can be an arbitrary integer, denoted by ω, satisfying, given any $n \in \mathbb{N}$, $\omega \pm n = \omega, \omega \times n = \omega$, $\omega \times 0 = 0$, and $n < \omega$. Its state space is approximated by a coverability set $CS(N, M_0) \subset (\mathbb{N} \cup \{\omega\})^h$. The previous works report that the coverability set includes all the markings of the reachability set [8, 9]. Therefore, there are two conditions hold [3]:

1. For the initial marking M_0, $M_0 \in CS(N, M_0)$;
2. $\forall M \in CS(N, M_0), \forall \sigma \in T^*$, and $\pi(\sigma) = y_\sigma$, it holds $M' \in CS(N, M_0)$, where $M' = M + C \cdot y_\sigma$.

A coverability graph $CG(N, M_0)$ can be constructed analogously to the reachability graph of a bounded Petri net [4, 5, 2, 6].

References

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