Preliminaries

Basic notions and notations of finite state automata and Petri nets

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1 Automata

A non-deterministic finite-state automaton (NFA) is a quintuple $\mathcal{A} = (X, E, \Delta, x_0, X_m)$, where X is the finite set of states, $E = \{a, b, \ldots, c\}$ is the alphabet (usually finite), $\Delta \subseteq X \times E_\varepsilon \times X$ is the transition relation with $E_\varepsilon = E \cup \{\varepsilon\}$, where ε is the empty string describing unobservable events, $x_0 \in X$ is the initial state, and X_m is the set of marker (or final) states. In this paper, we simply write an NFA as $\mathcal{A} = (X, E, \Delta, x_0)$ since the marker states are of no interest. Note that an NFA may have multiple initial states whose set is denoted by X_0. However, by introducing an additional initial state with an ε-transition to each state in X_0 from the added initial state, an NFA with multiple initial states can be equivalently converted into an NFA with only one initial state. In this sense, write $\mathcal{A} = (X, E, \Delta, x_0)$ without loss of generality.

The transition relation specifies the dynamics of an NFA $\mathcal{A} = (X, E, \Delta, x_0)$: if $(x, e, x') \in \Delta$, then from state x the occurrence of event $e \in E_\varepsilon$ yields the state x'. In particular, given any state $x \in X$, we have $(x, \varepsilon, x) \in \Delta$. The transition relation can be extended to $\Delta^* \subseteq X \times E^* \times X$, where $(x_0, \omega, x_k) \in \Delta^*$ if there exists a sequence of events and states $x_0 e_{j1} x_1 \ldots x_{k-1} e_{jk} x_k$ such that $\sigma = e_{j1} \ldots e_{jk}$ generates the word $\omega \in E^*$, $x_{ij} \in X$ for $i = 1, \ldots, k$, and $e_{ji} \in E_\varepsilon$, $(x_{ji-1}, e_{ji}, x_{ji}) \in \Delta$ for $i = 1, 2, \ldots, k$.

An event e is said to be defined at state x_i if there exists a state $x_j \in X$ such that $(x_i, e, x_j) \in \Delta$. An NFA is simply denoted as $\mathcal{A} = (X, E, \Delta)$ if the initial state could be any state in X or the initial state is of no interest. The generated language of an automaton $\mathcal{A} = (X, E, \Delta)$ from a state $x \in X$ is defined as

$$\mathcal{L}(\mathcal{A}, x) = \{\omega \in E^* \mid \exists x' \in X : (x, \omega, x') \in \Delta^*\}. \tag{1}$$

The generated language from a state can be extended to a set of states. Given a set of states $Y \subseteq X$, we define the language generated from the states in set Y as:

$$\mathcal{L}(\mathcal{A}, Y) = \bigcup_{x \in Y} \mathcal{L}(\mathcal{A}, x). \tag{2}$$

Given an NFA $\mathcal{A} = (X, E, \Delta, x_0)$, if the transition relation reduces to a partial function, i.e., $\Delta : X \times E^* \to X$, then $\mathcal{A}$ is said to be a deterministic finite automaton (DFA). For any NFA, there exists a DFA such that their languages are equivalent. From the lens of system modeling, their power is the same.

2 Petri nets

In this section, we review basic definitions of Petri nets. For more details, we refer readers to [1] and [2]. Let $\mathbb{N}$ be the set of non-negative integers.

Definition 1 (Petri net). A Petri net is a quadruple $N = (P, T, \mathrm{Pre}, \mathrm{Post})$, where P is a finite and non-empty set of places; T is a finite and non-empty set of transitions; $\mathrm{Pre} : P \times T \to \mathbb{N}$ and $\mathrm{Post} : P \times T \to \mathbb{N}$ are respectively the pre- and post- incidence functions that specify the arcs directed from places to transitions, and vice versa. By definition, the functions Pre and Post can be tabulated as a matrix indexed by P and T. We use $\mathcal{C} = \mathrm{Post} - \mathrm{Pre}$ to represent the incidence matrix of a Petri net. The cardinality of the set of places is denoted as $m = |P|$ and that of the set of transitions is $n = |T|$ by conventional mathematical notations. $\square$

A Petri net can be diagrammatically represented by a directed graph with two types of nodes: Places and transitions that are graphically portrayed by circles and boxes, respectively. The input and output sets of a node $x \in P \cup T$ are defined by ${}^\bullet x = \{y \mid (y, x) \in F\}$ and $x^\bullet = \{y \mid (x, y) \in F\}$, respectively, with $F \subseteq (P \times T) \cup (T \times P)$ being the set of all directed arcs from places to transitions and from transitions to places, specified by the pre- and post incidence functions, respectively. This notation can be extended to a set of nodes as follows: given $X \subseteq P \cup T$, one has ${}^\bullet X = \bigcup_{x \in X} {}^\bullet x$ and $X^\bullet = \bigcup_{x \in X} x^\bullet$.

A marking of a net is a mapping $M : P \to \mathbb{N}$ that assigns to each place a non-negative integer number of tokens, signified by black dots in a Petri net diagram. We denote by $M(p)$ the marking of a place p at the marking M. For computational convenience, a marking can be represented by a vector of size m which is the cardinality of the place set. A Petri net system, denoted by $\langle N, M_0 \rangle$, is a net N with initial marking M_0.

Token flows in a net system among the places via transition firing, following the enabling and firing rules of transitions (to be presented) define the dynamics of the Petri net system. In fact, the rules are to determine the flow of tokens between places, thus specifying how the initial marking can evolve.

Definition 2 (Firing rule). Given a Petri net system $\langle N, M_0 \rangle$, a transition $t \in T$ is said to be enabled at a marking M if $M \geq Pre(\cdot, t)$. An enabled transition can fire, yielding a marking M' such that $M' = M + \mathcal{C}(\cdot, t)$, where $\mathcal{C}(\cdot, t)$ is the column vector associated with the transition t in the matrix $\mathcal{C}$. We write $M_0[\sigma\rangle M$ to denote that the sequence of transitions $\sigma = t_{j_1} \cdots t_{j_k}$ is enabled at M_0 and $M[\sigma\rangle M'$ to denote that the firing of σ yields M', i.e., the transitions t_{j_1}, t_{j_2}, …, t_{j_k} fire sequentially such that the marking M' is finally reached. Given a sequence $\sigma \in T^*$, the function $\pi : T^* \to \mathbb{N}^n$ associates with a transition sequence σ in T^* the Parikh vector $y_\sigma = \pi(\sigma) \in \mathbb{N}^n$, i.e., $y(t) = k$ if transition t appears k times in σ.

Definition 3 (Reachability set). A marking M is reachable in $\langle N, M_0 \rangle$ if there exists a sequence $\sigma \in T^*$ such that $M_0[\sigma\rangle M$. The set of all markings reachable from M_0 defines the reachability set of $\langle N, M_0 \rangle$, denoted by $R(N, M_0)$, i.e., $R(N, M_0) = \{M \in \mathbb{N}^{|P|} \mid \exists \sigma \in T^* : M_o[\sigma\rangle M\}$.

A Petri net system is *bounded* if there exists a non-negative integer $k \in \mathbb{N}$ such that for any place $p \in P$ and for any reachable marking $M \in R(N, M_0)$, $M(p) \leq k$ holds. A marking M is said to be a *deadlock* or *dead marking* if no transition is enabled at M. A Petri net system is said to be *deadlock-free* if at least one transition is enabled at every reachable marking. A transition t is said to be *live* at M_0 if for any $M \in R(N, M_0)$, there exists a feasible sequence of transitions at M such that $M_0[\sigma\rangle M'$ and t is enabled at M'. A Petri net is said to be *live* at marking M_0 if all transitions are live at M_0.

Definition 4 *(Induced subnet).* Given a net $N = (P, T, \mathrm{Pre}, \mathrm{Post})$ and a subset $T' \subseteq T$ of its transitions, the T'–induced subnet of N is a net $N' = (P, T', \mathrm{Pre}', \mathrm{Post}')$, where Pre' (resp., Post') is the restriction of Pre (resp., Post) to T'.

The net N' can be thought of as a net obtained from N by removing all transitions in $T \setminus T'$, which is also written as $N' \prec_{T'} N$. An essential notion in Petri net structural analysis is siphon [3]. A nonempty set $\mathcal{S} \subseteq P$ is a siphon if ${}^\bullet\mathcal{S} \subseteq \mathcal{S}^\bullet$. A siphon is said to be minimal if there is no siphon contained in it as a proper subset. Siphons are a major structural technique for deadlock control in a Petri net system.

3 Labeled Petri nets

A labeled Petri net (LPN) is a quadruple $G = (N, M_0, E, \ell)$ [4], where

- E is the *alphabet*, i.e., the set of labels;
- $\langle N, M_0 \rangle$ is a Petri net system;
- $\ell : T \to E \cup \{\epsilon\}$ is the *labeling function* that assigns to a transition $t \in T$ either a symbol from E or the empty string ε.

As usual, T_{uo} is used to denote the set of *silent (unobservable) transitions*, i.e., the set of transitions labeled with the empty string ε. Hence, founded on this type of events, the set of transitions can be partitioned into two disjoint sets $T = T_o \cup T_{uo}$, where $T_o = \{t \in T \mid \ell(t) \in E\}$ is the set of observable transitions and $T_{uo} = \{t \in T \mid \ell(t) = \varepsilon\}$ is called the set of unobservable transitions. By extending the labeling function to firing sequences $\ell : T^* \to E^*$, we have that

$$\ell \text{ is recursively defined as } \begin{cases} \ell(\varepsilon) = \varepsilon \\ \ell(\sigma) = \varepsilon \text{ if } \sigma \in T_{uo}, \\ \ell(\sigma) = \sigma \text{ if } \sigma \in T_o, \\ \ell(\sigma t) = \ell(\sigma)\ell(t) \text{ with } \sigma \in T^* \text{ and } t \in T. \end{cases} \tag{3}$$

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References

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[2] Cassandras, C.G.; Lafortune, S. *Introduction to Discrete Event Systems*; 2nd ed., Springer: New York, USA, 2008.

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