STABLE AND TRANSFERABLE WIRELESS RESOURCE ALLOCATION POLICIES VIA MANIFOLD NEURAL NETWORKS

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ABSTRACT

We consider the problem of resource allocation in large scale wireless networks. When contextualizing wireless network structures as graphs, we can model the limits of very large wireless systems as manifolds. To solve the problem in the machine learning framework, we propose the use of Manifold Neural Networks (MNNs) as a policy parametrization. In this work, we prove the stability of MNN resource allocation policies under the absolute perturbations to the Laplace-Beltrami operator of the manifold, representing system noise and dynamics present in wireless systems. These results establish the use of MNNs in achieving stable and transferable allocation policies for large scale wireless networks. We verify our results in numerical simulations that show superior performance relative to baseline methods.

Index Terms— Resource allocation, manifolds, stability analysis, large scale wireless networks, deep learning

1. INTRODUCTION

With growing number of devices and high-load applications deployed in wireless systems, it becomes an increasingly difficult challenge to allocate resources that mitigate interference and meet finite resource limitations across very large networks. While the resource allocation problem can be easily formulated as an optimization problem, due to the non-convexity and large dimensionality, the exact solution is often non-tractable. Traditional heuristic methods have been used but require explicit model knowledge and large computation cost. Considering the difficulty in modeling large scale wireless systems, machine learning has become a valuable tool in tackling wireless resource allocation problems [1–5].

Modern machine learning techniques often involve training neural networks as a parametrization of the resource allocation policies. The specific parametric form imposed on the policy however plays an important role in both the overall performance and the generalization of the learning method. In particular, wireless systems require policies that are *stable* to perturbations of the channel or network states because (i) channel measurements are often noisy due to environmental factors and (ii) wireless network configurations change frequently in reality and it is impractical to retrain the neural networks for each configuration. Graph neural networks (GNNs) [6] have been recently considered in many wireless resource allocation problems due to their low dimensionality and invariance to network topological structure [4, 5, 7, 8]. While prior numerical results have been able to demonstrate transferability of GNN-based resource allocation policies [4, 5], existing theoretical stability bounds of GNNs fail to capture this property when the size of graph is very large [9, 10]. In this paper, we address resource allocation problems over large-scale wireless networks. We propose the modeling of resource allocation policies in large scale networks via Manifold Neural Networks (MNNs) [11], and further prove the stability of such policies to formally establish their stability with respect to environmental/measurement noise and demonstrate transferable performance across wireless networks with different scales.

Wireless network interference patterns can be modeled as discrete graphs with the edges representing the dynamic interference channel states. As the number of network devices increases in the limit, the discrete graph can be represented as a continuous manifold structure $\mathcal{M}$ [12, 13]. We then consider the large scale resource allocation policy as processing inputs over a manifold and subsequently propose the Manifold Neural Network (MNN)—in effect representing GNNs on large graphs—as a suitable parametrization model. By modelling the system noise and dynamic changes as a perturbation to the Laplace-Beltrami operator of the underlying manifold $\mathcal{M}$, we can analytically prove the stability of an MNN composed of frequency difference threshold (FDT) filters to such system perturbations. The learned MNN parametrization can moreover be transferred to both similar manifolds and to finitely sized graphs of increasing size [14]. These results demonstrate the ability of MNNs to provide stable and transferable resource allocation policies for large scale wireless networks.

The rest of this paper is organized as follows. We formulate resource allocation problems over large scale graphs and manifolds (Section 2). We define MNNs as a parametrization of such resource allocation policies (Section 2.1). In Section 3, we prove stability of MNN policies to absolute perturbations. In Section 4, we provide numerical analysis of GNNs in large scale wireless networks to validate the stability properties of MNNs and evaluate their transferability across networks.

2. OPTIMAL RESOURCE ALLOCATION

We consider a wireless network consisting of m pairs of transmitters and receivers with each pair given a label $i \in$

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$\{1, 2, \ldots, m\}$. The channel link state can be characterized by a matrix $\mathbf{S} \in \mathbb{R}^{m \times m}$ whose entries $s_{ij} := [\mathbf{S}]_{ij}$ reflect the fading state between pairs i and j. Note that the diagonal entries s_{ii} reflect the direct channel between the transmitter and receiver in pair i while off-diagonal entries s_{ij} reflect an interference caused by transmitter i at receiver j. We further consider the transmitter state $\mathbf{x} \in \mathbb{R}^m$ with each entry $[\mathbf{x}]_i$ representing some state of transmitter i.

The goal in resource allocation problems is to determine the optimal resource level p_i for each transmitter i under a given set of state observations $(\mathbf{S}, \mathbf{x})$. In particular, we seek a mapping, or resource allocation policy, $\mathbf{p}(\mathbf{S}, \mathbf{x}) := [p_1; p_2; \ldots; p_m]$. For an instantaneous state and resource allocation, each pair i experiences some level of performance $r_i(\mathbf{p}(\mathbf{S}, \mathbf{x}), \mathbf{S}, \mathbf{x})$ (e.g. capacity, packet error rate). In fast-fading environments, the channel and transmitter states vary fast and randomly over time, and thus we optimize performance over the long-term distribution of performance measures obtained by the policy $\mathbf{p}(\mathbf{S}, \mathbf{x})$. Due to the challenges inherent in optimizing an arbitrary function, the allocation policy $\mathbf{p}(\mathbf{S}, \mathbf{x})$ is typically restricted to a vector-valued function family $\boldsymbol{\phi}(\mathbf{H}, \mathbf{S}, \mathbf{x})$ parameterized by some parameter set $\mathbf{H}$. The optimal resource allocation policy is then formalized as determining the optimal policy parameter $\mathbf{H}^*$ as

$$\begin{aligned}
\mathbf{H}^* := \underset{\mathbf{H}}{\operatorname{argmax}} \quad & \mathbb{E}_{\mathbf{S},\mathbf{x}} \left[\sum_{i=1}^{m} r_i \left(\boldsymbol{\phi}(\mathbf{H}, \mathbf{S}, \mathbf{x}), \mathbf{S}, \mathbf{x} \right) \right], \\
\text{s.t.} \quad & \mathbb{E}_{\mathbf{S},\mathbf{x}} \left[\sum_{i=1}^{m} \phi_i(\mathbf{H}, \mathbf{S}, \mathbf{x}) \right] \leq P_{\max}, \\
& \phi_i(\mathbf{H}, \mathbf{S}, \mathbf{x}) \in \{0, p_0\}, \quad i = 1, .., m.
\end{aligned} \tag{1}$$

The optimal allocation policy $\boldsymbol{\phi}(\mathbf{H}^*, \mathbf{S}, \mathbf{x})$ is the one that maximizes the sum of performance measures under a constraint $P_{\max}$ on the total resource budget.

Note that, in problem (1), $\mathbf{S}$ can be seen as the adjacency matrix of a very large graph $\mathbf{G}$ where each transmitter-receiver pair is a node and the connecting edges reflect the interference channels between pairs. Similarly, the transmitter states $\mathbf{x}$ can be seen as a graph signal on the nodes of $\mathbf{G}$. In large scale wireless networks, where the number of transmitter/receiver pairs m is very large, the full network channel state can instead be modeled as a continuum of interfering links between devices. In this paper, we model a large wireless system structure as a manifold $\mathcal{M}$, i.e., the limit of graph $\mathbf{G}$ as $m \to \infty$.

Explicitly, we consider the channel state to be a smooth d-dimensional manifold $\mathcal{M}$, which is a topological space that is locally Euclidean. The transmitter states are modeled as a manifold signal $f : \mathcal{M} \to \mathbb{R}$, which can be seen as the limit of the graph signal $\mathbf{x}$ when the network grows very large.

Therefore, we can reformulate (1) in the limit sense as

$$\begin{aligned}
\mathbf{H}^* := \underset{\mathbf{H}}{\operatorname{argmax}} \ & \mathbb{E}_{\mathcal{M},f} \int_{u \in \mathcal{M}} r \left(\boldsymbol{\phi}(\mathbf{H}, \mathcal{M}, f)(u), \mathcal{M}, f(u) \right) \mathrm{d}u, \\
\text{s.t.} \quad & \mathbb{E}_{\mathcal{M},f} \int_{u \in \mathcal{M}} \boldsymbol{\phi}(\mathbf{H}, \mathcal{M}, f)(u) \mathrm{d}u \leq P_{\max}, \\
& \boldsymbol{\phi}(\mathbf{H}, \mathcal{M}, f)(u) \in [0, p_0], \quad u \in \mathcal{M}.
\end{aligned} \tag{2}$$

Observe in (2) that the resource allocation is processed over a manifold $\mathcal{M}$, while the utility and constraints are evaluated over the manifold rather than a discrete set of nodes. While continuous manifolds cannot be directly measured or observed in practice, the modeling of very large graphs as manifolds in (2) is used in this paper as an analytical tool necessary for establishing the desired stability properties of policies $\boldsymbol{\phi}(\mathbf{H}, \cdot, \cdot)$.

The challenge with problem (2) is that the manifold is infinite-dimensional, so the policy parametrization has to be independent of the manifold dimension. This requirement is satisfied by a convolutional parametrization; thus, in the following we propose to parametrize $\boldsymbol{\phi}(\mathbf{H}, \mathcal{M}, f)$ as a Manifold Neural Network (MNN).

2.1. Resource Allocation with Manifold Neural Networks

In this paper we consider d-dimensional manifolds $\mathcal{M}$ which are smooth and compact embedded submanifolds of Euclidean space. To each manifold $\mathcal{M}$, a unique operator $\mathcal{L}$ can be associated which characterizes how information propagates on $\mathcal{M}$. This operator, called Laplace-Beltrami operator, is defined as $\mathcal{L}f(u) = -\mathrm{div}(\nabla f)(u)$, which is the divergence of the gradient of manifold signal f in the local Euclidean space around the point $u \in \mathcal{M}$. Akin to the Laplacian matrix in graphs, the Laplace-Beltrami operator measures the total variation of the manifold signal f.

The Laplace-Beltrami operator is a self-adjoint and positive-semidefinite operator. Therefore, it has a discrete, real and non-negative spectrum $\{\lambda_i, \boldsymbol{\varphi}_i\}_{i \in \mathbb{N}^+}$ which satisfies $\mathcal{L}\boldsymbol{\varphi}_i = \lambda_i \boldsymbol{\varphi}_i$. Ordering the λ_i in increasing order, i.e., $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$, it can be shown that λ_i grows as $i^{2/d}$ where d is the manifold dimension [15]. The eigenfunctions $\{\boldsymbol{\varphi}\}_{i \in \mathbb{N}^+}$ form an orthonormal basis of signals $f : \mathcal{M} \to \mathbb{R}$. Thus, a square-integrable signal f can be represented as $f = \sum_{i=1}^{\infty} \langle f, \boldsymbol{\varphi}_i \rangle \boldsymbol{\varphi}_i$.

Leveraging the eigendecomposition of $\mathcal{L}$, the *spectral convolution* of a manifold signal can be expressed as the filter

$$\mathbf{H}(\mathcal{L}) f := \sum_{i=1}^{\infty} \sum_{k=0}^{K-1} h_k \lambda_i^k \langle f, \boldsymbol{\varphi}_i \rangle \boldsymbol{\varphi}_i, \tag{3}$$

where $h_0, \ldots, h_{K-1}$ are coefficients that define a filter function $h(\lambda) = \sum_{k=0}^{K-1} h_k \lambda^k$ determining the amplification of the signal's spectral components based on their eigenvalues. From (3), we see that the manifold convolution only depends on the filter function and the Laplace-Beltrami operator eigenpairs. Hence, a manifold convolutional filter can be easily *transferred* to other manifolds by replacing operator $\mathcal{L}$.

Given the convolutional filter in (3), Manifold Neural Networks (MNNs) are defined as a cascade of layers where each layer consists of a bank of manifold convolutional filters and a nonlinear activation function. Letting σ_l denote the activation function at layer l, the p-th output feature of the l-th layer of a MNN can be written as

$$f_l^p = \sigma_l \left(\sum_{q=1}^{F_{l-1}} \mathbf{H}_l^{qp}(\mathcal{L}) f_{l-1}^q \right) \tag{4}$$

where, for $1 \leq p \leq F_l$ and $1 \leq q \leq F_{l-1}$, $\mathbf{H}_l^{qp}$ is the filter mapping the q-th feature from layer $l-1$ to the p-th feature of layer l. The output features of the last layer, given by f_L^p for $1 \leq p \leq F_L$, are the MNN outputs g^p. The input features at the first layer, f_0^q, are the input data f^q for $1 \leq q \leq F_0$. Since in problem (2) the transmitter states and the policy are one-dimensional on the transmitter-receiver pairs, when parametrizing $\phi(\mathbf{H}, \mathcal{M}, f)$ as a GNN we have $F_0 = F_L = 1$. Letting $g = g^1$ and $f = f^1$, we can thus represent the MNN (4) more succinctly as $g = \phi(\mathbf{H}, \mathcal{M}, f)$.

3. STABILITY ANALYSIS

In this paper, we analyze the stability of MNNs to absolute perturbations of the Laplace-Beltrami operator in particular, which are presented in Definition 1.

Definition 1 (Absolute perturbations)**.** Let $\mathcal{L}$ be the Laplace-Beltrami operator of a manifold $\mathcal{M}$. An absolute perturbation of $\mathcal{L}$ is defined as

$$\mathcal{L}' = \mathcal{L} + \mathbf{A}, \tag{5}$$

where the absolute perturbation operator $\mathbf{A}$ is symmetric.

The absolute perturbation model introduced in Definition 1 is a rather general perturbation model including many different types of perturbations and it can give an approximation for the additive environmental noise in the wireless fading channel states.

3.1. Frequency difference threshold (FDT) filters

With the spectrum of the Laplace-Beltrami operator given, the absolute perturbation to $\mathcal{L}$ would naturally cause shifts in the eigenvalues and eigenfunctions. Proposition 1 shows that Weyl's law can help to deal with the infinite dimensional spectrum of $\mathcal{L}$.

Proposition 1. Let $\mathcal{M}$ be a d-dimensional embedded manifold with Laplace-Beltrami operator $\mathcal{L}$, and let λ_k denote the eigenvalues of $\mathcal{L}$. Let C_1 denote an arbitrary constant, C_d be the volume of the d-dimensional unit ball and Vol$(\mathcal{M})$ the volume of the manifold. For any $\alpha > 0$, there exists N_1 given by

$$N_1 = \lceil (\alpha d / C_1)^{d/(2-d)} (C_d \text{Vol}(\mathcal{M}))^{2/(2-d)} \rceil \tag{6}$$

such that, for all $k > N_1$, it holds that

$$\lambda_{k+1} - \lambda_k \leq \alpha.$$

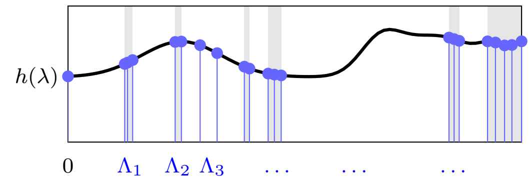

Fig. 1: An α-FDT filter that separates the spectrum of $\mathcal{L}$ by grouping eigenvalues that are less than α apart.

Proof. This is a direct consequence of Weyl's law [15]. □

This important property allows us to gather eigenvalues that are close enough, i.e. less than α apart, in a finite number of groups. This α-separated spectrum is specified in Definition 2, which is achieved by the defined Frequency Difference Threshold (FDT) manifold filter in Definition 3.

Definition 2 (α-separated spectrum)**.** The α-separated spectrum of a Laplace-Beltrami operator $\mathcal{L}$ is defined as the partition $\Lambda_1(\alpha) \cup \ldots \cup \Lambda_N(\alpha)$ such that, all $\lambda_i \in \Lambda_k(\alpha)$ and $\lambda_j \in \Lambda_l(\alpha)$, $k \neq l$, satisfy

$$|\lambda_i - \lambda_j| > \alpha.$$

Definition 3 (α-FDT filter)**.** The α-frequency difference threshold (α-FDT) filter is defined as a filter $\mathbf{h}(\mathcal{L})$ whose frequency response satisfies

$$|h(\lambda_i) - h(\lambda_j)| \leq \Delta_k \text{ for all } \lambda_i, \lambda_j \in \Lambda_k(\alpha) \tag{7}$$

while $\Delta_k \leq \Delta$ for $k = 1, \ldots, N$.

The eigenvalues belonging to different groups, i.e. $\lambda_i \in \Lambda_k(\alpha)$ and $\lambda_j \in \Lambda_l(\alpha)$ with $k \neq l$, are at least α apart from each other. The eigenvalues within the same group $\lambda_i, \lambda_j \in \Lambda_k(\alpha)$ are less than α apart from each other. To achieve the spectrum separation, the α-FDT filter gives similar frequency responses with difference less than Δ_k for all $\lambda_i \in \Lambda_k(\alpha)$.

3.2. Manifold Neural Network Stability

With the spectrum separated by banks of the proposed α-FDT manifold filters, we can show that manifold neural networks composed of these filters are stable under the absolute perturbations to the Laplace-Beltrami operator as specified in Definition 1. We can state and prove this stability property in Theorem 1 under Assumption 1 and Assumption 2.

Assumption 1. The filter function $h : \mathbb{R} \to \mathbb{R}$ is B- Lipschitz continuous and non-amplifying, i.e.,

$$|h(a) - h(b)| \leq B|a - b|, \quad |h(a)| < 1. \tag{8}$$

Assumption 2. The activation function σ is normalized Lipschitz continous, i.e., $|\sigma(a) - \sigma(b)| \leq |a - b|$, with $\sigma(0) = 0$.

Theorem 1 (Manifold Neural network stability)**.** Let $\mathcal{M}$ be a manifold with Laplace-Beltrami operator $\mathcal{L}$. Let f be a manifold signal and $\phi(\mathbf{H}, \mathcal{L}, f)$ an L-layer manifold neural network on $\mathcal{M}$ (4) with $F_0 = F_L = 1$ input and output features and $F_l = F, i = 1, 2, \dots, L-1$ features per layer, and where the filters $\mathbf{h}(\mathcal{L})$ are α-FDT filters with $\Delta = \pi\epsilon/(2\alpha - 2\epsilon)$[cf. Definition 3]. Consider an absolute perturbation $\mathcal{L}' = \mathcal{L} + \mathbf{A}$ of the Laplace-Beltrami operator $\mathcal{L}$ [cf. Definition 1] where $\|\mathbf{A}\| = \epsilon \leq \alpha$. Then, under Assumptions 1 and 2 it holds:

$$\begin{aligned} &\|\boldsymbol{\phi}(\mathbf{H}, \mathcal{L}, f) - \boldsymbol{\phi}(\mathbf{H}, \mathcal{L}', f)\| \\ &\qquad \leq LF^{L-1}\left(\frac{\pi N}{\alpha - \epsilon} + B\right)\epsilon\|f\|. \end{aligned} \tag{9}$$

where N is the number of spectrum partitions.

Provided that Assumption 1 and 2 are satisfied, MNNs with α-FDT filters are thus stable to absolute perturbations of the operator $\mathcal{L}$. The assumptions are reasonable in that no constraint is put on the Lipschitz constant B. Furthermore, most common activation functions, such as the ReLU, the modulus function and the sigmoid satisfy the normalized Lipschitz continuity condition. we can observe that stability can be improved if the Lipschitz constant B is small and α is large. However, these both cause the filter to become less discriminative. The stability bound also depends on the number of layers L and the number of features per layer F of the MNN, which is a natural dependence on the neural network architecture. Given the stability bound of MNNs presented in Theorem 1, such a parametrization thus provides a stable and transferable resource allocation policy for large scale wireless networks.

4. NUMERICAL EXPERIMENTS

In this section, we verify the transference and stability properties of the proposed MNN resource allocation policies by numerically evaluating such properties with learned GNN-based policies on large graphs. While dropping m transmitters randomly over a range of $\mathbf{a}_i \in [-m, m]^2$, the paired receivers are dropped within $\mathbf{b}_i \in [\mathbf{a}_i + [-m/4, m/4]]^2$. When considering the large-scale pathloss gain and a random fast fading gain, the link state can be written as $s_{ij} = \log(d_{ij}^{-2.2} h^f)$, where $d_{ij}$is the distance between pair i and j, while $h^f \sim$ Rayleigh(2) is the random fading. The GNN is constructed with $L = 10$ layers with a $K = 5$ tap filter and a ReLu nonlinear activation function in each layer.

When verifying the transferability of our proposed GNN methods, we compare with three existing baseline methods for solving this resource allocation problem. They are WMMSE [2], equal resource allocation (i.e. assign P_{max}/m to each transmitter) and random resource allocation (i.e. randomly select P_{max}/p_0 transmitters and assign p_0). We trained the GNN policy on a $m = 50$ wireless networks using unsupervised learning [5] and evaluate the trained policies on newly randomly generated wireless networks of larger size but the same overall network density. We can see that the GNNs trained on smaller wireless networks can still outperform other methods on larger size networks as shown in Figure 2.

To study the stability properties in large scale wireless system, we model environmental noise by adding a log-normal matrix to the original channel state $\mathbf{S}$ with 500 pairs of transmitters and receivers. With the original trained GNN employed, we measure the stability by the difference of the ratios of the final sum-of-rate achieved by the GNN on the noisy channel states and that achieved by WMMSE. We can observe from Figure 3 that the difference increases with the number of layers and the number of filters per layer in the constructed GNN but is still generally small. This further validates the stability result in Theorem 1 of GNN-based resource allocation policies in large scale networks.

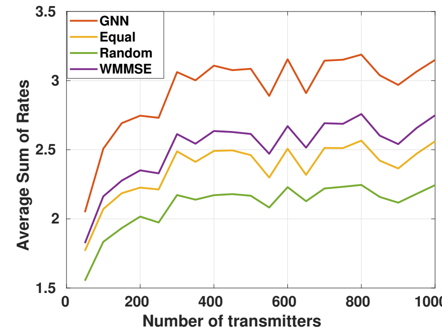

Fig. 2: Sum-of-rate achieved by GNN trained on small network and execute on larger networks compared with other baseline methods.

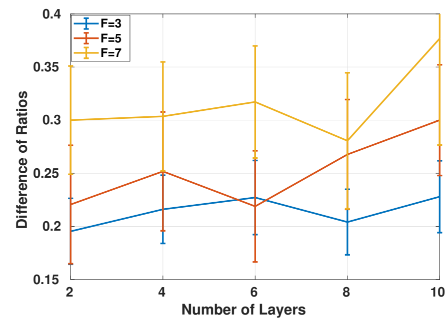

Fig. 3: Difference between the sum-of-rate ratios on the original wireless network setting and the perturbed one.

5. CONCLUSIONS

In this paper, we have formulated the constrained resource allocation problem in large scale wireless systems. While the wireless network structure can be modeled as discrete graphs, large wireless networks tend to converge to a manifold structure as the number of transmitters and receivers grow. We propose a Manifold Neural Network (MNN) method for solving and analyzing resource allocation policies in large wireless networks. We prove that MNN is stable under the absolute perturbations to the Laplace-Beltrami operator of the manifold which in turn provides a stable and transferable allocation parametrization for large scale wireless networks. We verified our results with practical large wireless network settings compared with other heuristic methods.

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