

Stability of Aggregation Graph Neural Networks

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Abstract—In this paper we study the stability properties of aggregation graph neural networks (Agg-GNNs) considering perturbations of the underlying graph. An Agg-GNN is a hybrid architecture where information is defined on the nodes of a graph, but it is processed block-wise by Euclidean CNNs on the nodes after several diffusions on the graph shift operator. We derive stability bounds for the mapping operator associated to a generic Agg-GNN, and we specify conditions under which such operators can be stable to deformations. We prove that the stability bounds are defined by the properties of the filters in the first layer of the CNN that acts on each node. Additionally, we show that there is a close relationship between the number of aggregations, the filter’s selectivity, and the size of the stability constants. We also conclude that in Agg-GNNs the selectivity of the mapping operators can be limited by the stability restrictions imposed on the first layer of the CNN stage, but this is compensated by the pointwise nonlinearities and filters in subsequent layers which are not subject to any restriction. This shows a substantial difference with respect to the stability properties of selection GNNs, where the selectivity of the filters in all layers is constrained by their stability. We provide numerical evidence corroborating the results derived, testing the behavior of Agg-GNNs in real life application scenarios considering perturbations of different magnitude.

Index Terms—Aggregation graph neural networks (Agg-GNNs), graph signal processing, graph neural networks (GNNs), convolutional neural networks (CNNs), stability to deformations.

I. INTRODUCTION

Convolutional neural networks (CNNs) have become essential tools in machine learning. Numerical evidence emerges every day in diverse applications exhibiting their strengths and limits, raising fundamental questions about why they perform well. In recent years stability analyses have been considered to provide some explanations about their good performance [1]–[6]. However, none of these results are applicable to hybrid architectures like aggregation graph neural networks (Agg-GNNs) [7]. This prompts the question of whether Agg-GNNs can be stable, and what role the properties of the two different domains involved in an Agg-GNN play in the stability analysis.

Agg-GNNs are convolutional architectures that allow the processing of information supported on graphs by means of regular or Euclidean CNNs. This is achieved using the operation of aggregation to capture the distinctive features of the signals on a graph, and afterwards this information is processed block-wise by a regular CNN [7]. This versatile architecture has been used successfully in the problems of source localization, authorship attribution, text classification, resource allocation, and flocking in distributed autonomous systems [7]–[9]. The good performance of Agg-GNNs has

been explained by how complex symmetries on the data are captured by a combination of operators in two different domains. However, as shown in [4], [7], the use of domain symmetries only explains partly why convolutional architectures work well. This follows from the fact that both filters and networks are equally good at leveraging symmetries. Additionally, unlike filters, networks incorporate pointwise nonlinearity functions, which points to additional properties that should explain the superiority of networks over filters.

In recent works, stability analyses have been considered to explain the good performance of convolutional architectures [3], [4]. Nevertheless, none of those results apply to hybrid architectures like Agg-GNNs. The main reason for this lies in the way information is mapped between the two different domains. In this paper we provide stability results for Agg-GNNs, considering the perturbation of the underlying graph relying on the deformation models used in [3]–[5]. The main contributions of our paper are:

- (C1) The derivation of stability bounds for Agg-GNNs with an arbitrary number of layers in the CNN stage.
- (C2) Proving that Agg-GNNs can be stable to perturbations in the underlying graph.
- (C3) Showing that there is a trade-off between stability and selectivity affecting only the filters in the first layer of the CNN stage.

The results presented in this paper have several implications among which we highlight the following:

- (I1) The selectivity lost in the first layer of the CNN of an Agg-GNN with stable filters is compensated by the nonlinearity functions *and the filters* in subsequent layers of the CNN stage.
- (I2) Increasing the number of aggregations in an Agg-GNN provides more flexibility for the selection of the filters in the CNN stage and at the same time increases directly the value of the stability constants.

This paper is organized as follows. In Section II we introduce Agg-GNNs discussing in full detail the properties of the aggregation operator and stating basic notions and terminology for the rest of the paper. In Section III we discuss the formal concepts of perturbations, stability in Agg-GNNs, and we derive the main results of the paper. To corroborate and visualize the implications of our results we performed a set of numerical experiments presented in Section IV. Finally, in Section V we discuss our main results and present some conclusions.

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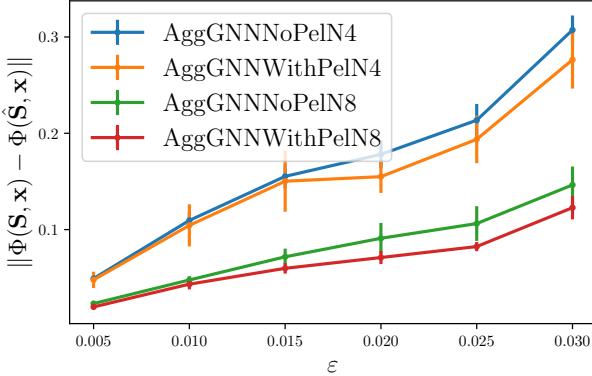


Figure 6. The output difference of Agg-GNNs with different number of aggregations under synthetic additive and multiplicative perturbations – both norms bounded by ϵ – in the underlying graph with N nodes. The learned filters in the Agg-GNNs are trained with or without Lipschitz and integral Lipschitz penalty terms.

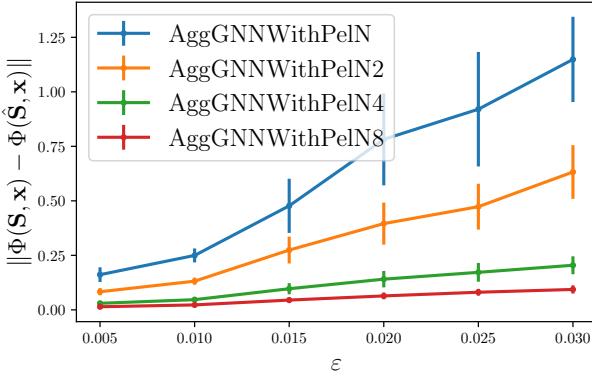


Figure 7. The output difference of Agg-GNNs with different number of aggregations under synthetic additive perturbations whose norms are bounded by ϵ in the underlying graph with N nodes. The learned filters in the Agg-GNNs are trained with Lipschitz penalty terms.

Lipschitz constants, AggGNNs are more stable compared to general filters with large Lipschitz and integral Lipschitz constants. Additionally, we can observe that AggGNNs with more aggregation steps are less stable as ϵ grows. This can be more clearly observed in Fig. 7, Fig. 8 and Fig. 9 for additive, multiplicative and combined perturbations. This verifies our result in Theorem 3 that the stability bounds of Agg-GNNs grow directly with the size of perturbations.

In Fig. 10, we study the relationship between the stability and the number of aggregations a of Agg-GNNs with different levels of both additive and multiplicative perturbations. We can observe that the output difference scales with $\sqrt{a+1}$ as we have proposed in Theorem 3.

B. Wireless Resource Allocation

We model the wireless network as a graph model with the transmitters seen as graph nodes and the channel links as edges. Based on the channel states, we parameterize the decentralized power allocation policy as Agg-GNN to maximize the sum of capacity. The stability of Agg-GNNs is shown under a synthetic absolute perturbation to the channel states,

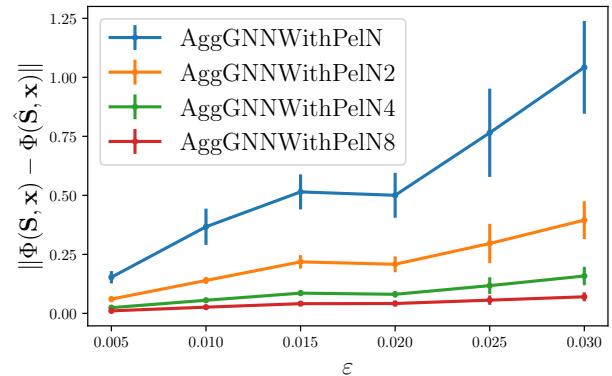


Figure 8. The output difference of Agg-GNNs with different number of aggregations under synthetic multiplicative perturbations whose norms are bounded with ϵ in the underlying graph with N nodes. The learned filters in the Agg-GNNs are trained with integral Lipschitz penalty terms.

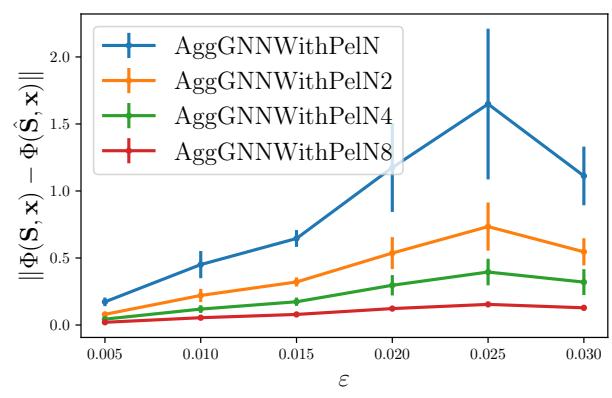


Figure 9. The output difference of Agg-GNNs with different number of aggregations under synthetic perturbations in the underlying graph with N nodes with the norm of both additive and multiplicative perturbation matrix norm bounded as ϵ . The learned filters in the Agg-GNNs are trained with Lipschitz and integral Lipschitz penalty terms.

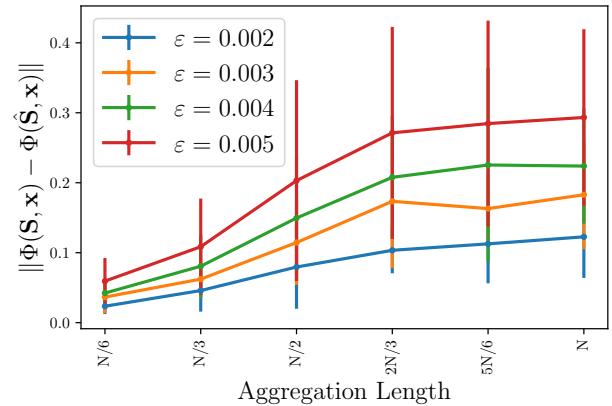


Figure 10. The output difference of Agg-GNNs under synthetic perturbations in the underlying graph with N nodes with respect to a growing number of aggregations. The filters in the Agg-GNNs are trained with Lipschitz and integral Lipschitz penalty terms. The labels of different perturbation levels ϵ indicate the norms of both the additive and multiplicative perturbation matrices.

