# 1. Introduction to Matching

## 1.1. Set and Strict Preference Order

Before the real matching comes, it is the point to have the common conventions about our topic. There are some rules, which this handout and the presentaion are based on. The participant is called *agent*, and the set of all agnets is denoted by $X$, so each agent $x \in X$. A *matching* $m$ in a set of machings $M$, so that $m \in M$, represents the output of a matching mechanism. The *strict preference order* is defined as preference, that a specific agent $x$ has on different matchings.

**Strict Preference** Agent $x$ has a *strict preference order* $\succ_x$, so that $m \succ_x m'$, for matchings $m$ and $m'$, indicates that agent $x$ strict prefers $m$ to $m'$.

**Example 1.1.1.** The man $x$ admiring the woman $y$ most in woman's set $Y$ can be formulated as: $\forall y' \in Y \setminus y : y \succ_x y'$

## 1.2. Three Types of Matching

**Two-Sided Matching** is the matching, where there are two distinct groups of agents, and the problem is to match each agent on one side of the market with an agent on the other side. Stable Marriage Problem belongs to this type.

**Assignment Problems** is the matching, where there are items and agents with preferences on items, and the problem is to assign a distinct item to each agent.

**Kidney-Paired Donation** is the matching, where patient-donor pairs arrive to the market, and the problem is to determine how to match pairs such that the donor in one pair is compatible with the patient in another pair and vice versa, so that such a match enables two kidney transplants. [1]

In this handout and the presentation we will only talk about the Two-Sided Matching and Stable Marriage Problem, which is a case based on the this type of matching.

## 1.3. Matching in Graph

**Unweighted Graphs** Let $G = (V, E)$ be any graph with $n$ vertices and $m$ edges. A matching $M$ of $G$ is a subset of $E$ such that no two edges in $M$ are adjacent. We say that a vertex $v$ is matched in $M$ if there is other some vertices $M(v) \in V$ such that

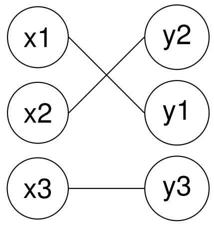
[1] David C. Parkes, Sven Seuken. Economics and Computation. Chapter 12: Matching Markets(pp. 291-293). 2017

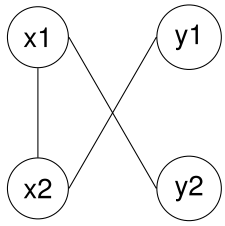
$\{v, M(v)\} \in M$. Otherwise, $v$ is unmatched in $M$.[2]

**Bipartite graph** A graph $G = (V, E)$ is *bipartite* if the vertex set $V$ can be partitioned into two disjoint subsets such that for every edge $(u, v) \in E$, $u$ and $v$ are in opposite subsets.[3]

### 1.3.1. Exercise

Following are two graphs representing the matching between 2 sides. Which is a bipartite graph?




[2] David J. Abraham. Algorithmics of Two-Sided Matching Problems. 2003

[3] https://en.wikibooks.org/wiki/Graph_Theory/Definitions

# 2. Two-Sided Matching and Stable Marriage Problem

## 2.1. Description and Definition

**Two-side-matching Markets - Definiton** [1] There are **two distinct groups of agents**. The problem is to match each agent in one group to an agent in the other group, such that no agent is matched to more than one other agent. This outcome is a matching.

**Stable Marriage Problem - Description** The Stable Marriage Problem is a matching problem in graph theory first introduced by Gale and Shapley[2]. An instance of this problem consists of two sets, a set of men and a set of women. Each person specifies the order of all members of opposite sex. A matching is stable if there is no pair where both man and woman prefer each other to their current partner in the matching. The goal of this problem is to find a stable matching for a given instance. Gale and Shapley proved that there always exists such a matching for any instance. They also provided a polynomial algorithm for solving the problem.

**Stable Marriage Problem - Definition** An instance of the Stable Marriage Problem consists of a set of $n$ men and a set of $n$ women where each man and each woman provides a lineary strict preference order lists of all members of the opposite sex. Such lists are called preference lists. Let $M$ be an one-to-one correspondence between the subset of men and the subset of women, then $M$ is a marriage. If $(x, y) \in M$, called $(x, y)$ as a pair in $M$, means $x$ is married with $y$ in $M$ and $y$ is married with $x$ in $M$. Furthermore, that a agent is unmatched in $M$ if he (or she) is not married with any woman (or man) in $M$, otherwise this agent is matched in $M$. Let $M(x)$ denote the woman who is married with the man $x$ in $M$ and let $M(y)$ denote the man married with the woman $y$ in $M$. Agents have only the strict preferences in this problem. [3]

## 2.2. Prefer to be Unmatched

An agent may prefer to be unmatched than matched with a particular agent on the other side.

**Example 2.2.1.** Preference $\emptyset \succ_y$ shows the intention of a woman $y$ who strictly prefers not to be matched than to be matched with man $x$. In this case, the man $x$ is said to be *unacceptable* to the woman $y$.

[1] David C. Parkes, Sven Seuken. Economics and Computation. Chapter 12: Matching Markets(pp. 293 - 294). 2017

[2] D. Gale and L. S. Shapley. College admissions and the stability of marriage. The American Mathematical Monthly. 1962

[3] Andrej Podhradsky. Stable Marriage Problem Algorithms. 2010

## 2.3. Blocking Pair and Stable Marriage

Let $X$ be a set of men, $Y$ be a set of women, and $M \subset X \times Y$ be a marriage. A pair $(x, y) \in X \times Y$ is a blocking pair [4] for $M$ if $(x, y) \notin M$ but:

- $y$ prefers $x$ to $M(y)$ and
- $x$ prefers $y$ to $M(x)$

If there is at least one blocking pair for $M$ then we say that $M$ a *unstable marriage*, otherwise a *stable marriage*. A blocking pair also exists when an agent prefers to be unmatched rather than assigned to his/her match.

**Yes-No Questions**: Are the following statements true?

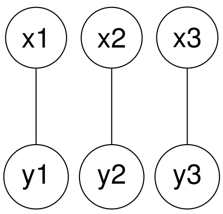
- A stable matching is one in which no pair of agents prefer each other over their respective matches.
- If $x$ is unmatched in $M$ and $y$ prefers $x$ to $M(y)$ means that: $x \succ_y M(y)$
- If $\emptyset \succ_x M(x)$ for a men $x$, then $(x, \emptyset)$ form a blocking pair.

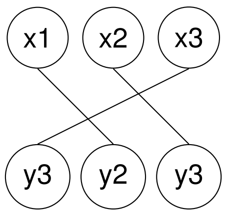
### 2.3.1. Exercise

Suppose that there are three men and three women, with strict preference orders:

$$y_2 \succ_{x_1} y_1 \succ_{x_1} y_3$$
$$y_1 \succ_{x_2} y_3 \succ_{x_2} y_2$$
$$y_1 \succ_{x_3} y_2 \succ_{x_3} y_3$$
$$x_1 \succ_{y_1} x_3 \succ_{y_1} x_2$$
$$x_3 \succ_{y_2} x_1 \succ_{y_2} x_2$$
$$x_1 \succ_{y_3} x_3 \succ_{y_3} x_2$$

Can you judge the following graphs, if they are stable?




[4] Dan Gusfield, Robert W.Irving. The Stable Marriage Problem: Structure and Algorithms. Chapter 1: Elementary Concept and Result(pp. 6). The MIT Press, 1989

## 2.4. Gale–Shapley Deferred Acceptance Algorithm

The following is an algorithm by Dale Gale and Lloyd Shapley [5] in 1962 to find such a stable matching. The algorithm is like actual human behavior. This algorithm takes a list preference order as its input and output of it is a stable marriage. With the help of that, a **stable marriage** could be found. Deferred Acceptance Algorithm has agents on one side make offers to agents on the other side. The agent on the other side will tentaively agree on acceptable offers unless a better offer comes along. There are two versions of Deferred Acceptance Algorithm, depending on which side makes the offers. In the running example about marriage problem, we refer to these versions as the men-proposing algorithm and the woman-proposing algorithm. We describe the men-proposing version in the Algorithm 1 below. The woman-proposing version is defined analogously. Noted that if the two versions of Deferred Acceptance Algorithm return different matching, there may be more than one stable matching.

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**Algorithm 1:** Gale–Shapley Deferred Acceptance Algorithm (men-proposing version) [a]

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**Input :** a list of preference order
**Output:** a stable marriage $M$

Initialize $M$ to empty matching.
**while** *Some man $x$ is unmatched and hasn't proposed to every woman* **do**
  $x$ proposes to the most preferred woman $y$ to whom he has not yet propose;
  **if** $y$ *is unmatched* **then**
    add $(x, y)$ to the Matching $M$;
  **else if** $y$ *prefers $x$ to current partner $x'$* **then**
    Replace $(x', y)$ with $(x, y)$ in Matching $M$;
  **else**
    $y$ rejects $x$;

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[a] See footnote 5 below

### 2.4.1. Exercise

Try to apply **Gale–Shapley Deferred Acceptance Algorithm** on Exercise $2.3.1$. Write down every step that you need.

[5] D. Gale and L. S. Shapley. College admissions and the stability of marriage. The American Mathematical Monthly (pp.69:9–15). 1962