Iterative Improvement

CS3230: Design and Analysis of Algorithms

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Chapter 10: Iterative Improvement

- Greedy vs. Iterative Improvement
  - Greedy strategy constructs a solution to an optimization problem piece by piece.
  - Iterative improvement starts with a feasible solution and improves it repeatedly through small, localized steps.

Iterative Improvement

- Goal:
  - Find a feasible solution with an improved objective function.
- Obstacles:
  - Need initial solution. Sometimes we can start with trivial solution or one obtained by greedy strategy. However, sometimes finding an initial solution is complex.
  - How to check efficiently if modified solution is better.
  - Local vs. global optimium.

Linear Programming

- Simplex Method
- Problem statement: optimize a linear function of several variables subject to a set of constraints.
  - Maximize (or minimize) $c_1x_1 + \ldots + c_nx_n$
  - subject to
    $a_{i1}x_1 + \ldots + a_{in}x_n \leq$ (or $\geq$ or $=$) b_i for $i = 1, \ldots, m$
  - $x_1 \geq 0, \ldots, x_n \geq 0$.

- Two researchers, L. V. Kantorovich and T. C. Koopmans, received the Nobel Price in Economics for their contributions to linear programming.

Simplex Method Example

- Example 1:

  maximize $3x + 5y$

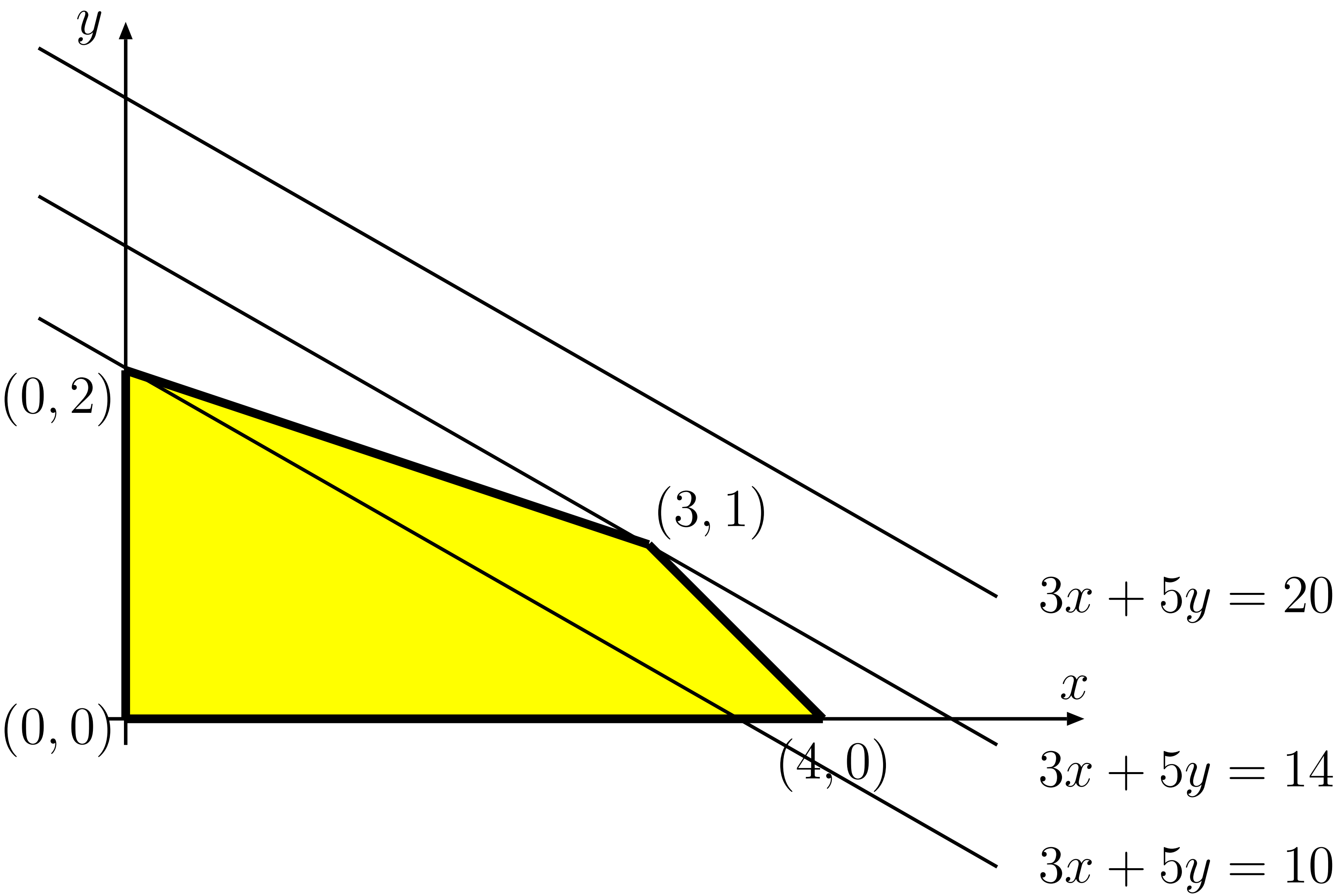
  subject to $x + y \leq 4$

  $x + 3y \leq 6$

  $x \geq 0, y \geq 0$

- A feasible solution is any point (x, y) that satisfies all constraints of the problem.
- A feasible region is the set of all feasible points.
- An optimal solution is a feasible point that maximizes the objective function.

Simplex Method Example (2)

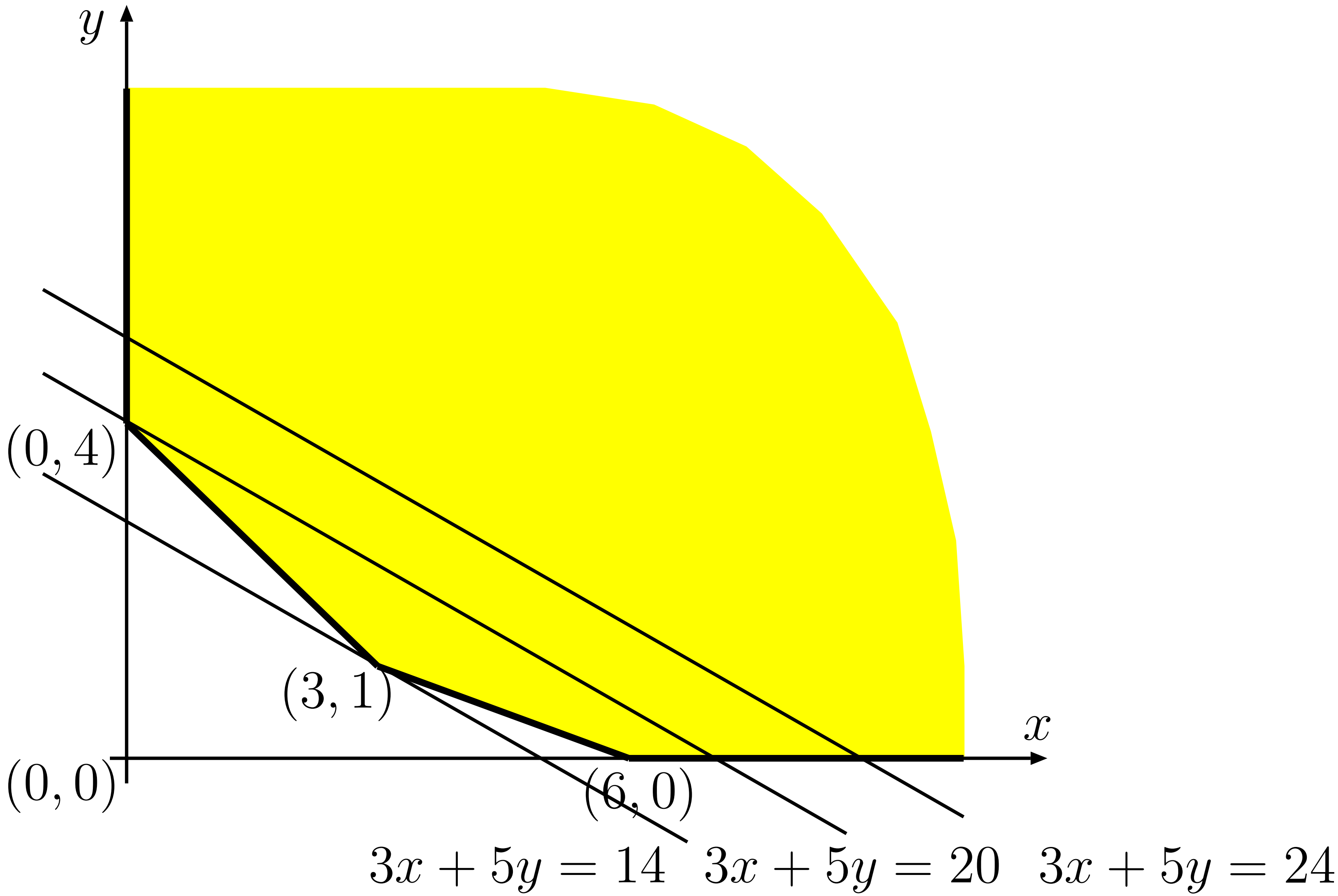


Simplex Method Example (3)

- Some linear programming problems are infeasible.
- Ex.: $x + y \leq 1$ and $x + y \geq 2$.

- Some linear programming problems result in an unbounded region.
- Ex.: $x + y \geq 4$ and $x + 3y \geq 6$.
- (If we want to find the minimum optimal value then the problem has a solution.)

Simplex Method Example (4)



Extreme Points

- THEOREM (Extreme Point Theorem)

  Any linear programming problem with a nonempty bounded feasible region has an optimal solution; moreover, an optimal solution can always be found at an extreme point of the problem's feasible region.

- Thus, to solve problem, inspect a finite number of points in the feasible region.
- Compute value of objective function at each extreme point. Select the one with the best value.

Simplex Method

- Standard form:

  It must be a maximization problem.

  All the constraints (except for the nonnegativity constraints) must be in the form of linear equations.

  All the variables must be required to be nonnegative.

- Thus, m constraints and n unknowns $(n \geq m)$:

  maximize $c_1 x_1 + \ldots + c_n x_n$

  subject to $a_{i1} x_1 + \ldots + a_{in} x_n = b_i \quad$ for $i = 1, 2, \ldots, m$

  $x_1 \geq 0, \ldots, x_n \geq 0$.

Simplex Method (2)

- Notes:
  - The standard form can also be written in matrix form.
  - If objective function needs to be minimized $\Rightarrow$ replace coefficients c_j with $-c_j, j = 1, 2, \ldots, n$.
  - If a constraint is given as an inequality $\Rightarrow$ add a slack variable.
  - Ex.: $x + y + u = 4$ where $u \geq 0$ and $x + 3y + v = 6$ where $v \geq 0$, instead of $x + y \leq 4$ and $x + 3y \leq 6$.
  - Variables must be nonnegative. If not, x_j can be replaced with two variables as follows: $x_j = x'_j - x''_j$, $x'_j \geq 0, x''_j \geq 0$.

Simplex Method (3)

- Identify extreme points as follows.
- Given: m equations in n unknowns $(n \geq m)$.
- Set $n - m$ variables to zero to obtain a system of m equations and m unkowns.
- If system has a unique solution we have a basic solution.
  - The coordinates set to zero are called nonbasic.
  - The coordinates obtained by solving the system are called basic.
- If all the coordinates of a basic solution are ≥ 0 the basic solution is called a basic feasible solution.

Simplex Method (4)

- Example:

  maximize $3x + 5y + 0u + 0v$

  subject to
  $$\begin{aligned} x + y + u &= 4 \\ x + 3y + v &= 6 \\ x, y, u, v &\geq 0. \end{aligned}$$

  - If $x, y = 0 \Rightarrow$ basic feasible solution $(0, 0, 4, 6)$.

    $u = 4; v = 6$
  - If $x, u = 0 \Rightarrow$ basic not feasible solution $(0, 4, 0, -6)$.

    $y = 4; 3y + v = 6; \Rightarrow v = -6$
  - $(0, 0, 4, 6)$ is an extreme point of the feasible region.

Simplex Method (5)

- Simplex tableau:

	x	y	u	v	
u	1	1	1	0	4
v	1	3	0	1	6
	-3	-5	0	0	0

- $m + 1$ rows and $n + 1$ columns.
- m rows with coefficients of corresponding constraint equation.
- Last row is the objective row. Initialize with coefficients of objective function with signs reversed.

Simplex Method (6)

- Objective row: used to check whether tableau represents an optimal solution (all coefficients ≥ 0, except possibly last column).
- Ex.: Basic feasible solution $(0, 0, 4, 6)$ is not optimal.
- Negative value in x-column means we can increase objective function $z = x + 3y + 0u + 0v$.
- Idea: increase value of x, but "compensate" with u and v to keep point feasible.

$$x + u = 4 \text{ where } u \geq 0$$
$$x + v = 6 \text{ where } v \geq 0$$
$$x \leq \min\{4, 6\} = 4.$$

- New extreme point $(4, 0, 0, 2)$ with $z = 12$.

Simplex Method (6b)

- Objective row: used to check whether tableau represents an optimal solution (all coefficients ≥ 0, except possibly last column).
- Another choice:
- Negative value in y-column means we can increase objective function $z = x + 3y + 0u + 0v$.
- Idea: increase value of y, but "compensate" with u and v to keep point feasible.

$$y + u = 4 \text{ where } u \geq 0$$
$$3y + v = 6 \text{ where } v \geq 0$$
$$y \leq \min\{4, \tfrac{6}{3}\} = 2.$$

- New extreme point $(0, 2, 2, 0)$ with $z = 10$.

Simplex Method (6c)

- Common Rule 1: Choose the most negative variable/value in the objective row. This hopefully leads to the largest increase in the objective function.
  Terminology: entering variable in pivot column.
- Rule 2: For each positive entry in the pivot column, compute the θ-ratio by dividing the row's last entry by the entry in the pivot column.

$$
\begin{array}{c|cccc|c|}
\hline
u & 1 & 1 & 1 & 0 & 4 \\
v & 1 & 3 & 0 & 1 & 6 \\
\hline
\end{array}
$$

- Ex.: $\theta_u = \frac{4}{1} = 4$, $\theta_v = \frac{6}{3} = 2$.

Simplex Method (7)

- The row with the smallest θ determines the departing variable $\rightarrow$ identifies pivot row.
- Departing variable: basic variable to become nonbasic in the next tableau.

$$
\begin{array}{c|cccc|c|}
u & 1 & 1 & 1 & 0 & 4 \\
v & 1 & 3 & 0 & 1 & 6
\end{array}
$$

Simplex Method (7b)

$$u \quad \left|\begin{array}{cccc|c} \hline 1 & 3 & 0 & 1 & 6 \\ \hline \end{array}\right|$$

- Divide all the entries in the pivot row by the pivot.

  $\overleftarrow{row}_{new}$: $\frac{1}{3} \;\; 1 \;\; 0 \;\; \frac{1}{3} \;\; 2$.

- Replace each of the other rows by the following difference:

  $row - c \cdot \overleftarrow{row}_{new}$ (c: row's entry in pivot column).

  $row\ 1 - 1 \cdot \overleftarrow{row}_{new}$: $\frac{2}{3} \;\; 0 \;\; 1 \;\; -\frac{1}{3} \;\; 2$.

  $row\ 3 - (-5) \cdot \overleftarrow{row}_{new}$: $-\frac{4}{3} \;\; 0 \;\; 0 \;\; -\frac{5}{3} \;\; 10$.

Simplex Method (8)

- Next Simplex tableau:

$$\begin{array}{c|cccc|c|}
 & x & y & u & v & \\
\hline
u & \frac{2}{3} & 0 & 1 & -\frac{1}{3} & 2 \\
v & \frac{1}{3} & 1 & 0 & \frac{1}{3} & 2 \\
\hline
 & -\frac{4}{3} & 0 & 0 & \frac{5}{3} & 10 \\
\hline
\end{array}$$

- Basic feasible solution $(0, 2, 2, 0)$ with $z = 10$.

Simplex Method (8b)

- Final Simplex tableau:

$$\begin{array}{c|cccc|c|}
 & x & y & u & v & \\
\hline
u & 1 & 0 & \frac{3}{2} & -\frac{1}{2} & 3 \\
v & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 1 \\
\hline
 & 0 & 0 & 2 & 1 & 14 \\
\hline
\end{array}$$

- Basic feasible solution $(3, 1, 0, 0)$ with $z = 14$.
- Optimal solution because all values in objective row are positive.

Maximum Matching

- Def.: A matching in a graph G is a subset of its edges with the property that no two edges share a vertex.
- Def.: A maximum matching is a matching with the largest number of edges.
- Maximum matching problem: find a maximum matching in a graph G.
- Simpler case: consider a bipartite graph where all the vertices can be partitioned into disjoint sets V and U, so that every edge connects a vertex in one of these sets in the other set.

Maximum Matching

- A graph is bipartite if it is 2-colorable.
- Def.: A perfect matching matches all vertices of a graph.

Stable Matching Problem

- Goal: Given n men and n women, find a “suitable” matching.

  $Y = \{m_1, m_2, \ldots, m_n\}$: set of n men.

  $X = \{w_1, w_2, \ldots, w_n\}$: set of n women.

  - Participants rate members of opposite sex (no ties).
  - Each man lists women in order of preference from best to worst.
  - Each woman lists men in order of preference from best to worst.

Stable Matching Problem (2)

- Men's preference profile:

	favorite $\downarrow$ 1^{st}	2^{nd}	least favorite $\downarrow$ 3^{rd}
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Stable Matching Problem (3)

- Women's preference profile:

	favorite ↓ 1^{st}	2^{nd}	least favorite ↓ 3^{rd}
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Stable Matching Problem (4)

- Representation in Ranking Matrix:

	Amy	Bertha	Claire
Xavier	1,2	2,1	3,1
Yancey	2,1	1,2	3,2
Zeus	1,3	2,3	3,3

Stable Matching Problem (5)

- Perfect matching: everyone is matched monogamously.
  - Each man is matched to exactly one woman.
  - Each woman is matched to exactly one man.
- Stability: no incentive for some pair of participants to undermine assignment by joint action.
  - In matching M, an unmatched pair $m-w$ is unstable if man m and woman w prefer each other to current partners.
  - Unstable pair $m-w$ could each improve by eloping.
- Stable matching: perfect matching with no unstable pairs.

Stable Matching Problem (6)

- Stable matching problem: Given the preference lists of n men and n women, find a stable matching if one exists.

Stable Matching Problem (7)

- Q: Is assignment $X - C$, $Y - B$, $Z - A$ stable?

	favorite $\downarrow$		least favorite $\downarrow$
	1^{st}	2^{nd}	3^{rd}
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Stable Matching Problem (8)

- Q: Is assignment $X - C$, $Y - B$, $Z - A$ stable?

	favorite $\downarrow$		least favorite $\downarrow$
	1^{st}	2^{nd}	3^{rd}
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Stable Matching Problem (9)

- Q: Is assignment $X - C$, $Y - B$, $Z - A$ stable?
- A: No. Bertha and Xavier will hook up.

	favorite $\downarrow$		least favorite $\downarrow$
	1^{st}	2^{nd}	3^{rd}
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Stable Matching Problem (10)

- Q: Is assignment $X-C$, $Y-B$, $Z-A$ stable?
- A: No. Bertha and Xavier will hook up.

	favorite $\downarrow$		least favorite $\downarrow$
	1^{st}	2^{nd}	3^{rd}
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Stable Matching Problem (11)

- Q: Is assignment $X-A$, $Y-B$, $Z-C$ stable?
- A: Yes.

	favorite ↓		least favorite ↓
	1^{st}	2^{nd}	3^{rd}
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Stable Matching Problem (12)

- Q: Is assignment $X-A$, $Y-B$, $Z-C$ stable?
- A: Yes.

	favorite $\downarrow$ 1^{st}	2^{nd}	least favorite $\downarrow$ 3^{rd}
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Propose-And-Reject Algorithm

- Propose-and-reject algorithm. [Gale-Shapley 1962]. Intuitive method that guarantees to find a stable matching.

Propose-And-Reject Algorithm

ALGORITHM $Propose - And - Reject$

Initialize each person to be free.

while (some man is free and hasn't proposed to every woman) do

Arbitrarily select a man m and choose $w =$ next woman on m's list to whom m has not yet proposed

if (w is free)

assign m and w to be engaged

else if (w prefers m to her fiance m')

assign m and w to be engaged, and m' to be free

else

w rejects m

Proof of Correctness: Termination

- Observation 1: Men propose to women in decreasing order of preference.
- Observation 2: Once a woman is matched, she never becomes unmatched; she only “trades up.”
- Claim: Algorithm terminates after at most n^2 interations of while loop.
- Proof: Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals.

Proof of Correctness: Termination

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

- $n(n-1)+1$ proposals required.

Proof of Correctness: Termination

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

- $n(n-1)+1$ proposals required.

Proof of Correctness: Termination

- Claim: All men and women get matched.
- Proof (by contradiction):
  - Suppose, for the sake of contradiction, that Zeus is not matched upon termination of the algorithm.
  - Then some woman, say Amy, is not matched upon termination.
  - By Observation 2, Amy was never proposed to.
  - But, Zeus proposes to everyone, since he ends up unmatched.

Proof of Correctness: Stability

- Claim: No unstable pairs.
- Proof (by contradiction):
  - Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S^*.
  - Case 1: Z never proposed to A.
    $\Rightarrow$ Z prefers his GS partner to A.
    $\Rightarrow$ A-Z is stable.
  - Case 2: Z proposed to A.
    $\Rightarrow$ A rejected Z (right away or later).
    $\Rightarrow$ A prefers her GS partner to Z.
    $\Rightarrow$ A-Z is stable.
  - In either case A-Z is stable, a contradiction.

Summary

- Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.
- Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.