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Deriving the Black-Scholes formula

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Assumptions

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Market characteristics:

- Assets can be infinitely divided.
- Assets can be traded at any time in any size.
- There are no transaction costs.
- There are no restrictions on borrowing and short selling.
- Borrowing and lending rates are both equal to risk-free rate r.

Assumptions

The risk-free asset

$$\begin{cases} \mathrm{d}Q_t &= rQ_t\,\mathrm{d}t \\ Q_0 &= 1 \end{cases} \qquad Q_t = e^{rt}$$

The risky asset

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t$$

The option

$$C_t = C(t, S_t)$$

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Black-Scholes partial differential equation

Portfolio

Build a portfolio consisting at time t of

- N_t^1 shares of the risky asset,
- N_t^2 options,
- Q_t monetary units in the risk-free asset.

The value at time t of the portfolio is:

$$V_t = N_t^1 S_t + N_t^2 C_t + Q_t$$

The self-financing condition gives:

$$\mathrm{d}V_t = N_t^1\,\mathrm{d}S_t + N_t^2\,\mathrm{d}C_t + \mathrm{d}Q_t$$

Evolution of the option price

Using Itō's formula:

$$\mathrm{d}C_t = \left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\,\mu S_t + \frac{1}{2}\,\frac{\partial^2 C}{\partial S^2}\,\sigma^2 S_t^2 \right) \mathrm{d}t + \frac{\partial C}{\partial S}\,\sigma S_t\,\mathrm{d}B_t$$

Let's write:

$$\mathrm{d}C_t = \mu_C C_t\,\mathrm{d}t + \sigma_C C_t\,\mathrm{d}B_t$$

Deriving the Black-Scholes PDE

Replacing in the diffusion equation for the value of the portfolio:

$$\mathrm{d}V_t = N_t^1 S_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t) + N_t^2 C_t(\mu_C\,\mathrm{d}t + \sigma_C\,\mathrm{d}B_t) + rQ_t\,\mathrm{d}t$$

Introducing the weights of the assets:

$$w_t^1 = \frac{N_t^1 S_t}{V_t} \qquad w_t^2 = \frac{N_t^2 C_t}{V_t} \qquad w_t^3 = \frac{Q_t}{V_t}$$

with

$$w_t^1 + w_t^2 + w_t^3 = 1$$

Deriving the Black-Scholes PDE

and rearranging the terms we get:

$$\frac{\mathrm{d}V_t}{V_t} = \left(w_t^1\mu + w_t^2\mu_C + w_t^3 r\right)\mathrm{d}t + \left(w_t^1\sigma + w_t^2\sigma_C\right)\mathrm{d}B_t$$

Set up the weights to obtain a risk-free portfolio:

$$w_t^1\sigma + w_t^2\sigma_C = 0$$

What should be the return of this portfolio ?

Deriving the Black-Scholes PDE

We must have

$$w_t^1 \mu + w_t^2 \mu_C + w_t^3 r = r$$

This condition leads to the equality

$$\frac{\mu - r}{\sigma} = \frac{\mu_C - r}{\sigma_C}$$

The excess return per unit of risk is independent of the asset.

Black-Scholes PDE

Replacing $C\mu_C$ and $C\sigma_C$ by their expressions we get the Black-Scholes PDE:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0$$

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Black-Scholes pricing formulas

Black-Scholes pricing formulas

For a call option:

$$C(t, S_t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

For a put option:

$$P(t, S_t) = K e^{-r(T-t)} N(-d_2) - S_t N(-d_1)$$

Black-Scholes pricing formulas

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\left(T - t\right)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

and N is the cumulative distribution function of a standardized normal distribution:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}}\, \mathrm{d}x$$

Properties of the Black-Scholes formula

- What happens when S_t becomes very large ?
- What happens when σ approaches 0 ?