- 8 -

Limitations of the Black-Scholes model

- 8.1 -

Assumptions of the Black-Scholes model

Assumptions of the Black-Scholes model

- Options are of the European type.
  $\Rightarrow$ Use of other formulas or numerical procedures.
- No dividends are payable before the option expiry date.
  $\Rightarrow$ Simple extensions of the Black-Scholes model exist. It is possible to adapt trees to dividend payments.
- It is possible to continuously rebalance a portfolio at no cost and to borrow at the risk-free rate.
  $\Rightarrow$ Practical solutions for management of options.

Assumptions of the Black-Scholes model

- The underlying asset's volatility is constant across all strikes.
  $\Rightarrow$ Existence of a smile or skew. Alternative models for the underlying asset.
- The underlying asset's volatility is constant over the life of the option.
  $\Rightarrow$ Time-varying volatility models.

Assumptions of the Black-Scholes model

- The risk-free rate of interest is constant over the life of the option.
  $\Rightarrow$ Models of the short term rate or the forward rate for interest rates derivatives.
- The underlying asset price follows a geometric brownian motion.
  $\Rightarrow$ Models with jumps.

- 8.2 -

Evidence of strike dependent and time varying volatilities

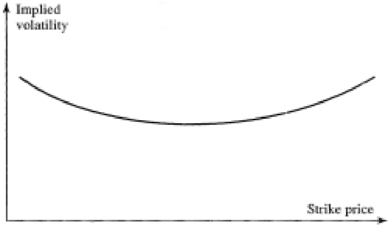
Implied volatility

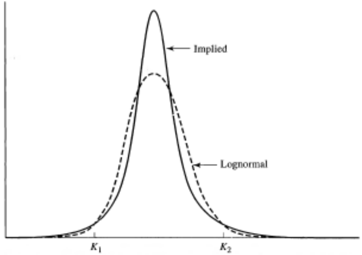
$$C(S, K, r, T, \sigma) = SN(d_1) - Ke^{-rT}N(d_2)$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad d_2 = d_1 - \sigma\sqrt{T}$$

- Values of the variables in the Black-Scholes formula are all known except for the volatility σ.
- The value $\hat{\sigma}$ such that $C(S, K, r, T, \hat{\sigma})$ is equal to the traded market price is the Black-Scholes implied volatility.

Smile

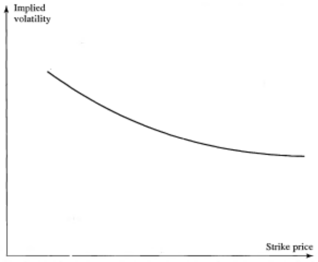


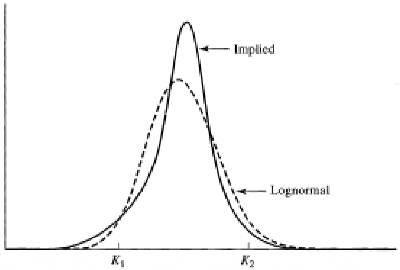


(Hull, "Options, Futures and Other Derivatives")

- With a constant volatility, ATM options are overpriced and ITM and OTM options are underpriced: sign of "fat-tails".

Skew





(Hull, "Options, Futures and Other Derivatives")

The “crash-o-phobia” explanation of the smile

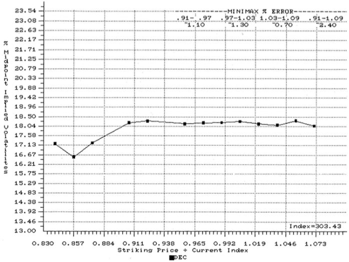


Figure 1. Typical precrash smile. Implied combined volatilities of S&P 500 index options (July 1, 1987; 9:00 A.M.).

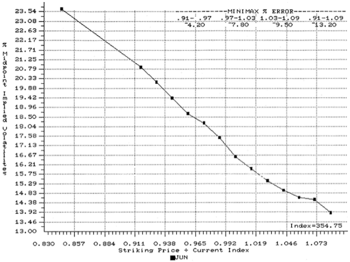
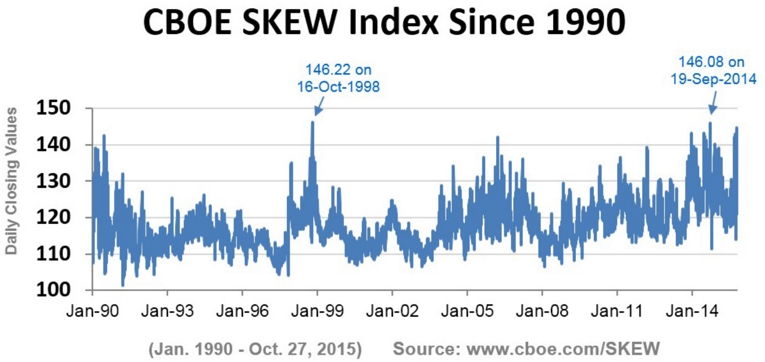


Figure 2. Typical postcrash smile. Implied combined volatilities of S&P 500 index options (January 2, 1990; 10:00 A.M.).

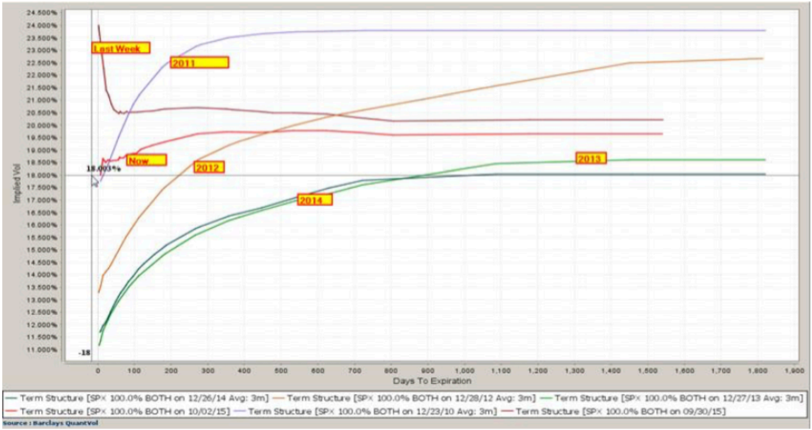
(Rubinstein (1994), Journal of Finance, 49)

CBOE SKEW Index

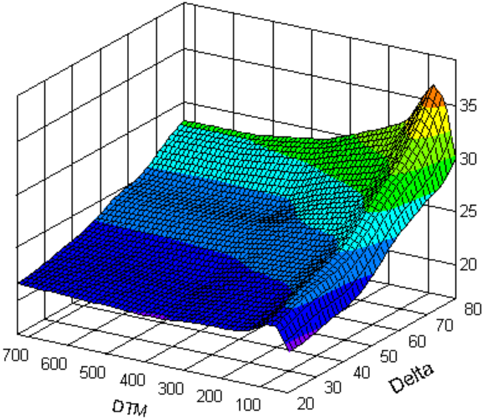


(Chicago Board Options Exchange)

Term structure of the volatility



The volatility surface



- 8.3 -

Models that generates a smile

The Geometric Brownian Motion model

Under the risk-neutral probability $\mathbb{Q}$, the price follow the Stochastic Differential Equation (SDE):

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t$$

or equivalently, the price is a Geometric Brownian Motion:

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma B_t}$$

CEV model (Cox, 1975)

Description

We now look at models that generalise the SDE to

$$dS_t = rS_t\, dt + \alpha S_t^{\beta}\, dB_t \quad 1/2 \leqslant \beta < 1$$

If we set $\nu_t = \alpha S_t^{\beta}$, it holds that

$$\frac{d\nu_t}{dS_t}\frac{S_t}{\nu_t} = \beta$$

This is the Constant Elasticity of Variance property.

CEV model (Cox, 1975)

Properties

- The randomness of the volatility derives from the randomness of the asset.
- There exist formulas for options prices which involve series expansions but can be computed.
- Using the two parameters α and β, it is possible to have a better fit of the observed volatility skew.

Local volatility models

Description

Dupire (1993, 1994), Derman and Kani (1994).

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma(S_t, t)S_t\,\mathrm{d}B_t$$

where the function $\sigma\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ which is such that the model's prices coincide with the market's prices is called the local volatility function.

Local volatility models

Properties

- It is possible to derive from the PDE that the prices of european options follow.
- There is a one-to-one correspondence between the volatility surface and the local volatility function.
- Practically, the calibration of the local volatility to the market prices is a complex task.
- The assumption that the proper dynamics of the underlying asset is captured by the local volatility is questionned.

Stochastic volatility models

Description

The general formulation:

$$\begin{cases} \mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma_t S_t\,\mathrm{d}B_t \\ \mathrm{d}\sigma_t = a(\sigma_t, t)\,\mathrm{d}t + b(\sigma_t, t)\,\mathrm{d}B'_t \end{cases}$$

- Hull and White (1987): B_t and B'_t are independent.

Stochastic volatility models

Some other specifications

- Heston (1993): B_t and B'_t have a constant correlation coefficient ρ and

$$\begin{cases} \mathrm{d}S_t = rS_t\,\mathrm{d}t + \sqrt{\sigma_t}S_t\,\mathrm{d}B_t \\ \mathrm{d}\sigma_t = \hat{\kappa}(\hat{\sigma}_t - \sigma_t)\,\mathrm{d}t + \eta\sqrt{\sigma_t}\,\mathrm{d}B'_t \end{cases}$$

- Hagan et al. (2002, the SABR model): B_t and B'_t have a constant correlation coefficient ρ and

$$\begin{cases} \mathrm{d}F_t = \hat{\alpha}_t F_t^{\beta}\,\mathrm{d}B_t \\ \mathrm{d}\hat{\alpha}_t = \nu\hat{\alpha}_t\,\mathrm{d}B'_t \end{cases}$$

where F_t is the forward price of the asset.

Stochastic volatility models

Properties

- Generic stochastic volatility models generate endogeneously a smile.
- Stochastic volatility models are incomplete.
- Advantages of the SABR model:
  - an approximating formula for the equivalent Black-Scholes volatility has been derived,
  - it is easier to calibrate to market data,
  - the predicted dynamic behavior of the smile is consistent with the observed behavior,
  - traders can easily interpret the parameters of the model and the greeks (vanna and volga) that derive from these parameters.

- 8.4 -

Models for interest rates derivatives

Short-Term Rates models

- Vasicek (1977)

$$\mathrm{d}r_t = (a - br_t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t$$

- Cox-Ingersoll-Ross (CIR, 1985)

$$\mathrm{d}r_t = (a - br_t)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}B_t$$

- Hull and White (1990)

$$\mathrm{d}r_t = (a(t) - b(t)r_t)\,\mathrm{d}t + \sigma(t)r_t^{\beta}\,\mathrm{d}B_t \quad 0 \leqslant \beta \leqslant 1$$

Other models

- Models of Instantaneous Forward Rates:
  Heath-Jarrow-Morton (HJM, 1990, 1992).
- Market LIBOR Models: Brace-Gatarek-Musiela (BGM, 1997)

- 8.5 -

Models with jumps

Models with jumps

- The Geometric Brownian Motion model predicts that large price changes are much less likely than is actually the case.
- In Jump-diffusion models, discontinuity are added through a Poisson component.

Poisson distribution

Poisson distribution of parameter (intensity) λ

$$\mathbb{P}(n) = e^{-\lambda}\frac{\lambda^n}{n!}$$

Probability of having n arrivals in a given time interval, knowing that the average number of arrivals during this time interval is λ and that arrivals occur independently of the length of time since the previous event.

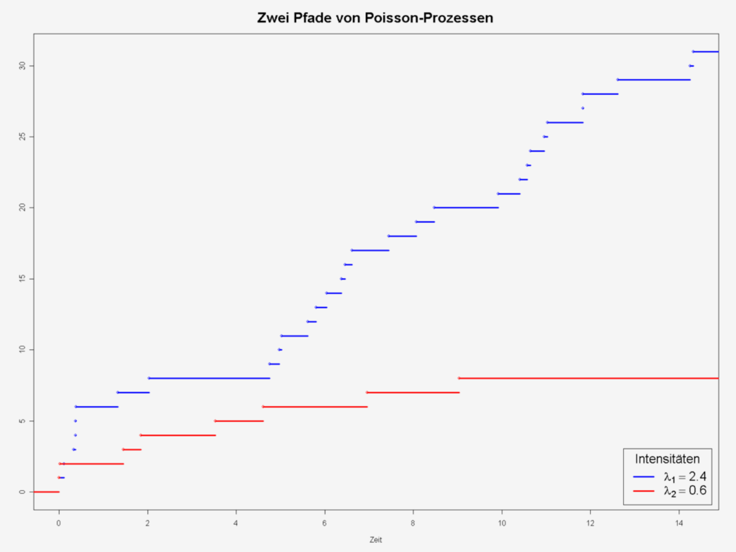
Poisson processes

N is a Poisson process of parameter (intensity) λ if

$$\mathbb{P}[N(t+h) - N(t) = n] = e^{-\lambda h} \frac{(\lambda h)^n}{n!}$$

Probability of having n events in the time interval h.

Poisson processes



Merton's model (1976)

- The stock price S follows a Geometric Brownian Motion but Poisson distributed random events can happen.
- λ is the mean number of arrivals per unit of time.
- If an event happens, then the stock price jumps from S to SY
- where the process Y is i.i.d. and $k = \mathbf{E}[Y - 1]$.

Merton's model (1976)

The stock price follows the SDE:

$$\frac{dS_t}{S_t} = \begin{cases} (\alpha - \lambda k)\,dt + \sigma\,dB_t & \text{no event} \\ (\alpha - \lambda k)\,dt + \sigma\,dB_t + (Y-1) & \text{event during } dt \end{cases}$$

The stock price is given by:

$$S_t = Y_n S_0 e^{(\alpha - \lambda k - \sigma^2/2)t + \sigma\,dB_t}$$

where $Y_n = \prod_{j=1}^{n} Y_j$ and n is Poisson distributed with parameter λt.

Lévy processes