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Brownian motion: a model for  
stock prices behaviour

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## Introduction

# A model for stock prices behaviour

- ▶ We start with the simplest possible model : a one period binomial model. We introduce the main notions of mathematical finance in this setup.
- ▶ Then we extend the binomial model to  $n$  periods.
- ▶ Finally we move to the continuous time setting and detail the arithmetic Brownian Motion and the geometric BM.

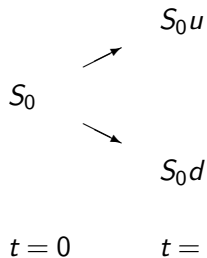
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## A one period binomial model

# Construction of the one period binomial model

A risk-free asset and a risky asset.

$$1 \rightarrow R = (1 + r)$$



## A probabilistic model

- ▶ **States** of the world:  $\Omega = \{\omega_u, \omega_d\}$ .
- ▶ **Filtration**:  $\mathcal{F}_0 = \{\Omega, \emptyset\}$ ,  $\mathcal{F}_1 = \{\Omega, \emptyset, \{\omega_u\}, \{\omega_d\}\}$ .  $\mathcal{F}_0 \subset \mathcal{F}_1$ .
- ▶ A filtration generated by the observation of a random variable:  $\mathcal{F}_1 = \sigma(S_1)$ .
- ▶ The **probability**  $\mathbb{P}$  in the “real” world:

$$\mathbb{P}(\omega_u) = p$$

$$\mathbb{P}(\omega_d) = 1 - p$$

- ▶ A **probability space**:

$$(\Omega, \mathcal{F}, \mathbb{P})$$

## Self-financing portfolio strategy

- ▶ A **self-financing portfolio strategy** is given by the couple  $(x, \Delta)$ .
- ▶ Let  $V_t(x, \Delta)$  be the value at time  $t$  of the portfolio corresponding to the strategy  $(x, \Delta)$ .
- ▶ Then

$$V_0(x, \Delta) = \Delta S_0 + (x - \Delta S_0)1 = x$$

$$V_1(x, \Delta) = \Delta S_1 + (x - \Delta S_0)R$$

- ▶ **Self-financed.**

## Replicating the payoff of a derivative

A **self-financing portfolio strategy**  $(x, \Delta)$  that replicates the payoff of a derivative  $\phi(S_1)$  with  $\phi_u = \phi(uS_0)$  and  $\phi_d = \phi(dS_0)$  is given by:



## Replicating the payoff of a derivative

A **self-financing portfolio strategy**  $(x, \Delta)$  that replicates the payoff of a derivative  $\phi(S_1)$  with  $\phi_u = \phi(uS_0)$  and  $\phi_d = \phi(dS_0)$  is given by:

$$\Delta = \frac{\phi_u - \phi_d}{(u - d)S_0} \quad x = \frac{1}{R} \left( \frac{R - d}{u - d} \phi_u + \frac{u - R}{u - d} \phi_d \right)$$

A market is **complete** if any contingent claim can be replicated by a self-financing strategy.

## No arbitrage condition

An **arbitrage** is a strategy  $(0, \Delta)$  such that

$$V_1(0, \Delta) \geq 0 \text{ and } \mathbb{P}(V_1(0, \Delta) > 0) > 0$$

**No arbitrage** condition:

$$\forall (0, \Delta), \quad V_1(0, \Delta) \geq 0 \Rightarrow V_1(0, \Delta) = 0 \quad \mathbb{P} - \text{a.s.}$$

### Proposition

There exist no arbitrage if and only if  $d < R < u$ .

## Martingales and risk-neutral probability

A random variable  $X$  is a **martingale** under a probability  $\mathbb{Q}$  if

$$X_0 = \mathbf{E}_{\mathbb{Q}}[X_1]$$

A **risk-neutral probability** is any probability  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  that makes the discounted value of any self-financing strategy a martingale:

$$\tilde{V}_0(x, \Delta) = \mathbf{E}_{\mathbb{Q}}[\tilde{V}_1(x, \Delta)]$$

## Existence of a risk-neutral probability

We've obtained that

$$V_0(x, \Delta) = \frac{1}{R} \left( \frac{R-d}{u-d} V_1(x, \Delta)(\omega_u) + \frac{u-R}{u-d} V_1(x, \Delta)(\omega_d) \right)$$

If  $d < R < u$ , then

$$\mathbb{Q}(\omega_u) = \frac{R-d}{u-d} \qquad \mathbb{Q}(\omega_d) = \frac{u-R}{u-d}$$

is a probability equivalent to  $\mathbb{P}$  such that  $\tilde{V}(x, \Delta)$  is a martingale.

## Pricing a derivative

Then no-arbitrage implies the existence of a risk-neutral probability  $\mathbb{Q}$ . As any derivative is replicable by a self-financing strategy whose discounted value is a martingale under  $\mathbb{Q}$  we have that

### Proposition

In the absence of arbitrage,

$$\begin{aligned}\phi(S_0) &= \frac{1}{R} \mathbf{E}_{\mathbb{Q}}[\phi(S_1)] \\ \phi(S_0) &= \frac{1}{R} \left( \frac{R-d}{u-d} \phi_u + \frac{u-R}{u-d} \phi_d \right)\end{aligned}$$

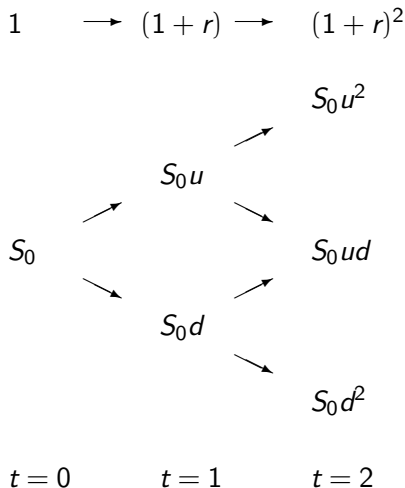
## Exercises

1. Show that the discounted price of the risky asset is a martingale under  $\mathbb{Q}$ .
2. Prove that the existence of a risk-neutral probability implies no-arbitrage.
3. Prove that there exist no arbitrage if and only if  $d < R < u$
4. Prove that if the market is complete then the risk-neutral probability is unique.

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## A two periods binomial model

## Graphical representation





# The probabilistic model

- ▶ States space:  $\Omega = \{(\omega_1, \omega_2) : \omega_i = \omega_i^d \text{ or } \omega_i = \omega_i^u\}$
- ▶ Filtration generated by a random variable:  $\mathcal{F}_0 = \sigma(S_0)$ ,  
 $\mathcal{F}_1 = \sigma(S_0, S_1)$ ,  $\mathcal{F}_2 = \sigma(S_0, S_1, S_2)$ .
- ▶ Historical probability  $\mathbb{P}$ :  $\mathbb{P}(\omega_i = \omega_i^u) = p$  and  
 $\mathbb{P}(\omega_i = \omega_i^d) = 1 - p$

## Self-financing portfolio strategy

A self-financing portfolio strategy:  $(V_0 = x, \Delta_0, \Delta_1)$ .

$$V_0 = \Delta_0 S_0 + (V_0 - \Delta_0 S_0)1$$

$$V_1 = \Delta_0 S_1 + (V_0 - \Delta_0 S_0)(1 + r)$$

$$V_1 = \Delta_1 S_1 + (V_1 - \Delta_1 S_1)1$$

$$V_2 = \Delta_1 S_2 + (V_1 - \Delta_1 S_1)(1 + r)$$

The **self-financing condition** is equivalent to

$$\tilde{V}_1 - \tilde{V}_0 = \Delta_0(\tilde{S}_1 - \tilde{S}_0)$$

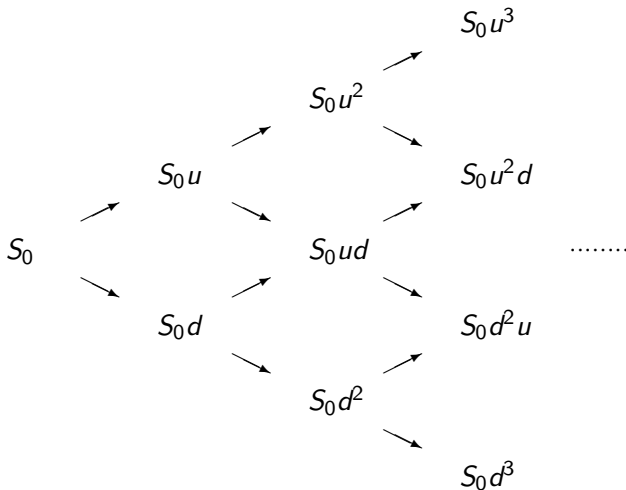
$$\tilde{V}_2 - \tilde{V}_1 = \Delta_1(\tilde{S}_2 - \tilde{S}_1)$$

$$\tilde{V}_2 = \tilde{V}_0 + \Delta_0(\tilde{S}_1 - \tilde{S}_0) + \Delta_1(\tilde{S}_2 - \tilde{S}_1)$$

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## A n periods binomial model

## Graphical representation



## Self-financing portfolio strategy

A self-financing portfolio strategy:  $(V_0 = x, \Delta_0, \dots, \Delta_{n-1})$ .

The **self-financing condition** is equivalent to

$$\tilde{V}_t - \tilde{V}_{t-1} = \Delta_{t-1}(\tilde{S}_t - \tilde{S}_{t-1})$$

$$\tilde{V}_t = \tilde{V}_0 + \sum_{k=0}^{t-1} \Delta_k(\tilde{S}_{k+1} - \tilde{S}_k)$$

## No arbitrage and the risk-neutral probability

No arbitrage is equivalent to the existence of a risk-neutral probability and is equivalent to  $d < 1 + r < u$ .

The risk-neutral probability is given by

$$\mathbb{Q}(\omega_i^u) = q^{\#\{i, \omega_i = \omega_i^u\}} \cdot (1 - q)^{\#\{i, \omega_i = \omega_i^d\}}$$
$$\text{with } q = \frac{(1 + r) - d}{u - d}$$

## Returns in the binomial model

Returns are given by  $Y_t = \frac{S_t}{S_{t-1}}$ .

- ▶  $Y_t$  is a random variable that takes value  $u$  or  $d$  with probabilities  $\mathbb{P}(Y_t = u) = p$  and  $\mathbb{P}(Y_t = d) = 1 - p$ .
- ▶ The  $Y_t$  are **independent**.

The price can be written as  $S_t = S_0 \prod_{k=1}^t Y_k$ .

Hence we have a **random walk**:

$$\ln \left( \frac{S_t}{S_0} \right) = \sum_{k=1}^t Z_k$$

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## Arithmetic Brownian Motion



# Introduction

- ▶ We want a **continuous time** model for stock prices behaviour.
- ▶ The model will be built around the **Brownian Motion**.
- ▶ It can be proved that it is the continuous time limit of a properly specified random walk or of a properly specified binomial model.
- ▶ History: Brown (1828), Einstein (1905), Wiener (1923).
- ▶ Applications to finance: Bachelier (1900), Samuelson (1965), Black & Scholes (1973), Merton (1973), ...

## Definition of the Arithmetic Brownian Motion

On  $(\Omega, \mathcal{F}, \mathbb{P})$ , a Brownian Motion starting at 0 is a continuous stochastic process with independent, stationary and Gaussian increments. That is  $(B_t)_{t \geq 0}$  is a BM if

1.  $\mathbb{P}(B_0 = 0) = 1$ .
2. For all  $0 \leq s \leq t$ ,  $B_t - B_s \sim \mathcal{N}(0, \sqrt{t-s})$ .
3. For all  $0 < t_1 < t_2 < t_3 < t_4$ , the random variables  $B_{t_4} - B_{t_3}$  and  $B_{t_2} - B_{t_1}$  are independent.

# Properties of the Arithmetic Brownian Motion

The change  $\Delta B_t = B_{t+\Delta t} - B_t$  during a small period of time  $\Delta t$  is

$$\Delta B_t = \epsilon \sqrt{\Delta t}$$

where  $\epsilon \sim \mathcal{N}(0, 1)$ . Hence

$$\Delta B_t \sim \mathcal{N}(0, \sqrt{\Delta t})$$

The BM is obtained by taking the limit  $\Delta t \rightarrow 0$ .

# Properties of the Arithmetic Brownian Motion

Almost every **sample paths**  $t \rightarrow B_t(\omega)$  of a BM are:

- ▶ continuous,
- ▶ not differentiable at any point,
- ▶ not monotone in any interval,
- ▶ of infinite expected length in any interval ! (fractal property)

# Properties of the Arithmetic Brownian Motion

The independence of the increments implies that a BM is **markovian**, that is:

for all  $0 \leq s \leq t$ , the conditional distribution of  $B_t$  given  $\mathcal{F}_s$  is the same as the conditional distribution of  $B_t$  given  $B_s$ :

$$\mathbb{P}(B_t \leq y \mid \mathcal{F}_s) = \mathbb{P}(B_t \leq y \mid B_s)$$

The BM is a **martingale**:

for all  $0 \leq s \leq t$ ,

$$\mathbf{E}_{\mathbb{P}}[B_t \mid \mathcal{F}_s] = B_s$$

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## Geometric Brownian Motion

# Using BM to model stock prices

- Bachelier's model:

$$S_t = S_0 + \mu t + \sigma B_t$$

can lead to negative prices.

- Samuelson's model: the Geometric Brownian Motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

## The deterministic part of the GBM

Consider the deterministic model:

$$dS_t = \mu S_t dt$$

The price is then given by



## The deterministic part of the GBM

Consider the deterministic model:

$$dS_t = \mu S_t dt$$

The price is then given by

$$S_t = S_0 e^{\mu t}$$

The price compounds at the continuous rate  $\mu$ .

## Adding noise

Starting from the diffusion equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

we can write

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

But  $S_t$  is a stochastic process hence

$$\frac{dS_t}{S_t} \neq d(\ln S_t)!$$

We need a specific tool for stochastic calculus.

## Itô's Formula

When  $X_t$  is an Itô process<sup>1</sup>

$$dX_t = \mu(X_t, t) dt + \sigma(X_t, t) dB_t$$

and  $Y_t = f(X_t)$ , then

$$dY_t = \left[ f'(X_t)\mu(X_t, t) + \frac{1}{2}f''(X_t)\sigma^2(X_t, t) \right] dt + f'(X_t)\sigma(X_t, t) dB_t$$

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<sup>1</sup>Kiyoshi Itô (1915-2008)

## A more general Itô's Formula

With  $Y_t = f(t, X_t)$ , then

$$\begin{aligned} dY_t = & \frac{\partial f}{\partial X}(t, X_t) dX_t \\ & + \left( \frac{\partial f}{\partial X}(t, X_t) \mu(X_t, t) + \frac{1}{2} \frac{\partial^2 f}{\partial X^2}(t, X_t) \sigma^2(X_t, t) \right) dt \\ & + \frac{\partial f}{\partial X}(t, X_t) \sigma(X_t, t) dB_t \end{aligned}$$

## Application of Itô's Formula to the GBM diffusion

$$f(S_t) = \ln(S_t) \quad \mu(S_t, t) = \mu S_t \quad \sigma(S_t, t) = \sigma S_t$$

$$f'(S_t) = \frac{1}{S_t} \quad f''(S_t) = -\frac{1}{S_t^2}$$

$$d(\ln(S_t)) = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dB_t$$

# The stock price dynamics

$$S_t = S_0 \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma B_t \right)$$

- ▶ We have the exponential of an arithmetic MB.
- ▶ The drift is adjusted by a **quadratic term**  $-\frac{1}{2}\sigma^2 t$ . It is a consequence of the rules of stochastic calculus.  
**BE CAREFUL NOT TO FORGET THIS TERM !**
- ▶ The resulting dynamics is called **log-normal**.

## Distribution of returns

$$S_{t+\Delta t} = S_t \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma (B_{t+\Delta t} - B_t) \right)$$

$$\ln \left( \frac{S_{t+\Delta t}}{S_t} \right) = \ln \left( \frac{S + \Delta S}{S} \right) = \ln \left( 1 + \frac{\Delta S}{S} \right) \approx \frac{\Delta S}{S}$$

$$r = \ln \left( \frac{S_{t+\Delta t}}{S_t} \right) = \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma (B_{t+\Delta t} - B_t)$$

$$r \sim \mathcal{N} \left( \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t, \sigma \sqrt{\Delta t} \right)$$

## Exercises

1. What is the price dynamics for the diffusion given by

$$dS_t = \mu S_t dt - \sigma S_t dB_t?$$

2. Let

$$Z_t = \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma B_t$$

and  $S_t = e^{Z_t}$ . What is  $dS_t$  ?