- 2 -

Brownian motion: a model for stock prices behaviour

- 2.1 -

Introduction

A model for stock prices behaviour

- We start with the simplest possible model : a one period binomial model. We introduce the main notions of mathematical finance in this setup.
- Then we extend the binomial model to n periods.
- Finally we move to the continuous time setting and detail the arithmetic Brownian Motion and the geometric BM.

- 2.2 -

A one period binomial model

Construction of the one period binomial model

A risk-free asset and a risky asset.

$$1 \longrightarrow R = (1 + r)$$

$$\begin{array}{ccc} & & S_0 u \\ & \nearrow & \\ S_0 & & \\ & \searrow & \\ & & S_0 d \end{array}$$

$$t = 0 \qquad t = 1$$

A probabilistic model

- States of the world: $\Omega = \{\omega_u, \omega_d\}$.
- Filtration: $\mathcal{F}_0 = \{\Omega, \emptyset\}$, $\mathcal{F}_1 = \{\Omega, \emptyset, \{\omega_u\}, \{\omega_d\}\}$. $\mathcal{F}_0 \subset \mathcal{F}_1$.
- A filtration generated by the observation of an random variable: $\mathcal{F}_1 = \sigma(S_1)$.
- The probability $\mathbb{P}$ in the "real" world:

$$\mathbb{P}(\omega_u) = p \qquad \mathbb{P}(\omega_d) = 1 - p$$

- A probability space:

$$(\Omega, \mathcal{F}, \mathbb{P})$$

Self-financing portfolio strategy

- A self-financing portfolio strategy is given by the couple (x, Δ).
- Let $V_t(x, \Delta)$ be the value at time t of the portfolio corresponding to the strategy (x, Δ).
- Then

$$V_0(x, \Delta) = \Delta S_0 + (x - \Delta S_0)1 = x$$
$$V_1(x, \Delta) = \Delta S_1 + (x - \Delta S_0)R$$

- Self-financed.

Replicating the payoff of a derivative

A self-financing portfolio strategy (x, Δ) that replicates the payoff of a derivative $\phi(S_1)$ with $\phi_u = \phi(uS_0)$ and $\phi_d = \phi(dS_0)$ is given by:

Replicating the payoff of a derivative

A self-financing portfolio strategy (x, Δ) that replicates the payoff of a derivative $\phi(S_1)$ with $\phi_u = \phi(uS_0)$ and $\phi_d = \phi(dS_0)$ is given by:

$$\Delta = \frac{\phi_u - \phi_d}{(u-d)S_0} \qquad x = \frac{1}{R}\left(\frac{R-d}{u-d}\phi_u + \frac{u-R}{u-d}\phi_d\right)$$

A market is complete if any contingent claim can be replicated by a self-financing strategy.

No arbitrage condition

An arbitrage is a strategy $(0, \Delta)$ such that

$$V_1(0, \Delta) \geqslant 0 \text{ and } \mathbb{P}(V_1(0, \Delta) > 0) > 0$$

No arbitrage condition:

$$\forall (0, \Delta), \quad V_1(0, \Delta) \geqslant 0 \Rightarrow V_1(0, \Delta) = 0 \quad \mathbb{P} - \text{a.s.}$$

Proposition

There exist no arbitrage if and only if $d < R < u$.

Martingales and risk-neutral probability

A random variable X is a martingale under a probability $\mathbb{Q}$ if

$$X_0 = \mathbf{E}_{\mathbb{Q}}[X_1]$$

A risk-neutral probability is any probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ that makes the discounted value of any self-financing strategy a martingale:

$$\widetilde{V}_0(x, \Delta) = \mathbf{E}_{\mathbb{Q}}[\widetilde{V}_1(x, \Delta)]$$

Existence of a risk-neutral probability

We've obtained that

$$V_0(x,\Delta) = \frac{1}{R}\left(\frac{R-d}{u-d}V_1(x,\Delta)(\omega_u) + \frac{u-R}{u-d}V_1(x,\Delta)(\omega_d)\right)$$

If $d < R < u$, then

$$\mathbb{Q}(\omega_u) = \frac{R-d}{u-d} \qquad \mathbb{Q}(\omega_d) = \frac{u-R}{u-d}$$

is a probability equivalent to $\mathbb{P}$ such that $\widetilde{V}(x,\Delta)$ is a martingale.

Pricing a derivative

Then no-arbitrage implies the existence of a risk-neutral probability $\mathbb{Q}$. As any derivative is replicable by a self-financing strategy whose discounted value is a martingale under $\mathbb{Q}$ we have that

Proposition

In the absence of arbitrage,

$$\phi(S_0) = \frac{1}{R}\mathbf{E}_{\mathbb{Q}}[\phi(S_1)]$$

$$\phi(S_0) = \frac{1}{R}\left(\frac{R-d}{u-d}\phi_u + \frac{u-R}{u-d}\phi_d\right)$$

Exercises

1. Show that the discounted price of the risky asset is a martingale under $\mathbb{Q}$.
2. Prove that the existence of a risk-neutral probability implies no-arbitrage.
3. Prove that there exist no arbitrage if and only if $d < R < u$
4. Prove that if the market is complete then the risk-neutral probability is unique.

- 2.3 -

A two periods binomial model

Graphical representation

$$
\begin{array}{ccccc}
1 & \longrightarrow & (1+r) & \longrightarrow & (1+r)^2 \\
 & & & & S_0 u^2 \\
 & & & \nearrow & \\
 & & S_0 u & & \\
 & \nearrow & & \searrow & \\
S_0 & & & & S_0 ud \\
 & \searrow & & \nearrow & \\
 & & S_0 d & & \\
 & & & \searrow & \\
 & & & & S_0 d^2 \\
t=0 & & t=1 & & t=2
\end{array}
$$

The probabilistic model

- States space: $\Omega = \{(\omega_1, \omega_2) : \omega_i = \omega_i^d \text{ or } \omega_i = \omega_i^u\}$
- Filtration generated by a random variable: $\mathcal{F}_0 = \sigma(S_0)$, $\mathcal{F}_1 = \sigma(S_0, S_1)$, $\mathcal{F}_2 = \sigma(S_0, S_1, S_2)$.
- Historical probability $\mathbb{P}$: $\mathbb{P}(\omega_i = \omega_i^u) = p$ and $\mathbb{P}(\omega_i = \omega_i^d) = 1 - p$

Self-financing portfolio strategy

A self-financing portfolio strategy: $(V_0 = x, \Delta_0, \Delta_1)$.

$$
\begin{aligned}
V_0 &= \Delta_0 S_0 + (V_0 - \Delta_0 S_0)1 \\
V_1 &= \Delta_0 S_1 + (V_0 - \Delta_0 S_0)(1+r) \\
V_1 &= \Delta_1 S_1 + (V_1 - \Delta_1 S_1)1 \\
V_2 &= \Delta_1 S_2 + (V_1 - \Delta_1 S_1)(1+r)
\end{aligned}
$$

The self-financing condition is equivalent to

$$
\widetilde{V}_1 - \widetilde{V}_0 = \Delta_0(\widetilde{S}_1 - \widetilde{S}_0)
$$

$$
\widetilde{V}_2 - \widetilde{V}_1 = \Delta_1(\widetilde{S}_2 - \widetilde{S}_1)
$$

$$
\widetilde{V}_2 = \widetilde{V}_0 + \Delta_0(\widetilde{S}_1 - \widetilde{S}_0) + \Delta_1(\widetilde{S}_2 - \widetilde{S}_1)
$$

- 2.4 -

A n periods binomial model

Graphical representation

$$
\begin{tikzcd}
 & & & S_0 u^3 \\
 & & S_0 u^2 \arrow[ur] \arrow[dr] & \\
 & S_0 u \arrow[ur] \arrow[dr] & & S_0 u^2 d \\
S_0 \arrow[ur] \arrow[dr] & & S_0 ud \arrow[ur] \arrow[dr] & & \cdots\cdots \\
 & S_0 d \arrow[ur] \arrow[dr] & & S_0 d^2 u \\
 & & S_0 d^2 \arrow[ur] \arrow[dr] & \\
 & & & S_0 d^3
\end{tikzcd}
$$

Self-financing portfolio strategy

A self-financing portfolio strategy: $(V_0 = x, \Delta_0, \ldots, \Delta_{n-1})$.
The self-financing condition is equivalent to

$$\widetilde{V}_t - \widetilde{V}_{t-1} = \Delta_{t-1}(\widetilde{S}_t - \widetilde{S}_{t-1})$$

$$\widetilde{V}_t = \widetilde{V}_0 + \sum_{k=0}^{t-1} \Delta_k(\widetilde{S}_{k+1} - \widetilde{S}_k)$$

No arbitrage and the risk-neutral probability

No arbitrage is equivalent to the existence of a risk-neutral probability and is equivalent to $d < 1 + r < u$.
The risk-neutral probability is given by

$$\mathbb{Q}(\omega_i^u) = q^{\#\{i, \omega_i = \omega_i^u\}} \cdot (1-q)^{\#\{i, \omega_i = \omega_i^d\}}$$

$$\text{with } q = \frac{(1+r)-d}{u-d}$$

Returns in the binomial model

Returns are given by $Y_t = \dfrac{S_t}{S_{t-1}}$.

- Y_t is a random variable that takes value u or d with probabilities $\mathbb{P}(Y_t = u) = p$ and $\mathbb{P}(Y_t = d) = 1 - p$.
- The Y_t are independent.

The price can be written as $S_t = S_0 \prod_{k=1}^{t} Y_k$.

Hence we have a random walk:

$$\ln\left(\frac{S_t}{S_0}\right) = \sum_{k=1}^{t} Z_k$$

- 2.5 -

Arithmetic Brownian Motion

Introduction

- We want a continuous time model for stock prices behaviour.
- The model will be built around the Brownian Motion.
- It can be proved that it is the continuous time limit of a properly specified random walk or of a properly specified binomial model.
- History: Brown (1828), Einstein (1905), Wiener (1923).
- Applications to finance: Bachelier (1900), Samuelson (1965), Black & Scholes (1973), Merton (1973), ...

Definition of the Arithmetic Brownian Motion

On $(\Omega, \mathcal{F}, \mathbb{P})$, a Brownian Motion starting at 0 is a continuous stochastic process with independent, stationary and Gaussian increments. That is $(B_t)_{t \geqslant 0}$ is a BM if

1. $\mathbb{P}(B_0 = 0) = 1$.
2. For all $0 \leqslant s \leqslant t$, $B_t - B_s \sim \mathcal{N}(0, \sqrt{t-s})$.
3. For all $0 < t_1 < t_2 < t_3 < t_4$, the random variables $B_{t_4} - B_{t_3}$ and $B_{t_2} - B_{t_1}$ are independent.

Properties of the Arithmetic Brownian Motion

The change $\Delta B_t = B_{t+\Delta t} - B_t$ during a small period of time Δt is

$$\Delta B_t = \epsilon\sqrt{\Delta t}$$

where $\epsilon \sim \mathcal{N}(0, 1)$. Hence

$$\Delta B_t \sim \mathcal{N}(0, \sqrt{\Delta t})$$

The BM is obtained by taking the limit $\Delta t \to 0$.

Properties of the Arithmetic Brownian Motion

Almost every sample paths $t \to B_t(\omega)$ of a BM are:

- continuous,
- not differentiable at any point,
- not monotone in any interval,
- of infinite expected length in any interval ! (fractal property)

Properties of the Arithmetic Brownian Motion

The independence of the increments implies that a BM is markovian, that is:
for all $0 \leqslant s \leqslant t$, the conditional distribution of B_t given $\mathcal{F}_s$ is the same as the conditional distribution of B_t given B_s:

$$\mathbb{P}(B_t \leqslant y \mid \mathcal{F}_s) = \mathbb{P}(B_t \leqslant y \mid B_s)$$

The BM is a martingale:
for all $0 \leqslant s \leqslant t$,

$$\mathbf{E}_{\mathbb{P}}[B_t \mid \mathcal{F}_s] = B_s$$

- 2.6 -

Geometric Brownian Motion

Using BM to model stock prices

- Bachelier's model:

$$S_t = S_0 + \mu t + \sigma B_t$$

  can lead to negative prices.
- Samuelson's model: the Geometric Brownian Motion

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}B_t$$

The deterministic part of the GBM

Consider the deterministic model:

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t$$

The price is then given by

The deterministic part of the GBM

Consider the deterministic model:

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t$$

The price is then given by

$$S_t = S_0 e^{\mu t}$$

The price compounds at the continuous rate μ.

Adding noise

Starting from the diffusion equation

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}B_t$$

we can write

$$\frac{\mathrm{d}S_t}{S_t} = \mu \,\mathrm{d}t + \sigma \,\mathrm{d}B_t$$

But S_t is a stochastic process hence

$$\frac{\mathrm{d}S_t}{S_t} \neq \mathrm{d}(\ln S_t)!$$

We need a specific tool for stochastic calculus.

Itō's Formula

When X_t is an Itō process[1]

$$dX_t = \mu(X_t, t)\,dt + \sigma(X_t, t)\,dB_t$$

and $Y_t = f(X_t)$, then

$$dY_t = \left[f'(X_t)\mu(X_t, t) + \frac{1}{2} f''(X_t)\sigma^2(X_t, t) \right] dt + f'(X_t)\sigma(X_t, t)\,dB_t$$

[1] Kiyoshi Itō (1915-2008)

A more general Itō's Formula

With $Y_t = f(t, X_t)$, then

$$
\begin{aligned}
\mathrm{d}Y_t = {} & \frac{\partial f}{\partial X}(t, X_t)\,\mathrm{d}t \\
& + \left(\frac{\partial f}{\partial X}(t, X_t)\mu(X_t, t) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(t, X_t)\sigma^2(X_t, t) \right)\mathrm{d}t \\
& \qquad + \frac{\partial f}{\partial X}(t, X_t)\sigma(X_t, t)\,\mathrm{d}B_t
\end{aligned}
$$

Application of Itō's Formula to the GBM diffusion

$$f(S_t) = \ln(S_t) \qquad \mu(S_t, t) = \mu S_t \qquad \sigma(S_t, t) = \sigma S_t$$

$$f'(S_t) = \frac{1}{S_t} \qquad f''(S_t) = -\frac{1}{S_t^2}$$

$$\mathrm{d}(\ln(S_t)) = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma\,\mathrm{d}B_t$$

The stock price dynamics

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right)$$

- We have the exponential of an arithmetic MB.
- The drift is adjusted by a quadratic term $-\frac{1}{2}\sigma^2 t$. It is a consequence of the rules of stochastic calculus.
  BE CAREFUL NOT TO FORGET THIS TERM !
- The resulting dynamics is called log-normal.

Distribution of returns

$$S_{t+\Delta t} = S_t \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)\Delta t + \sigma(B_{t+\Delta t} - B_t)\right)$$

$$\ln\left(\frac{S_{t+\Delta t}}{S_t}\right) = \ln\left(\frac{S+\Delta S}{S}\right) = \ln\left(1 + \frac{\Delta S}{S}\right) \approx \frac{\Delta S}{S}$$

$$r = \ln\left(\frac{S_{t+\Delta t}}{S_t}\right) = \left(\mu - \tfrac{1}{2}\sigma^2\right)\Delta t + \sigma(B_{t+\Delta t} - B_t)$$

$$r \sim \mathcal{N}\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)\Delta t, \sigma\sqrt{\Delta t}\right)$$

Exercises

1. What is the price dynamics for the diffusion given by

$$dS_t = \mu S_t\,dt - \sigma S_t\,dB_t?$$

2. Let

$$Z_t = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t$$

and $S_t = e^{Z_t}$. What is dS_t ?