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Numerical Procedures

Examples of numerical procedures

It is not possible to derive closed-form solutions for all types of options. Hence the need for numerical procedures that allow to approximate the prices.

- Trees
- Monte-Carlo simulations
- Finite difference methods

- 4.1 -

Trees

The Cox-Ross-Rubinstein binomial model

Starting from a binomial tree with time interval of Δt between two consecutive nodes.

- Under the no-arbitrage condition, we can assume that the world is risk-neutral.
- Then, at any node, the price at t of the stock is the discounted expected value of its price at time $t + \Delta t$:

$$S_t e^{r\Delta t} = pS_t u + (1-p)S_t d$$

- The variance of the return of the stock is $\sigma^2 \Delta t$, hence

$$pu^2 + (1-p)d^2 - e^{2r\Delta t} = \sigma^2 \Delta t$$

The Cox-Ross-Rubinstein binomial model

Adding the Cox, Ross and Rubinstein (1979) condition that

$$u = \frac{1}{d}$$

and using a first order approximation, the solution is to set:

$$u = e^{\sigma\sqrt{\Delta t}} \qquad d = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \frac{a-d}{u-d} \qquad a = e^{r\Delta t}$$

Graphical representation

$$\begin{array}{ccccccccc}
 & & & & & & S_0u^3 & & \\
 & & & & & \nearrow & & & \\
 & & & & S_0u^2 & & & & \\
 & & & \nearrow & & \searrow & & & \\
 & & S_0u & & & & S_0u & & \\
 & \nearrow & & \searrow & & \nearrow & & & \\
S_0 & & & & S_0 & & & & \cdots\cdots \\
 & \searrow & & \nearrow & & \searrow & & & \\
 & & S_0d & & & & S_0d & & \\
 & & & \searrow & & \nearrow & & & \\
 & & & & S_0d^2 & & & & \\
 & & & & & \searrow & & & \\
 & & & & & & S_0d^3 & &
\end{array}$$

- 4.2 -

Monte Carlo

Random Sampling

We look at the change of the stock price over Δt in the risk-neutral world ($\mu = r$):

$$S_{t+\Delta t} = S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma(B_{t+\Delta t} - B_t)\right)$$

and we know that

$$B_{t+\Delta t} - B_t = \epsilon\sqrt{\Delta t}$$

Random Sampling

Therefore, if we generate values for

$$\epsilon \sim \mathcal{N}(0,1)$$

we can construct a sample path for S:

$$S_{t+\Delta t} = S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\epsilon\sqrt{\Delta t}\right)$$

The Box-Muller algorithm

- From two independent random variables U_1 and U_2 with uniform distribution in $[0, 1]$ (Rnd function in VBA)
- it is possible to generate two independent random variables with standard normal distribution $\mathcal{N}(0, 1)$:

$$Z_1 = \sqrt{-2 \ln U_1} \cos(2\pi U_2)$$
$$Z_2 = \sqrt{-2 \ln U_1} \sin(2\pi U_2)$$

The Finite Difference Method

- Divide the time to maturity in N equal intervals.
- Choose a sufficiently high price S_{max} of the underlying for the price of the option to be known with certainty.
- Divide the price scale from 0 to S_{max} in M equal intervals.
- The boundary values for $t = T$, $S = 0$ and $S = S_{max}$ are known.
- Work backward solving systems of difference approximations of the differential equation.