- 5 -

Greeks

- 5.1 -

Delta

Delta

$$\Delta = \frac{\partial C}{\partial S}$$

- Delta is the rate of change of the option's price with respect to the price of the underlying asset.
- For a european call $\Delta = N(d_1)$.
- For a european put $\Delta = N(d_1) - 1$.
- Link with the Black-Scholes PDE. BS PDE

Delta hedging

- Delta neutral portfolio.
- Dynamic delta hedging.
- Transaction costs.

- 5.2 -

Theta and Gamma

Theta

$$\Theta = \frac{\partial C}{\partial t}$$

- Theta is the rate of change of the option's price with respect to the passage of time.
- Unit: usually in years, divide by 365 to have theta per calendar day or by 252 to have theta per business day.
- Intrinsic and optional value of the option.
- Time decay is a certainty.

Gamma

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 C}{\partial S^2}$$

- Gamma is the rate of change of the option's Delta with respect to the price of the underlying asset.
- For a european call or put

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}} \qquad \text{with} \qquad N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- Options are not "linear" products.

Relationship between Gamma and Theta

From the Black-Scholes PDE we have that the value Π of a portfolio of options must satisfy

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

When the portfolio is delta neutral

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

Also, by a Taylor series expansion we obtain

$$\mathrm{d}\Pi = \Theta\,\mathrm{d}t + \frac{1}{2}\Gamma(\mathrm{d}S)^2$$

- 5.3 -

Vega and related greeks

Vega

$$\mathcal{V} = \frac{\partial C}{\partial \sigma}$$

- Vega is the rate of change of the option's price with respect to the volatility of the underlying asset.
- For a european call or put

$$\mathcal{V} = S_0 \sqrt{T} N'(d_1)$$

- The volatility surface.

Vanna and Volga

$$Vanna = \frac{\partial \mathcal{V}}{\partial S} \qquad Volga = \frac{\partial \mathcal{V}}{\partial \sigma} = \frac{\partial^2 C}{\partial \sigma^2}$$

- Vanna is the rate of change of the option's vega with respect to the price of the underlying asset.
- Volga is the rate of change of the option's vega with respect to the volatility of the underlying asset.

- 5.4 -

Rho

Rho

$$\rho = \frac{\partial C}{\partial r}$$

- Rho is the rate of change of the option's price with respect to the interest rate.

- 5.5 -

Numerical simulations

Using a tree

$$
\begin{tikzcd}
 & & \begin{matrix} S_0 u^2 \\ C_{22} \end{matrix} \\
 & \begin{matrix} S_0 u \\ C_{11} \end{matrix} \arrow[ur] \arrow[dr] & \\
\begin{matrix} S_0 \\ C_{00} \end{matrix} \arrow[ur] \arrow[dr] & & \begin{matrix} S_0 \\ C_{21} \end{matrix} \\
 & \begin{matrix} S_0 d \\ C_{10} \end{matrix} \arrow[ur] \arrow[dr] & \\
 & & \begin{matrix} S_0 d^2 \\ C_{20} \end{matrix}
\end{tikzcd}
$$

Using a tree

$$\Delta = \frac{C_{11} - C_{10}}{S_0 u - S_0 d}$$

$$\Gamma = \left(\frac{C_{22} - C_{21}}{S_0 u^2 - S_0} - \frac{C_{21} - C_{20}}{S_0 - S_0 d^2} \right) \cdot \frac{2}{(S_0 u^2 - S_0 d^2)}$$

$$\Theta = \frac{C_{21} - C_{00}}{2\Delta t}$$

$$\mathcal{V} = \frac{C' - C}{\Delta\sigma}$$

where C' is the price of the option obtained on a new tree built after changing the volatility to $\sigma + \Delta\sigma$.

Using Monte-Carlo

- The sensitivity is the difference between the price computed after changing the variable S, t, σ by a small amount δ and the initial price, difference divided by δ.
- To minimise the estimation error, the number of steps, number of trials and the values of the realisations of the standard normal law should be the same.