1. Prove and recall the Put-Call parity relationship.

$C + Ke^{-rT} = P + S_0$

At time 0

$P_1(0) = 1 \text{ call} + Ke^{-rT} \text{ (cash)}$

$P_2(0) = 1 \text{ put} + 1 \text{ stock}$

At time t

$P_1(t) = (S_t - K)^+ + K = (S_t, K)^+$

$P_2(t) = (K - S_t)^+ + S_t = (K, S_t)^+$

$P_1(t) = P_2(t) \xrightarrow{\text{No Arbitrage}} P_1(0) = P_2(0) \Rightarrow C + Ke^{-rT} = P + S_0$

.

2. Prices and Values of Forward and Future Contracts.

Forward: $F_{(0,T)} = S_0 \cdot e^{rT}$

Future: $\phi(0) = E_Q(S_T)$, ~~price~~ Value $= 0$, 因为 mark to market.

Constant interest rate $\Longrightarrow F(0) = \phi(0)$

证: $e^{-rT}\phi(0) = E_Q(e^{-rT}S_T)$ (prices discounted at risk-free rate are Q-Mar)

$= e^{-r\cdot 0}S_0$

$= S_0$

$\phi(0) = S_0 e^{rT} = F(0)$

3. Asian options. Pricing and hedging considerations.

Asian Options.

· Arithmetic Asian Option $(S_T, S_{T-1} \cdots S_{T-N+1})$

$$\text{Payoff} = \left(\frac{\sum_{i=0}^{N-1} S_{T-i}}{N} - K \right)^+ \text{ for a call.}$$

4. Valuation of Barrier Options with Monte-Carlo Simulations.

```
Function GaussRand()
GaussRand = Sqr(-2 * Log(Rnd())) * Cos(2 * Application.WorksheetFunction.Pi * Rnd())
End Function

Function DOC(S0, K, r, Sigma, T, B, N, M)
Dim epsilon As Double: epsilon = T / N
Dim i As Integer, j As Integer
Dim intermS, intermM, intermP As Double
intermP = 0
For j = 1 To M
    intermS = S0
    intermM = S0
    For i = 1 To N
        intermS = intermS * Exp((r - Sigma * Sigma / 2) * epsilon + Sigma * Sqr(epsilon) * GaussRand())
        intermM = Application.WorksheetFunction.Min(intermM, intermS)
    Next i
If intermM > B Then
    intermP = intermP + Application.WorksheetFunction.Max(intermS - K, 0)
    End If
Next j
DOC = intermP / M * Exp(-r * T)
End Function
```

以上是 code，如果要改 in or out 就改红色箭头上 intermM 和 B(Barrier)的关系，cal or put 就改第二行最后的 Max 后面括号里的就行这个大家都懂。

原理就是用 Gaussian random 来生成 M 个 stocks 在 N 天内的走势，然后模拟 stock 和 Barrier 以及 Strike 的关系来定价。

这个 code 不一定语法一定要对，老师在乎的是关键地方，也就是划黄色地方的两处 for 和 next 的 loop, loop 里的 code 一定要放对地方，不然运算会出错

5. Valuation of American Options with Trees.

这道题的 code 特别复杂，而且十分不好记，个人觉得考到的话我们可以画一个两 periods 的二叉树把公式写上去告诉老师我们会算就好，这题不会专门让我们写 code 应该，如果有时间可以画 3 个 periods 的树哈哈

$$u = e^{\sigma\sqrt{\delta t}} \qquad \text{Denoting } a = e^{r\delta t}$$

$$d = e^{-\sigma\sqrt{\delta t}} \qquad p = \frac{a-d}{u-d}$$

大家记得公式就好，Delta t 是一段 period 的长度，即 Maturity/段数； a 在这里相当于之前其他课里的 r，p 是上涨的概率(吧)， q 下降的概率是(u-a)/(u-d)

C= 上涨的 payoff*p+下降的 payoff*q, voila 搞定

6. Limits of the Black and Scholes model.

1) The stock pays no dividends during the option's life

Most companies pay dividends to their share holders, so this might seem a serious limitation to the model considering the observation that higher dividend yields elicit lower call premiums. A common way of adjusting the model for this situation is to subtract the discounted value of a future dividend from the stock price.

2) European exercise terms are used

European exercise terms dictate that the option can only be exercised on the expiration date. American exercise term allow the option to be exercised at any time during the life of the option, making American options more valuable due to their greater flexibility. This limitation is not a major concern because very few calls are ever exercised before the last few days of their life. This is true because when you exercise a call early, you forfeit the remaining time value on the call and collect the intrinsic value. Towards the end of the life of a call, the remaining time value is very small, but the intrinsic value is the same.

3) Markets are efficient

This assumption suggests that people cannot consistently predict the direction of the market or an individual stock. The market operates continuously with share prices following a continuous Itô process. To understand what a continuous Itô process is, you must first know that a Markov process is "one where the observation in time period t depends only on the preceding observation." An Itô process is simply a Markov process in continuous time. If you were to draw a continuous process you would do so without picking the pen up from the piece of paper.

4) No commissions are charged

Usually market participants do have to pay a commission to buy or sell options. Even floor traders pay some kind of fee, but it is usually very small. The fees that Individual investor's pay is more substantial and can often distort the output of the model.

5) Interest rates remain constant and known

The Black and Scholes model uses the risk-free rate to represent this constant and known rate. In reality there is no such thing as the risk-free rate, but the discount rate on U.S. Government Treasury Bills with 30 days left until maturity is usually used to represent it. During periods of rapidly changing interest rates, these 30 day rates are often subject to change, thereby violating one of the assumptions of the model.

6) Returns are lognormal distributed

This assumption suggests, returns on the underlying stock are normally distributed, which is reasonable for most assets that offer options.

划黄色的是老师上课讲的点，其他也有道理可以了解一下

7. Gamma Trading.

Gamma Trading

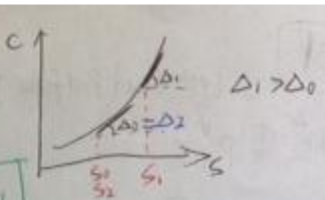
at $\boxed{t_0}$ Hedged Port

Value $= C_0 - \Delta_0 S_0$

$\boxed{t_1}$

Value $= C_1 - \Delta_0 S_1$

Maintein Δ hedged



但Wo 要 $C_1 - \Delta_1 S_1$

$\Downarrow$

卖更多 stock $(\Delta_1 - \Delta_0)$ 补差

$\Downarrow$

Cash-in $(\Delta_1 - \Delta_0) \times S_1$

$\boxed{t_2}$

$C_2 - \Delta_1 S_2$

$\Downarrow$ rehedging

$C_2 - \Delta_2 S_2$

[illegible] $\boxed{\text{Cash-out}}$ $(\Delta_1 - \Delta_2)$ stock

$\hookrightarrow (\Delta_1 - \Delta_2) \times S_2$

P&L

$(\Delta_1 - \Delta_0) S_1 - (\Delta_1 - \Delta_2) S_2 > 0$

[illegible]用Δ赚, $\therefore \Gamma \uparrow \Rightarrow$ Profit $\uparrow$

8. Give and Prove the Relationship between a Forward's Price and its Underlying's Value.

It is such that Value (·) = 0.

$$F_{0,T} = S_0 e^{rT}$$

forward price determined at t=0 for a contract maturity at t=T.

· assume $F_{0,T} > S_0 e^{rT}$

at t=0,
- borrow S_0 €.
- buy 1 stock.
- sell a forward (enter in a forward position to sell the underlying)

at t=T, sell the stock through the forward

get $F_{0,T}$ € from the counterparty of the forward

reimburse $S_0 e^{rT}$ € to close the loan.

P&L : $+ F_{0,T} - S_0 e^{rT} > 0$

Abritrage : Contradiction.

• assume now $F_{0,T} < S_0 e^{rT}$

at $t=0$, sell the stock S_0.

get S_0 € and invest them

buy forward at $F_{0,T}$

at $t=T$, you now have $S_0 e^{rT}$ € in cash

buy a stock through a forward and buy $F_{0,T}$ €

Give back the stock to your broker to close that short sale.

P&L : $S_0 e^{rT} - F_{0,T} > 0$. → Arbitrage Contradiction.

Conclusion : $F_{0,T} = S_0 e^{rT}$

这个证明基本是反正，要没有arbitrage的话必须相等， 这个题很重要，出过好多次，各种说法，但都还是要让我们证明forward price 和 underlying的关系

9. Prove that for a self-financed portfolio V on N1 stocks S, and N2 bonds B : dV = N1dS + N2dB.

Stochastic process in full generality

$N_1(t)$ Stocks S_t

$N_2(t)$ Option C_t.

$Q(t)$ Cash. Q_t.

Portfolio V such that.

$(uv)' = u'v + uv' + u'v'$ if both u&v are stochastic.

$d(uv) = udv + vdu + dudv$.

$V_t = N_1(t)S_t + N_2(t)C_t + Q_t$.

$dV_t = N_1 dS_t + dN_1 S_t + dN_1 dS_t + N_2 dC + dN_2 C + dN_2 dC + dQ_t$

Homework, assuming a self-financial portfolio

$dV_t = N_1 dS_t + N_2 dC_t + dQ_t$.

① use the cash only to buy S_t, or C_t

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dZ_t. \\ dQ_t = rQ_t dt \\ C_t = C(S_t, t) \end{cases}$$

Itô's lemma.

$$dC_t = \Big[\overbrace{\frac{\partial C}{\partial t} + \mu S_t \frac{\partial C}{\partial S_t} + \frac{1}{2}(\sigma \cdot S_t)^2 \cdot \frac{\partial^2 C}{\partial S_t^2}}^{\mu_c \cdot C}\Big]dt + \Big[\overbrace{\sigma S_t \frac{\partial C}{\partial S_t}}^{\sigma_c \cdot C}\Big]dZ_t$$

$$\frac{dC_t}{C_t} = \mu_c dt + \sigma_c dZ_t.$$

$$dV_t = N_1 dS + N_2 dC + dQ.$$

$$dV_t = N_1 \mu S_t dt + N_1 \sigma S_t dZ_t + N_2 \mu_c C_t dt + N_2 \sigma_c C_t dZ_t + rQ_t dt$$

$$\frac{dV_t}{V_t} = \left(\frac{N_1 \mu S + N_2 \mu_c C + rQ}{V}\right)dt + \left(\frac{N_1 \sigma S + N_2 \sigma_c C}{V}\right)dZ_t$$

这是笔记里的，不过这题我不是特别清楚该怎么证明，是很早之前考过一次的，个人感觉再考可能性不是特别大，不过最好也不要大意了，经过Insurance Market我已经不太相信自己的判断了。

10. Prove that a European and an American options, written on the same underlying paying no dividends, have the same value.

when dividend $= 0$: European call = American call.

$q = 0. \Rightarrow \cancel{C = q}. \quad c = C.$

$r > 0.$

$-rT < 0.$

$e^{-rT} < 1.$

$Ke^{-rT} < K.$

$-Ke^{-rT} > -K.$

$S_0 - Ke^{-rT} > S_0 - K.$

$C \geq c \geq S_0 - Ke^{-rT}$

$C_{(0)} > S_0 - K.$ $\quad \therefore$ keep the option, not early exercise.

Same thing at any time $\therefore$ American call = European call.

American Value > Early Exercise Value.

$\Rightarrow$ No Early Exercise.

$\Rightarrow C = c.$

证法2：

Assume we are in the "in the money" part of the stock tree in the American call Option.

$$S \begin{array}{l} \xrightarrow{p} uS \\ \xrightarrow{1-p} dS \end{array}$$

$(uS-K)^+ = uS-K$

$(dS-K)^+ = dS-K.$

discount

$$\frac{p(uS-K)+(1-p)(dS-K)}{a}$$

$$= \frac{(pu+(1-p)d)S-K}{a}$$

$$p = \frac{a-d}{u-d} \quad = \frac{(a-d+d)S-K}{a}$$

$pu-pd = a-d.$

$$= S-\frac{K}{a} > S-K. \qquad a>1, \quad \frac{1}{a}<1 \quad -\frac{1}{a}>-1$$

Value of continuating > Value of Stopping.

$\Rightarrow$ No early Exercise of American Call on underlying without dividends.

11. Price an European Call Option

$$C(0) = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2} - q\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

背公式吧

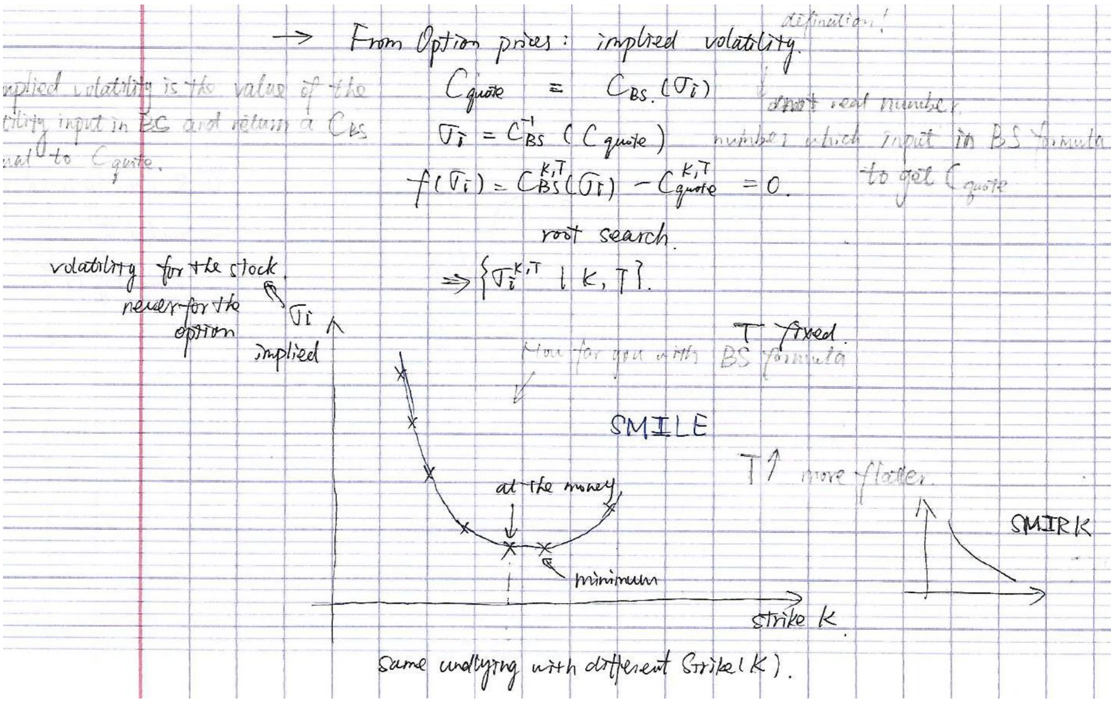
```
Option Explicit

Function BSCall(S0, K, r, Sigma, T, q)
Dim d1, d2 As Double
d1 = (Log(S0 / K) + (r - q + 0.5 * Sigma * Sigma)) / (Sigma * Sqr(T))
d2 = d1 - Sigma * Sqr(T)
BSCall = S0 * Exp(-q * T) * Application.WorksheetFunction.NormSDist(d1) - K * Exp(-r * T) * Application.WorksheetFunction.NormSDist(d2)
End Function
```

VBA code as above

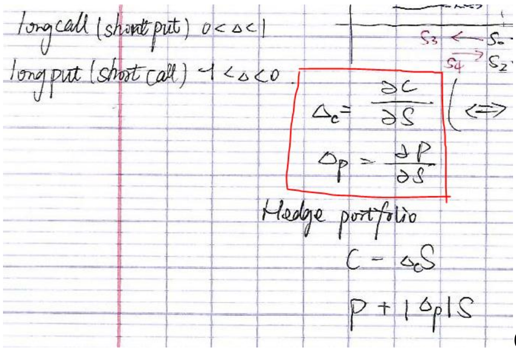
12. The Smile

A plot of the implied volatility of an option as a function of its strike price is known as a volatility smile.



13. Computation of Greeks in a tree.

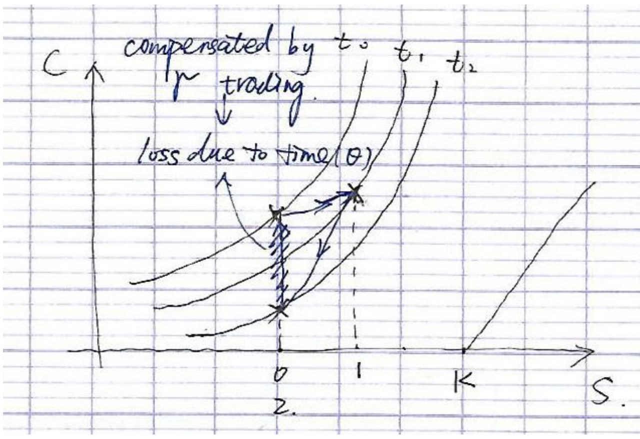
先介绍一下各个Greek，

Delta: 

Option 与 Stock的变化关系

Gamma: $\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S}$ Gamma

是Delta的与stock的变化关系，用法见Gamma Trading题



$$\theta = \frac{\partial c}{\partial t} \quad \text{typically} < 0.$$

$$\Gamma \quad \text{typically} > 0.$$

Gamma Trading赚的钱一般会被Theta赔回去，但一般还是要做，不然亏更多，Theta是option与时间t的变化关系

$$\text{RHO}: \frac{\partial c}{\partial r}$$

$$\text{VEGA}: \frac{\partial c}{\partial \sigma}$$

另两个Greeks，不过这俩考得不太多。这些要了解下因为有两次考试题目就是"The Greeks"，估计能多说点更好

$$\text{Black-Scholes} \quad \frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 c}{\partial S^2} - rC = 0$$

$$\Rightarrow \theta + rS\Delta + \frac{\sigma^2 S^2}{2}\Gamma - rC = 0.$$

这个柿子很重要

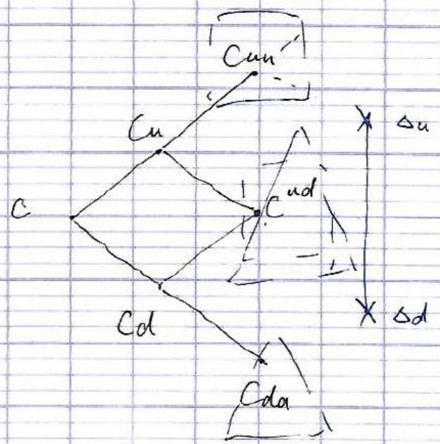
好开始算树上的希腊人

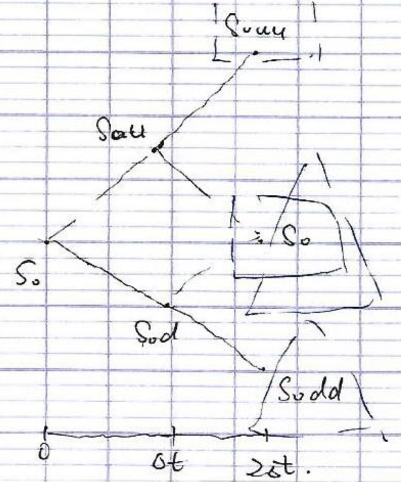
ex. European Call. $\Delta = N(d_1)$

$$\frac{\partial}{\partial S}\left(S_0 N(d_1) - K e^{-rT} N(d_2)\right).$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$





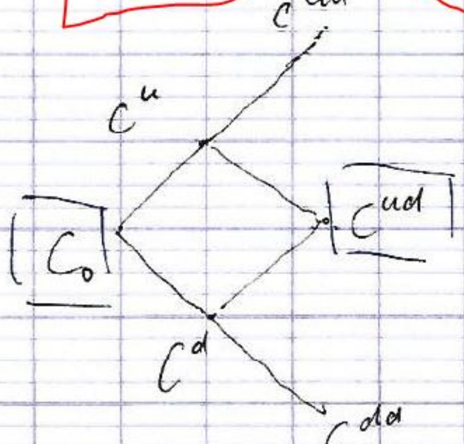
n't comput Δ at time 0.

$$\Delta_{\Delta t} \# \frac{C^u - C^d}{S_0^u - S_0^d}$$

don't mix value ...

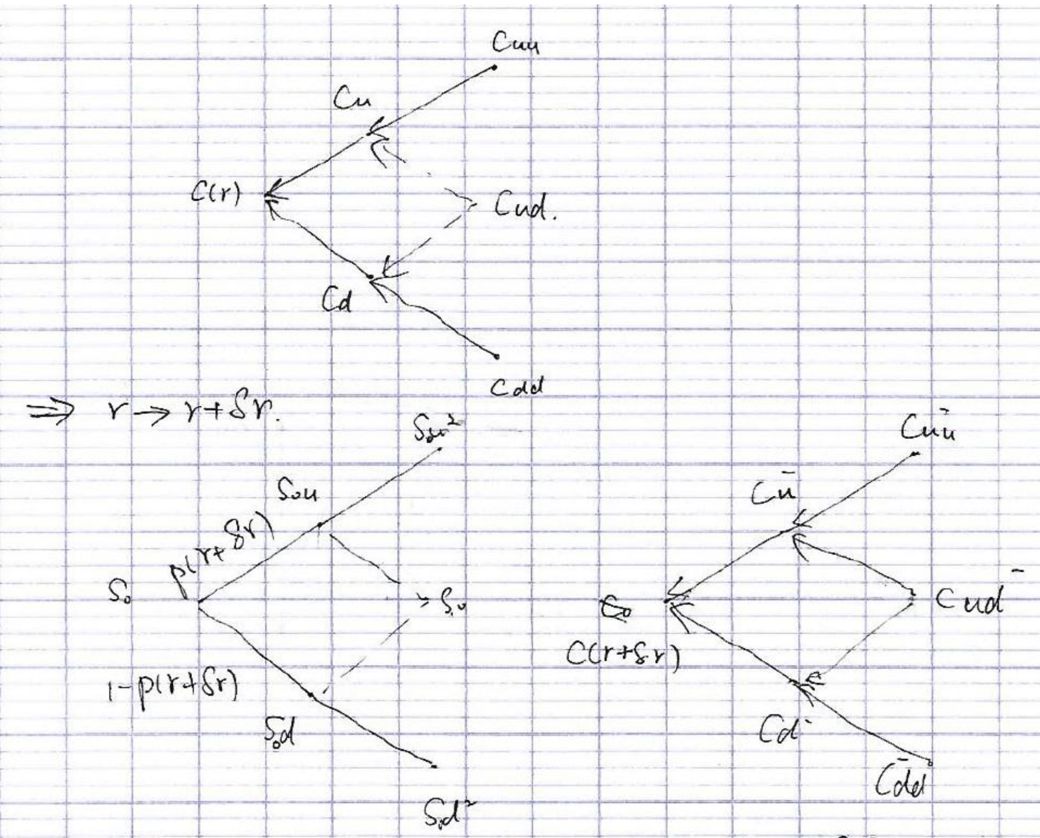
Make $\Delta t \to 0$. by increasing N.

$$\Gamma_{2\Delta t} \# \frac{\dfrac{C^{uu} - C^{ud}}{S_0 u^2 - S_0} - \dfrac{C^{ud} - C^{dd}}{S_0 - S_0 d}}{\dfrac{S_0 u^2 - S_0 d^2}{2}}.$$



$$\Theta = \frac{C^{ud} - C_0}{\boxed{2}\Delta t}$$

Gamma的公式有点复杂，大家对着最上面的两个树一起看很快就明白了，不难，大神请无视我，因为这节课我没来，感觉看懂了老开心了…



$$\text{RHO} : \rho \approx \frac{c(r+\delta r) - c(r)}{r+\delta r - r} = \frac{c(r+\delta r) - c(r)}{\delta r}.$$

Estimation of σ ?

→ Historical data + econometrics

$$\hat{\sigma} = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} \left(S_i - \frac{\sum_{i=1}^{N} S_i}{N}\right)^2}$$

Sigma太复杂个人感觉不会考

14. Risk-neutral Pricing.

Trajectries should be simulated in the risk-neutral world

$$\frac{dS}{S} = r\,dt + \sigma\, dZ_t^Q \rightarrow \text{risk neutral}$$

$\Downarrow$ whatever σ, always get r.

$\sigma_1 > \sigma_2 \Rightarrow \mu_1 > \mu_2$.

risk-neutral world $\leftarrow$ option pricing.

Historical $\rightarrow$ Risk-neutral

$$S_t = S_0\, e^{(r-\frac{\sigma^2}{2}-q)t + \sigma Z_t^Q}$$

$$S_{t+\varepsilon} = S_0\, e^{(r-\frac{\sigma^2}{2}-q)(t+\varepsilon) + \sigma Z_{t+\varepsilon}^Q}$$

$$\frac{S_{t+\varepsilon}}{S_t} = e^{(r-\frac{\sigma^2}{2}-q)\varepsilon + \sigma(Z_{t+\varepsilon}^Q - Z_t^Q)}$$

$$Z_{t+\varepsilon}^Q - Z_t^Q = Z_{0+\varepsilon}^Q - Z_0^Q = Z_\varepsilon^Q \sim N(0, \sqrt{\varepsilon})$$

$$= \sqrt{\varepsilon}\cdot\tilde{x} \quad \text{where } \tilde{x} \sim N(0,1)$$

$$\boxed{S_{t+\varepsilon} = S_t \cdot e^{(r-\frac{\sigma^2}{2}-q)\varepsilon + \sigma\sqrt{\varepsilon}\cdot\tilde{x}}}$$

where $\tilde{x} \sim N(0,1)$

$$S_0 \xrightarrow{\tilde{x}^{(1)}} S_\varepsilon = S_0\, e^{(r-\frac{\sigma^2}{2}-q)\varepsilon + \sigma\sqrt{\varepsilon}\,\tilde{x}^{(1)}}$$

$$S_\varepsilon \longrightarrow S_{2\varepsilon} = S_\varepsilon\, e^{(r-\frac{\sigma^2}{2}-q)\varepsilon + \sigma\sqrt{\varepsilon}\,\tilde{x}^{(1)}}$$

用VBA算的话用Box-Muller Model

Box-Muller: simulate 2 uniforms U_1 and U_2.

$$\tilde{x} = \cos(2\pi U_1)\cdot\sqrt{-2\ln(U_2)} \sim N(0,1)$$

```
Function GaussRand()
GaussRand = Cos(2 * Application.WorksheetFunction.Pi * Rnd()) * Sqr(-2 * Log(Rand()))
End Function
```

```
Function NextStock(St, r, Sigma, q, epsilon)
NextStock = St * Exp((r - q - 0.5 * Sigma * Sigma) * epsilon + Sigma * Sqr(epsilon) * GaussRand())
End Function
```

15. Stochastic volatility and implied volatility (the volatility smile). Heston’s Model.

Heston – Merton.

$$\begin{cases} \frac{dS}{S} = r\,dt + \sigma\, dz_1^Q + dJ \\ d\sigma^2 = a(b-\sigma^2)\,dt + \xi\sqrt{\sigma^2}\, dz_2^Q \end{cases}$$

$$\langle dz_1^Q, dz_2^Q \rangle = \rho\, dt$$

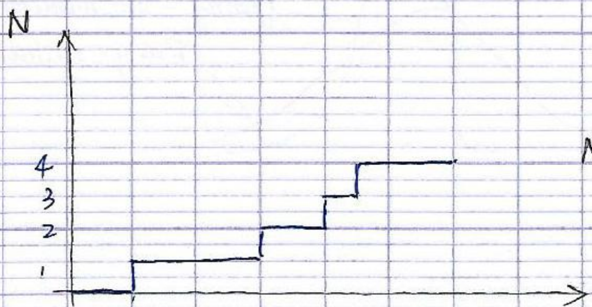
Typically $\rho < 0$

$$J_t = \sum_{i=1}^{N_t} G_i$$

Rho<0意味着Stock Price下降以后Volatility反而上升，too bad for investors!

关于Jump

Simple Poisson Process.



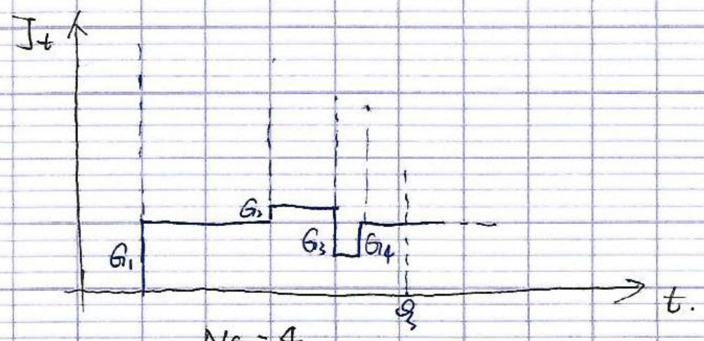
$$N \begin{cases} N(0) = 0. \\ \text{independed increment} \\ N_t - N_s \sim P(\lambda(t-s)) \end{cases}$$

counting process used for counting events.

$s<t$, $P(N_t - N_s = k) = \dfrac{\lambda^k (t-s)^k e^{-\lambda(t-s)}}{k!}$

$\downarrow$

probability of observe k events between s and t.



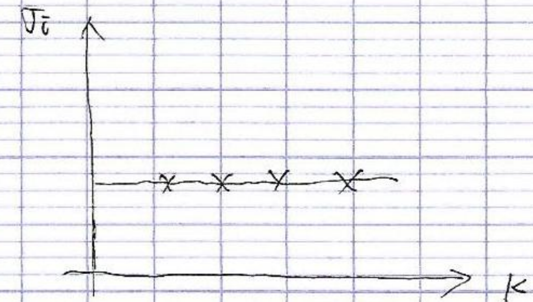
$N_\theta = 4$

$J_Q = G_1 + G_2 + G_3 + G_4$

λ: intensity: average number of jumps / year.

16. The Volatility Surface. Smile and Models beyond Black and Scholes.

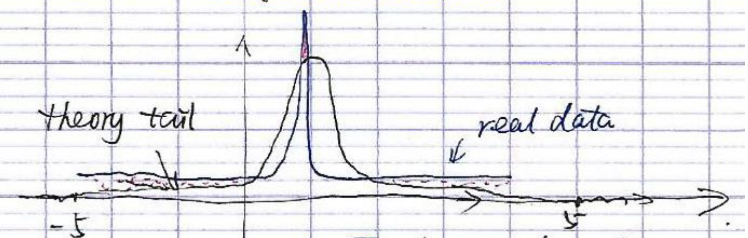
Remark: If BS prevail on the market



Explaination why usually BS does not prevail:

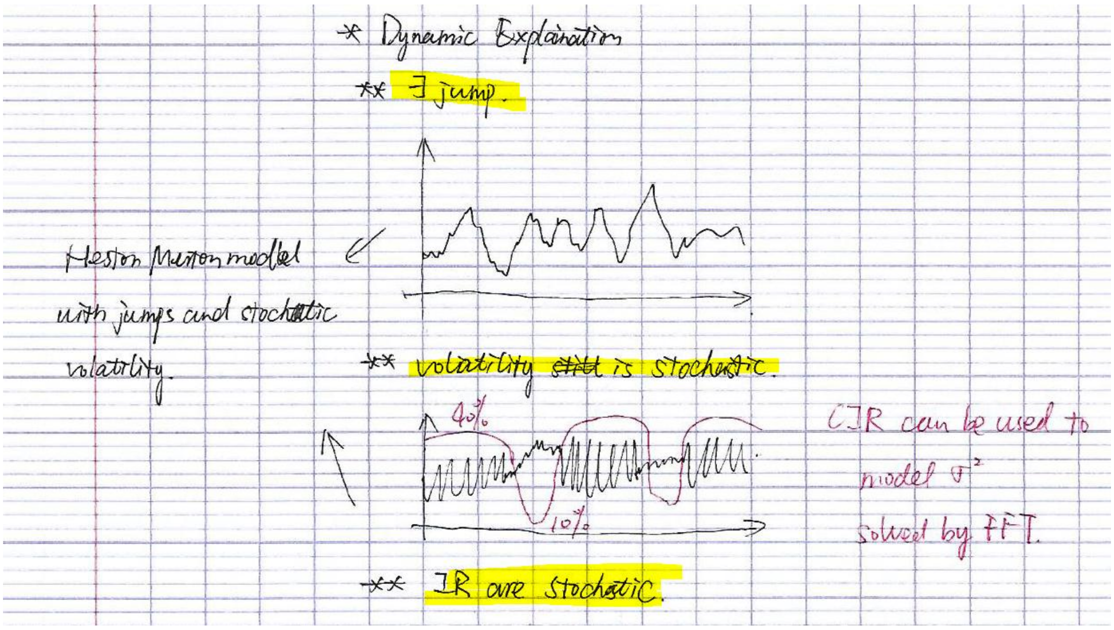
* Static explanation:

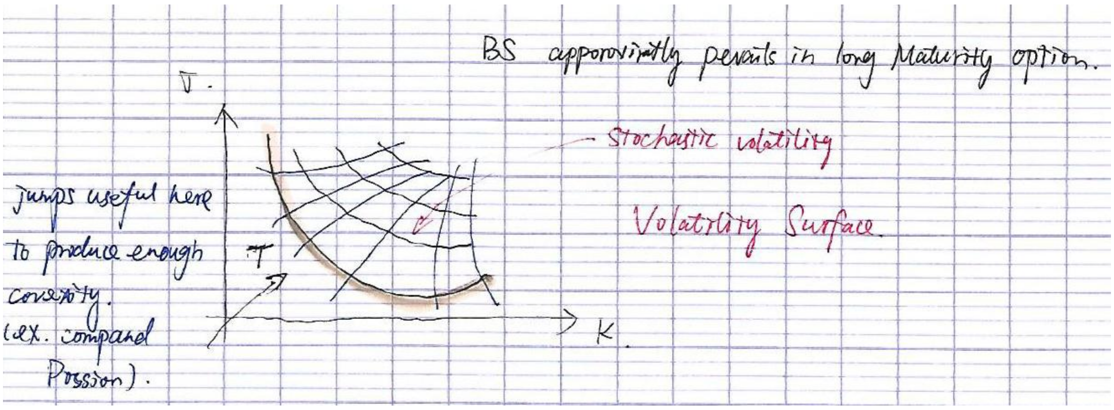
stock returns are not Gaussian.



∃ skewness kurtosis

$e^{-25} \simeq 10^{-11}$

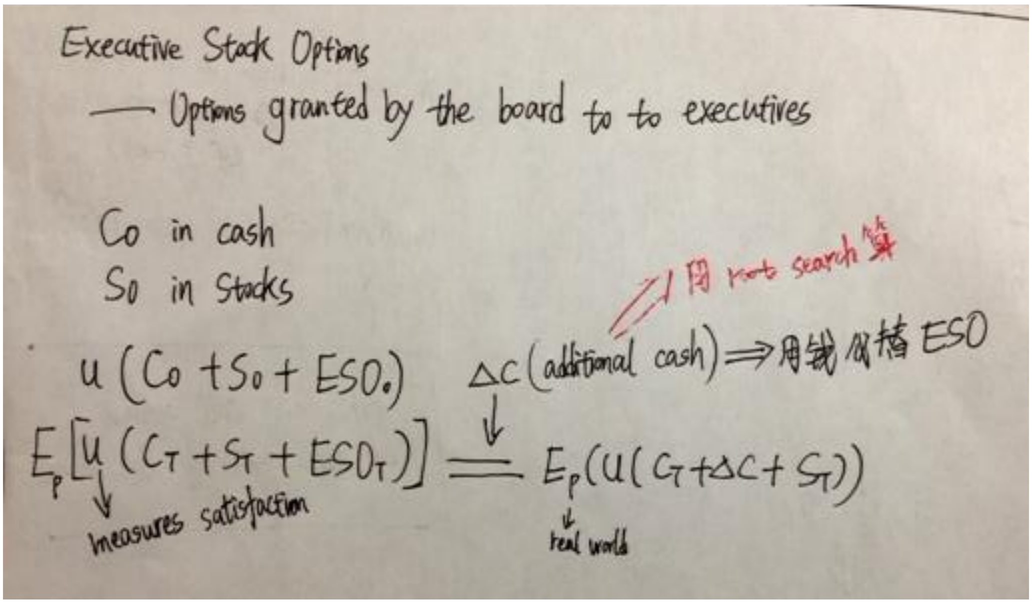




17. Options on Futures

An option on a futures contract gives the holder the right to enter into a specified futures contract. If the option is exercised, the initial holder of the option would enter into the long side of the contract and would buy the underlying asset at the futures price. A short option on a futures contract lets an investor enter into a futures contract as the short who would be required to sell the underlying asset on the future date at the specified price.

18. Stock Options



即用DeltaC代替这个StockST，是的两边的Utility Function的Expectation一样，这个可以用Excel里的Root Search或者叫Goal Seek算，code比较复杂，因为不知道Utility的方程，个人觉得让老师知道我们会算也好，这老师是够任性的…