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# Numerical Procedures

## Examples of numerical procedures

It is not possible to derive closed-form solutions for all types of options. Hence the need for numerical procedures that allow to approximate the prices.

- ▶ Trees
- ▶ Monte-Carlo simulations
- ▶ Finite difference methods

- 4.1 -

Trees

# The Cox-Ross-Rubinstein binomial model

Starting from a binomial tree with time interval of  $\Delta t$  between two consecutive nodes.

- ▶ Under the no-arbitrage condition, we can assume that the world is risk-neutral.
- ▶ Then, at any node, the price at  $t$  of the stock is the discounted expected value of its price at time  $t + \Delta t$  :

$$S_t e^{r\Delta t} = pS_t u + (1 - p)S_t d$$

- ▶ The variance of the return of the stock is  $\sigma^2 \Delta t$ , hence

$$pu^2 + (1 - p)d^2 - e^{2r\Delta t} = \sigma^2 \Delta t$$

# The Cox-Ross-Rubinstein binomial model

Adding the Cox, Ross and Rubinstein (1979) condition that

$$u = \frac{1}{d}$$

and using a first order approximation, the solution is to set:

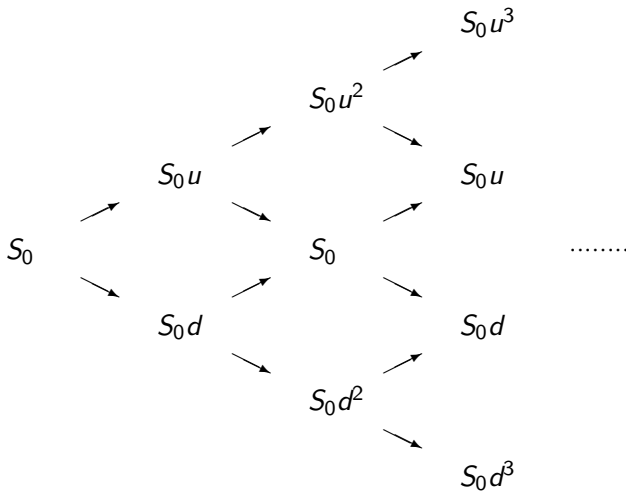
$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \frac{a - d}{u - d}$$

$$a = e^{r\Delta t}$$

# Graphical representation



- 4.2 -

Monte Carlo

# Random Sampling

We look at the change of the stock price over  $\Delta t$  in the risk-neutral world ( $\mu = r$ ):

$$S_{t+\Delta t} = S_t \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma (B_{t+\Delta t} - B_t) \right)$$

and we know that

$$B_{t+\Delta t} - B_t = \epsilon \sqrt{\Delta t}$$



# Random Sampling

Therefore, if we generate values for

$$\epsilon \sim \mathcal{N}(0, 1)$$

we can construct a sample path for  $S$ :

$$S_{t+\Delta t} = S_t \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma \epsilon \sqrt{\Delta t} \right)$$

# The Box-Muller algorithm

- ▶ From two independent random variables  $U_1$  and  $U_2$  with uniform distribution in  $[0, 1]$  (Rnd function in VBA)
- ▶ it is possible to generate two independent random variables with standard normal distribution  $\mathcal{N}(0, 1)$ :

$$Z_1 = \sqrt{-2 \ln U_1} \cos(2\pi U_2)$$

$$Z_2 = \sqrt{-2 \ln U_1} \sin(2\pi U_2)$$

# The Finite Difference Method

- ▶ Divide the time to maturity in  $N$  equal intervals.
- ▶ Choose a sufficiently high price  $S_{max}$  of the underlying for the price of the option to be known with certainty.
- ▶ Divide the price scale from 0 to  $S_{max}$  in  $M$  equal intervals.
- ▶ The boundary values for  $t = T$ ,  $S = 0$  and  $S = S_{max}$  are known.
- ▶ Work backward solving systems of difference approximations of the differential equation.