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Risk-Neutral valuation

A model for the market

- A probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- A finite time T.
- A filtration $(\mathcal{F}_t)_{t=0}^T$ on Ω.
- Assets prices: $(\mathcal{F}_t)$-adapted stochastic processes $(S_t^i)_{t=0}^T$, $i = 0, \ldots, n$.
- We will suppose asset $i = 0$ is a risk-free asset with deterministic return r and we will use the discounted prices of the assets $(\widetilde{S}_t^i)_{t=0}^T$, $i = 1, \ldots, n$.

Definitions

- A derivative, or a contingent claim, is an $\mathcal{F}_T$-measurable random variable.
- A trading strategy is an $\mathbb{R}^{d+1}$-valued process $(H_t)_{t=0}^{T}$ which is predictable, that is H_t is $\mathcal{F}_{t-1}$-measurable.
- A market is complete if any contingent claim is replicable by a trading strategy.

Fundamental Theorem of Asset Pricing

A risk-neutral probability, or equivalent martingale measure, is a probability $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, equivalent to $\mathbb{P}$ and such that all the discounted assets prices are martingales under $\mathbb{Q}$, that is

$$\mathbf{E}_{\mathbb{Q}}[\widetilde{S_{t+1}}|\mathcal{F}_t] = \widetilde{S}_t$$

Fundamental Theorem of Asset Pricing

There are no arbitrage in the market if and only if there exists at least one risk-neutral probability.

If the market is complete then the risk-neutral probability is unique.

Pricing by No-Arbitrage

Pricing a derivative

For any contingent claim X, its no-arbitrage price is given by

$$\pi(X) = e^{-rT}\mathbf{E}_{\mathbb{Q}}[X]$$

- This price is not only valid in the risk-neutral world but also in the "real world" of probability $\mathbb{P}$. When valuing derivatives we can assume that all investors are risk-neutral.
- In the risk-neutral world, the return on any asset is the risk-free rate and the discount rate is the risk-free rate.

Applications to a forward contract

- What is f_t the price at time t of a forward contract with a delivery price F_0 and maturity $T > t$?
- The payoff at T is $S_T - F_0$, therefore the price at t is

$$\begin{aligned} f_t &= e^{-r(T-t)}\, \mathbf{E}_{\mathbb{Q}}[S_T - F_0] \\ &= e^{-r(T-t)}\, \mathbf{E}_{\mathbb{Q}}[S_T] - F_0 e^{-r(T-t)} \\ &= S_t - F_0 e^{-r(T-t)} \end{aligned}$$

- This is consistent with calculus of the delivery price at time 0 which is such that $f_0 = 0$: $F_0 = S_0 e^{rT}$ and the value at time t_1 of this forward contract: $f_1 = S_{t_1} - F_0 e^{-r(T-t_1)} = (S_{t_1} e^{r(T-t_1)} - F_0) e^{-r(T-t_1)} = (F_1 - F_0) e^{-r(T-t_1)}$.

Applications to european options

We have that

$$C = e^{-rT}\mathbf{E}_{\mathbb{Q}}[(S_T - K)^+]$$

Under the assumptions of the Black-Scholes model, the underlying price S_T is log-normally distributed with drift $\left(r - \frac{\sigma^2}{2}\right)T$ and standard deviation $\sigma\sqrt{T}$. It is therefore possible to compute the price using the probability density function of S_T. The result is

$$C = e^{-rT}\big(S_0 N(d_1) e^{rT} - KN(d_2)\big)$$

where $S_0 N(d_1) e^{rT}$ is the expected value under $\mathbb{Q}$ of a random variable that is worth S_T if $S_T > K$ and 0 otherwise and $N(d_2)$ is the probability the option will be exercised in the risk-neutral world.

Application with a time varying risk-free rate

The discount factor from time t to T is

$$e^{-\int_t^T r(s)\,\mathrm{d}s}$$

To evaluate the risk-neutral price of an asset:

$$X_0 = \mathbf{E}_{\mathbb{Q}}\left[\left(e^{-\int_t^T r(s)\,\mathrm{d}s}\right) X_T\right]$$