Technische Universität München
ESPACE Master Program
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Exercise 1 of Orbit Mechanics

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Tutorial Number: Exercise 1

Topic: Keplerian Orbits in Space-fixed, Earth-fixed and Topocentric systems

Study Program: ESPACE (Earth Oriented Space Science and Technology)

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1. Introduction of tasks

The whole exercise consists of four tasks.

1. Creating kep2orb.m function that computes polar coordinates of satellite position. Using inputs: a, e, t, T0 to obtain outputs: r (position), v (true anomaly), M, E.
2. Creating kep2cart.m function to build a space-fixed system. Using a, e, I, Ω, w, t, T0 as inputs to obtain r2 (position) and v2 (velocity).
3. Creating cart2efix.m to build an earth-fixed system. Using t, r2 and v2 as inputs to obtain r3 (position) and v3(velocity) in the earth fixed system.
4. Creating efix2topo.m to build a topo-centric system. Using r3, v3 and $\overrightarrow{r_{wettzell}}$ as inputs to obtain r4(position), v4(velocity), azimuth and elevation.

2. Analysis and Method of Each Task.

This chapter have five sub-parts: parameters known and analysis of each task.

2.1 Basic parameters of each satellite

Satellite	a[km]	e	i[deg]	Ω [deg]	ω [deg]	T_0 [h]
GOCE	6629	0.004	96.6	210	144.2	01:00
GPS	26560	0.01	55	30	30	11:00
Molniya	26554	0.7	63	200	270	05:00
GEO	Geostationary	0	0	0	50	00:00
Michibiki	Geosynchronous	0.075	41	200	270	19:00

Table.1 basic parameters of each satellite.

Table.1 demonstrates the given parameters of each satellite from the given sheet. Except from the table, there are also other known parameters: The Earth fixed reference system rotates with an angular rate of ω_{Earth} = 2π/86164s

Geocentric gravitational constant GM = $398.6005 \cdot 10^{12} m^3/s^2$.

The siderealtime was 03:00 am on the sixth day of November.

The earth's radius R_E= 6371km. Thus, the geostationary and geosynchronous satellites have the same semi-major at 42169km.

The r_wettzell vector = $(407.553022, 931.78130, 480.161819)^T$km.

```
GM=398600500000000;
eps=0.00001;
omegaE=2*pi/86164;
siderealtime=2*pi*(3*60*60)/(24*3600);
r_wettzell=[4075530.22;931781.30;4801618.19]; %unit (m)
x1=r_wettzell(1,1);y1=r_wettzell(2,1);z1=r_wettzell(3,1);
beta=atan2(y1,x1);alpha=(pi/2)-atan(z1/sqrt(x1*x1+y1*y1));
Q1=[-1,0,0;0,1,0;0,0,1];
%S(1)=GOCE
s(1).a=6629000; s(1).e=0.004; s(1).i=96.6*2*pi/360; s(1).o=210*2*pi/360;
s(1).w=144*2*pi/360;s(1).T0=1;
%s(2)=GPS
s(2).a=26560000; s(2).e=0.01; s(2).i=55*2*pi/360; s(2).o=30*2*pi/360;
s(2).w=30*2*pi/360;s(2).T0=11;
%s(3)=Molniya
s(3).a=26554000; s(3).e=0.7; s(3).i=63*2*pi/360; s(3).o=200*2*pi/360;
s(3).w=270*2*pi/360;s(3).T0=5;
%s(4)=GEO
s(4).a=42164000; s(4).e=0; s(4).i=0*2*pi/360; s(4).o=0*2*pi/360;
s(4).w=50*2*pi/360;s(4).T0=0;
%s(5)=Michibiki
s(5).a=42164000; s(5).e=0.075; s(5).i=41*2*pi/360; s(5).o=200*2*pi/360;
s(5).w=270*2*pi/360;s(5).T0=19;
```

Figure.2.1 The code of restoring given data

As shown in figure 2.1, according to the given data, a struct s is created in MATLAB to restore the given data of each satellite.

2.2 Calculation process of task 1

In this task, the most important mission is to create the 2-D orbits together and present the result figure. The steps are below.

Firstly, using GM and a to calculate the rotation rate n of each orbit, then using the n and time interval to calculate M(t) at each time. Equation (1) and codes are followed.

```
t=48; %t as the input
for j=1:5
    s(j).n=sqrt(GM/(s(j).a*s(j).a*s(j).a)); %calculate each n
    s(j).interval=t-s(j).T0;          %calculate time interval
    s(j).M=s(j).n*s(j).interval*3600; %When calculate M, the unit of time interval should be s
    s(j).matrix=zeros(7,s(j).interval*60); %value matrix for satellites
    number=s(j).interval*60;  % When going through t, I choose i, the interval between i and i+1 is 60s (1min)
    ecc=s(j).e;
```

Figure.2.2 The code of calculating M and n

$$n = \sqrt{\frac{GM}{a^3}}; M(t) = n \cdot (t - T0) \quad (1)$$

$$M(t) = E - e \cdot sinE \quad (2)$$

Secondly, using Kepler's equation, as shown in equation (2) to calculate E at each time. Due to the complexity of the equation, itineration method can be applied.

```
for i=1:number
    s(j).matrix(1,i)=i*60;  %unit second
    s(j).matrix(2,i)=s(j).n*s(j).matrix(1,i); % M at each t
    E=s(j).matrix(2,i);   %fisrt value of E at each time.
    M0=E;
    for k=1:100  %itineration of E at each t
        dE=(M0-E+ecc*sin(E))./(1-ecc*cos(E));
        if(max(abs(dE))<eps)
            break
        end
        E=E+dE;
    end
    s(j).matrix(3,i)=E; %E at each time is calculated.
```

Figure.2.3 creating a loop to calculate the E at each time epoch.

As shown in figure.2.3, the itineration was set to be 100. The loop will automatically break out when the absolve error (abs) is less than eps (eps=0.001).

$$r = a(1 - e \cdot cosE) \tag{3}$$

$$tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \cdot tan\frac{E}{2} \tag{4}$$

After obtaining the angle E, the equation (3) can be used to calculate radius r. The true anomaly can be calculated by equation (4), in which E and eccentricity were used.

$$x = rcosv \ and \ y = rsinv \tag{5}$$

Finally, the vector [x,y] can be calculated by using radius r and true anomaly in equation (5).

By using the vector [x,y], the 2-D plane orbits of each satellites can be present. The code of above process is demonstrated in Figure.2.4.

```
for i=1:number
    s(j).matrix(1,i)=i*60;  %unit second
    s(j).matrix(2,i)=s(j).n*s(j).matrix(1,i); % M at each t
    E=s(j).matrix(2,i);   %fisrt value of E at each time.
    M0=E;
    for k=1:100  %itineration of E at each t
        dE=(M0-E+ecc*sin(E))./(1-ecc*cos(E));
        if(max(abs(dE))<eps)
            break
        end
        E=E+dE;
    end
    s(j).matrix(3,i)=E; %E at each time is calculated.
    s(j).matrix(4,i)=s(j).a*(1-ecc*cos(E)); % r at each time.
    r_temporal=s(j).matrix(4,i);
    s(j).matrix(5,i)=2*atan(tan(E/2)*sqrt((1+ecc)/(1-ecc))); %v at each time
    v_temporal=s(j).matrix(5,i);
    s(j).matrix(6,i)=r_temporal*cos(v_temporal); %x at eachtime
    s(j).matrix(7,i)=r_temporal*sin(v_temporal); %y at eachtime
end
```

Figure.2.4 The code of calculating radius, E, true anomaly v, x and y

2.3 Calculation process of task 2

To transfer the 2D coordinates into the space fixed system, the following steps are applied.

Firstly, equation (6) defines the vector r2' to represent the coordinates of x and y.

$$\overrightarrow{r2'} = r \cdot \begin{bmatrix} cosv \\ sinv \\ 0 \end{bmatrix} \tag{6}$$

Secondly, as shown in equation (7), using the transferring matrix to change the coordinates of vector and obtain vector r2 in space-fixed system.

$$\overrightarrow{r2} = R3(-\Omega) \cdot R1(-i) \cdot R3(-\omega) \cdot r \begin{bmatrix} cosv \\ sinv \\ 0 \end{bmatrix} \tag{7}$$

```
    v_temporal=s(j).matrix(5,i);
    s(j).rmatrix(1,i)=r_temporal*cos(v_temporal);         %vector r line 1
    s(j).rmatrix(2,i)=r_temporal*sin(v_temporal);         %vector r line 2
    s(j).rdotmatrix(1,i)=r_temporal*sin(v_temporal)*(-1);         %vector rdot line 1
    s(j).rdotmatrix(2,i)=r_temporal*cos(v_temporal)+ecc;          %vector rdot line 2
    s(j).matrix(6,i)=r_temporal*cos(v_temporal); %x at eachtime
    s(j).matrix(7,i)=r_temporal*sin(v_temporal); %y at eachtime
  end
  s(j).rdotmatrix = s(j).rdotmatrix*sqrt(GM/(s(j).a*(1-ecc*ecc)));
  s(j).r_equator=s(j).omatrix*s(j).imatrix*s(j).wmatrix*s(j).rmatrix;      %position
  s(j).rdot_equator=s(j).omatrix*s(j).imatrix*s(j).wmatrix*s(j).rdotmatrix; %velocity
  x=s(j).r_equator(1,:)';
  y=s(j).r_equator(2,:)';
  z=s(j).r_equator(3,:)';
end
```

Figure.2.5 the code of calculating x,y,z in the space-fixed system

The code of above process is demonstrated in figure.2.5. It is an extension from the code in section.2.2.

2.4 Calculation process of task 3

To transfer the space fixed system coordinates into the earth fixed system, the following steps are applied.

Firstly, the equation (8) uses the earth rotation angle rate and sidereal angle to calculate $\theta(t)$, which is the side real time.

$$\theta(t) = \omega_{earth} \cdot t + sidereal\ angle\ (03{:}00\ in\ rad) \tag{8}$$

Secondly, the equation (9) uses R3 matrix to transfer r2 into vector r3. Hence the space fixed position vector r2 is changed to earth fixed position vector r3.

$$\overrightarrow{r3} = R3\big(\theta(t)\big) \cdot \overrightarrow{r2} \tag{9}$$

After obtaining the position vector in the earth fixed system, the pictures must be used to present the orbits. To draw the figure, the longitude and latitude need to be calculated.

In MATLAB, the longitude should be calculated by atan2 and latitude should be calculated by atan. The equation (10) demonstrates the process.

$$longitude = \arctan\left(\frac{x3}{y3}\right);\ latitude = \arctan\left(\frac{z3}{\sqrt{x3 \cdot x3 + y3 \cdot y3}}\right) \tag{10}$$

```
        v_temporal=s(j).matrix(5,1);
        s(j).rmatrix(1,i)=r_temporal*cos(v_temporal);           %vector r line 1
        s(j).rmatrix(2,i)=r_temporal*sin(v_temporal);           %vector r line 2
        s(j).rdotmatrix(1,i)=r_temporal*sin(v_temporal)*(-1);           %vector rdot line 1
        s(j).rdotmatrix(2,i)=r_temporal*(cos(v_temporal)+ecc);            %vector rdot line 2
        s(j).matrix(6,i)=r_temporal*cos(v_temporal); %x at eachtime
        s(j).matrix(7,i)=r_temporal*sin(v_temporal); %y at eachtime
        s(j).r_equator(:,i)=s(j).omatrix*s(j).imatrix*s(j).wmatrix*s(j).rmatrix(:,i);
        s(j).theta0(1,i)=omegaE*(i*60+s(j).T0*3600)+2*pi/8;   %theta0
        theta=s(j).theta0(1,i);
        thetamatrix=[cos(theta),sin(theta),0;(-1)*sin(theta),cos(theta),0;0,0,1];  %R3(theta0)
        s(j).r3matrix(:,i)=thetamatrix*s(j).r_equator(:,i);    %r3=R3(theta0(t))*r2
        x3=s(j).r3matrix(1,i);y3=s(j).r3matrix(2,i);z3=s(j).r3matrix(3,i);  %the vector of r3
        s(j).latiandlongti(1,i)=360*atan2(y3,x3)/(2*pi); %longitude
        s(j).latiandlongti(2,i)=360*atan(z3/sqrt(x3*x3+y3*y3))/(2*pi); %laditude
    end
    s(j).rdotmatrix = s(j).rdotmatrix*sqrt(GM/(s(j).a*(1-ecc*ecc)));
    s(j).r_equator=s(j).omatrix*s(j).imatrix*s(j).wmatrix*s(j).rmatrix;      %position
    s(j).rdot_equator=s(j).omatrix*s(j).imatrix*s(j).wmatrix*s(j).rdotmatrix; %velocity
    %resultr3(j).matrix=s(j).r3matrix;
end
```

Figure.2.6 The code of calculating vector r3 and its x,y,z, latitude and longitude.

The figure.2.6 demonstrated the process to calculate the parameters of vector r3.

2.5 Calculation process of task 4

To transfer the earth fixed system coordinates into the topo-centric system, the following steps are applied.

$$\overrightarrow{r_{trans}} = \overrightarrow{r3} - \overrightarrow{r_{wettzell}} \tag{11}$$

Firstly, equation (11) used given vector $\overrightarrow{r_{wettzell}}$ to calculate vector $\overrightarrow{r_{trans}}$.

$$\overrightarrow{r4} = Q1 \cdot R2(90 - latitude_{wettzell}) \cdot R3(longitude_{wettzell}) \cdot \overrightarrow{r_{trans}} \tag{12}$$

Secondly, equation (12) is used to obtain position vector r4. The latitude and longitude of wettzell can be obtained through equation (10).

$$azimuth = \arctan\left(\frac{x4}{y4}\right); elevation = \arctan\left(\frac{z4}{\sqrt{x4 \cdot x4 + y4 \cdot y4}}\right) \tag{13}$$

```
    s(j).latiandlongti(1,i) = longitude; s(j).latiandlongti(2,i) = latitude; %longitude and latitude of r3
    xtrans=s(j).rtransmatrix(1,i);ytrans=s(j).rtransmatrix(2,i);ztrans=s(j).rtransmatrix(3,i);  %the vector of rtrans
    [latitude, longitude, height] = xyzblh(xtrans, ytrans, ztrans);
    s(j).translatiandlongti(1,i) = longitude; s(j).translatiandlongti(2,i) = latitude;
    R2=[cos(alpha),0,(-1)*sin(alpha);0,1,0;sin(alpha),0,cos(alpha)];
    R3=[cos(beta),sin(beta),0;(-1)*sin(beta),cos(beta),0;0,0,1];
    s(j).r4matrix(:,i)=Q1*R2*R3*s(j).rtransmatrix(:,i); %calculation of r4
    x4=s(j).r4matrix(1,i);y4=s(j).r4matrix(2,i);z4=s(j).r4matrix(3,i);  %the vector of r4
    [azimuth, elevation] = xyzazel(x4, y4, z4);
    s(j).r4latiandlongti(1,i) = azimuth; s(j).r4latiandlongti(2,i) = elevation;   %azimuth and elevation from r4
    if s(j).r4latiandlongti(2,i)<0
        s(j).r4latiandlongti(3,i) = NaN;
        s(j).r4latiandlongti(4,i) = 0;  %visuality
    else
        s(j).r4latiandlongti(3,i) = s(j).r4latiandlongti(2,i);
        s(j).r4latiandlongti(4,i) = 1;  %visuality
    end

end
```

Figure.2.7 The code of calculating vector r4 and its x,y,z, elevation and azimuth

The longitude and latitude of r4 should also be calculated through equation (13). And figure.2.7 present the process to calculate r4 and its azimuth and elevation angle.

3. Result and Interpretation

3.1 The result and interpretation of Task 1

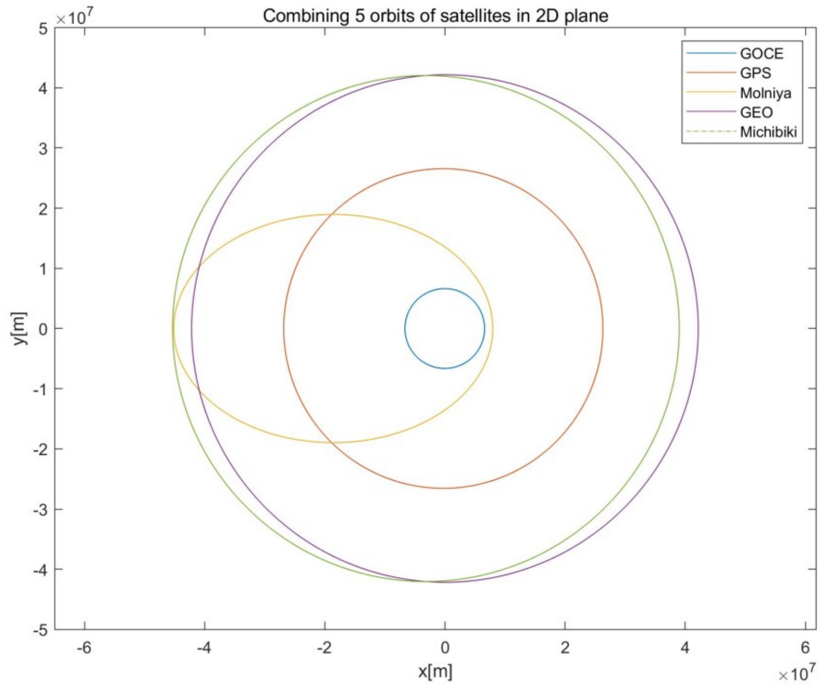

Figure 3.1 Combining five orbits of satellites in a 2D plane

As figure 3.1 shown, five orbits are integrated in to one figure. GEO and Michibiki are geostationary and geosynchronous respectively. This indicates that the semi-major ‘a’ of these orbits are the same. The difference in inclination angle and eccentricity results the difference in orbits. The semimajor of GPS and Molniya are close, but the huge difference in eccentricity (0.01 of GPS and 0.7 of Molniya) results the difference in the shape of orbits. Hence orbit of GPS is approximately a circle while Molniya has an ellipse orbit.

The GOCE has the smallest orbit and the second lowest eccentricity, which makes it is the closest orbit to the earth. Comparing the semimajor of GOCE (6629km) and the radius of earth (6371km), it is obvious that GOCE is a Low Earth Orbit (LEO) Satellite.

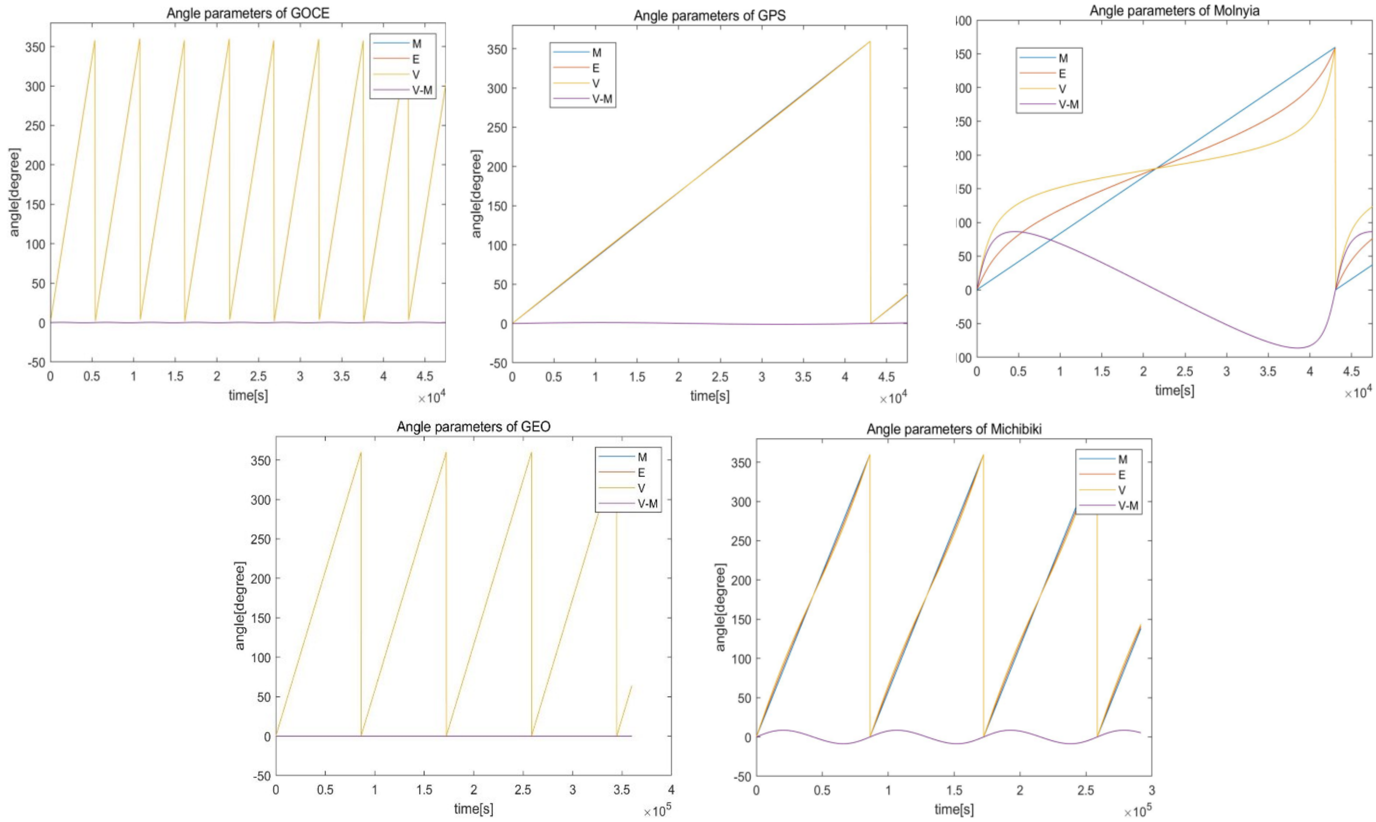

Figure 3.2 Angle parameters (M,E,v,v-M) of five satellites according to time

As shown in figure.3.2, the four angle parameters of five satellites and how they change according to time are present. Among four parameters, M is the angle of the rotation of earth, as the rotation rate w is constant, hence the increase of M is linear, and in this figure the range of M is [0,360] (unit: degrees). E is the eccentric anomaly, as shown in equation (2), the difference between the value of M and E will not exceed eccentricity (e) in radians. The v is the true anomaly.

Based on the analysis and the figure above, it can be observed that only the Molniya satellite exhibits a difference between M and E ranging from 50 degrees to 100 degrees. This is attributed to the ellipticity e of the orbit being 0.7. For the other satellites, the eccentricity e of their orbits is less than 0.1, and for GEO orbits, e is 0, implying a complete overlap between M and E.

Additionally, examining v-M, fluctuations near 0 are observed only for the Molniya and Michibiki satellites, the value can even reach -100 degrees. This is due to their respective orbit eccentricities of 0.7 and 0.075, which are larger than those of the other satellites.

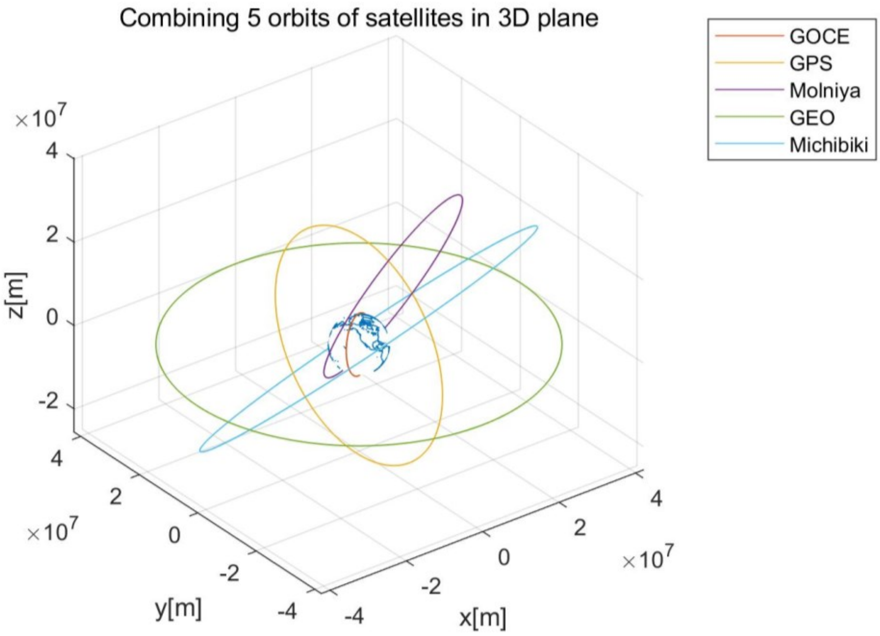

Figure 3.3 Combining five orbits of satellites in the space fixed system

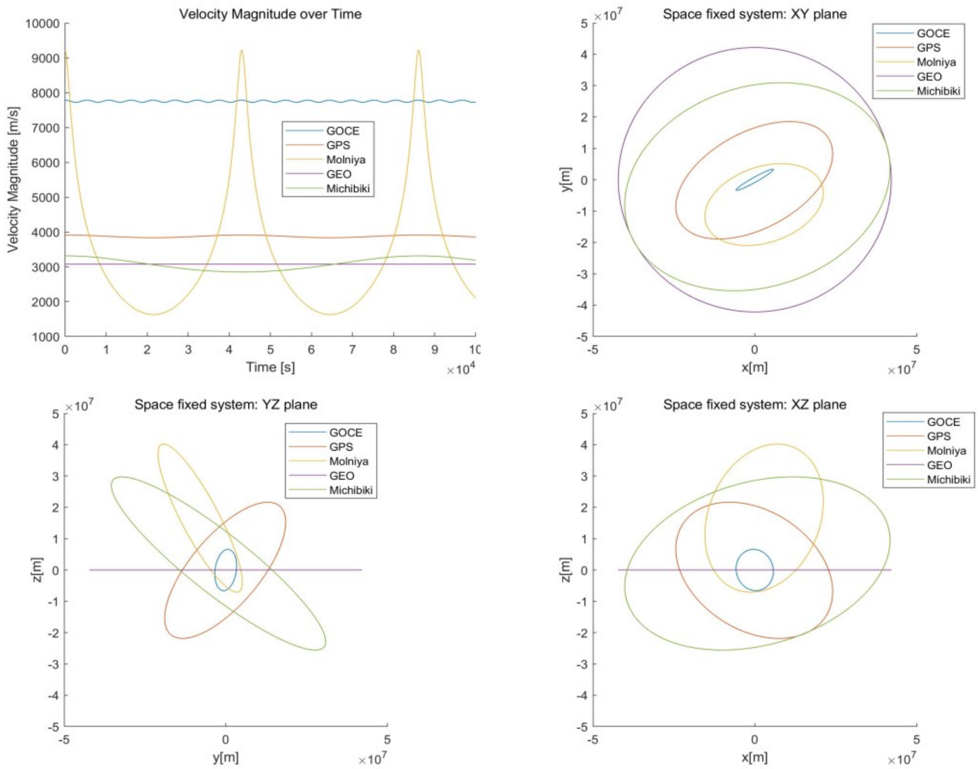

Figure 3.4 Combining five orbits of satellites in the space fixed system (2D and velocity)

3.2 The result and interpretation of Task 2

The result figures are present in figure 3.3 and figure.3.4. Figure.3.3 describes the 3D model of the orbits of five satellites in the space-fixed system.

According to figure.3.3, the observation reveals that GOCE, as a low Earth orbit satellite, closely adheres to the Earth, with its orbit inclination of 96.6 degrees represented in the figure as the red near-Earth orbit perpendicular to the x-y plane.

In contrast, Michibiki and GEO, as geostationary satellites, share the same orbit radius and semi-major axis. However, the orbit inclination of the latter is 0 degrees, resulting in a synchronous orbit parallel to the X-Y plane.

While GPS satellites and Molniya satellites have similar semi-major axes, the eccentricity of the former is 0.01, only 1/70th of the latter's 0.7. Consequently, in the yellow and purple orbits in the figure, there is a significant difference in the distance from the Earth for the perigee due to this substantial difference in eccentricities.

According to figure.3.4, In the top-left corner, the plot illustrates the variation of the magnitude of orbital velocity for five satellites over time. The velocity magnitude is related to the semi-major axis a and the true anomaly v. As per the conclusions drawn in Task 1, the eccentricity of the Molniya satellite results in a significant disparity between M and E at different times. Consequently, the calculated v based on E exhibits pronounced fluctuations over time, leading to periodic oscillations in its velocity magnitude, ranging from 1500 to over 8000 meters per second.

Furthermore, the velocity magnitude is inversely proportional to the square root of the semi-major axis. The figure indicates that GOCE, having the smallest semi-major axis, exhibits the highest average velocity. The satellites with the longest semi-major axes, GEO and Michibiki, have average velocities of approximately 3000 meters per second, which are the lowest among the five satellites.

Figure 3.4 also illustrates the projections of the five orbits onto the xy-plane, xz-plane, and yz-plane in space-fixed frames. Notably, GOCE and GEO exhibit distinctive features. The orbit inclination of GOCE is 96.6 degrees, resulting in an almost linear projection on the xy-plane. In contrast, GEO has an orbit inclination of 0 degrees, leading to a complete orbit projection on the xy-plane, while only a straight line is observable in the other planes.

3.3 The result and interpretation of Task 3

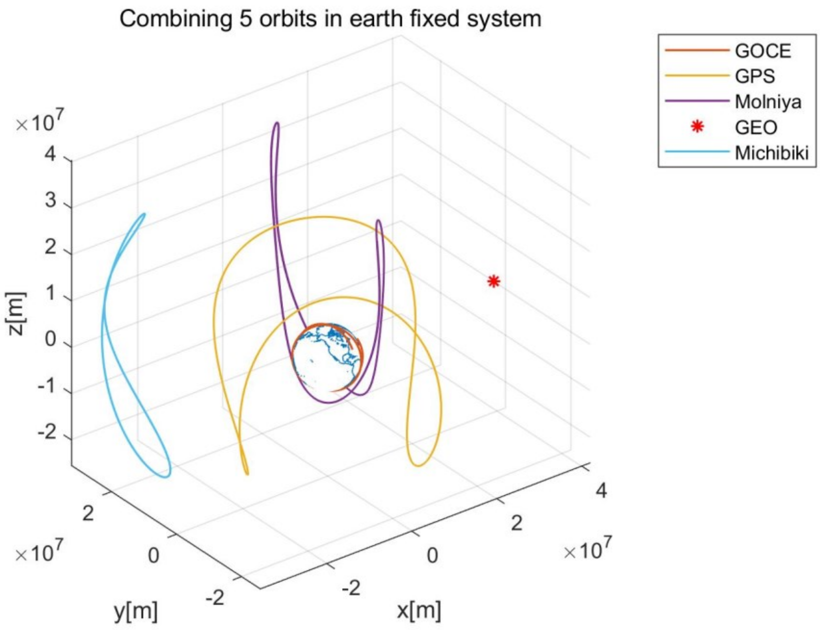

Figure 3.5 Combining five orbits of satellites in the earth fixed system

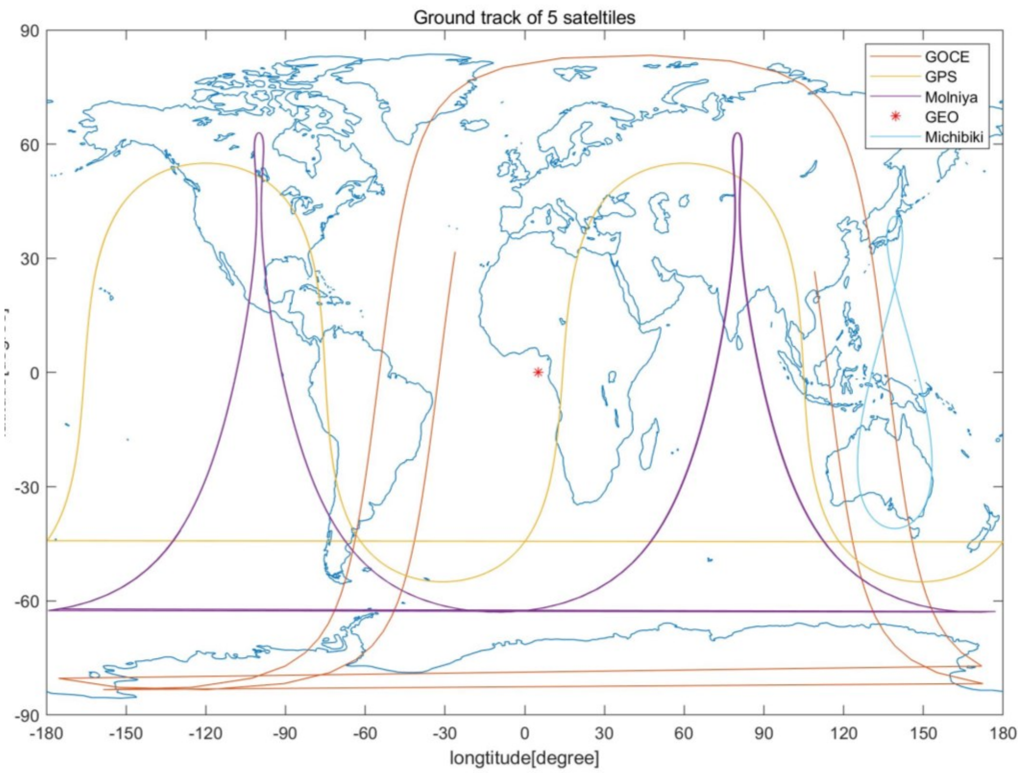

Figure 3.6 Ground track of five satellites

Figures 3.5 and 3.6 depict the orbital coverage of the five satellites around the Earth when the Earth is considered stationary. Figure 3.6 is presented in a 2D format, showcasing the periodic projection of satellite orbits onto the Earth. As observed from the graphics, the GEO satellite, being geostationary, is represented as a single point in both figures.

Following the conversion of the earth-fixed system's three-dimensional coordinates to latitude and longitude, Figure 3.6 provides a clearer depiction, highlighting the variation in the ground projection of the satellite trajectories with each orbit around the Earth. Michibiki, a satellite of Japanese origin, follows an orbit in the shape of a figure-eight pattern, covering only a portion of East Asia in terms of longitude. On the other hand, GOCE, GPS, and Molniya satellites orbit the Earth, and with each orbit, the projected trajectory on the Earth's plane exhibits a fixed offset in longitude.

It can also be observed, in figure 3.5, that GOCE, as a low Earth orbit satellite, continues to closely follow a circular path around the Earth, exhibiting no significant variation across different coordinate systems. The orbits of GPS and Molniya, on the other hand, manifest as saddle-shaped surfaces.

3.4 The result and interpretation of Task 4

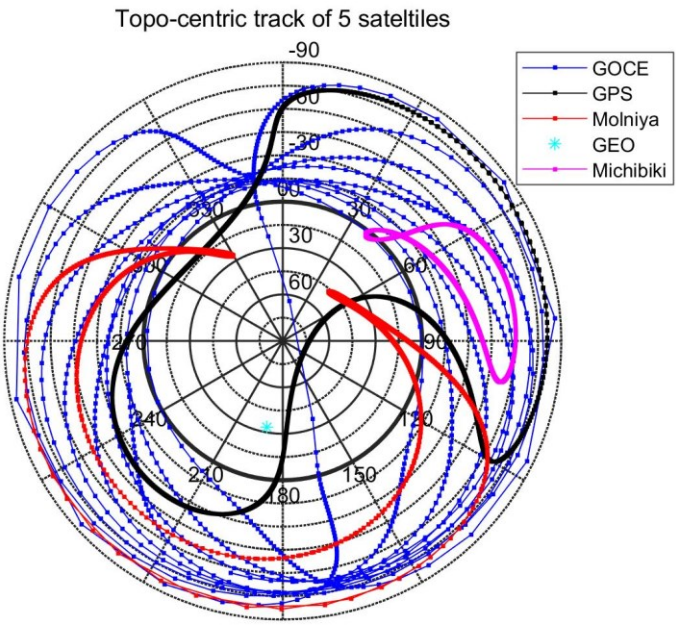

Figure 3.7 Topocentric tracks of five satellites

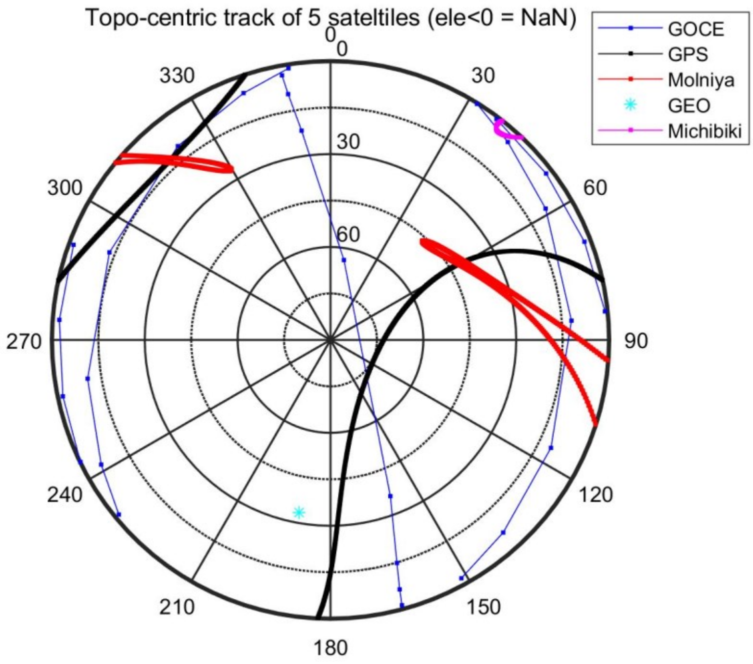

Figure 3.8 Topocentric tracks of five satellites (elevation >0)

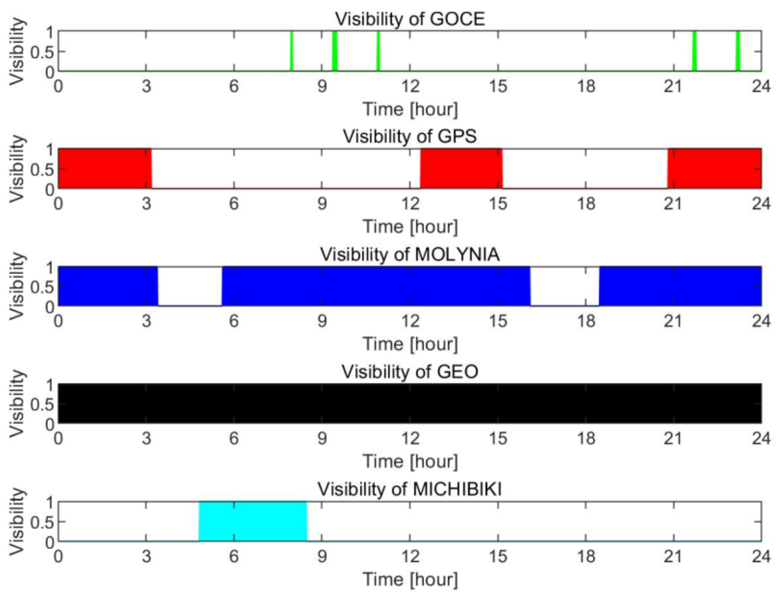

Figure 3.9 Visibility of five satellites (elevation>0)

The figures 3.7-3.9 depict the orbits of five orbits in the topo-centric system. A topo-centric coordinate system is a variant of the geocentric coordinate system, with the observer (a point on the Earth's surface) as the coordinate origin, taking into account the Earth's rotation. In the topo-centric coordinate system, the observer is located at the coordinate origin, and the axes are typically aligned with the horizon, zenith, and north-south directions.

After calculating the three-dimensional position vector (r4) in this coordinate system, a 3D to 2D conversion is still performed to obtain elevation and azimuth. As shown in Figure 3.7, it can be observed that due to GOCE having the shortest period, its trajectory is the longest within the same timeframe, composed of several closed curves. GEO, maintaining its geostationary characteristic, still appears as a single point. The orbit of Michibiki continues to exhibit a figure-eight pattern, approximating the overhead view of its trajectory from above the Earth. The orbits of GPS and Molniya, shown as the black and red closed curves, respectively, resemble saddle surfaces. This characteristic remains consistent with the curves in the earth-fixed system.

Figures 3.8 and 3.9 present further processed results, wherein segments of the five orbits with elevation less than 0 have been removed. Figure 3.8 directly eliminates the portions with elevation less than 0 based on Figure 3.7. It can be observed that, except for the GEO satellite, each orbit's points show a significant reduction. The most notable change is in the Michibiki curve, where only a portion of the trajectory near the upper right corner (0,35) remains. GEO continues to appear as a single point. The curves of GPS and Molniya, representing saddle surfaces, are truncated, leaving only two unconnected segments on the largest radius circle. The GOCE orbit still retains a curve passing through the center, and across multiple periods, not every period intersects the center. This characteristic remains consistent with the features in the earth-fixed system.

Figure 3.9 illustrates the time intervals within 24 hours during which the elevation of each satellite orbit is greater than 0. It is evident that GEO maintains elevation greater than 0 throughout the entire period, while GOCE has five very short intervals (less than 0.5 hours). The GPS, MICHIBIKI, and Molniya satellites exhibit time intervals with elevation less than 0, corresponding to Figure 3.8.