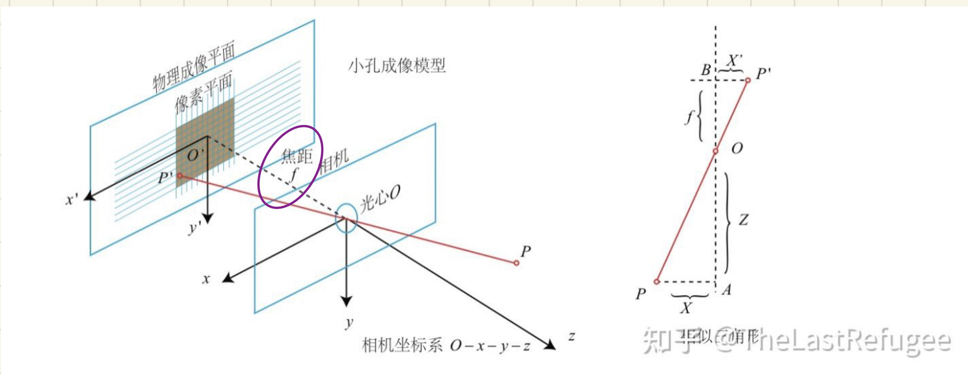
Intrinsic matrix and Exterinsic matrix

$$\text{World} \xleftrightarrow{\text{Ext}} \text{Camera cord} \xleftrightarrow{\text{Int}} \text{Pixel cord}$$

$$s\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_x & 0 & u_0 \\ 0 & \alpha_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{32} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{array}\right] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$= \underbrace{K}_{\text{Int}} \underbrace{[R \mid t]}_{\text{Ext}} \underbrace{Q}_{\text{世界座標系下的某一点}}$$

Int :



$\because$ Similar triangle

$$\therefore \frac{Z}{f} = \frac{X}{X'} = \frac{Y}{Y'}$$

Suppose a point $P=[X,Y,Z]^T$ in the world cord and a $P'=[X',Y',Z']^T$ in the pixel cord.

So, we can get

$$X' = X\frac{f}{Z} \quad , \quad Y' = Y\frac{f}{Z}$$

The above funtions describe the relation between the world cord and pixel cord.

But the original point of pixel cord is at the upper left corner.

So, we suppose P' cord is $[u,v]^T$ in the pixel cord, and we can get

$$u = \underline{\alpha} X' + C_x$$

$$v = \underline{\beta} Y' + C_y$$

α.β 表示縮放倍數

C_x, C_y 表示平移

$$u = \alpha X\frac{f}{Z} + C_x = X\frac{f_x}{Z} + C_x$$

$$v = \beta Y\frac{f}{Z} + C_y = Y\frac{f_y}{Z} + C_y$$

f_x and f_y unit is pixel

We can use homogeneous cord rewritten the formula.

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{Z} \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Ext:

Int matrix only tells us how to transform between camera cord and pixel cord. We still need a matrix to transform between camer cord and world cord.

$$P = \overset{\text{旋轉矩陣}}{R}\,\underbrace{P_w}_{\text{世界座標系下的点}} + \overset{\text{平移矩陣}}{t}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\therefore P_{uv} = \frac{1}{Z} \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \overbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}}^{\text{World cord to Camera cord}} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

繞 X 軸:

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}$$

繞 Y 軸

$$R_y(\theta_y) = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix}$$

繞 Z 軸

$$R_z(\theta_z) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = R_x(\theta_x)\,R_y(\theta_y)\,R_z(\theta_z)$$

5個未知數

3個未知數 $\theta_x, \theta_y, \theta_z$

3個未知數 t_x, t_y, t_z

$$P' = \underset{3\times4}{M}\,\underset{4\times1}{P_w} = \underbrace{K}_{\text{內參}}\underbrace{[R\ \ T]}_{\text{外參}} P_w$$

3x4　　4x4　　4x1

$5+3+3=11$

總共有11個未知數

$$M = \underbrace{\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}}_{\text{投影矩陣}} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \begin{array}{l} 1\times4 \\ 1\times4 \\ 1\times4 \end{array}$$

$$P' = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \underbrace{\begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}}_{P_i,\ 4\times1}$$

$$= \begin{bmatrix} m_{11}X_w + m_{12}Y_w + m_{13}Z_w + m_{14} \\ m_{21}X_w + m_{22}Y_w + m_{23}Z_w + m_{24} \\ m_{31}X_w + m_{32}Y_w + m_{33}Z_w + m_{34} \end{bmatrix}$$

$$= \begin{bmatrix} m_1 P_i \\ m_2 P_i \\ m_3 P_i \end{bmatrix} \xRightarrow{\ H \to E\ } \begin{bmatrix} \dfrac{m_1 P_i}{m_3 P_i} \\[2ex] \dfrac{m_2 P_i}{m_3 P_i} \end{bmatrix}$$

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}_{3\times 4}$$ 共有 11 個未知數

而一對点 $\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \frac{m_1 P_i}{m_3 P_i} \\ \frac{m_2 P_i}{m_3 P_i} \end{bmatrix}$ 可以提供 2 個

方程，所以至少需要 6 個点

$$u_i = \frac{m_1 P_i}{m_3 P_i}, \quad u_i m_3 P_i = m_1 P_i, \quad u_i m_3 P_i - m_1 P_i = 0$$

$$v_i = \frac{m_2 P_i}{m_3 P_i}, \quad v_i m_3 P_i = m_2 P_i, \quad v_i m_3 P_i - m_2 P_i = 0$$

若有 n 個点，則：

$$\begin{bmatrix} \boxed{\begin{matrix} u_1 m_3 P_1 - m_1 P_1 = 0 \\ v_1 m_3 P_1 - m_2 P_1 = 0 \end{matrix}} \\ \boxed{\begin{matrix} u_2 m_3 P_2 - m_1 P_2 = 0 \\ v_2 m_3 P_2 - m_2 P_2 = 0 \end{matrix}} \\ \vdots \\ \boxed{\begin{matrix} u_n m_3 P_n - m_1 P_n = 0 \\ v_n m_3 P_n - m_2 P_n = 0 \end{matrix}} \end{bmatrix}$$

$$P \stackrel{\text{def}}{=} \begin{bmatrix} P_1^T & 0^T & -u_1 P_1^T \\ 0^T & P_1^T & -v_1 P_1^T \\ & \vdots & \\ P_n^T & 0^T & -u_n P_n^T \\ 0^T & P_n^T & -v_n P_n^T \end{bmatrix}_{2n \times 12}$$

1x4 1x4 1x4

$$M = \begin{bmatrix} m_1^T \\ m_2^T \\ m_3^T \end{bmatrix}_{12\times 1}$$

4x1 4x1 4x1

$$
PM = \begin{bmatrix} p_1^T & 0^T & -u_1 p_1^T \\ & \vdots & \\ & & \end{bmatrix} \begin{bmatrix} m_1^T \\ m_2^T \\ m_3^T \end{bmatrix}
$$

$$
= \begin{bmatrix} \overset{1\times4}{p_1^T}\,\overset{4\times1}{m_1^T} - u_1 \overset{1\times4}{p_1^T}\,\overset{4\times1}{m_3^T} \\ \vdots \\ \\ \end{bmatrix}
$$

已知 P $M = 0$ 齊性線性方程

但不要 0 解，所以只能求非0解.

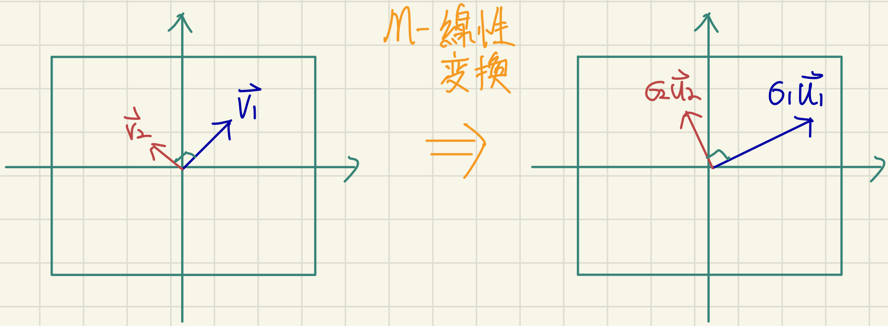
即 minmize $\|PM\|$

SVD:

https://m.youtube.com/watch?v=OIe48iAqh8E

$$M = \underbrace{U}_{\text{旋轉}}\ \underbrace{\Sigma}_{\text{拉伸}}\ \underbrace{V^T}_{\text{旋轉}}$$

M為一個矩陣，所以可以將其看成 linear transformation. 假設 M為一 2x2 矩陣.



$V = [\vec{v_1}, \vec{v_2}]$

$U = [\vec{u_1}, \vec{u_2}]$

$$\underbrace{\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\text{拉伸作用}}$$

$$U\Sigma = [\vec{u_1}\sigma_1, \vec{u_2}\sigma_2]$$

$$MV = U\Sigma$$

$$M = U\Sigma \underbrace{V^{-1}}_{\text{orthogonal matrix}}$$

$$= U\Sigma V^T$$

solve SVD：求 (V)

$$M = U\Sigma V^T$$

$$M^TM = \cancel{(U\Sigma V^T)^T} U\Sigma V^T$$

$$= \cancel{(V^T)^T \Sigma^T U^T} U\Sigma V^T$$

$$= \cancel{V\Sigma U^T} U\Sigma V^T$$

$\boxed{(AB)^T = B^TA^T}$

$\boxed{U \text{ 是一個正交基底所以 } U^T = U^{-1},\ U^TU = I}$

$\boxed{\because \text{ 是一個對稱矩陣 } \Sigma^T = \Sigma}$

$$= V\Sigma\Sigma V^T$$

Let $L = \Sigma\Sigma$, $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$

$$M^TM = VLV^T$$

$$M^TMV = VL \longrightarrow M^TM \text{ 特征向量公式 } M^TM\vec{v} = \lambda\vec{v}$$

所以說，V 就是 M^TM 的 eigen vector，

L 為 eigen value.

求 U：

$$M = U\Sigma V^T$$

$$MM^T = U\Sigma V^T(U\Sigma V^T)^T$$

$$= U\Sigma V^TV\Sigma U^T$$

$= U\Sigma\Sigma U^T$,

let $\Sigma\Sigma = B$

$MM^T = UBU^T$

$MM^TU = UB$

所以說，U 就是 MM^T 的 eigen vector，

B 為 eigen value.

利用SVD求解 $\|PM\| = 0$:

$\min\|PM\|^2 = \min\|\underbrace{U}_{\text{正交矩陣，不影響 norm}}DV^TM\|^2 = \min\|DV^TM\|^2$

約束條件 $\|X\| = 1$:

將 P 進行 svd 分解後，由於奇異值越往右越小，可將其視為 0. 而當奇異值中最小的數值為 0 的話，則代表 $Pm = 0$ 有無限多組解。

proof:

$\because P = UDV^T$，D為 PP^T or P^TP 中的 eigen value，且若矩陣的 eigen value 有 0 出現的話，則可證明 $rank(A) < m$，具有無限多組解.

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \quad Ax = 0，求 x = ?$$

Eigen value of A :

$$\det(A - \lambda I) = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda) - 2 = 0$$

$$\Rightarrow 2 - 2\lambda - \lambda + \lambda^2 - 2 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda = 0$$

$$\Rightarrow \lambda(\lambda - 3) = 0, \quad \lambda = \boxed{0}, 3$$

求 x :

$$2x_1 + x_2 = 0, \quad x_1 = -\frac{1}{2}x_2$$

$$null(A) = \left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

由上可知，這樣會有無限多組解，所以在求解時，
會為了方便計算、求解，設定一約束條件 $\|x\|=1$

證明 P 矩陣的最小奇異值對應的 Eigen vector 為
$\min\|Ax\|$ 的最優解：

$$\min\|Ax\|^2 = \min\|\underbrace{\overset{m\times r}{U}}_{\text{正交矩陣，不影響 norm}}\overset{r\times r}{D}\overset{r\times n}{V^T}\overset{n\times 1}{x}\|^2 = \min\|\overset{r\times r}{D}\overset{r\times n}{V^T}\overset{n\times 1}{x}\|^2$$

令 $\overset{r\times 1}{y} = \overset{r\times n}{V^T}\overset{n\times 1}{x}$ ，$\because \|x\|=1 \quad \therefore \|y\|=1$

$$\Rightarrow \min\|\overset{r\times r}{D}\overset{r\times 1}{y}\|^2 = \min\|(Dy)^T Dy\| = \min\|\underset{1\times r}{y^T}\underbrace{D^T D}_{r\times r}\underset{r\times 1}{y}\|$$

$$D^T D = \begin{bmatrix} \sigma_0^2 & 0 & \cdots & \cdot \\ 0 & \sigma_1^2 & 0 & \cdots \\ \vdots & \vdots & \sigma_2^2 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

$$y^T D^T D y = [y_0 \;\; y_1 \;\; y_2]\begin{bmatrix} \sigma_0^2 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_2^2 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

$$= y_0^2\sigma_0^2 + y_1^2\sigma_1^2 + y_2^2\sigma_2^2, \quad \sigma_0 > \sigma_1 > \sigma_2$$

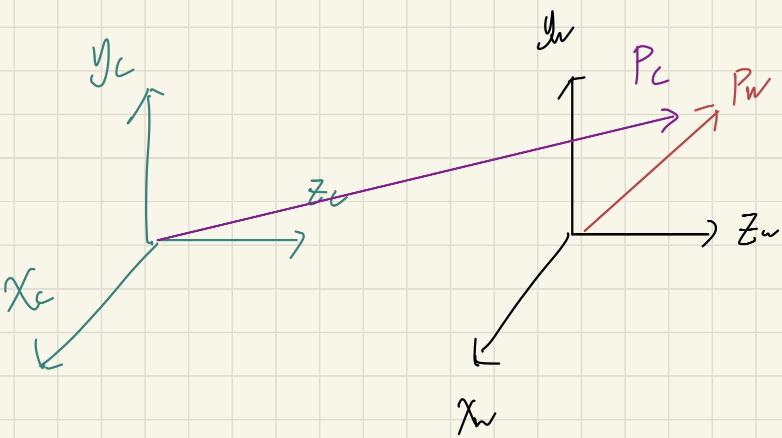
let $y=[0,0,1]$，這樣做既可以滿足 $\|y\|=1$

又可以讓 $\|y^T D D^T y\|$ 最

分解出内、外參矩陣：

$$m = \underbrace{K}_{\text{内參}}\underbrace{[R \mid t]}_{\text{外參}}$$

$$= [\underbrace{KR}_{3\times 3} \mid \underbrace{Kt}_{3\times 1}]$$



review：當想要將世界座標系下的一点，轉換到相机座標系下的一点，則：

$$P_c = [R \mid t] P_w$$

$$= E\, P_w$$

Q：那世界座標到相机座標系的距離呢？

要先变成齊次矩陣後再求inv

$$dis = \left\| E \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{世界座標的原点}} \right\| = \left\| \boxed{E^T} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{相机座標的原点}} \right\|$$

吉文斯 (Givens) 旋轉：

利用一個旋轉矩陣將一向量剛剛好轉移到某一軸上，即

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad A = \begin{bmatrix} a \\ b \end{bmatrix}$$

s.t. $RA = \begin{bmatrix} r \\ 0 \end{bmatrix}$，令 $\cos\theta = c$，$\sin\theta = s$

$$\therefore \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

$\because$ R為旋轉矩陣 $\therefore \|A\| = r = \sqrt{a^2+b^2}$

$$\therefore \begin{cases} ca - sb = \sqrt{a^2+b^2} & ① \\ sa + cb = 0 & ② \end{cases}$$

$$sa = -cb, \quad s = -\frac{cb}{a} \quad ③$$

③ → ① :

$$ca + \frac{cb^2}{a} = \sqrt{a^2+b^2}$$

$$c\left(a + \frac{b^2}{a}\right) = c\left(\frac{a^2+b^2}{a}\right) = \sqrt{a^2+b^2}$$

$$c = \frac{\sqrt{a^2+b^2}}{\frac{a^2+b^2}{a}} = \frac{a\sqrt{a^2+b^2}}{a^2+b^2} = \frac{a}{\sqrt{a^2+b^2}} \ \#$$ ④

④ → ③ :

$$s = -\frac{cb}{a} = -\frac{a}{\sqrt{a^2+b^2}} \times \frac{b}{a} = -\frac{b}{\sqrt{a^2+b^2}} \ \#$$

QR分解得出內外參:

將一矩陣分解成一上三角矩陣和一正交矩陣的內積.

$$A = \underbrace{R}_{\text{上三角矩陣}}\ \underbrace{Q}_{\text{正交矩陣}}$$

ex:

$$A = \begin{bmatrix} 0 & -15 & 14 \\ 4 & 32 & 2 \\ 3 & -1 & 4 \end{bmatrix}$$ ，利用Givens進行QR分解.

想將這邊的元素都變0

繞 x 軸:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

繞 y 軸:

$$\begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix}$$

繞 z 軸:

$$\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

step 1: 先將 A_{21} 變為 0 (利用繞 z 軸)

$$R_{21}A = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{c} a \\ b \\ \\ \end{array} \begin{bmatrix} 0 & -15 & 14 \\ 4 & 32 & 2 \\ 3 & -1 & 4 \end{bmatrix}$$

$$c = \frac{a}{\sqrt{a^2+b^2}} = \frac{0}{\sqrt{0^2+4^2}} = 0$$

$$s = -\frac{b}{\sqrt{a^2+b^2}} = \frac{-4}{4} = -1$$

$$R_{21}A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -15 & 14 \\ 4 & 32 & 2 \\ 3 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 32 & 2 \\ \boxed{0} & 15 & -14 \\ 3 & -1 & 4 \end{bmatrix}$$

step 2: 將 A_{31} 變為 0 (利用繞 y 軸)

$$R_{31}R_{21}A = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix} \begin{array}{c} a \\ \\ b \end{array} \begin{bmatrix} 4 & 32 & 2 \\ 0 & 15 & -14 \\ 3 & -1 & 4 \end{bmatrix}$$

$$C = \frac{a}{\sqrt{a^2+b^2}} = \frac{4}{\sqrt{4^2+3^2}} = \frac{4}{5}$$

$$S = -\frac{b}{\sqrt{a^2+b^2}} = -\frac{3}{\sqrt{4^2+3^2}} = -\frac{3}{5}$$

$$R_{31}R_{21}A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ 0 & 1 & 0 \\ -\frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 4 & 32 & 2 \\ 0 & 15 & -14 \\ 3 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 25 & 4 \\ 0 & 15 & -14 \\ 0 & -20 & 2 \end{bmatrix}$$

step 3：將 A_{32} 變為 0 (繞 x 軸)

$$R_{32}R_{31}R_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C & -S \\ 0 & S & C \end{bmatrix} \begin{bmatrix} 5 & 25 & 4 \\ 0 & {\scriptstyle a}\,15 & -14 \\ 0 & {\scriptstyle b}\,-20 & 2 \end{bmatrix}$$

$$C = \frac{15}{\sqrt{15^2+20^2}} = \frac{3}{5}$$

$$S = \frac{20}{\sqrt{15^2+20^2}} = \frac{4}{5}$$

$$R_{32}R_{31}R_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 & 25 & 4 \\ 0 & 15 & -14 \\ 0 & -20 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 25 & 4 \\ 0 & 25 & -10 \\ 0 & 0 & -10 \end{bmatrix} = B$$

$\therefore$

$$R_{32}R_{31}R_{21}A = B$$

$$A = R_{32}^{-1}R_{31}^{-1}R_{21}^{-1}B$$

$$= R_{32}^{T}R_{31}^{T}R_{21}^{T}B$$

$$= \underbrace{(R_{21}R_{31}R_{32})^{T}}_{\text{正交矩陣}}\underbrace{B}_{\text{上三角矩陣}} = QR$$

RQ分解:

由剛剛的QR分解得到的是正交矩陣，上三角矩陣，但我們要的是反過來的，所以需要利用以下方法。

def: $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$，可得知 $P^{T} = P = P^{-1}$，

$$PP = I$$

$$\mathbf{M} = \begin{bmatrix} 0 & m1 & m2 \\ 0 & 0 & m3 \\ 0 & 0 & 0 \end{bmatrix}$$

则：

$$\mathbf{MP} = \begin{bmatrix} 0 & m1 & m2 \\ 0 & 0 & m3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} m2 & m1 & 0 \\ m3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{PM} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & m1 & m2 \\ 0 & 0 & m3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & m3 \\ 0 & m1 & m2 \end{bmatrix}$$

step1:

$$A' = PA$$

step2:

$$A'^T = QR$$

step3: 将 step1 代入 step2

$$(PA)^T = QR$$

$$P^T A^T = QR$$

$$A^T = QRP$$

$$A = (QRP)^T$$

$$= P^T R^T Q^T$$

$$= P\underbrace{R^T}_{\text{下三角矩陣}}\underbrace{Q^T}_{\text{正交矩陣}}$$

要將一個下三角矩陣轉為一上三角矩陣，則只要將 R^T 先左右翻再上下顛倒，

$$PR^TP : \text{上三角矩陣}$$

$\therefore$

$$A = \underbrace{(PR^TP)}_{\text{上三角矩陣}}\underbrace{(\underbrace{P}_{\text{為了抵掉多出来的}P}Q^T)}_{\text{正交矩陣}}$$