Gravitational Waves

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Word count:

1 Part 1

Part 1 of the project with generation of the model template, followed by the calculation of the amplitude spectral density of the signal. The antenna pattern functions are then defined, followed by the calculation of the optimum SNR and the average SNR of the signal.

1.1 Waveform modelling

The waveform is modelled using the `get_fd_waveform` function from the `PyCBC` library. The function takes the following arguments:

1. `approximant`: The waveform approximant to use. In this project, we use the `IMRPhenomD` approximant.
2. `mass1 and mass2`: The masses of the two compact objects in the binary system. In this project, we use $m_1 = 30M_\odot$ and $m_2 = 30M_\odot$.
3. `distance`: The distance to the source. $d = 500$ Mpc.
4. `f_low`: The lower frequency cutoff of the waveform. $f_{\text{low}} = 10$ Hz.
5. δf: The frequency step size of the waveform. $\delta f = 1/16$ Hz.
6. `inclination`: The inclination angle of the source. $\iota = \pi/3$.
7. `coalescence_phase`: The coalescence phase of the source. $\phi_c = 0$.

The waveform generated, in the frequency domain, is shown in Figure 1.

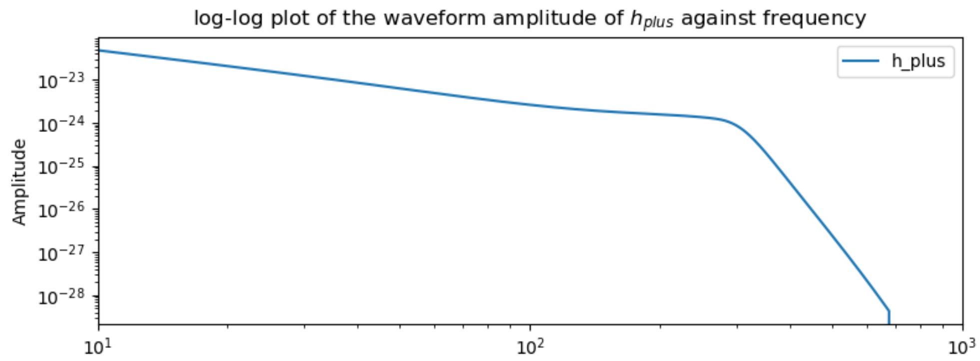

Fig. 1: log-log plot of waveform of the signal.

Figure 1 shows the waveform of the signal. The approximate frequency at which the merger of the two black holes occurs is around 600 Hz.

1.2 Amplitude spectral density

From the file `ASD.txt`, the amplitude spectral density of the signal is loaded and plotted against frequency, shown as the orange line in Figure 2.

On the same axis, the square root of the frequency multiplied by the plus polarization of the waveform:

$$\sqrt{f} \times |h_+(f)| \tag{1}$$

is plotted as the blue line in Figure 2.

Given that the unit of strain measured from the detector is dimensionless, the unit of the amplitude spectral density is $\frac{1}{\sqrt{\text{Hz}}}$.

The generated waveform has a unit of 1/Hz, therefore, the expression $\sqrt{f} \times |h_+(f)|$ also has unit of $1/\sqrt{\text{Hz}}$.

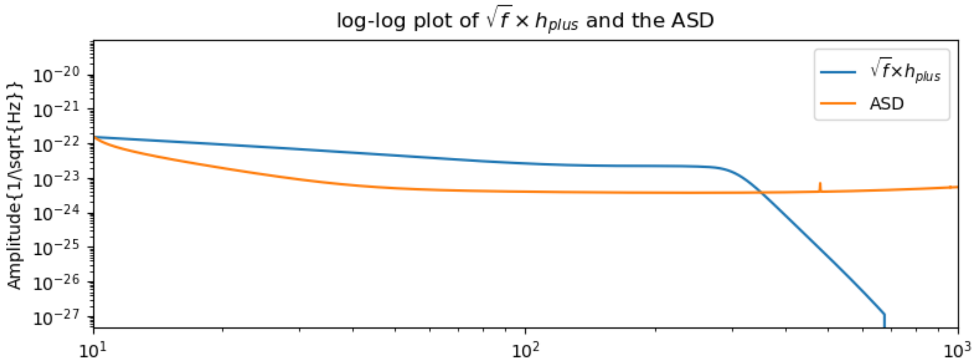

Fig. 2: log-log plot of the amplitude spectral density of the signal and the square root of the frequency multiplied by the plus polarization of the waveform.

1.3 Antenna Pattern Functions

For the L-shaped detector, The antenna pattern functions are defined as[2]:

$$F_+(\theta, \phi, \psi) = \frac{1}{2}(1 + \cos^2(\theta)) \cos(2\phi) \cos(2\psi) - \cos(\theta) \sin(2\phi) \sin(2\psi) \tag{2}$$

$$F_\times(\theta, \phi, \psi) = \frac{1}{2}(1 + \cos^2(\theta)) \cos(2\phi) \sin(2\psi) + \cos(\theta) \sin(2\phi) \cos(2\psi) \tag{3}$$

where $F_+(\theta, \phi, \psi)$ and $F_\times(\theta, \phi, \psi)$ are the plus and cross antenna pattern functions respectively. The parameters θ, ϕ and ψ are the angle between the line of sight to the source and the normal to the detector plane, the azimuthal angle of the source in the detector plane, and the polarization angle of the source respectively.

These function determines how sensitive a detector is to each of the two polarizations of the gravitational waves coming from different points in the sky.

The detector output $\tilde{h}(t)$ is given by:

$$\tilde{h}(t) = F_+(\theta, \phi, \psi)\tilde{h}_+(t) + F_\times(\theta, \phi, \psi)\tilde{h}_\times(t) \tag{4}$$

where $\tilde{h}_+(t)$ and $\tilde{h}_\times(t)$ are the plus and cross polarizations of the waveform respectively.

1.4 Optimum SNR

The optimum SNR is given by:

$$(\text{optimum SNR})^2 = 4 \int_{f_\text{lower}}^{f_\text{upper}} \frac{|\tilde{h}(f)|^2}{S(f)} \, \mathrm{d}f \tag{5}$$

The optimum SNR is the inner product of the signal with itself, weighted by the inverse of the density spectral density.

There are five parameters that can ßbe varied to maximize the SNR, three are parameters of the antenna pattern functions, θ, ϕ and ψ, and the other two are the inclination angle, ι, and the polarization angle, Ψ.

1. θ is the angle between the line of sight to the source and the normal to the detector plane. To maximize the SNR, θ should be set to 0 or π,representing the source being directly overhead or directly below the detector.
2. ϕ is the azimuthal angle of the source in the detector plane. In the case of the L-shaped detector, and with the source directly overhead, ϕ has no effect on the SNR.
3. ψ is the polarization angle of the source. In the case of the L-shaped detector, and with the source directly overhead, ψ has no effect on the SNR.
4. ι is the inclination angle of the source, which describes the normal of the plane of detector arms to the normal of the orbital plane of the binary system. The SNR is maximise when $\iota = 0$ or π, representing the two planes are parallel or anti-parallel to each other.
5. Ψ is the phase factor of the waveform, which has no effect on the SNR.

In summary, we choose the following parameters to maximize the SNR:

$$\theta = 0 \tag{6}$$
$$\phi = 0 \tag{7}$$
$$\psi = 0 \tag{8}$$
$$\iota = 0 \tag{9}$$
$$\Psi = 0 \tag{10}$$

The optimum SNR of the signal, using the optimum parameters mentioned above is calculated to be 37.933.

1.5 Average SNR

The average SNR is achieved by averaging the SNR over all possible values of θ, ϕ, ψ, ι and Ψ:

$$(\text{average SNR})^2 = 4\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \text{SNR}(\theta,\phi,\psi,\iota,\Psi)P(\theta,\phi,\psi,\iota,\Psi)\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\iota\,\mathrm{d}\Psi \tag{11}$$

The average SNR is a function of the probability distribution function(PDF) of the parameters, $P(\theta,\phi,\psi,\iota,\Psi)$. The PDF of each parameter is assumed to be independent, i.e. $P(\theta,\phi,\psi,\iota,\Psi) = P(\theta)P(\phi)P(\psi)P(\iota)P(\Psi)$. The distribution of each parameter:

$$\begin{aligned}
P(\theta) &= X \sim \text{Uniform}(0,\pi)\\
P(\phi) &= X \sim \text{Uniform}(0,2\pi)\\
P(\psi) &= X \sim \text{Uniform}(0,2\pi)\\
P(\iota) &= X \sim \text{Uniform}(0,\pi)\\
P(\Psi) &= X \sim \text{Uniform}(0,2\pi)
\end{aligned}$$

In this project, a sample of 10^6 points, using the distribution functions mentioned above. A histogram of the SNR is plotted, to sense check the distribution and maximum value of the SNR, shown in Figure 3.

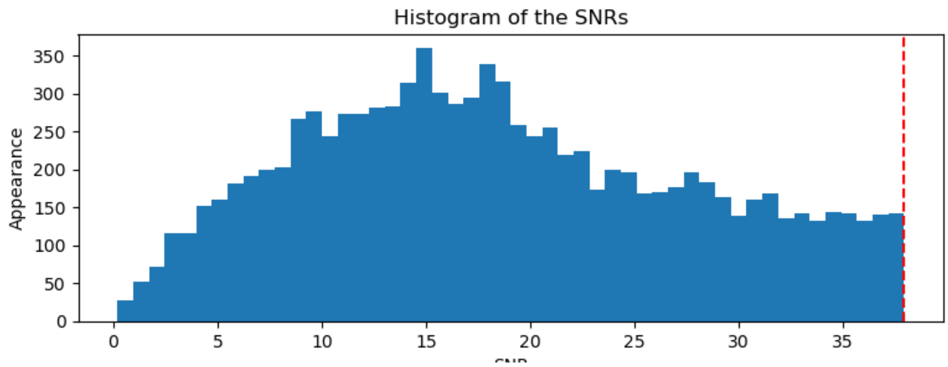

Fig. 3: Histogram of distribution of the SNR, with the optimum SNR marked as a vertical line.

Figure 3 shows that the SNR has a distribution from 0 to 38, with the maximum value of the average SNR being 18.404.

2 Part 2:

Part 2 of the coursework involved implementing a matched filter to detect the presence of a signal using a known signal templated. We first whitened the data using the expected power spectral density of the signal. The whitened data was then used to calculate the matched filter signal-to-noise ratio (SNR) at each time step in the time series. The time at which the SNR is maximised is the time at which the signal is detected.

2.1 Data Processing

The simulated data series is loaded from `data.txt` and plotted against time, shown in Figure 4

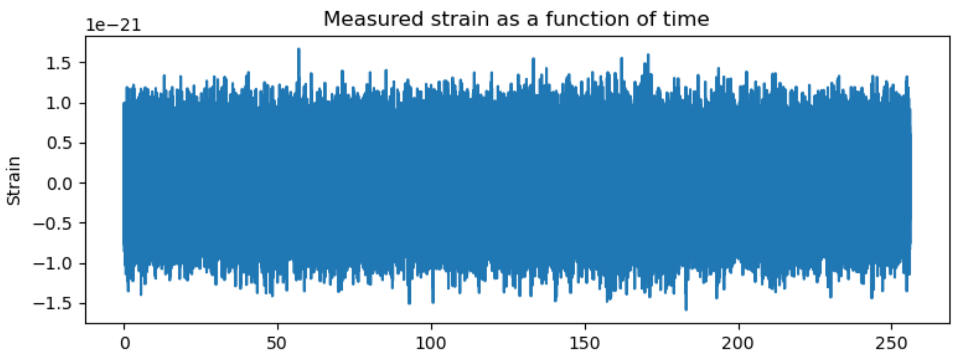

Fig. 4: Simulated data series plotted against time

Here the data is a time series of a signal mixed with noise, therefore the signal is not easily distinguishable.

By using a Welch periodogram, the PSD of the signal is calculated. The ASD is then calculated by taking the square root of the PSD and plotted against frequency, shown in Figure 5.

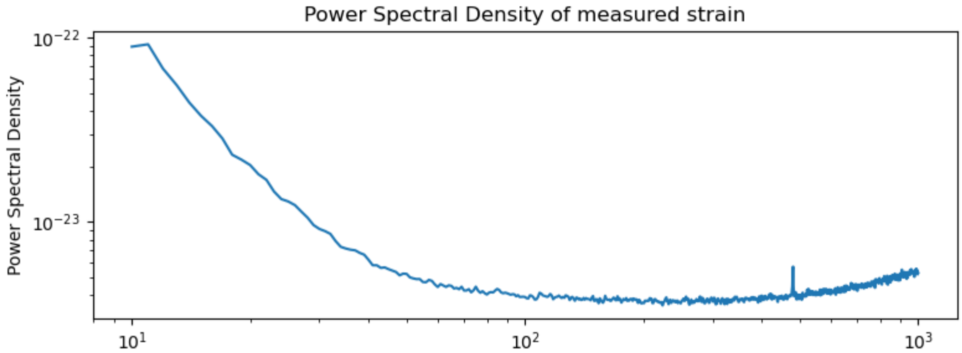

Fig. 5: Power spectral density of the simulated data series

The measured ASD is then compared with the expected amplitude spectral density of the signal from Part 1, shown in Figure 6. The expected ASD from the previous part is the ASD without the signal. The residuals are calculated as the difference between the measured ASD and the expected ASD. The residuals are then plotted against frequency in the bottom plot of Figure 6.

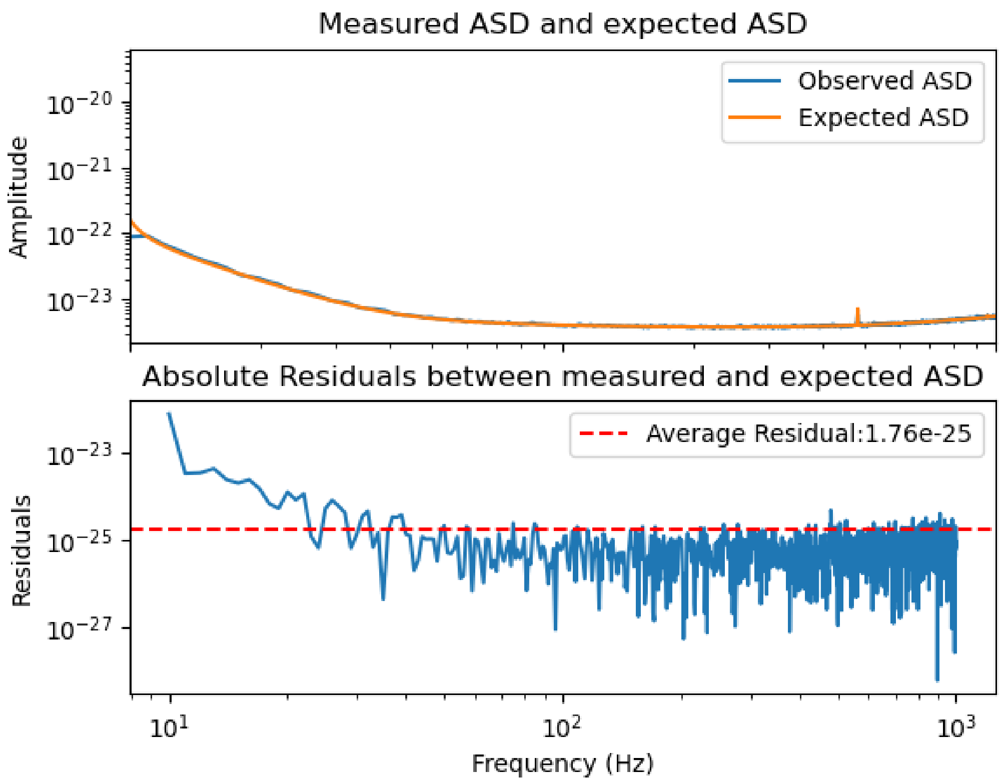

Fig. 6: Upper: comparison of the observed power spectral density with the expected power spectral density. The expected ASD(ASD without the signal) is largely overlapping with the measured ASD.
Lower: the absolute residuals are shown in the bottom plot, the residuals are within the expected range of 10^{-25}, which is about 1 order of magnitude smaller than the expected ASD.

The Figure 6 shows that the measured ASD is largely overlapping with the expected ASD. The residuals are within the expected range of 10^{-25}, which is about 1 order of magnitude smaller than the expected ASD. This indicates that the measured ASD is consistent with the expected ASD, while the residual is accounted for by the signal.

2.2 Whitening the data

The expected PSD is used to whiten the data. The whitened filter is defined as:

$$\text{whitened filter}(f) = \frac{1}{\sqrt{PSD(f)}} \tag{12}$$

The whitened filter is then normalised such that the maximum value of the filter is 1.

$$\text{whitened filter} = \frac{\text{whitened filter}}{\max(\text{whitened filter})} \tag{13}$$

The Fourier transform of the data series, $\tilde{d}(f)$ is calculated. The filter is then applied to the $\tilde{d}(f)$, before taking the inverse Fourier transform to obtain the whitened data series.

The whitened data is then plotted against time, shown in Figure 7.

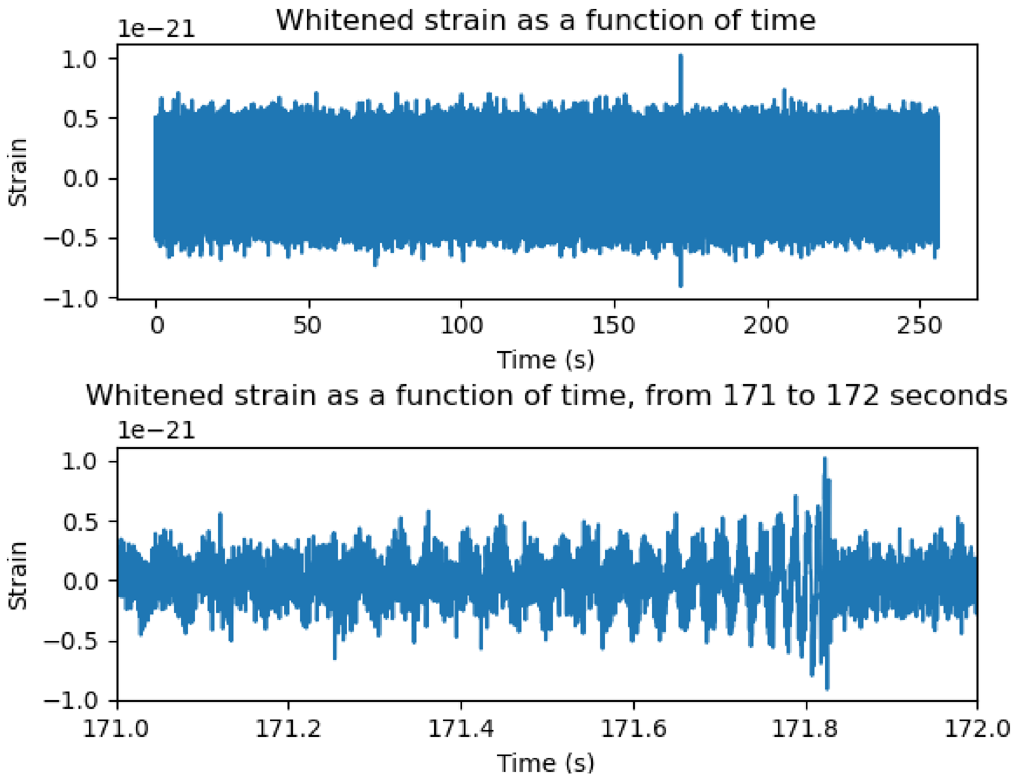

Fig. 7: Upper: the whitened data series is plotted against time. A signal is observed at around 172s.
Lower: the whitened data series is plotted against time zoomed in to time domain from 171.0s to 172.0s. A signal cam be observed at around 171.8s. It fits the expected shape of a waveform: initially, the amplitude is small; as the orbiting objects draw closer and their orbital speed increases, the amplitude grow, and the frequency gets higher.

From the lower graph in Figure 7, a signal is observed at around 171.8s. We observed the expected shape of a waveform: initially, the amplitude is small; as the black holes draw closer and their orbital speed increases, the amplitude grow, and the frequency gets higher. At time around 171.8, we observed the "chirp" signal: the frequency increases rapidly before the merger.

2.3 Matched Filter

2.3.1 Background Theory on Matched Filter

The observed data is mixed with noise.

$$d(t) = s(t) + n(t) \tag{14}$$

where $d(t)$ is the observed data, $s(t)$ is the signal, and $n(t)$ is the noise.

The output of the matched filter, matched filter SNR, is the convolution of template $h(t)$ with the whitened data $d(t)$. In Fourier space this is equivalent to the product of the Fourier transform of the template $\tilde{h}(f)$ and the Fourier transform of the whitened data $\tilde{d}(f)$, weighted by the inverse of the power spectral density of the noise $S(f)$.

$$(\text{matched filter SNR})^2 = \int_{-\infty}^{\infty} \mathrm{d}f \frac{\tilde{h}(f)\tilde{d}^*(f)}{S(f)} \tag{15}$$

This integral can be breakdown into the sum of two terms, using equation 14:

$$(\text{matched filter SNR})^2 = \int_{-\infty}^{\infty} \mathrm{d}f \frac{\tilde{h}(f)\tilde{s}^*(f)}{S(f)} + \int_{-\infty}^{\infty} \mathrm{d}f \frac{\tilde{h}(f)\tilde{n}^*(f)}{S(f)} \tag{16}$$

where the first term is the signal-to-noise ratio (SNR) of the signal and the second term averages to zero as the noise is uncorrelated with the waveform model.

Therefore, equation 15 provides a way of detecting the signal within the noisy data, by eliminating the noise using the inner product of the template with the data.

2.3.2 Normalisation of the Matched Filter

The matched filter SNR is then normalised by the square root of the inner product of the template with itself, i.e.

$$\text{normalised matched filter SNR} = \frac{\text{matched filter SNR}}{\sqrt{\langle h, h\rangle}} \tag{17}$$

where $\langle h, h\rangle$ is the inner product of the template with itself:

$$\langle h, h\rangle = 4\int_{-\infty}^{\infty} \mathrm{d}f \frac{|\tilde{h}(f)|^2}{S(f)} \tag{18}$$

We can absorb the normalisation factor into the template, so that the normalised matched filter SNR is equivalent to the matched filter SNR, i.e.

$$\tilde{h}_{norm}(f) = \frac{\tilde{h}(f)}{\sqrt{\langle h, h\rangle}} \tag{19}$$

Instead of integrating over all frequencies, we integrate over a range of frequencies where the signal is expected to be present, namely from f_{lower} to f_{upper}. Therefore, the normalised matched filter SNR is calculated as:

$$(\text{matched filter SNR})^2 = 4\left|\int_{f_{\text{lower}}}^{f_{\text{upper}}} \mathrm{d}f \frac{\exp(2\pi i f t)\tilde{h}_{norm}(f)\tilde{d}^*(f)}{S(f)}\right| \tag{20}$$

where f_{lower} and f_{upper} are the lower and upper frequency bounds of the signal.

Instead of searching through the time series at every time step, the matched filter SNR is the inverse Fourier transform of the product of the signal template and the whitened data in the frequency domain. The matched filter SNR is then plotted against time, shown in Figure 8.

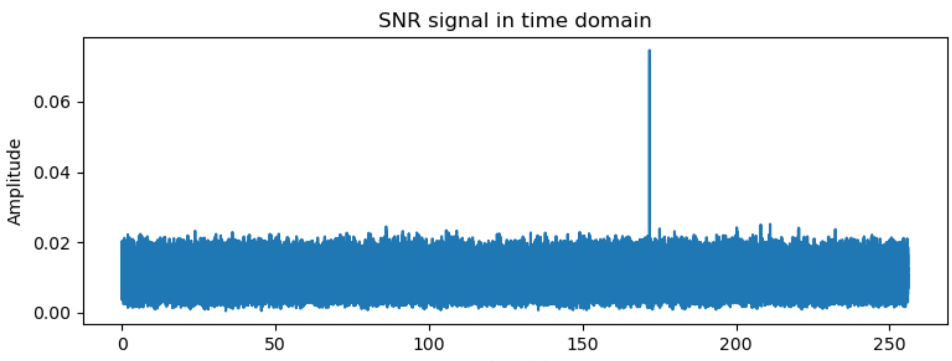

Fig. 8: Matched filter SNR plotted against time. The time at which the SNR is maximised is the time at which the signal is detected. The signal is detected at 171.8s.

The signal is detected at 171.8s, the time at which the SNR is maximised. The result is consistent with the time at which the signal is present in the whitened data series, shown in Figure 7.

3 Part 3:

In part 3, the Bayesian analysis of the gravitational wave signal is conducted. The analysis starts with the definition of the parameter space, followed by the waveform model, the likelihood function, and the prior distribution of the parameters. The nested sampling algorithm is then used to estimate the posterior distribution of the parameters.

3.1 Parameter Space

The parameter space θ is the set of parameters that describes the patten of the IMRPhenomD model. The parameters contains:

- The masses of the binary system m_1 and m_2. Their values are restricted to be greater than 0 and in the unit of solar mass, and their bound should reflect the range of possible masses of black holes[1], i.e. $m_1, m_2 \in [5, 50]$.
- The spin of the two black holes along the z-axis χ_1 and χ_2. Their values are restricted to be in the range of $[-1, 1]$.
- The distance to the binary system d. Its value is restricted to be greater than 0 and in the unit of Mpc. i.e. $d \in [0, \infty)$.
- The angle of inclination of the binary system ι. Its value is restricted to be in the range of $[0, \pi]$.
- The coalescence phase ϕ_c. Its value is restricted to be in the range of $[0, 2\pi]$.

The other parameter describes the location and orientation of the source:

- The angle between the line of sight to the source and the normal to the detector plane θ. Its value is restricted to be in the range of $[0, \pi]$.
- The azimuthal angle of the source ϕ. Its value is restricted to be in the range of $[0, 2\pi]$.
- The polarization angle ψ of the source. Its value is restricted to be in the range of $[0, 2\pi]$.

The assumption made is that the two sources have aligned spin along the z-axis. In the complete parameter space, the spin parameters are defined as a 3-dimensional vector $\vec{S}$. In this project, the aligned spin is assumed for the purpose of reducing the dimensionality of the parameter space and faster computation.

3.2 Waveform Model

The waveform model is obtained using the IMRPhenomD model, which describes the inspiral, merger, and ringdown of a binary black hole system. The model is described by the following equation:

$$h_+(f), h_\times(f) = \text{IMRPhenomD}(m_1, m_2, \chi_1, \chi_2, d, \iota, \phi_c) \tag{21}$$

The waveform model h, which describes the strain of the gravitational wave, is a linear combination of the two polarizations h_+ and $h_\times$ via antenna patterns F_+ and $F_\times$, the antenna patterns of the detector,

$$h_\theta(f) = F_+ h_+(f) + F_\times h_\times(f) \tag{22}$$

3.3 Likelihood Function

The likelihood function is defined as such:

$$\mathcal{L}(d|\vec{\theta}, H) = p(d - h_\theta|H_{\text{noise}}) = \mathcal{N} \exp\left[-\int_{-\infty}^{\infty} \mathrm{d}f \, \frac{|\tilde{d}(f) - \tilde{h}(f;\vec{\theta})|^2}{S_n(f)}\right] \tag{23}$$

where d is the data, $\vec{\theta}$ is the set of parameters that describe the signal, H is the hypothesis,h_θ is the waveform model, $\mathcal{N}$ is the normalization constant, $\tilde{d}(f)$ is the Fourier transform of the data, $\tilde{h}(f;\vec{\theta})$ is the Fourier transform of the waveform model, and $S_n(f)$ is the power spectral density (PSD) of the noise.

A good estimation of the parameters $\vec{\theta}$ means that the residual $d - h_\theta$ is close to the noise H_{noise}. Hence, the inner product of the residual will be close to zero, giving a high likelihood value, and vice versa.

The normalization constant $\mathcal{N}$ is the normalization factor that ensures the likelihood function integrates to 1. Its numerical value is trivial in the case of Bayesian inference.

3.4 Prior Transformation

The prior distribution of the parameters, and the rationale behind the choice of the prior distribution are as follows:

1. $m_1, m_2 \sim$ log-uniform$(5, 50)$. The log-uniform distribution is used to ensure that the masses are positive and scale invariant.
2. $\chi_1, \chi_2 \sim$ uniform$(-1, 1)$, with -1 representing a black hole with a negative spin, 0 representing a non-spinning black hole, and 1 representing a black hole with a positive spin.
3. $d \sim$ log-uniform$(0, 1000)$. The log-uniform distribution is used to ensure that the distance is positive and scale invariant.
4. $\cos(\iota) \sim$ uniform$(-1, 1)$. The inclination angle ι is then calculated as $\iota = \cos^{-1}(\cos(\iota))$.

5. $\phi_c \sim \text{uniform}(0, 2\pi)$.
6. $\theta \sim \text{uniform}(0, \pi)$.
7. $\phi \sim \text{uniform}(0, 2\pi)$.
8. $\psi \sim \text{uniform}(0, 2\pi)$.

3.5 Bayesian analysis

An 8 second-long segment of data is used to conduct the basic Bayesian analysis.

3.5.1 Bayes' Theorem

Bayes' Theorem is then used to update the prior distribution of the parameters to the posterior distribution of the parameters. The Bayesian formula is given by:

$$\mathcal{P}(\vec{\theta}|d) = \frac{\mathcal{L}(d|\vec{\theta})\pi(\vec{\theta})}{\mathcal{Z}} \tag{24}$$

where the posterior distribution $\mathcal{P}(\vec{\theta}|d)$ is the probability of the parameters $\vec{\theta}$ given the data d, the likelihood function $\mathcal{L}(d|\vec{\theta})$ is the probability of observing the data d given the parameters $\vec{\theta}$, the prior distribution $\pi(\vec{\theta})$ is the probability of the parameters $\vec{\theta}$, and the evidence $\mathcal{Z}$ is the normalization constant.

3.5.2 Nested Sampling

Nested sampling is used to estimate the posterior distribution of the parameters. Nested sampling starts by generating a set of "live" points from the prior distribution. The number of "live" points are will affect the accuracy and the computational cost of the analysis[5].

The algorithm will iteratively remove the point with the lowest likelihood and replace it with a new point with a higher likelihood.

As the algorithm progresses, the live points will be restricted to regions of increasingly higher likelihood, and eventually converge to the maximum likelihood point.

3.5.3 Implementation of Nested Sampling

The nested sampling algorithm is implemented using the `dynesty` package in Python[5].

3.6 Reparameterization

There is a negative correlation between the masses of the binary system m_1 and m_2. This motivates the reparameterization of the masses, in seek of a more informative representation of the 2-D space that describes the masses of the binary system.

One choice of reparameterization is the chirp mass $\mathcal{M}$ and the mass ratio q. The chirp mass $\mathcal{M}$ is defined as:

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} \tag{25}$$

and the mass ratio q is defined as:

$$q = \frac{m_2}{m_1} \tag{26}$$

The prior define for the chirp mass $\mathcal{M}$ and the mass ratio q are:

- The chirp mass $\mathcal{M}$ is log-uniformly distributed. i.e. $\mathcal{M} \sim \text{log-uniform}(10, 100)$.
- The mass ratio q is uniformly distributed. i.e. $q \sim \text{uniform}(0, 1)$.

3.7 Posterior Distribution

The result of the Bayesian analysis is shown in Figure 9 and Table 10.

Figure 9 shows the posterior distribution of the parameters. The diagonal shows the marginal posterior distribution of each parameter, while the off-diagonal shows the correlation between each pair of parameters.

Table 10 summaries the mean and 1 standard deviation i.e. 68.3% interval of the parameters.

The pair plot of the masses of the binary system m_1 and m_2 shows a negative correlation between the two masses.

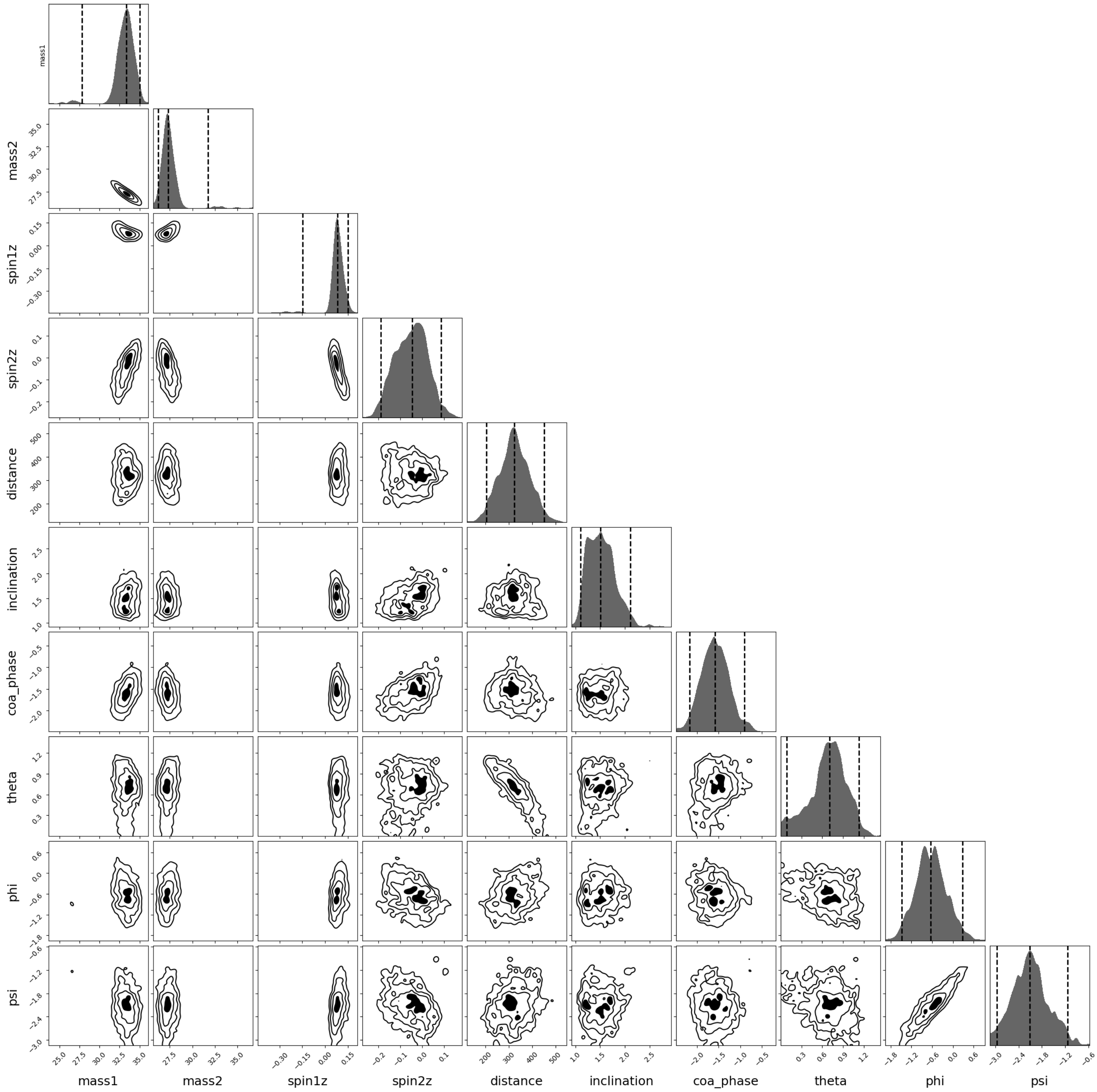

Fig. 9: Posterior distribution of the parameters, using the IMRPhenomD model with the priors defined in the previous section, and reparameterization of masses

Parameter	Mean	1 Std Interval
m_1	33.16	[31.74, 34.58]
m_2	27.47	[26.31, 28.63]
χ_1	0.08	[0.02, 0.14]
χ_2	-0.05	[-0.12, 0.02]
d	323.48	[259.15, 387.81]
ι	1.53	[1.26, 1.80]
ϕ_c	-1.58	[-2.04, -1.12]
Array Pattern	Mean	90 %Interval
θ	0.68	[0.14, 1.22]
ϕ	-0.63	[-1.53, 0.27]
ψ	-2.09	[-2.55, -1.63]

Fig. 10: Bayesian Analysis Results

4 Conclusion and Key Concepts

4.1 Inner Product

Inner product is an essential part of the analysis of signals such as gravitational waves. The inner product measures the similarity between two inputs, and is used to calculate the signal-to-noise ratio (SNR) of the signal.

In part 1, the inner product of the model waveform with itself gives an estimation of the SNR of the signal. Signal of higher SNR is more likely to be detected in the presence of noise.

In part 2, the inner product of the waveform model with the data, with a shift in time, gives an estimation of the SNR of the signal in the presence of noise as a function of time. This is used to estimate the time of occurrence of the signal.

In part 3, the inner product of the residual

$$\tilde{d}(f) - \tilde{h}_\theta(f) \tag{27}$$

with itself gives a measurement of likelihood of the parameters of the signal. In the presence of a good estimation of the parameters, the residual will be close to the noise, leading to a lower inner product value, an indication of a high likelihood value.

4.2 Bayesian Analysis

Bayesian inference and nested sampling is another essential part of the analysis of signals such as gravitational waves.

Bayesian inference gives the framework of updating the prior distribution of the parameters to the posterior distribution of the parameters, given the data.

Nested sampling algorithm is a robust method that constructs the posterior distribution of the parameters, which is relevant in the analysis of gravitational wave signals. In this project, nested sampling is used to estimate the posterior distribution of 10 parameters of the signal, whereas many other parameters are omitted for the purpose of reducing the dimensionality of the parameter space. With a higher computational power, and proper parallelization, the nested sampling algorithm can be used to estimate the posterior distribution of the complete parameter space.

References

[1] *Black Holes*. 2024. URL: https://www.cfa.harvard.edu/research/topic/black-holes.

[2] J. D. E. Creighton and W. G. Anderson. *Gravitational-Wave Physics and Astronomy: An Introduction to Theory, Experiment and Data Analysis*. Wiley-VCH, 2011. ISBN: 978-3-527-40886-4.

[3] *Michele Maggiore: Gravitational waves. Volume 1: theory and experiments*. Feb. 2009.

[4] *PyCBC Inference: A Python-based parameter estimation toolkit for compact binary coalescence signals*. 2019.

[5] Joshua S Speagle. *dynesty: a dynamic nested sampling package for estimating Bayesian posteriors and evidences*. Vol. 493. 3. Oxford University Press (OUP), Feb. 2020. DOI: 10.1093/mnras/staa278. URL: http://dx.doi.org/10.1093/mnras/staa278.