Gravitational Waves

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Word count:

1 Background

We are working with two input data files:

- ASD.txt - The expected amplitude spectral density of the noise in the strain data against frequency $f \in [0, 500]$. This is the noise that is expected to be present in the data, with or without the presence of a gravitational wave signal.
- data.txt - Measured strain against sampled time $t \in [0, 250]$. This is noisy data, so no gravitational wave signal is expected to be observed easily with the naked eye.

1.1 Overview of the project

The project is separated into three main parts.

1. In the first part, we constructed a waveform model IMRPhenomD, which is a model for the Gravitational Wave signal from the inspiral-merger-ringdown of a binary black hole. We simulated the optimum SNR for such a signal in the data, using a detector with expected noise from ASD.txt.
2. In the second part, we used the IMRPhenomD model to construct a likelihood function to search for a signal in the noisy data from data.txt. We are able to identify the presence of a signal in the data, with the knowledge of the parameters of the signal.
3. We search through the parameter space of the signal to find the best estimate of the parameters of the signal in the data. We use the IMRPhenomD model to construct a likelihood function, and use a Markov Chain Monte Carlo (MCMC) method to sample the posterior distribution of the parameters of the signal.

2 Part 1

In part 1, we start with generation of the model template, followed by the calculation of the amplitude spectral density of the signal. The antenna pattern functions are then defined, and the optimum SNR is calculated. Finally, the average SNR is calculated.

2.1 Waveform modelling

The template of the waveform is generated using the function get_fd_waveform from the PyCBC library. The function takes the following arguments:

2.2 Amplitude spectral density

From the file, ASD.txt the amplitude spectral density of the signal is loaded and plotted against frequency, shown as the orange line in Figure 1.

On the same axis, the square root of the frequency multiplied by the plus polarization of the waveform:

$$\sqrt{f} \times |h_+(f)| \tag{1}$$

is plotted as the blue line in Figure 1. Note that this is a weighted waveform in the frequency domain with the weight of $\sqrt{f}$.

Given that the unit of strain measured from the detector is dimensionless, the unit of the amplitude spectral density is $\frac{1}{\sqrt{\text{Hz}}}$.

The generated waveform has no unit, therefore, the expression in Equation 1 has unit of $\sqrt{\text{Hz}}$.

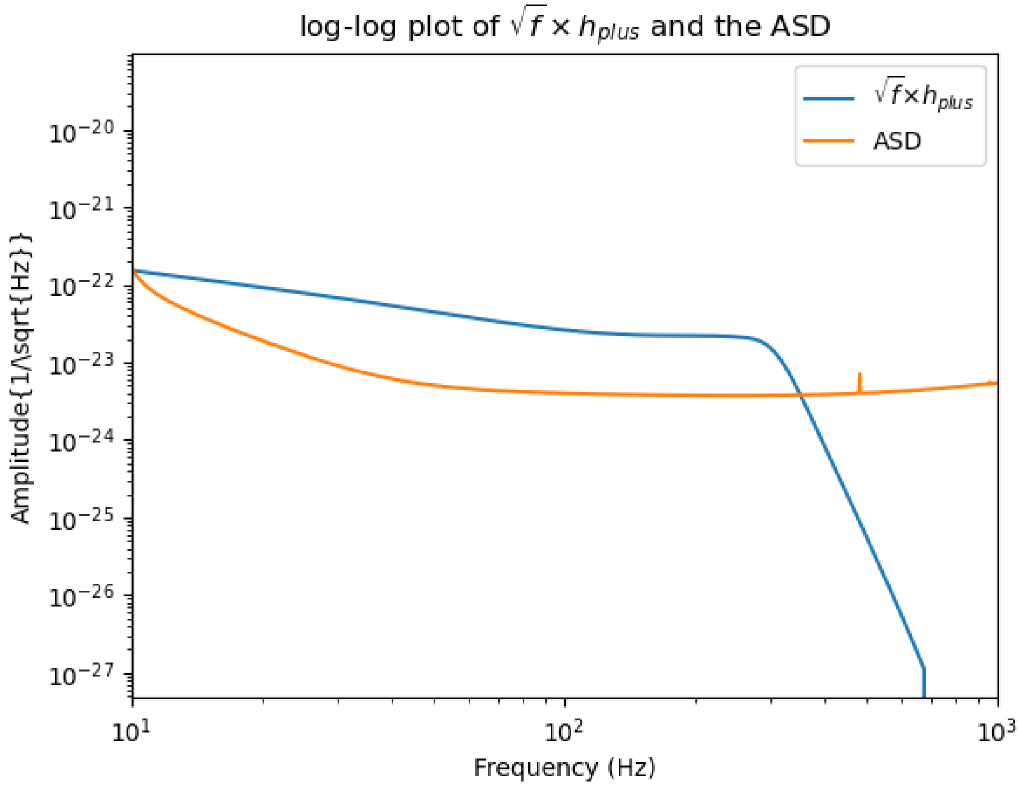

Fig. 1: log-log plot of the amplitude spectral density of the signal and the square root of the frequency multiplied by the plus polarization of the waveform.

2.3 Antenna Pattern Functions

For the L-shaped detector, The antenna pattern functions are defined as:

$$F_{+}(\theta, \phi, \psi) = \frac{1}{2}(1 + \cos^2(\theta)) \cos(2\phi) \cos(2\psi) - \cos(\theta) \sin(2\phi) \sin(2\psi) \tag{2}$$

$$F_{\times}(\theta, \phi, \psi) = \frac{1}{2}(1 + \cos^2(\theta)) \cos(2\phi) \sin(2\psi) + \cos(\theta) \sin(2\phi) \cos(2\psi) \tag{3}$$

There are are The detector output $\tilde{h}(t)$ is given by:

$$\tilde{h}(t) = F_{+}(\theta, \phi, \psi)\tilde{h}_{+}(t) + F_{\times}(\theta, \phi, \psi)\tilde{h}_{\times}(t) \tag{4}$$

where $\tilde{h}_{+}(t)$ and $\tilde{h}_{\times}(t)$ are the plus and cross polarizations of the waveform respectively.

2.4 Optimum SNR

The optimum SNR is given by:

$$(\text{optimum SNR})^2 = 4 \int_{f_{\text{lower}}}^{f_{\text{upper}}} \frac{|\tilde{h}(f)|^2}{S(f)} \, \mathrm{d}f \tag{5}$$

The optimum SNR is the inner product of the signal with itself, weighted by the inverse of the density spectral density.

There are five parameters that can be varied to maximize the SNR, three are parameters of the antenna pattern functions, θ, ϕ and ψ, and the other two are the inclination angle, ι, and the polarization angle, Ψ.

1. θ is the angle between the line of sight to the source and the normal to the detector plane. To maximize the SNR, θ should be set to 0 or π,representing the source being directly overhead or directly below the detector. We will set $\theta = 0$ for the rest of the analysis.

2. ϕ is the azimuthal angle of the source in the detector plane. In the case of the L-shaped detector, and with the source directly overhead, ϕ has no effect on the SNR.

3. ψ is the polarization angle of the source. In the case of the L-shaped detector, and with the source directly overhead, ψ has no effect on the SNR.

4. ι is the inclination angle of the source, which describes the normal of the plane of detector arms to the normal of the orbital plane of the binary system. The SNR is maximise when $\iota = 0$ or π, representing the two planes are parallel or anti-parallel to each other.

5. Ψ is the phase factor of the waveform, which has no effect on the SNR.

In summary, the SNR is maximized at:

$$\begin{aligned} \theta &= 0 \text{ or } \pi \\ \phi &= \text{arbitrary} \\ \psi &= \text{arbitrary} \\ \iota &= 0 \text{ or } \pi \\ \Psi &= \text{arbitrary} \end{aligned}$$

In this project, we will set the parameters to the following values:

$$\theta = 0 \tag{6}$$
$$\phi = 0 \tag{7}$$
$$\psi = 0 \tag{8}$$
$$\iota = 0 \tag{9}$$
$$\Psi = 0 \tag{10}$$

The optimum SNR of the signal, using the optimum parameters mentioned above is calculated to be 37.933.

2.5 Average SNR

The average SNR is achieved by averaging the SNR over all possible values of θ, ϕ, ψ, ι and Ψ.

The average SNR is a function of the probability distribution function(PDF) of the parameters, $P(\theta, \phi, \psi, \iota, \Psi)$. The PDF of each parameter is assumed to be independent, i.e. $P(\theta, \phi, \psi, \iota, \Psi) = P(\theta)P(\phi)P(\psi)P(\iota)P(\Psi)$. The distribution of each parameter:

$$\begin{aligned} P(\theta) &= X \sim \text{Uniform}(0, 2\pi) \\ P(\phi) &= X \sim \text{Uniform}(0, \pi) \\ P(\psi) &= X \sim \text{Uniform}(0, 2\pi) \\ P(\iota) &= X \sim \text{Uniform}(0, \pi) \\ P(\Psi) &= X \sim \text{Uniform}(0, 2\pi) \end{aligned}$$

In this project, a sample of 10^6 points is generated for each parameter, using the distribution functions mentioned above. A histogram of the SNR is plotted, to sense check the distribution and maximum value of the SNR, shown in Figure 2.

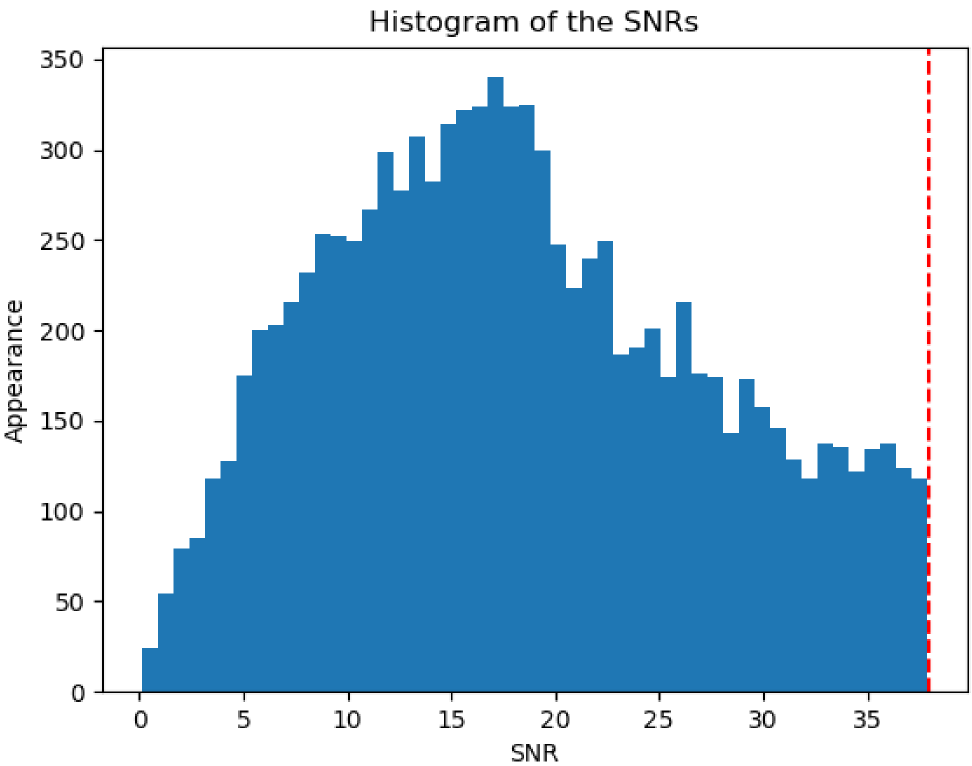

Fig. 2: Histogram of the SNR, with the optimum SNR marked as a vertical line.

Figure 2 shows that the SNR has a distribution that is approximately Gaussian with a tail on the right side. The average SNR is calculated to be 18.404. The Figure also demonstrates that the maximum SNR is 37.933, which is consistent with the calculated optimum SNR.

3 Part 2:

Part 2 of the coursework involved implementing a matched filter to detect the presence of a signal using a known signal templated. We first whitened the data using the expected power spectral density of the signal. The whitened data was then used to calculate the matched filter signal-to-noise ratio (SNR) at each time step in the time series. The time at which the SNR is maximised is the time at which the signal is detected.

3.1 Data Processing

The simulated data series is loaded from `data.txt` and plotted against time, shown in Figure 3

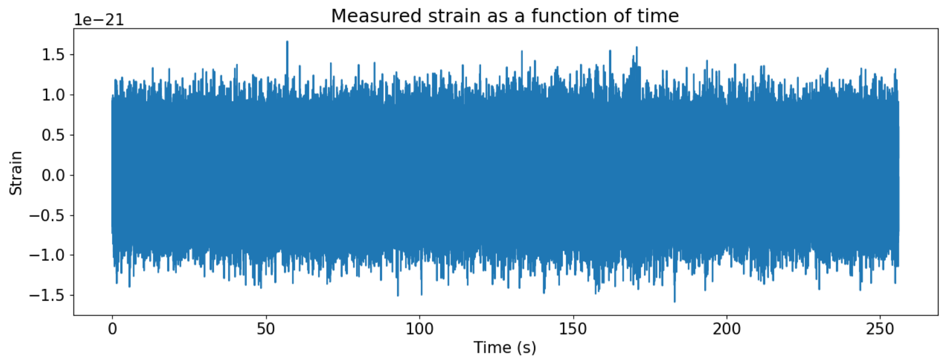

Fig. 3: Simulated data series plotted against time

Here the data is a time series of a signal mixed with noise, therefore not easily distinguishable. The signal is expected to be a chirp signal with a frequency that increases linearly with time.

By using a Welch periodogram, the power spectral density of the signal is calculated. The amplitude of the signal is then calculated by taking the square root of the power spectral density and plotted against frequency, shown in Figure 4.

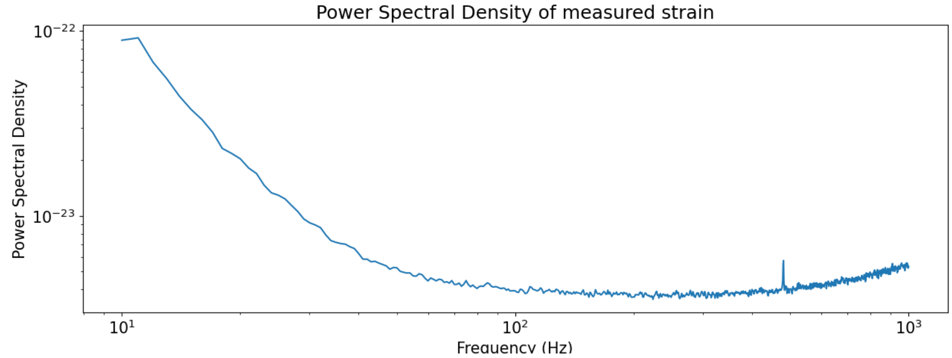

Fig. 4: Power spectral density of the simulated data series

The measured ASD is then compared with the expected amplitude spectral density of the signal from Part 1, shown in Figure 5. The expected ASD from the previous part is the ASD without the signal. The residuals are calculated as the difference between the measured ASD and the expected ASD. The residuals are then plotted against frequency in the bottom plot of Figure 5.

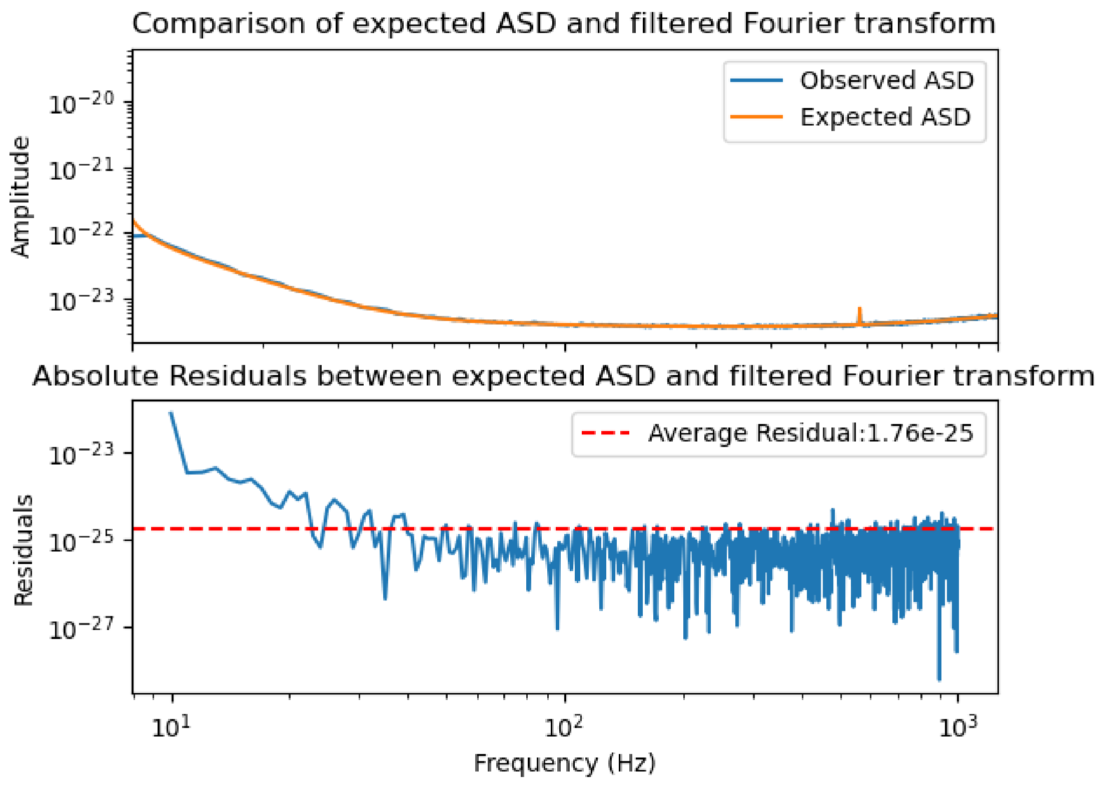

Fig. 5: Upper: comparison of the observed power spectral density with the expected power spectral density. The expected ASD(ASD without the signal) is largely overlapping with the measured ASD.
Lower: the absolute residuals are shown in the bottom plot, the residuals are within the expected range of 10^{-25}, which is about 1 order of magnitude smaller than the expected ASD.

The Figure 5 shows that the measured ASD is largely overlapping with the expected ASD. The residuals are within the expected range of 10^{-25}, which is about 1 order of magnitude smaller than the expected ASD. This indicates that the measured ASD is consistent with the expected ASD, while the residual is accounted for by the signal.

3.2 Whitening the data

The expected power spectral density of the signal is used to whiten the data. The whitened filter is defined as:

$$\text{whitened filter} = \frac{1}{\sqrt{PSD}} \tag{11}$$

where the PSD is the power spectral density of the signal. The whitening filter is then applied The filter is then applied to the Fourier transformed measurement in the frequency domain. Then the inverse Fourier transform is applied to the whitened data to obtain the whitened data in the time domain.

The whitened data is then plotted against time, shown in Figure 6.

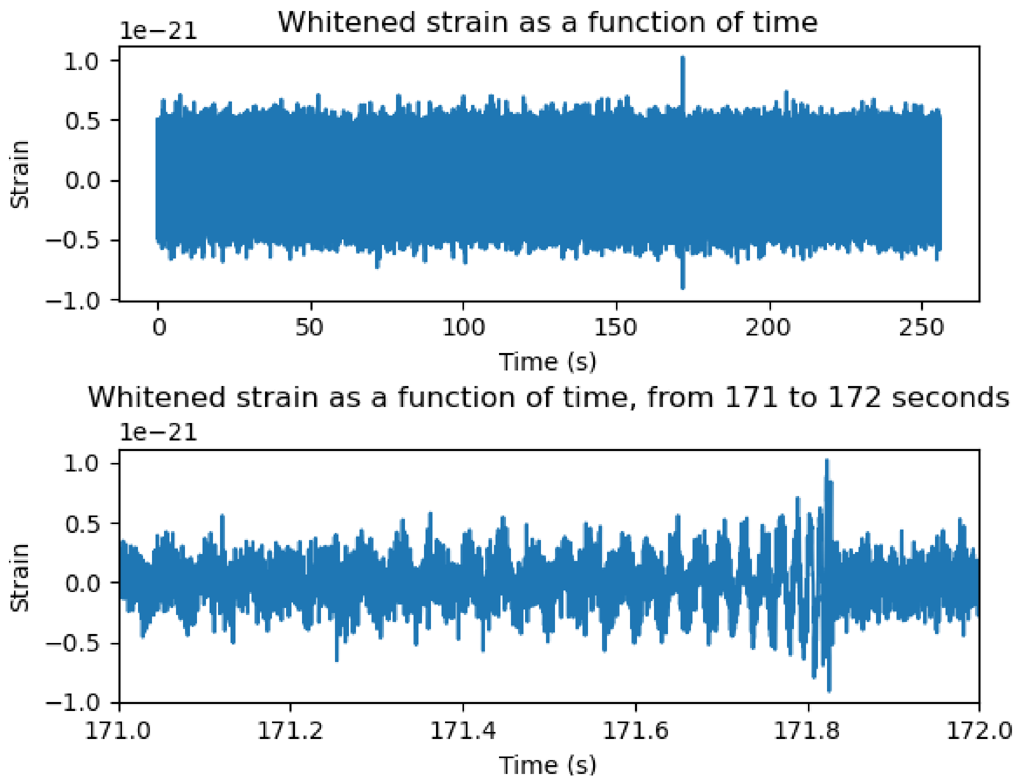

Fig. 6: Upper: the whitened data series is plotted against time. The whitened data series is the result of applying the whitened filter to the Fourier transformed data series.
Lower: the whitened data series is plotted against time zoomed in to time domain from 171.0s to 172.0s. A signal cam be seen from 171.0s to 171.8s.

3.3 Matched Filter

The observed data is mixed with noise.

$$x(t) = s(t) + n(t) \tag{12}$$

where $x(t)$ is the observed data, $s(t)$ is the signal, and $n(t)$ is the noise. The signal is expected to be a chirp signal with a frequency that increases linearly with time, as observed in the whitened data in Figure 6.

The output of the matched filter, matched filter SNR, is the convolution of template $h(t)$ with the whitened data $d(t)$. In Fourier space this is equivalent to the product of the Fourier transform of the template $\tilde{h}(f)$ and the Fourier transform of the whitened data $\tilde{d}(f)$, weighted by the inverse of the power spectral density of the noise $S(f)$.

$$\text{matched filter SNR} = \int_{-\infty}^{\infty} \mathrm{d}f \frac{\tilde{h}(f)\tilde{d}^*(f)}{S(f)} \tag{13}$$

The matched filter SNR is then normalised by the square root of the inner product of the template with itself, i.e.

$$\text{normalised matched filter SNR} = \frac{\text{matched filter SNR}}{\sqrt{\langle h, h\rangle}} \tag{14}$$

We can absorb the normalisation factor into the template, so that the normalised matched filter SNR is equivalent to the matched filter SNR, i.e.

$$\tilde{h}_{normalised}(f) = \frac{\tilde{h}(f)}{\sqrt{\langle h, h\rangle}} \tag{15}$$

Instead of integrating over all frequencies, we integrate over a range of frequencies where the signal is expected to be present, namely from f_{lower} to f_{upper}. Therefore, the normalised matched filter SNR is calculated as:

$$(\text{matched filter SNR})^2 = 4\left|\int_{f_{\text{lower}}}^{f_{\text{upper}}} \mathrm{d}f \frac{\exp(2\pi i f t)\tilde{h}_{normalised}(f)\tilde{d}^*(f)}{S(f)}\right| \tag{16}$$

where f_{lower} and f_{upper} are the lower and upper frequency bounds of the signal, $\tilde{h}_{normalised}(f)$ is the normalised Fourier transform of the signal template, $\tilde{d}(f)$ is the Fourier transform of the whitened data, and $S(f)$ is the power spectral density of the noise. The exponential term is the Fourier transform of delta function centred at t. It represents a phase shift of the signal template.

We need to search through the time series to find the time at which the SNR is maximised. The matched filter SNR is calculated for each time step in the time series. The time at which the SNR is maximised is the time at which the signal is detected. The matched filter SNR is plotted against time, shown in Figure 7.

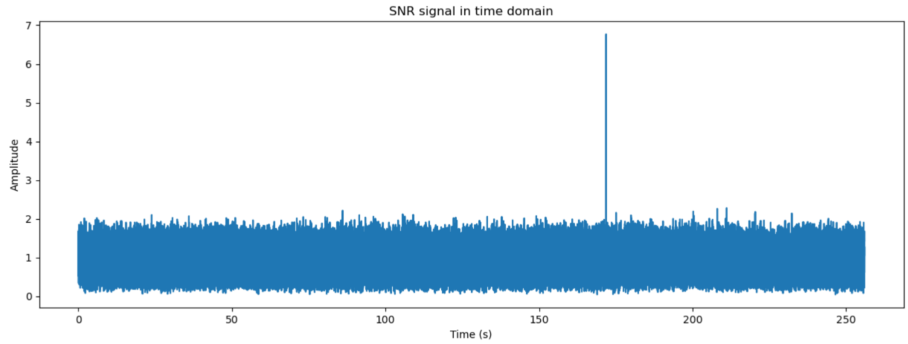

Fig. 7: Matched filter SNR plotted against time. The time at which the SNR is maximised is the time at which the signal is detected. The signal is detected at 171.8s.

The signal is detected at 171.8s, which is the time at which the SNR is maximised. The result is consistent with the time at which the signal is present in the whitened data series, shown in Figure 6.

4 Part 3:

In part 3, we defined the parameters space θ as the set of parameters that describe the signal. We then defined a likelihood function $\mathcal{L}(d|\theta)$, which is the probability of observing the data d given the parameters θ. With a predetermined prior distribution $\mathcal{P}(\theta)$, we can calculate the posterior distribution $\mathcal{P}(\theta|d)$ using Bayesian inference. We then can use the posterior distribution to estimate the parameters of the signal.

4.1 Parameter Space

The parameter space θ is the set of parameters that describes the patten of the IMRPhenomD model. The parameters contains:

- The masses of the binary system m_1 and m_2,
- The symmetric mass ratio η,
- The time of coalescence t_c,
- The phase ϕ_c,
- The amplitude A,
- The inclination angle ι,
- The polarization angle ψ,
- The right ascension α,
- The declination δ,
- The distance d.

4.2 Waveform Model

4.3 Likelihood Function

4.4 Prior Transformation

4.5 Bayesian analysis

4.6 Posterior Distribution