Examples Classes for WS2 this afternoon 4-5 & 5-6.

Posterior $$P(x \mid d) = \frac{\mathcal{L}(d \mid x)\, \pi(x)}{Z}$$

MCMC alg. gives $x \sim P(x \mid d)$

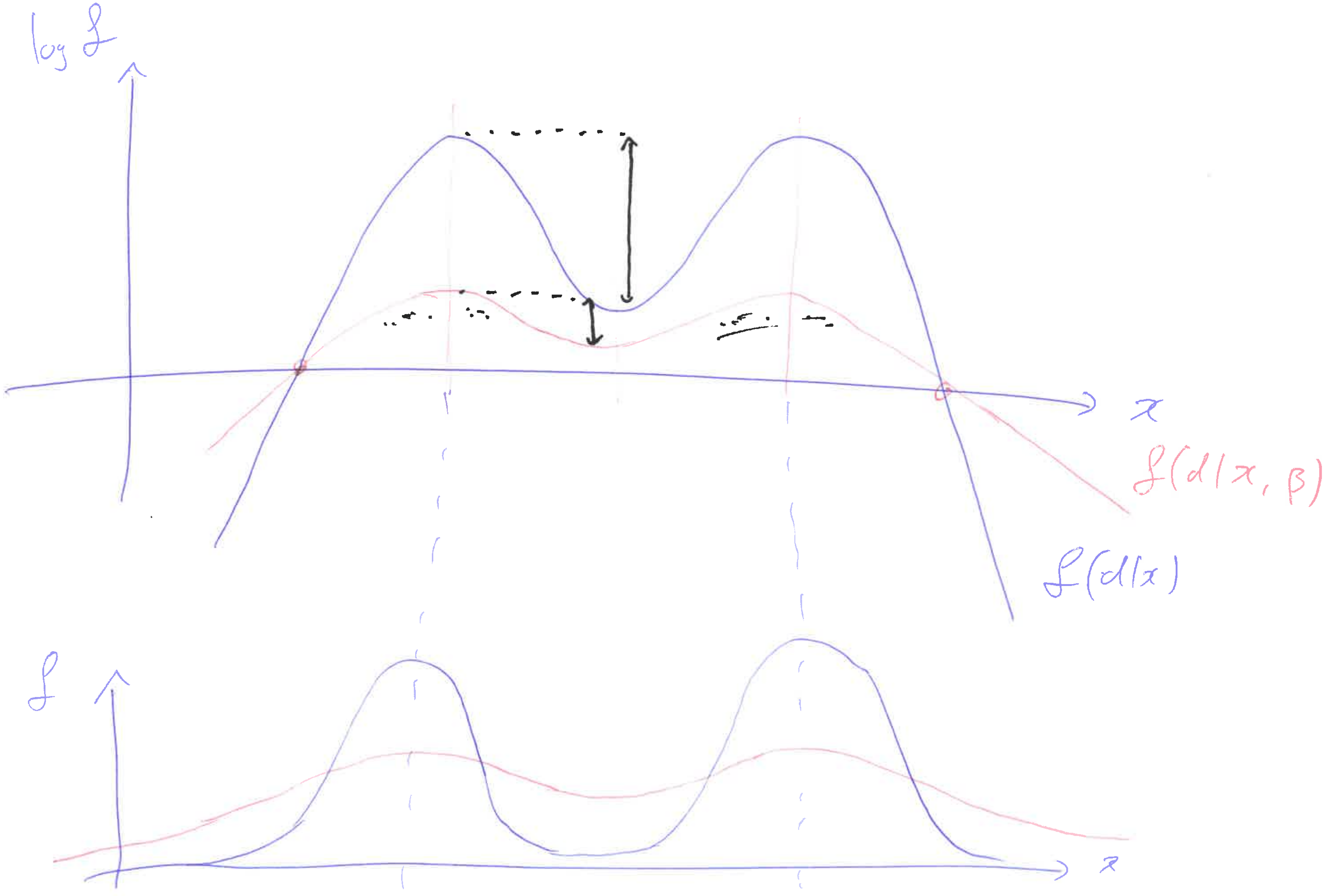
The one outstanding thing is $Z = \int dx \, \mathcal{L}(d \mid x)\, \pi(x)$

Thermodynamic Integration a method that use MCMC techniques to get the evidence.

define modified, annealed likelihood $\mathcal{L}(d \mid x, \beta) = \mathcal{L}(d \mid x)^{\beta}$

β is the inverse, annealing temperature. c.f. ~~Thermal~~ Statistical physics $\beta = \frac{1}{k_B T}$

$0 < \beta < 1$



modified Bayes' theorem

$$P(x \mid d, \beta) = \frac{\mathcal{L}(d \mid x)^{\beta} \, \pi(x)}{Z(\beta)}$$

where $Z(\beta) = \int dx \; \mathcal{L}(d \mid x)^{\beta} \, \pi(x)$

low T limit

$\downarrow$

$\beta = 1$ recovers the ~~log~~ original likelihood $\qquad Z(\beta = 1) = Z$

$\beta = 0$ gives a flat likelihood $\qquad Z(\beta = 0) = 1$

$\uparrow$

high T limit

$$\Downarrow$$

$$\log Z(\beta = 0) = 0$$

consider
$$\frac{d}{d\beta}\log Z(\beta) = \frac{1}{Z(\beta)}\frac{d Z(\beta)}{d\beta}$$
$$= \frac{1}{Z(\beta)} \frac{d}{d\beta}\int dx\, \underline{\mathcal{L}(d|x)^{\beta}}\, \pi(x)$$
performed β differentiation $\rightarrow$
$$= \frac{1}{Z(\beta)}\int dx\, \pi(x)\, \mathcal{L}(d|x)^{\beta} \log \mathcal{L}(d|x)$$
modified Bayes' theorem $\rightarrow$
$$= \int dx\, P(x|d,\beta) \log \mathcal{L}(d|x)$$
$$= E_x\left[\log \mathcal{L}(d|x) \;\middle|\; \beta\right]$$
where $x \sim P(x|d,\beta)$

this can be calculated using MCMC methods.
$$E_x\left[\log \mathcal{L}(d|x) \,|\, \beta\right] \approx \frac{1}{N}\sum_{i=1}^{N} \log \mathcal{L}(d|x_i)$$
where x_i is our thinned MCMC chain.

$$\log Z = \int_0^1 d\beta \quad E_x\left[\log f(d|x) \mid \beta\right]$$

pick a series of temperatures (temperature ladder)

$$\beta_1 = 0, \quad \beta_2 < \beta_3 < \beta_4 \ldots < \beta_n = 1$$

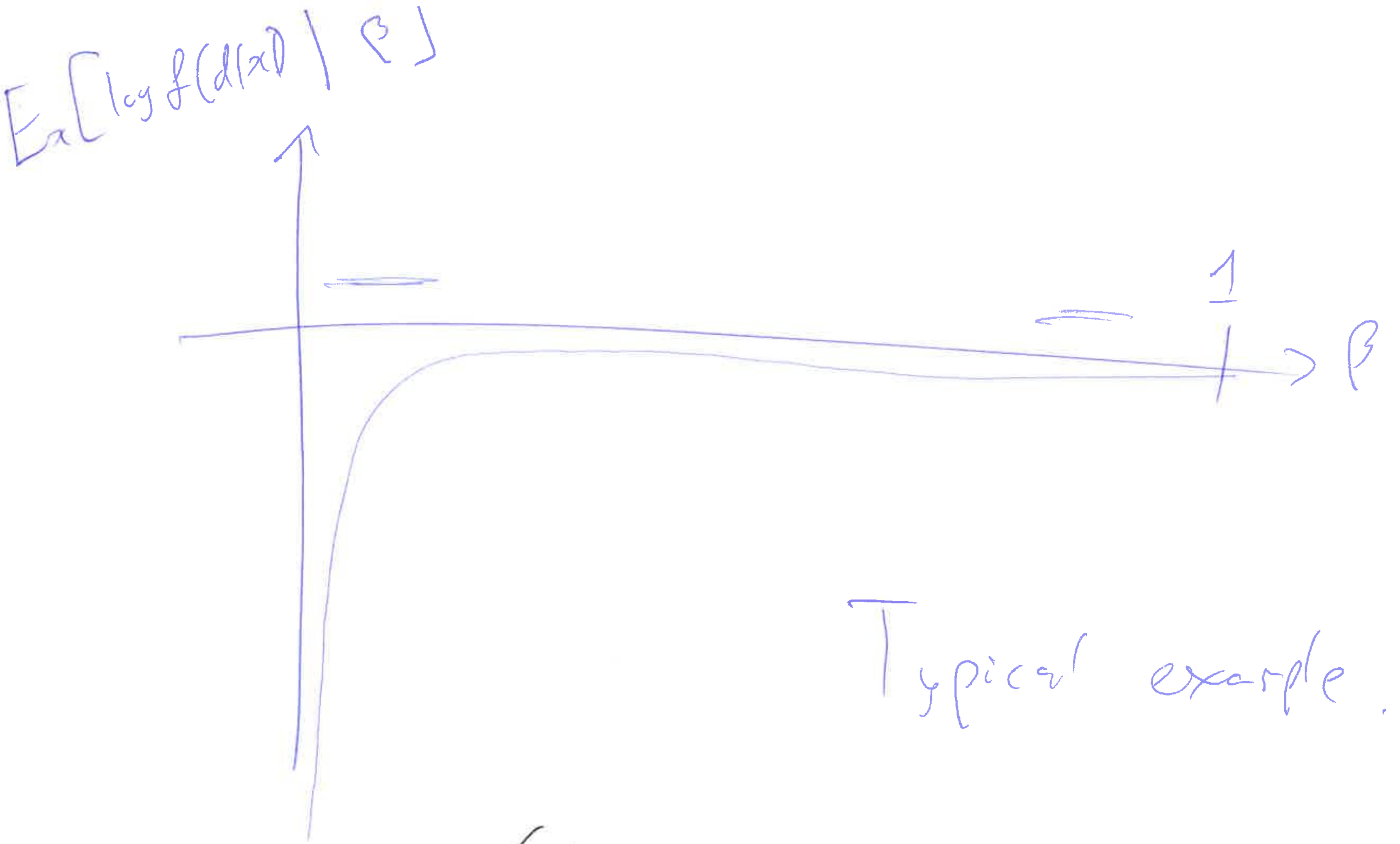
$$E_x\left[\log f(d|x) \mid \beta_\mu\right]$$

then trapezium rule for the final β integral

$$\log Z \approx \frac{1}{2} \sum_{\mu=1}^{\mu-1} \Delta\beta_\mu \left(E_x\left[\log f(d|x) \mid \beta_{\mu+1}\right] - E_x\left[\text{—} \mid \beta_\mu\right] \right)$$

$$\Delta\beta_\mu = \beta_{\mu+1} - \beta_\mu$$

Thermodynamic integration



Typical example.

$$\int_0^1 \frac{d}{d\beta} \log Z(\beta) \, d\beta$$

limits come from $Z(\beta=1) = Z$

$Z(\beta=0) = 1$.

The Savage - Dickey density ratio

Two models M_1 & M_2 this gives $B_{1,2} = \frac{P(d|M_1)}{P(d|M_2)} = \frac{Z_{M_1}}{Z_{M_2}}$

evidence ratio
a.k.a. Bayes factor.

where M_1 is <u>nested</u> inside M_2.

what is meant by nested?

suppose M_2 has parameters (ϵ, ϕ) $\epsilon \in \mathbb{R}$ $\phi \in \mathbb{R}^d$

if I set $\epsilon = 0$.

M_1 just has parameters ϕ

precisely

- $\mathcal{L}(d \mid \phi, M_1) = \mathcal{L}(d \mid \varepsilon = 0, \phi, M_2)$

$\longrightarrow \pi(\phi \mid M_1) = \pi(\phi \mid M_2, \varepsilon = 0)$

e.g. $\pi(\phi \mid M_1) = g(\phi)$

$\pi(\phi, \varepsilon \mid M_2) = g(\phi)\, f(\varepsilon)$

consider the simpler model

$$\begin{aligned} Z_{M_1} &= P(d \mid M_1) \\ &= \int d\phi\; \mathcal{L}(d \mid \phi, M_1)\, \pi(\phi \mid M_1) \\ &= \int d\phi\; \mathcal{L}(d \mid \phi, \varepsilon = 0, M_2)\, \pi(\phi \mid \varepsilon = 0, M_2) \\ &= P(d \mid \varepsilon = 0, M_2) \end{aligned}$$

using Bayes' theorem

$$= \frac{P(\varepsilon=0 \mid d, M_2)}{P(\varepsilon=0 \mid M_2)} \underbrace{P(d \mid M_2)}_{Z_{M_2}}$$

$$\frac{Z_{M_1}}{Z_{M}} = B_{1,2} = \frac{P(\varepsilon=0 \mid d, M_2)}{P(\varepsilon=0 \mid M_2)}$$

only work in M_2

$$= \frac{\text{Posterior PDF } (\varepsilon=0 \mid \cancel{M_2})}{\text{Prior PDF } (\varepsilon=0 \mid \cancel{M_2})}$$