Fisher Information

Define **Score** gradient of log-likelihood w.r.t. model parameter

$$S(\theta) = \frac{\partial}{\partial\theta} \log \mathcal{L}(x|\theta)$$

function of model parameter θ.

(If $\theta \to \theta_n$ the score becomes gradient vector $\frac{\partial}{\partial\theta_n} \log \mathcal{L}(x|\theta^n)$)

$$E_x\left[S(\theta) \mid @ \underset{\theta'}{\text{true params}} \right] = \boxed{\int dx \; S(\theta') \; \mathcal{L}(x|\theta')}$$

this is (unknown) true value of θ.

$$S(\theta') = \frac{\partial}{\partial\theta}\Big|_{\theta=\theta'} \log \mathcal{L}(x|\theta)$$

$$= \boxed{\int dx \frac{1}{\mathcal{L}(x|\theta)} \frac{\partial \mathcal{L}(x|\theta)}{\partial\theta} \mathcal{L}(x|\theta')}$$

$$= \int dx \frac{\partial \mathcal{L}(x|\theta)}{\partial\theta}$$

$$E_x\left[\, S(\theta) \mid \theta' \,\right] \quad = \frac{\partial}{\partial\theta}\int dx\, \mathcal{L}(x|\theta)$$

$$= \frac{\partial}{\partial\theta}\, 1$$

$$= 0.$$

$$\mathrm{I}$$

Define Fisher Information as variance of score. (@ θ')

$$\mathrm{I} \geq 0 \quad \Longleftarrow \text{ as variance.}$$

$$\mathrm{I}(\theta) = \mathrm{Var}_x\left(S(\theta) \right)$$

$$= E_x\left[S(\theta)^2 \mid @\theta' \right] - E_x\left[S(\theta) \mid @\theta' \right]^2$$

$$= \int dx\; S(\theta)^2\, \mathcal{L}(x|\theta) \quad \leftarrow \underline{\text{Def 1}}$$

$[s] = [\theta]^{-1}$ $\qquad$ $[I] = [\theta]^{-2}$

$$I(\theta) = -\int dx \, \frac{\partial^2}{\partial\theta^2} \log f(x|\theta) \, f(x|\theta) \quad \leftarrow \underline{\text{Def 2}}$$

I is related to curvature in $\log f$ (w.r.t. θ)

left as exercise.

if ϕ is an alternative param, related by $\theta(\phi)$

$$I(\phi) = I(\theta)\left(\frac{\partial\theta}{\partial\phi}\right)^2$$

$$I_{nv}(\theta) = -\int dx \left[\frac{\partial^2}{\partial\theta_n \partial\theta_v} \log f(x|\theta)\right] f(x|\theta) \leftarrow \text{FIM in } n \text{ dimension}$$

$$\log P(\theta|x) = \log \pi(\theta) + \log \mathcal{L}(x|\theta) - \log Z$$

expand about θ'

close to θ' $\quad \frac{\partial}{\partial\theta} \log \pi \approx 0$

$$\log P(\theta|x) = \text{const} + (\theta-\theta') \frac{\partial}{\partial\theta} \log \mathcal{L}(x|\theta) + \frac{1}{2}(\theta-\theta')^2 \frac{\partial^2}{\partial\theta^2} \log \mathcal{L}(x|\theta) + \dots$$

Expectation x

$$= \text{const} - \frac{1}{2}(\theta-\theta')^2 I(\theta)$$

width $\sim \frac{1}{\sqrt{I(\theta)}}$

$n_1 = 5$ stars in $A_1 = 1 \deg^2$

$$I_1(S) = \int dx \left(\frac{\partial}{\partial \theta} \log \mathcal{L}(x|\theta) \right)^2 \mathcal{L}(x|\theta)$$

$$= \sum_{n_1=0}^{\infty} \left(\frac{n_1}{S} - A_1 \right)^2 \frac{(A_1 S)^{n_1} e^{-A_1 S}}{n_1!}$$

$$= \sum_{n_1=0}^{\infty} \left(\frac{n_1^2}{S^2} - \frac{2 n_1 A_1}{S} + A_1^2 \right) \mathcal{L}(n_1|S)$$

$$= \left(\left[\frac{A_1}{S} + A_1^2 \right] - \frac{2A_1}{S} A_1 S \quad A_1^2 \right)$$

$$= \frac{A_1}{S}$$

let m be actual # stars in area A_2

let n_2 be # stars detected

$$m \mid S \sim \text{Poisson}(A_2 S)$$

$$n_2 \mid m \sim \text{Binomial}(m, p)$$

PGM