Bayes' theorem

$$P(M \mid D, I) = \frac{\mathcal{L}(D \mid M, I)\,\pi(M \mid I)}{Z}$$

where $Z = P(D \mid I)$

I = any other prior assumptions.

e.g. $P(s \mid n, \underline{A}, \underline{\text{threshold}})$

normally not write explicitly unless

- v. important, change result.
- assumption that you might relax later.

Scale invariant distribution $\pi(x)\,dx$

$$= \pi(\alpha x)\, d(\alpha x)$$

$$\Rightarrow \pi(x) \propto \frac{1}{x}$$ at least in region w. support

$$\pi(S) = \dots$$

- uniform
- log-uniform (Jeffrey's)
- $\boxed{\text{conjugate prior.}}$

Gamma Distribution

$$x \sim \text{Gamma}(k, \theta) \qquad k > 0 \quad \text{"shape"} \qquad \theta > 0 \quad \text{"scale"}$$

$$P(x) = \begin{cases} \dfrac{x^{k-1} e^{-x/\theta}}{\theta^k \Gamma(k)} & \text{if } x > 0 \\ 0 & x \leq 0 \end{cases}$$

use Gamma prior for S

$$\pi(S) = \frac{S^{k-1} e^{-S/\theta}}{\theta^k \Gamma(k)} \quad \text{for} \quad S > 0.$$

$$\mathcal{L}(n|S) = \frac{(AS)^n e^{-AS}}{n!}$$

Bayes' theorem $\Rightarrow$ $$P(S|n) = \frac{A^n S^{k+n-1} \exp\left(-[A\theta+1] S/\theta\right)}{\theta^k n! \, \Gamma(k) \, Z}$$

(1)

Posterior has same functional form as prior.

$$P(s|n) = \text{Gamma dist with } k' = k+n, \quad \theta' = \frac{\theta}{A\theta+1}$$

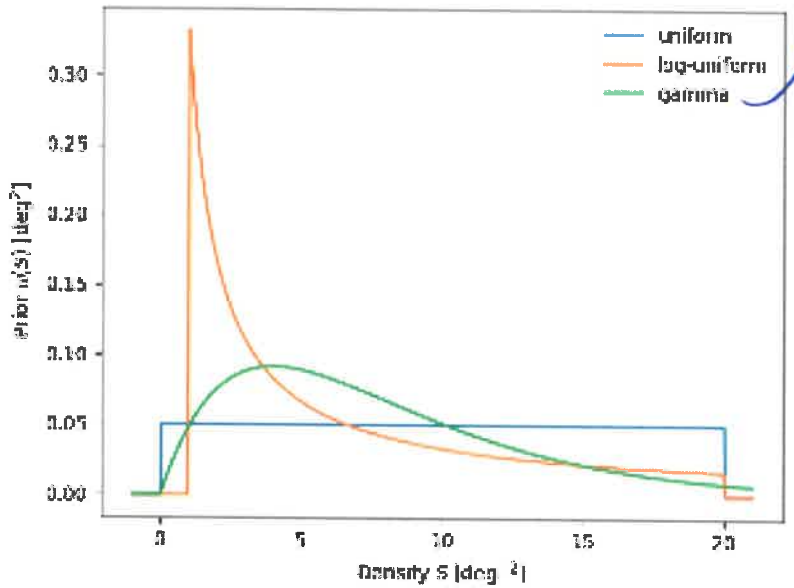
$$P(s|n) = \frac{s^{k+n-1} e^{-\left(\frac{A\theta+1}{\theta}\right)s}}{\left(\frac{\theta}{A\theta+1}\right)^{k+n} \Gamma(k+n)} \quad \text{for } s>0 \quad (3)$$

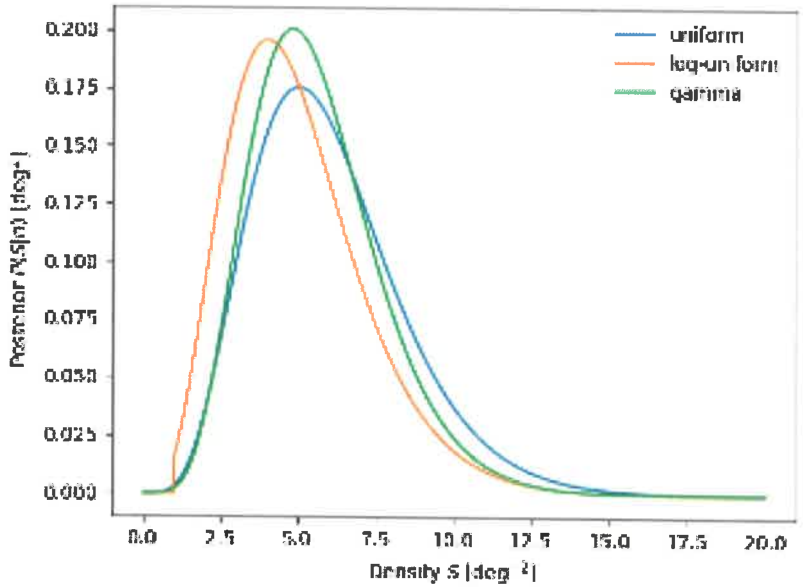
Equate (1) and (2)

$$\frac{A^n}{\theta^k n!\,\Gamma(k)\,Z} = \frac{1}{\left(\frac{\theta}{A\theta+1}\right)^{k+n} \Gamma(k+n)}$$

$$\Rightarrow \quad Z = \frac{\Gamma(k+n)\,(A\theta)^n}{\Gamma(k)\,(A\theta+1)^{k+n}\,n!}$$

$k = 2, \quad \theta = 4\ deg^{-2}$





Recap of Bayesian inference

Bayes' theorem

$$S \mid k, \theta \sim \text{Gamma}(k, \theta) \qquad \text{(Prior)}$$

$$n_1 \mid S, A \sim \text{Poisson}(A_1 S) \qquad \text{(Likelihood)}$$

$$S \mid n_1, A, k, \theta \sim \text{Gamma}(k', \theta') \qquad (\text{~~Posterior~~ Prior})$$

from before

$$k \to \boxed{k'} = k + n_1$$

$$\theta \to \boxed{\theta'} = \frac{\theta}{A_1 \theta + 1}$$

$$n_2 \mid S \sim \text{Poisson}(A_2 S) \qquad \text{(likelihood 2)}$$

$$S \mid n_2, n_1 \sim \text{Gamma}(k'', \theta'')$$

$$k'' = k' + n_2 = k + n_1 + n_2$$

$$\theta'' = \frac{\theta'}{A_2\theta' + 1} = \cdots = \frac{\theta}{(A_1 + A_2)\theta + 1}$$

little algebra left as exercise.