Markov Chain $x_0, \ x_1, \ x_2, \ x_3, \ \dots$

Transition probability

$$P(x_i \mid x_{i-1}, x_{i-2}, \dots, x_1, x_0) = P(x_i \mid x_{i-1})$$
$$= \rho(x_i, x_{i-1})$$

(Time homogeneous)

Long term behaviour

$$P(x_i \mid x_0) = \int dx_{i-1} \int dx_{i-2} \dots \int dx_1 \ \rho(x_i, x_{i-1}) \times \rho(x_{i-1}, x_{i-2}) \times \dots \times \rho(x_1, x_0)$$

Limit distribution

$$\lambda(x_n) = \lim_{n \to \infty} P(x_n \mid x_0)$$

Stationary distribution

$$x_0 \sim \pi$$

$$x_1 | x_0 \sim \rho$$

we want $x_1 \sim \pi$ $\leftarrow$ defines <u>stationary dist</u> π

$$P(x_0) = \pi(x_0)$$

$$P(x_1) = \int \underbrace{P(x_1 | x_0)}_{\rho} \underbrace{P(x_0)}_{\pi} \, dx_0 \overset{!}{=} \pi(x_1)$$

condition to be stationary

$$\pi(x) = \int dx' \, \pi(x') \, \rho(x', x)$$

if true then π is stationary dist

the limit dist must be a stationary dist.

Goal is design ρ s.t. P (target) is stationary.

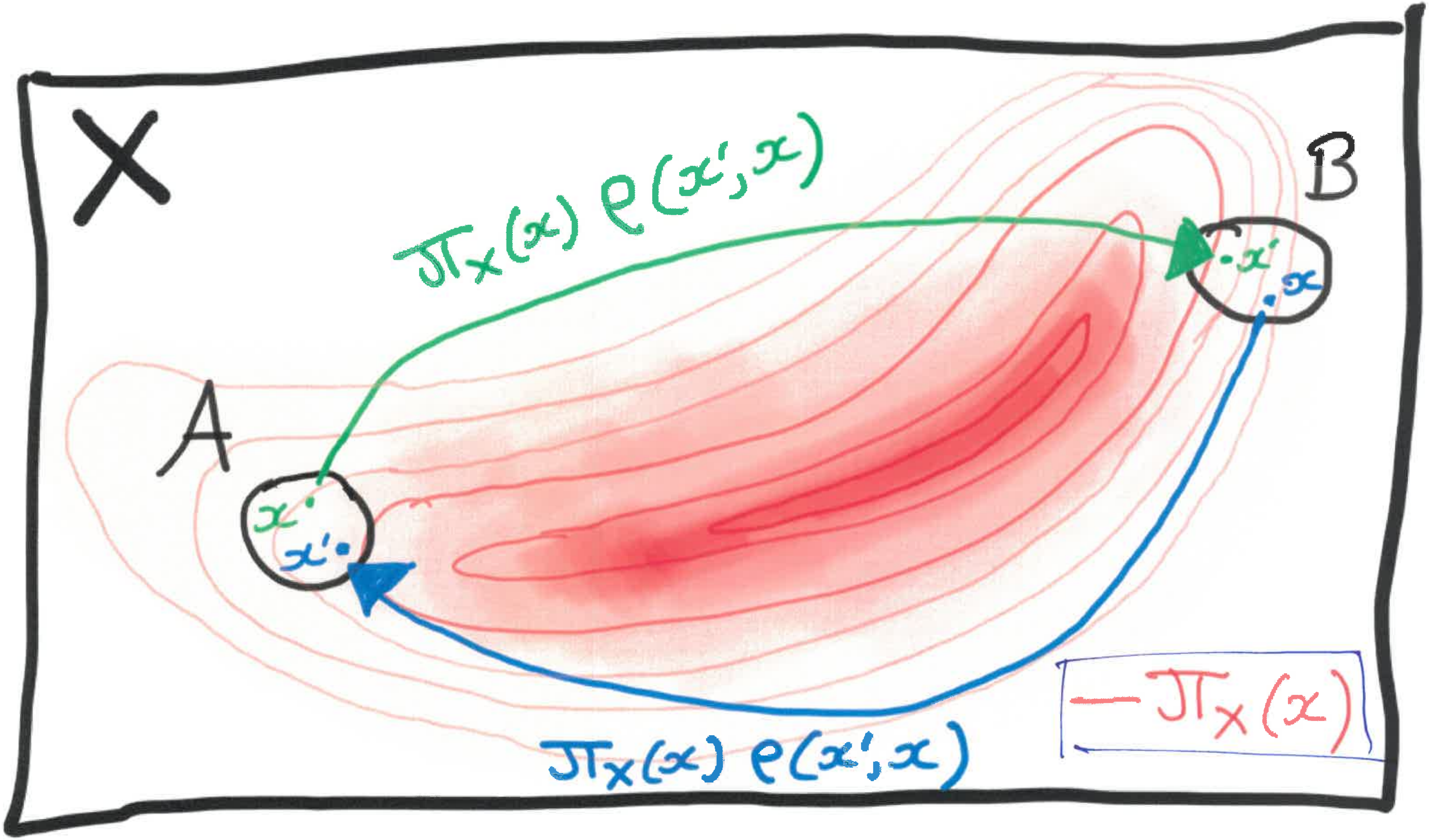
Detailed Balance condition

$$\boxed{\pi(x)\,\rho(x', x) = \pi(x')\,\rho(x, x')}$$

int above DB condition w.r.t. x' over whole sample space $\mathcal{X}$.

$$\pi(x) \underbrace{\int dx'\, \rho(x', x)}_{1} = \int dx'\, \pi(x')\, \rho(x, x')$$

stat condition. $\quad \pi(x) = \int dx'\, \pi(x')\, \rho(x, x')$



X
A
B
$\pi_X(x)\,\rho(x',x)$
$\pi_X(x)\,\rho(x',x)$
x
x'
x'
x
$\pi_X(x)$

two regions $A, B \subset X$

$$P_{\text{res}}(x \in A \text{ and } x' \in B) = \int_A dx \; \pi(x) \int_B dx' \; \rho(x'; x)$$

$$= \int_A dx \int_B dx' \; \pi(x)\, \rho(x'; x)$$

$$P_{\text{res}}(x \in B \text{ and } x' \in A) = \int_B dx \; \pi(x) \int_A dx' \; \rho(x'; x)$$

$$= \int_B dx \int_A dx' \; \pi(x)\, \rho(x', x)$$

$$= \int_B dx' \int_A dx \; \pi(x')\, \rho(x, x')$$

$$= \int_A dx \int_B dx' \; \pi(x')\, \rho(x, x')$$

Recall DB condition $\pi(x)\,\rho(x',x) = \pi(x')\,\rho(x,x')$

we want $\pi = P$

MH alg defines $\rho(y,x) = a\,Q(y|x) + (1-a)\,\delta^{(d)}(y-x)$

where $\overset{a(y,x)}{\cancel{a}} = \min\left(1, \frac{P(y)}{P(x)} \frac{Q(x|y)}{Q(y|x)}\right)$

2 cases : 1 $a(x,y) = \frac{P(x)}{P(y)} \frac{Q(y|x)}{Q(x|y)}$

$a(y,x) = 1$

2 $a(x,y) = 1$

$a(y,x) = \frac{P(y)}{P(x)} \frac{Q(x|y)}{Q(y|x)}$

consider Case #1 LHS DB condition $P(x)\,\rho(y,x) = P(x)\,Q(y|x)$ — $*$

RHS DB condition $P(y)\,\rho(x,y) = P(y)\left[\frac{P(x)Q(y|x)}{P(y)Q(x|y)} Q(x|y) + \left(1 - \frac{P(x)Q(y|x)}{P(y)Q(x|y)}\right)\delta^{(d)}(y-x)\right]$

2^{nd} term is zero.

int w.r.t. x $\quad 2^{nd}$ term vanishes

$$P(y)\,\rho(x,y) = P(x)\,Q(y|x) \quad - \; *$$

MCMC design alg. that

- practical to implement
- only requires the evaluate $f(x) \propto P(x)$
- produces TH, Markov chain with...
- ... ρ that satisfies DB with $\pi = P$.

Therefore running for many $x_n \sim P$ i.e we have 1 stochastic sample.