S2: Statistical Methods for Data Science

Example Sheet 3

Question 1

Let X be the magnitude of a star. (Magnitude is measured on a logarithmic scale, so negative numbers are allowed.) The magnitude is measured to have a value $x = 4.5$ with a Gaussian uncertainty with a known variance of $\sigma^2 = 1$.

1. Write down an expression for the likelihood $\mathcal{L}(x|X)$.

The magnitudes of stars from a certain population (pop1) are distributed as $\pi(X|\text{pop1})$ which is a Gaussian distribution with mean $\mu = 5$ and standard deviation $\Delta = 1$.

2. Using this population as a prior, find the evidence $P(x|\text{pop1})$.

The magnitudes of stars from another population (pop2) are distributed as $\pi(X|\text{pop2})$ which is a Gaussian distribution with mean $\mu = 6$ and standard deviation $\Delta = 1$.

3. Using this second population as a prior, find the evidence $P(x|\text{pop2})$.

In reality a mixture of stars from both populations are expected to be present in the ratio 3:1. I.e. there is a 75% *a priori* chance that the star comes from pop1, and a 25% chance it comes from pop2. You should now use this mixture as a prior. Let $i \in \{1, 2\}$ label which population the star comes from.

4. Write down the new prior $\pi(i, X)$ and find the joint posterior $P(i, X|x)$ (You do not need to find the evidence for this new model, you can work with the unnormalised posterior throughout).

5. By marginalising over X, find the posterior probability mass function (PMF) $P(i|x)$ on the discrete latent variable i.
6. How is $P(i|x)$ related to the values $P(x|\text{pop1})$ and $P(x|\text{pop2})$ calculated above?

Question 2

The lifetime x of a lightbulb is described by an exponential distribution with scale parameter θ which has probability density function

$$\mathcal{L}(x|\theta) = \frac{\exp(-x/\theta)}{\theta}.$$

The value of the scale parameter θ is unknown, and the following prior probability distribution with $\alpha = 1$ and $\beta = 1$ is used,

$$\pi(\theta|\alpha, \beta) = \frac{\beta^{\alpha}\theta^{-(\alpha+1)}\exp\left(\frac{-\beta}{\theta}\right)}{\Gamma(\alpha)}.$$

For a sample of $N = 3$ bulbs, the lifetimes were measured to be $x_1 = 5/2$, $x_2 = 1$, and $x_2 = 3/2$.

1. Use Laplace's approximation to estimate the Bayesian evidence $Z = P(\{x_1, x_2, x_2\})$ for this data.
2. Show that the evidence is given analytically by

$$Z = \frac{\beta^{\alpha}}{(\beta + \sum_{i=1}^{N} x^i)^{(\alpha+N)}} \frac{\Gamma(\alpha + N)}{\Gamma(\alpha)}.$$

3. Comment on the accuracy of Laplace's approximation.

For a larger sample of 50 bulbs, the following lifetimes were measured. (The same data is also provided in the file `lifetimes.txt` for convenience.)

$$
\begin{aligned}
\{x_i\} = \{&1.22543, 2.28995, 0.0765578, 0.273483, 0.0326272,\\
&2.21291, 0.691608, 1.5394, 0.190957, 0.588204,\\
&0.214819, 0.526133, 0.241459, 0.140623, 0.628795,\\
&0.52382, 0.535621, 0.12292, 2.12524, 0.0482703,\\
&1.67426, 0.00806649, 3.79265, 0.208549, 0.599776,\\
&0.168408, 1.31072, 0.253883, 0.373873, 1.29221,\\
&0.0727334, 2.08186, 1.81474, 1.9293, 0.165165,\\
&0.275968, 0.850653, 1.00761, 0.130893, 0.591911,\\
&0.754965, 0.760503, 0.990734, 0.833527, 0.925841,\\
&0.0110312, 0.0456368, 0.570373, 1.78373, 0.157806\}
\end{aligned}
$$

4. Use Laplace's approximation to numerically estimate the evidence for this new data.

5. Plot the posterior $P(\theta|\{x_i\})$ and comment on the accuracy of Laplace's approximation in this case.