Office hour: today (Wednesday 28^{th})
15-00 to 15·45
Kavli building, room K02.

Nested Sampling

Numerical integration to find $Z = \int_X dx\, \mathcal{L}(d|x)\, \pi(x)$

(also gives weighted samples $w_i, x_i \sim P(x|d)$ from posterior)

Let $\mathcal{L}_{max} = \max_x \mathcal{L}(d|x)$ peak value. (assumed unique)

Define

$$\xi(L) = \int_{\{x : \mathcal{L}(d|x) > L\}} dx \; \pi(x)$$

$$= \int_{\mathcal{X}} dx \; \pi(x) \; \mathbb{1}_{\{x : \mathcal{L}(d|x) > L\}}(x)$$

$$= \int_{\mathcal{L} > L} dx \; \pi(x)$$

shorthand notation.

interpretation

if $x \sim \pi$

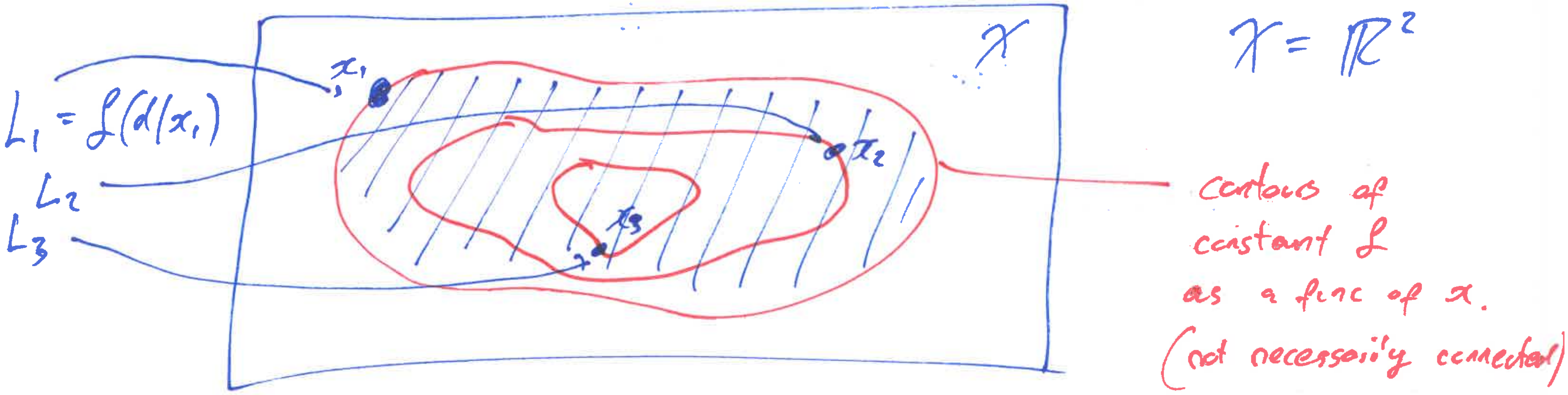
then $\xi(L) = \mathrm{Prob}(\mathcal{L}(d|x) > L)$.

"fraction of the prior, associated with higher values of the likelihood".

$\xi(0) = 1$ ← normalisation of π

$\mathcal{L}$ is never greater than $\mathcal{L}_{max}$ → $\xi(\mathcal{L}_{max}) = 0$

$\mathcal{X} = \mathbb{R}^2$



imagine, for simplicity, π is uniform over rectangle

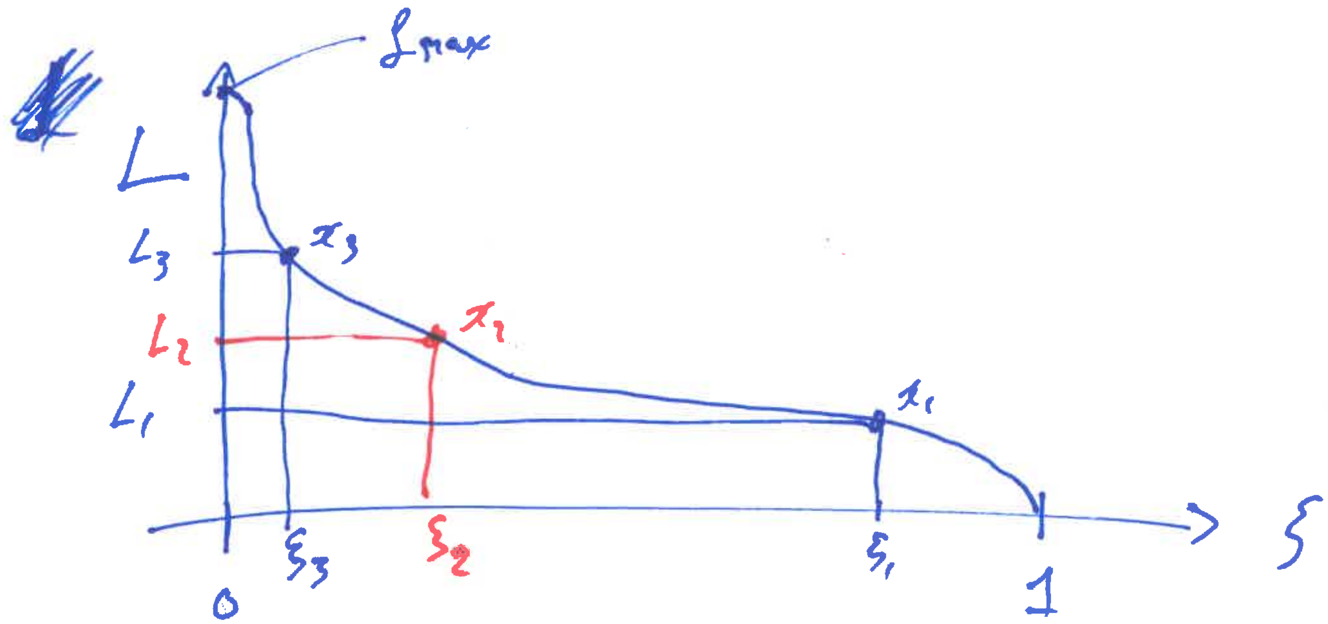
function of parameters $\mathcal{L}(d|x)$

$$x \sim \pi$$

$$L = \mathcal{L}(d|x)$$

$$\xi(L) = P_{\pi}(\mathcal{L} > L)$$

$\hookrightarrow$ inverting this gives $L(\xi)$ – Likelihood as a function of the enclosed prior "mass"



$$0 < \xi < 1$$

$$0 < L < \mathcal{L}_{max}$$

...

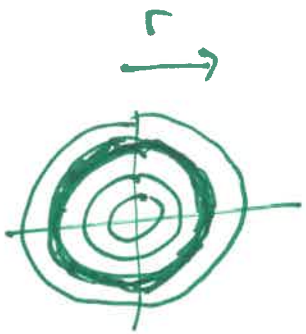
<u>example</u> : $\vec{x} \in \mathcal{X} \equiv \mathbb{R}^n$ n- dimensional param space.

choose prior uniform on the n-ball

$$\pi(\vec{x}) = \frac{1}{V_n R^n} \begin{cases} 1 & \text{if } |\vec{x}| < R \\ 0 & \text{otherwise} \end{cases}$$

V_n ↑ volume of unit ball in n dimensions

$L = e^{-r^2/2}$



suppose we have likelihood

$\longrightarrow \mathcal{L}(d|\vec{x}) = \exp\left(-\frac{1}{2} |\vec{x}|^2\right)$

no $\mathcal{L}$?

calculate $\xi(L) = \int_{\mathcal{L}>L} \pi(x) \, dx$

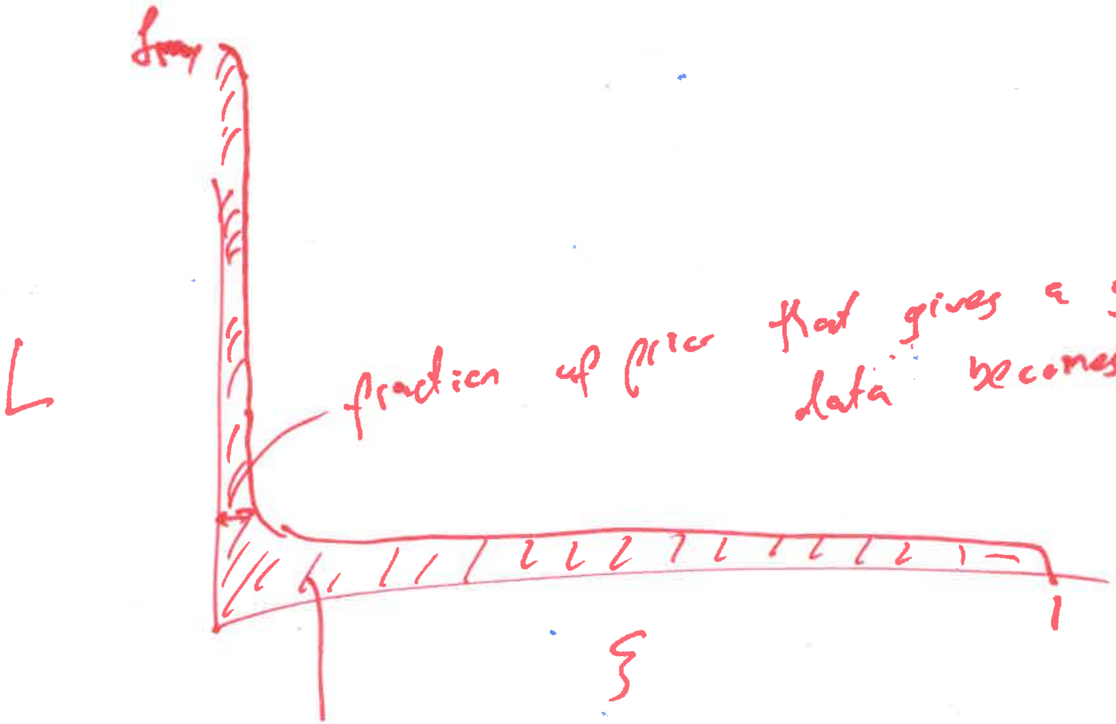
$0 < r < R$

$$= \frac{\text{Volume sphere radius } r}{\text{Volume sphere radius } R} = \frac{r^n}{R^n} = \left(\frac{\sqrt{-2\log L}}{R}\right)^n$$

$$\xi(L) = \left(\frac{\sqrt{-2 \log L}}{R} \right)^n$$

find inverse

$$L(\xi) = \exp\left(\frac{-R^2 \xi^{2/n}}{2} \right)$$



fraction of prior that gives a good fit to data becomes small in high dimensions

recall

$$Z = \int dx\, f(d|x)\, \pi(x)$$

Z is the area under this curve

$$Z = \int_0^1 d\xi\, L(\xi)$$

given a sequence of points

$$\xi_{\mu+1} < \xi_\mu$$

$$\xi_\mu \quad \text{for} \quad \mu = 1, 2, \ldots M$$

$$L_\mu = L(\xi_\mu)$$

integrate using trapezium rule to find Z

$$\int_0^1 d\xi\, L(\xi) = Z \approx \frac{1}{2} \sum_{\mu=1}^{M} (L_{\mu-1} + L_\mu)(\xi_\mu - \xi_{\mu-1})$$

exercise: show that $\longrightarrow$ $Z \approx \sum_{\mu=1}^{M} w_\mu L_\mu$ where $w_\mu = \dfrac{\xi_{\mu-1} - \xi_{\mu+1}}{2}$

nested sampling — $N_{live} = 1000$ e.g.

live points

$x_j \sim \pi(x)$ for $j = 0, 1, 2, \dots, N_{live}-1$

e.g. easy if the prior is some "nice" function.

$L_j = \mathcal{L}(d(x_j))$

while True

$idx = \mathrm{argmin}(L_0, L_1, L_2, \dots L_{N_{live}-1})$

$x_{idx} \sim \pi(x \mid \mathcal{L}(d(x)) > L_{idx})$

e.g. by rejection sampling from prior.

$L_{idx} = \mathcal{L}(d \mid x_{idx})$