General Relativity

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April 11, 2023

Contents

Abstract

Abstract of this course

1 Particle Dynamics

1.1 4-velocity of massive particle

Definition 4-velocity of massive particle:

$$u^\mu = \frac{dt}{d\tau}(c, \vec{u})$$

Note that $\frac{dt}{d\tau} = \left(1 - \frac{|\vec{u}|^2}{c^2}\right)^{-1/2} = \gamma_u$

1. 4-velocity is the tangent vector to the worldline of the particle.
2. The length of 4 velocity is constant:

$$\eta_{\mu\nu} u^\mu u^\nu = \left(\frac{ds}{d\tau}\right)^2 = c^2$$

1.1.1 Velocity Transform Laws

1.2 4-acceleration of massive particle

Definition 4-acceleration :

$$a^\mu = du^\mu / d\tau$$

Expanding, we have $a^\mu = \gamma_u^2 \left(\frac{\gamma_u^2}{c} \vec{u} \cdot \vec{a}, \vec{a} + \frac{\gamma_u^2}{c^2}(\vec{u} \cdot \vec{a})\vec{u}\right)$

1. In an inertial frame, a free particle has $d^2 x^i / dt^2 = 0$, so that $\vec{u} = \text{const.}$ and $\gamma_u = \text{const.}$
2. It follows that the components of the 4 -velocity are also constant in Cartesian coordinates so

$$\frac{du^\mu}{d\tau} = 0$$

3. The acceleration 4 -vector is always orthogonal to the 4-velocity: in Cartesian inertial coordinates

$$\eta_{\mu\nu} a^\mu u^\nu = \eta_{\mu\nu} \frac{du^\mu}{d\tau} u^\nu = \frac{1}{2} \frac{d}{d\tau} \left(\eta_{\mu\nu} u^\mu u^\nu\right) = 0,$$

   so, generally, $\boldsymbol{g}(\boldsymbol{a}, \boldsymbol{u}) = 0$.

4. In the instantaneous rest frame of the particle, $\vec{u} = \overrightarrow{0}$, and the components of the 4-acceleration in that frame are simply $a^\mu = (0, \vec{a}_{\mathrm{IRF}})$, where $\vec{a}_{\mathrm{IRF}}$ is the 3-acceleration in the instantaneous rest frame.
5. Note that the magnitude of $\vec{a}_{\mathrm{IRF}}$ determines the (invariant) magnitude of the 4-acceleration:

$$|\boldsymbol{a}|^2 = -\left|\vec{a}_{\mathrm{IRF}}\right|^2$$

   which shows that the 4 -acceleration is a spacelike vector.(4-velocity is timelike)

1.3 4-momentum

Definition 4-momentum vector:

$$\boldsymbol{p} = m\boldsymbol{u}$$

In some inertial frame, the components of $\boldsymbol{p}$ are

$$p^\mu = (\gamma_u mc, \gamma_u m\vec{u})$$

or, simply

$$p^\mu = (E/c, \vec{p})$$

1. At any point along the worldline of the particle, the (squared) magnitude of the 4-momentum is

$$\boldsymbol{p}|^2 = m^2c^2$$

   compare that to the (squared) magnitude of the 4-velocity:
2. The time component of the 4 -momentum is the total energy E of the particle (i.e., the sum of the rest-mass energy and kinetic energy):

$$E = \gamma_u mc^2$$

3. Forming the invariant $|\boldsymbol{p}|^2$ in an inertial frame, we find the energy-momentum invariant

$$E^2 - |\vec{p}|^2c^2 = m^2c^4$$

   (squared)4-momentum in variant is equivalent to energy-(3)momentum invariant
4. In isolated system, (squared)4 - invariant is conserved, which is equivalent to the conservation of energy and momentum.

1.4 4-force

Definition 4-force:

$$\tfrac{Dp^\mu}{D\tau} = f^\mu$$

..

1. Since $|\boldsymbol{p}|^2 = m^2c^2$ is constant, p^μ is orthogonal to $Dp^\mu/D\tau$ and so the 4 -velocity and 4 -force are necessarily orthogonal:

$$\boldsymbol{g}(\boldsymbol{f}, \boldsymbol{u}) = 0$$

2. In some inertial frame,

$$f^\mu = \gamma_u \frac{d}{dt}\left(\frac{E}{c}, \vec{p}\right) = \gamma_u \left(\frac{\vec{f}\cdot\vec{u}}{c}, \vec{f}\right)$$

   where we have used $dE/dt = \vec{f}\cdot\vec{u}$.
3. $\eta_{\mu\nu} f^\mu u^\nu = 0$
4. Finally, note that the 4 -force can be related to the 4 acceleration via $\boldsymbol{f} = m\boldsymbol{a}$

1.5 massless particle

1.6 Lorentz transformation

Definition homogeneous Lorentz transformation:

$$\vec{u}^1 = \frac{(\vec{u}^1 - v)}{(1 - \vec{u}^1 v/c^2)}$$
$$\vec{u}'^2 = \frac{\vec{u}^2}{\gamma_v\,(1 - \vec{u}^1 v/c^2)}$$
$$\vec{u}^3 = \frac{\vec{u}^3}{\gamma_v\,(1 - \vec{u}^1 v/c^2)}$$

... *Definition transformation matrix:*

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

This is the end

2 Electromagnetism

Definition Electromagnetic field tensor:

$$F\mu\nu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$F\mu\nu$ is antisymmetric by construction and contains four independents fields

3 Spacetime Curvature

Definition Riemann Curvature Tensor:

$$R^d_{abc} = \nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c$$

The reason for doing so is that covariant derivative of dual vector field is not commutative

3.1 Subsection 1

4 Gravitation field equations

4.1 Energy Momentum Tensor

Definition Energy Momentum Tensor:

$$T^{\mu\nu}(x) = \rho_0(x) u^\mu(x) u^\nu(x)$$

T^{00}: energy density
T^{i0}: i-th component of 3-momentum density (times c)
T^{ij}: flux of i-component of 3-momentum in j-direction

4.1.1 Properties of Energy-momentum tensor

- Always symmetric

4.2 Energy-momentum Tensor in Ideal Fluid

Consider Ideal Fluid:
We can find a local inertial frame so that $T^{i0} = 0$; and the spatial components are isotropic: $T^{ij} \propto \delta^{ij}$. Hence, in **instantaneous rest frame**:

$$T^{\mu\nu} = \mathrm{diag}(\rho c^2, p, p, p)$$

where ρc^2 is the rest frame energy density and p is isotropic pressure. While in general:

$$T^{\mu\nu} = (\rho + \frac{p}{c^2})u^\mu u^\nu - pg^{\mu\nu} \tag{1}$$

Equation 1 is the energy momentum tensor, valid in any coordinate system.

4.2.1 Continuity Equation

The energy-momentum tensor satisfies continuity equation:

$$\nabla_\mu T^{\mu\nu} = 0$$

4.3 Einstein Field Equation

Definition Einstein Field Equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}$$

Constant proportionality and negative sign on the right is required for consistency with the weak field limit

5 The Schwarzchild Solution

Adopting a passive view point: change the coordinate system without changing the functional form of the fields on our coordinates.

5.1 Geodesics in Schwarzchild spacetime

In this section we study the equation of motion for 4 coordinates, t, r, θ, ϕ

1. Equation of motion for θ:
$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\psi}^2 = 0 \tag{2}$$
A possible solution is $\theta = \pi/2$, planar motion in the equatorial plane; given the spherical symmetry.

2. Equation of motion for t:
$$(1 - \frac{2\mu}{r})\dot{t} = k \tag{3}$$
$k = (1 - 2\mu/r)\dot{t}$ is related to the energy of the particle as measured by stationary observer.

3. Equation of motion for ϕ:
$$r^2\dot{\phi} = h \tag{4}$$
Here, we assume in the plane $\theta = \pi/2$. h arises from the symmetry of the spacetime under ratotion about z-axis, can be interpreted as *specific angular momentum.*

4. Equation of motion for r:
$$(1 - \frac{2\mu}{r})c^2\dot{t}^2 - (1 - \frac{2\mu}{r})^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = \begin{cases} c^2 & \text{massive} \\ 0 & \text{massless} \end{cases} \tag{5}$$

5.2 Effective potential energy

1. In spherical coordinates, the Newtonian effective potential energy is

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{h^2}{2r^2} \tag{6}$$

- It has a centrifugal barrier at small r, preventing particles reaching $r = 0$
- Bound orbits have $E_N < 0$, two turning points for r w.r.t V at $V = E_N$
- Effective potential have one turning point at $r = h^2/GM$ It is a minimum, corresponding to stable circular orbit.

2. massive particle in general relativity:

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{h^2}{2r^2}(1 - \frac{2\mu}{r}) \tag{7}$$

- the centrifugal term is slightly modified by a factor of $(1 - 2\mu/r)$
- solving for the extrema of V gives:

$$r_\pm = \frac{h}{2\mu c^2}(h \pm \sqrt{h^2 - 12\mu^2 c^2}) \tag{8}$$

- two stationary points for $h > \sqrt{\mu c}$, r_- is a maximum and $r_+$is minimum
- none for smaller h i.e V is increasing with r
-

6 Shapes of orbits for massive and massless particles

6.1 Shape of Orbit

1. Massive article gives

$$\frac{d^2u}{d\phi^2} + u - 3\mu u^2 = \frac{GM}{h^2} \tag{9}$$

2. Massless article gives

$$\frac{d^2u}{d\phi^2} + u - 3\mu u^2 = 0 \tag{10}$$

7 Cosmology

Fundamental observers agrees waht thay observe at any given proper time.

7.1 Robertson-Walker Metric

Robertson–Walker form: $ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{(1-Kr^2)} + r^2 d\Omega^2\right]$

7.2 Geometry of the 3D spaces

properties of the 3D maximally-symmetric spaces with line element depend on K

7.3 Cosmological field equations

The Robertson-Walker metric contains a single function of time a(t) whose evolution is determined by the Einstein field equations

7.4 Friedmann Equation

The solution for the time dependent part of R-W metric with Einstein field equations:

Friedmann Equation 1:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda c^2$$

2:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2$$