VI. ELECTROMAGNETISM

Electromagnetism is a relativistic theory, despite being conceived before special relativity.

Maxwell's equations in free space, expressed in terms of Cartesian coordinates in some inertial frame S, are

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}\,, \tag{1}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,, \tag{2}$$

$$\vec{\nabla} \cdot \vec{B} = 0\,, \tag{3}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\,, \tag{4}$$

where $\vec{\nabla}$ is the usual 3D derivative, ρ and $\vec{J}$ are the charge and current density in S, and $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields in S.

Maxwell's equations are supplemented by the *Lorentz force law*,

$$\vec{f} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)\,, \tag{5}$$

which gives the electromagnetic 3-force on a particle of charge q and 3-velocity $\vec{u}$.

Charge conservation is built into Maxwell's equations: taking the divergence of Eq. (4) and using Eq. (1) gives the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0\,. \tag{6}$$

It is not obvious when written in their 3D form, but Maxwell's equations do satisfy the principle of relativity: it is possible to find transformation laws for the electric and magnetic fields such that the equations are unchanged in form under Lorentz transformations.

You will already have seen hints of this in previous courses, where you have shown that Maxwell's equations can be reduced to wave equations with solutions that propagate at the speed of light.

In this topic, we shall develop electromagnetism as a relativistic field theory on Minkowski spacetime.

In particular, we shall show how the 3D Maxwell's equations can be combined into 4D tensor equations, which, by virtue of being tensor equations, show that the theory is relativistically covariant.

Moreover, by writing the equations in tensor form, we can express the theory in a form that is independent of the particular coordinate system that is used.

1 Lorentz force law

Recall that the 3-force $\vec{f}$ acting on a particle with 3-velocity $\vec{u}$ in some inertial frame is related to the spatial components of the force 4-vector,

$$f^\mu = \gamma_u \left(\frac{\vec{f} \cdot \vec{u}}{c}, \vec{f} \right) . \tag{7}$$

Since the 3-force depends linearly on the 3-velocity, we might expect the force 4-vector to depend linearly on the 4-velocity, i.e.,

$$f_\mu = q F_{\mu\nu} u^\nu . \tag{8}$$

Here, we have lowered the index of f^μ (to form the associated dual vector), and introduced a type-$(0,2)$ tensor with components $F_{\mu\nu}$ called the *Maxwell field-strength tensor.*

The charge q is a scalar quantity (i.e., all observers agree on its value), so $F_{\mu\nu}$ must be the components of a tensor since f_μ and u^μ are.

The equation of motion for the 4-velocity in the presence of this force is

$$\frac{Du^\mu}{D\tau} = \frac{q}{m} F^\mu{}_\nu u^\nu \,, \tag{9}$$

where we have raised the first index on $F_{\mu\nu}$.

The 4-force has to be orthogonal to the 4-velocity, $f_\mu u^\mu = 0$, which requires the field-strength tensor to be antisymmetric:

$$f_\mu u^\mu = qF_{\mu\nu}u^\mu u^\nu = 0 \quad \Rightarrow \quad F_{\mu\nu} = -F_{\nu\mu} \,. \tag{10}$$

If we raise both indices to form $F^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}$ (working in general coordinates), we preserve antisymmetry: $F^{\mu\nu} = -F^{\nu\mu}$.

We can relate the components $F_{\mu\nu}$ in Cartesian inertial coordinates to the electric and magnetic field components in that inertial frame by comparing Eq. (8) to the 3D Lorentz force law.

We have

$$f_0 = q\sum_i F_{0i}u^i = \frac{q}{c}\gamma_u \vec{E}\cdot\vec{u} \,. \tag{11}$$

From this, we can read off that, numerically, $F_{0i} = \vec{E}^i/c$.

For the spatial components, we have

$$f_i = qF_{i0}u^0 + q\sum_j F_{ij}u^j = -q\gamma_u\left(\vec{E}^i + (\vec{u}\times\vec{B})^i\right) \,, \tag{12}$$

where we used $f_i = -\gamma_u \vec{f}^i$ in Cartesian inertial coordinates.

The $-q\gamma_u\vec{E}^i$ term equals $qF_{i0}u^0$, so we are left with

$$q\gamma_u \sum_j F_{ij}\vec{u}^j = -q\gamma_u(\vec{u}\times\vec{B})^i \,. \tag{13}$$

Setting $i = 1$, 2 and 3, we can read off

$$F_{12} = -\vec{B}^3 \,, \quad F_{13} = \vec{B}^2 \,, \quad F_{23} = -\vec{B}^1 \,. \tag{14}$$

We arrive at the following components of the field-strength tensor in terms of the electric and magnetic fields:

$$F_{\mu\nu} = \begin{pmatrix} 0 & \vec{E}^1/c & \vec{E}^2/c & \vec{E}^3/c \\ -\vec{E}^1/c & 0 & -\vec{B}^3 & \vec{B}^2 \\ -\vec{E}^2/c & \vec{B}^3 & 0 & -\vec{B}^1 \\ -\vec{E}^3/c & -\vec{B}^2 & \vec{B}^1 & 0 \end{pmatrix} . \tag{15}$$

For reference, if we raise both indices with $\eta^{\mu\nu}$, we obtain

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\vec{E}^1/c & -\vec{E}^2/c & -\vec{E}^3/c \\ \vec{E}^1/c & 0 & -\vec{B}^3 & \vec{B}^2 \\ \vec{E}^2/c & \vec{B}^3 & 0 & -\vec{B}^1 \\ \vec{E}^3/c & -\vec{B}^2 & \vec{B}^1 & 0 \end{pmatrix} . \tag{16}$$

As $F^{\mu\nu}$ are the components of a type-$(2, 0)$ tensor, we know how they transform under a Lorentz transformation $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$:

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} F^{\rho\sigma} \,. \tag{17}$$

This tells us how the electric and magnetic fields are related in different inertial frames; for a standard Lorentz boost we have (exercise!)

$$\vec{E}' = \begin{pmatrix} \vec{E}^1 \\ \gamma(\vec{E}^2 - v\vec{B}^3) \\ \gamma(\vec{E}^3 + v\vec{B}^2) \end{pmatrix} , \; \vec{B}' = \begin{pmatrix} \vec{B}^1 \\ \gamma(\vec{B}^2 + v\vec{E}^3/c^2) \\ \gamma(\vec{B}^3 - v\vec{E}^2/c^2) \end{pmatrix} . \tag{18}$$

2 Maxwell's equations

The relativistic form of the Lorentz force law has led us to introduce the field-strength tensor, whose components in any inertial frame encode the electric and magnetic fields there.

We now seek to express Maxwell's equations in terms of the spacetime derivative of the field-strength tensor (since the 3D Maxwell's equations have first-order space and time derivatives).

However, before we do this we need to consider the source terms ρ and $\vec{J}$.

2.1 Current 4-vector

Since a static charge density in some inertial frame S will transform to a moving charge density, i.e., possess a current, under a change in inertial frame, we might expect to be able to assemble a 4-vector from ρ and $\vec{J}$.

Consider a current distribution $\vec{J}$ formed from a charge density ρ moving with 3-velocity $(v, 0, 0)$ in inertial frame S, so that $\vec{J} = \rho(v, 0, 0)$.

Let S' be the rest-frame of the charges, which is in standard configuration with S, and let the charge density in this frame be ρ_0.

The current vanishes in S' since the charges are at rest.

We can relate the charge density and current in the two frames by the following physical argument.

Take some given volume V' in S' and mark those charges that lie within V'.

The same charges occupy a smaller volume V'/γ in S by length contraction (see Fig. 1), but the amount of charge is the same.

It follows that charge density is larger in S by a fac-

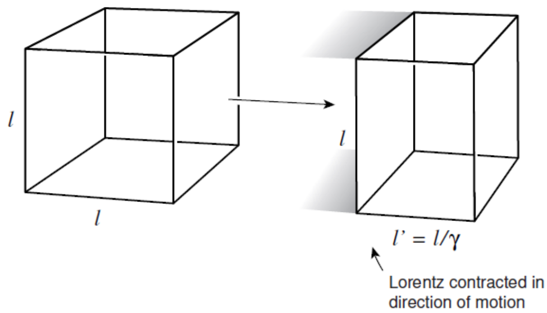

Figure 1: Length contraction of the volume occupied by a given set of charges between their rest frame volume (left) and that in a frame in which they are moving with speed v (right).

tor of γ so

$$\rho = \gamma\rho_0 \quad \text{and} \quad \vec{J} = \gamma\rho_0(v, 0, 0)\,. \tag{19}$$

However, these are just the transformations that would follow if ρc and $\vec{J}$ were the time and space components of a *current 4-vector*

$$\boxed{j^\mu = \left(c\rho, \vec{J}\right)\,.} \tag{20}$$

This is because in S' we would have $j'^\mu = (c\rho_0, \vec{0})$, and, on Lorentz transforming,

$$\begin{aligned} j^0 &= \gamma\left(j'^0 + \beta j'^1\right) \\ \Rightarrow \quad \rho &= \gamma\rho_0\,, \end{aligned} \tag{21}$$

and

$$\begin{aligned} j^1 &= \gamma\left(j'^1 + \beta j'^0\right) \\ \Rightarrow \quad \vec{J}^1 &= \gamma v\rho_0\,. \end{aligned} \tag{22}$$

2.2 Relativistic field equations

We want to relate the field-strength tensor to the current 4-vector and we expect this relation to be linear in spacetime derivatives.

If we contract the type-$(2,0)$ tensor $F^{\mu\nu}$ with the covariant derivative (i.e., form the covariant divergence), we necessarily form a 4-vector.

We therefore consider a tensor equation of the form

$$\nabla_\mu F^{\mu\nu} = kj^\nu \,, \tag{23}$$

for some constant scalar k.

Let us consider this equation in Cartesian inertial coordinates, so that the covariant derivative becomes a partial derivative and we have

$$\partial_\mu F^{\mu\nu} = kj^\nu \,. \tag{24}$$

Taking the divergence and using the antisymmetry of $F^{\mu\nu}$ gives

$$\begin{aligned} k\partial_\nu j^\nu &= \partial_\nu \partial_\mu F^{\mu\nu} = 0 \\ \Rightarrow \quad \partial_\mu j^\mu &= 0 \,. \end{aligned} \tag{25}$$

This is the just the continuity equation in 4D language since, using $j^\mu = (c\rho, \vec{J})$,

$$\frac{\partial j^0}{\partial(ct)} + \sum_i \frac{\partial j^i}{\partial x^i} = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \,. \tag{26}$$

Equation (23) therefore looks promising, but we still need to show that it does return the correct Maxwell's equations.

Consider the 0-component:

$$\begin{aligned} \frac{\partial F^{00}}{\partial(ct)} + \sum_i \frac{\partial F^{i0}}{\partial x^i} &= kj^0 \\ \Rightarrow \quad \frac{1}{c}\vec{\nabla} \cdot \vec{E} &= kc\rho \,. \end{aligned} \tag{27}$$

Recalling that $\epsilon_0\mu_0 = 1/c^2$, we see that we recover the Maxwell equation $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ if we take $k = \mu_0$.

Consider now the spatial components:

$$\frac{\partial F^{0i}}{\partial(ct)} + \sum_j \frac{\partial F^{ji}}{\partial x^j} = \mu_0 j^i\,. \tag{28}$$

Taking $i = 1$, we have

$$-\frac{1}{c^2}\frac{\partial \vec{E}^1}{\partial t} + \underbrace{\frac{\partial \vec{B}^3}{\partial x^2} - \frac{\partial \vec{B}^2}{\partial x^3}}_{\left(\vec{\nabla}\times\vec{B}\right)^1} = \mu_0 \vec{J}^1\,. \tag{29}$$

Repeating for the other components, and rearranging, we recover

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}\,. \tag{30}$$

Equation (23) is therefore the tensor version of the two sourced Maxwell's equations, but what of the other two?

We need to introduce a further (homogeneous) tensor equation to capture the two source-free equations,

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}\,, \tag{31}$$

$$\vec{\nabla}\cdot\vec{B} = 0\,. \tag{32}$$

These are four equations in total, so we look for a tensor equation involving only the covariant derivative of the field-strength that has four independent components.

In 4D, a totally-antisymmetric type-$(0,3)$ tensor has $4\times3\times2/3! = 4$ independent components, so we consider

$$\nabla_{[\mu}F_{\nu\rho]} = 0\,, \tag{33}$$

where, recall, the square brackets denote the antisymmetric part.

Since $F_{\mu\nu}$ is itself antisymmetric, this can be written explicitly as

$$\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0\,. \tag{34}$$

Again, let us consider this tensor equation in Cartesian inertial coordinates, in which case it reduces to[1]

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0\,. \tag{35}$$

The four independent choices of indices are

$$(\mu, \nu, \rho) = (0, 1, 2)\,,\ (0, 1, 3)\,,\ (0, 2, 3)\,,\ (1, 2, 3)\,. \tag{36}$$

Consider these in turn; for $(\mu, \nu, \rho) = (0, 1, 2)$ we have

$$-\frac{\partial \vec{B}^3}{\partial(ct)} - \frac{1}{c}\underbrace{\left(\frac{\partial \vec{E}^2}{\partial x^1} - \frac{\partial \vec{E}^1}{\partial x^2}\right)}_{(\vec{\nabla}\times\vec{E})^3} = 0\,, \tag{37}$$

which is the 3-component of Eq. (31), and the other two cases with $\mu = 0$ give the remaining components.

For $(\mu, \nu, \rho) = (1, 2, 3)$, we have

$$-\frac{\partial \vec{B}^1}{\partial x^1} - \frac{\partial \vec{B}^2}{\partial x^2} - \frac{\partial \vec{B}^3}{\partial x^3} = 0\,, \tag{38}$$

which is $\vec{\nabla} \cdot \vec{B} = 0$.

To summarise, the four Maxwell's equations in any inertial frame are the components in Cartesian inertial coordinates of the two tensor equations

$$\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu\,, \tag{39}$$

$$\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0\,. \tag{40}$$

Being tensor equations, these are valid in any coordinate system covering Minkowski space.

[1] The equation actually takes this form in any coordinate system due to the symmetry of the metric connection.

Charge conservation is built into these equations: in Cartesian inertial coordinates the conservation equation takes the form $\partial_\mu j^\mu = 0$, and in general coordinates this becomes the tensor equation[2]

$$\nabla_\mu j^\mu = 0 \,. \tag{41}$$

2.3 The 4-vector potential

The Maxwell equation $\partial_{[\mu} F_{\nu\rho]} = 0$, written in Cartesian inertial coordinates, implies that the type-$(0,2)$ field strength tensor can be derived from a (dual)-vector potential A_μ.

If we take

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{42}$$

then $\partial_{[\mu} F_{\nu\rho]} = 0$ is identically satisfied since

$$\begin{aligned}
\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} &= \partial_\mu \left(\partial_\nu A_\rho - \partial_\rho A_\nu\right) \\
&\quad + \partial_\nu \left(\partial_\rho A_\mu - \partial_\mu A_\rho\right) \\
&\quad + \partial_\rho \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\
&= 0 \,.
\end{aligned} \tag{43}$$

[2] A direct proof of this result in a general coordinate system is as follows. From Eq. (23), we have

$$\begin{aligned}
\mu_0 \nabla_\nu j^\nu &= \nabla_\nu \nabla_\mu F^{\mu\nu} \\
&= \partial_\nu \left(\nabla_\mu F^{\mu\nu}\right) + \Gamma^\nu_{\nu\rho} \left(\nabla_\mu F^{\mu\rho}\right) \\
&= \partial_\nu \left(\nabla_\mu F^{\mu\nu}\right) + \frac{1}{2}(\partial_\rho \ln|g|) \left(\nabla_\mu F^{\mu\rho}\right) \,,
\end{aligned}$$

where we used the result $\Gamma^\nu_{\nu\rho} = (\partial_\rho \ln|g|)/2$ (see Handout V). We now use

$$\begin{aligned}
\nabla_\mu F^{\mu\nu} &= \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} + \underbrace{\Gamma^\nu_{\mu\rho} F^{\mu\rho}}_{0} \\
&= \partial_\mu F^{\mu\nu} + \frac{1}{2}(\partial_\rho \ln|g|) F^{\rho\nu}
\end{aligned}$$

to find

$$\begin{aligned}
\mu_0 \nabla_\nu j^\nu &= \partial_\nu \partial_\mu F^{\mu\nu} + \frac{1}{2}(\partial_\nu \partial_\rho \ln|g|) F^{\rho\nu} + \frac{1}{2}(\partial_\rho \ln|g|) \partial_\nu F^{\rho\nu} + \frac{1}{2}(\partial_\rho \ln|g|) \partial_\mu F^{\mu\rho} \\
&\quad + \frac{1}{4}(\partial_\rho \ln|g|)(\partial_\sigma \ln|g|) F^{\sigma\rho} \\
&= 0 \,,
\end{aligned}$$

where we have used the commutativity of partial derivatives and the antisymmetry of $F^{\mu\nu}$.

This is analogous to a (globally) curl-free vector being expressible as the gradient of a scalar in 3D Euclidean space.

The 4-vector potential is not uniquely determined by the field-strength tensor – there is a residual *gauge freedom* to add the gradient of a scalar ψ to A_μ without altering $F_{\mu\nu}$:

$$\begin{aligned} F_{\mu\nu} &\to \partial_\mu \left(A_\nu + \partial_\nu \psi\right) - \partial_\nu \left(A_\mu + \partial_\mu \psi\right) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \,. \end{aligned} \tag{44}$$

A convenient gauge choice, which has the virtue of being Lorentz invariant, is to take A_μ to be divergence free,

$$\partial_\mu A^\mu = 0 \,. \tag{45}$$

This gauge choice is called the *Lorenz gauge* (Lorenz was different to Lorentz!).

The sourced 4D Maxwell equation can be written directly in terms of the 4-vector potential.

It is convenient to lower the index and write

$$\nabla^\mu F_{\mu\nu} = \mu_0 j_\nu \,, \tag{46}$$

or, in Cartesian inertial coordinates,

$$\eta^{\mu\rho} \partial_\rho \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = \mu_0 j_\nu \,. \tag{47}$$

In the Lorenz gauge, this simplifies further to

$$\nabla^2 A_\nu = \mu_0 j_\nu \,, \tag{48}$$

where $\nabla^2 A_\nu = \eta^{\mu\rho}\partial_\mu \partial_\rho A_\nu$ is the 4D Laplacian in Cartesian inertial coordinates.

The Laplacian in Minkowski spacetime is the wave operator, with wave speed c:

$$\nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \,. \tag{49}$$

In the absence of charges and currents, Eq. (48) therefore admits wavelike solutions that travel at the speed of light.

In the presence of a variable source, Eq. (48) describes the generation of electromagnetic fields that asymptotically far from the source describe radiation fields.

(Those of you taking Part-II Physics will study the generation of electromagnetic radiation from variable sources in detail in the course *Optics and Electrodynamics*.)

In Cartesian inertial coordinates, the components of A^μ are the familiar scalar and magnetic-vector potentials of 3D electromagnetism:

$$A^\mu = \left(\phi/c, \vec{A}\right) \quad \text{or} \quad A_\mu = \left(\phi/c, -\vec{A}\right) . \tag{50}$$

Using the relation (Eq. 15) between the components of the field-strength tensor and the electric and magnetic fields, we find

$$\begin{aligned} F_{0i} &= -\frac{\partial \vec{A}^i}{\partial(ct)} - \frac{1}{c}\frac{\partial \phi}{\partial x^i} \\ \Rightarrow \quad \vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi . \end{aligned} \tag{51}$$

Similarly, the spatial components of $F_{\mu\nu}$ give, for example,

$$F_{12} = -\frac{\partial \vec{A}^2}{\partial x^1} + \frac{\partial \vec{A}^1}{\partial x^2} = -\left(\vec{\nabla} \times \vec{A}\right)^3 , \tag{52}$$

so that

$$\vec{B} = \vec{\nabla} \times \vec{A} . \tag{53}$$

The source-free 3D Maxwell equations (31) and (32) are identically satisfied by the relations (51) and (53).

Finally, we note that the generalisation of Eq. (42) to arbitrary coordinates is

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu . \tag{54}$$

However, due to the symmetry of the metric connection, this is actually equivalent to $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in all coordinate systems.

3 Electromagnetism in curved spacetime

We have seen that in Minkowski spacetime, the tensor equations

$$\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu \,, \tag{55}$$

$$\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0 \tag{56}$$

are equivalent to the usual 3D Maxwell equations when expressed in global Cartesian coordinates.

In curved spacetime, electromagnetism is described by exactly the same tensor equations.

This is because if we express these tensor equations in local inertial coordinates at some point, they reduce to the same form there as in Minkowski space in inertial coordinates, as required by the equivalence principle.