Astrofluid Dynamics

Xinyu Zhong
Wolfson College

February 2, 2023

Contents

Abstract

Abstract of this course

1 Some basic concepts

Collisional v collisionless fluids
Eulerian and Lagrangian framework
Concepts of streamlines, particle paths and streaklines:
They coinside if the flow is steady.i.e.

2 Formulation of the Fluid Equations

This chapter talked about the conservation of mass and momentum.

2.1 Conservation of mass

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{u} = 0 \tag{1}$$

2.2 Conservation of momentum

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla}) vbu = -\boldsymbol{\nabla} p + \rho \mathbf{g} \tag{2}$$

We consider 4 different which contribute to the change of momentum?

3 Gravitation

In this section, we used $\vec{\mathbf{g}}$ to denote gravitational acceleration; Ψ to denote gravitational potential; and Ω to denote the energy required to take the system of point masses to infinity

Example: Spherical distribution of mass

Example: Infinitely cylindrical symmetrical mass

Example: Infinite planar distribution of masses

Example: Finite axisymmetric disk

3.1 Potential of a Spherical Mass Distribution

Ψ is affected by any matter outside r through our choice of setting Psi at infinity. i.e. We **can't** say that $\Psi = -GM/r$

3.2 Gravitational Potential Energy

3.3 Virial Theorem

Virial Theorem: states that for a system in steady state, $I \equiv mr^2 = constant$, $2T + \Omega = 0$
Kinetic energy T has contribution from local flows and random/thermal motions.
A result of virial theorem is that the gravitational potential sets the temperature or velocity dispersion of the system.

4 Equation of state and the energy equation

So far we have 4 variables, density ρ, pressure,p, gravitational potential Ψ and velocity which is $\boldsymbol{v}$. In terms of equations to solve them, we have the scalar equation of mass conservation and the vector equation for conservation of momentum. We also have Poisson equation for the potential term. Now what we need is another equation: 'Equation of state' to determinate pressure (which is a thermodynamic property). While doing so we might Introduce another equation, the energy equation when the system is not barotropic, i.e. p is a function of T.

4.1 Equation of state

1. Astrophysical fluid are treated as ideal gas, and corresponding EoS is:

$$p = nk_BT = \frac{k_B}{\mu m_p}\rho T \tag{3}$$

   where ρ is the mean particle mass.

2. This EoS introduces another scalar field, temperature, T

4.1.1 Barotropic

Barotropic means that p independent of T. i.e. only a function of ρ. This comes in two cases: Isothermal and Adiabatic.

1. Isothermal: Constant T
2. Adiabatic: Ideal has undergoes reversible thermodynamic changes

$$p = K\rho^\gamma \tag{4}$$

3. Derivation of C_p and C_v for both cases.

4.1.2 Energy equation for non-Barotropic case

1. It starts from first law

$$\bar{d}Q = d\mathcal{E} + pdV \tag{5}$$

2. total energy per unit volume is:

$$E = \rho(\frac{1}{2}u^2 + \Psi + \mathcal{E}) \tag{6}$$

3. take material derivative, note that $\frac{DE}{Dt} = \frac{\partial E}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} E$, gives Energy Equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E + p)\mathbf{u}] = \rho\frac{\partial \Psi}{\partial t} - \rho\dot{Q}_{cool} \tag{7}$$

   In many settings $\partial\Psi/\partial\Psi = 0$, If there is no cooling, the equation expresses the conservation for energy in which the total energy density E is driven by the divergence of the enthalpy flux $(E + p)\mathbf{u}$

4.2 Heating and Cooling Processes

Combining cooling and heating effects, we can parametrise $\dot{Q}_{cool}$

$$\dot{Q}_{cool} = A\rho T^{\alpha} - H \tag{8}$$

where the first and second term on RHS means radiative cooling and cosmic ray heating

4.3 Energy Transport process

Transport processes move energy through the fluid, via Thermal conduction, convention and radiation transport.

5 A lot of different system

5.1 Full set of equations describing the dynamics of an ideal non-relativisitc fluid

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{u} = 0 \tag{Continuity Equation}$$

(Momentum Equation)

(Poisson's Equation)

(Energy Equation)

(Definition fo Total energy)

(EoS of total energy)

(Internal Energy)

5.2 Hydrostatic Equilibrium

A System of hydrostatic equilibrium if

$$\mathbf{u} = \frac{\partial}{\partial t} = 0 \tag{9}$$

Continuity equation is trivially satisfied Sub into momentum equation gives:

$$\frac{1}{\rho}\boldsymbol{\nabla} p = -\boldsymbol{\nabla}\Psi \tag{Equation of Hydrostatic equilibirum}$$

Example: Isothermal atmosphere

Isothermal atmosphere with constant $\mathbf{g} = -g\hat{\mathbf{z}}$

$$\rho = \rho_0 \exp\left\{-\frac{\mu g}{R_* T} z\right\} \tag{10}$$

i.e. exponential atmosphere

Example: Isothermal self-gravitating slab

Isothermal atmosphere with constant $\mathbf{g} = -g\hat{\mathbf{z}}$

$$\nabla^2 \Psi = 4\pi G\rho \tag{11}$$

Is gives

$$\Psi - \Psi_0 = 2A \ln \cosh \sqrt{\frac{2\pi G\rho}{A}} z \tag{12}$$

$$\rho = \frac{\rho_0}{\cosh^2 \sqrt{\frac{2\pi G\rho}{A}} z} \tag{13}$$

6 Stars/Self-Gravitating Polytropes