V. MINKOWSKI SPACETIME AND PARTICLE DYNAMICS

Now that we have the machinery of tensor algebra and calculus in place, in this topic we shall first apply this to special relativity and consider how to express this theory in a more formal manner.

We shall also develop the theory of relativistic mechanics, which is best expressed in terms of 4D vectors in spacetime ("4-vectors").

The spacetime of special relativity is a pseudo-Euclidean manifold, over which we can globally define Cartesian coordinates.

Most of our treatment of special relativity will make use of such coordinates, which correspond to the coordinates of inertial frames.

However, by expressing our equations in tensor form, we can easily write them in arbitrary coordinates; we shall illustrate this with the specific example of a rotating frame of reference.

1 Minkowski spacetime in Cartesian coordinates

Minkowski spacetime is a 4D pseudo-Euclidean manifold.

We can therefore adopt a global system of Cartesian coordinates x^μ ($\mu = 0, 1, 2, 3$) such that the line element is everywhere

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,, \tag{1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ is the *Minkowski metric*.

Note that for applications to spacetime, we shall usually use Greek coordinate labels rather than the Roman a, b, c etc. that we have used so far on general manifolds, and allow them to run from 0–3 rather than 1–N.

These Cartesian coordinates correspond to the coordinates (ct, x, y, z) as defined by some inertial frame, with

$$x^0 = ct\,, \quad x^1 = x\,, \quad x^2 = y\,, \quad x^3 = z\,. \tag{2}$$

The components of the inverse metric in Cartesian coordinates are denoted $\eta^{\mu\nu}$ and are simply

$$\eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)\,. \tag{3}$$

As the components of the metric are constant, the metric connection vanishes in Cartesian coordinates: $\Gamma^{\mu}_{\nu\sigma} = 0$.

1.1 Lorentz transformations

Physically, Lorentz transformations relate Cartesian coordinates assigned to events (spacetime points) in different inertial frames.

Mathematically, they correspond to the residual freedom in our choice of global Cartesian coordinates in Minkowski spacetime, i.e., to coordinate transformations $x^\mu \to x'^\mu$ that leave the Minkowski metric unchanged:

$$\eta_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma}\,. \tag{4}$$

Multiplying through by the inverse of the transformation matrix twice we have the equivalent requirement for a Lorentz transformation,

$$\boxed{\eta_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\eta_{\rho\sigma}\,.} \tag{5}$$

By differentiating this condition, it can be shown (see e.g., Chapter 2 of Weinberg's *Gravitation and Cosmology*) that Lorentz transformations must be linear:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}\,, \tag{6}$$

for suitable constant $\Lambda^{\mu}{}_{\nu}$, with

$$\eta_{\mu\nu} = \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu} \eta_{\rho\sigma}\,, \tag{7}$$

and constant a^{μ}.

Equation (6) is known as an *inhomogeneous Lorentz transformation* or *Poincare transformation.*

The constant a^{μ} just corresponds to changing the spacetime origin; dropping this term gives what are called *homogeneous Lorentz transformations.*

1.2 Homogeneous Lorentz transformations

The constants $\Lambda^{\mu}{}_{\nu}$ of a homogeneous Lorentz transformation depend on the relative velocity and orientation of the two inertial frames.

Their form for a standard Lorentz boost with speed $v = \beta c$ along the x-axis is

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

where $\gamma = (1-\beta^2)^{-1/2}$.

The inverse of the transformation matrix is denoted by $(\Lambda^{-1})^{\mu}{}_{\nu}$ and is given by

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}}\,. \tag{9}$$

The inverse can be found from Eq. (7) to be

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \eta^{\mu\rho} \eta_{\nu\sigma} \Lambda^{\sigma}{}_{\rho}\,. \tag{10}$$

The notation $(\Lambda^{-1})^\mu{}_\nu$ is cumbersome so it is usual to define a new matrix $\Lambda_\mu{}^\nu$ (note the index positioning!) with

$$\Lambda_\mu{}^\nu = (\Lambda^{-1})^\nu{}_\mu = \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma \,. \tag{11}$$

Here, we are using the same kernel letter (Λ) to denote two different matrices, with these being distinguished by their index positions.

Note that Eq. (11) looks like raising and lowering indices on $\Lambda^\mu{}_\nu$ with the Minkowski metric and this is the motivation for the notation $\Lambda_\mu{}^\nu$.

However, the transformation matrix $\Lambda^\mu{}_\nu$ does not contain the components of a tensor so the similarity with raising and lowering indices on tensors is really just a useful mnemonic.

1.3 Proper Lorentz transformations

Proper Lorentz transformations form a subgroup of the full Lorentz transformations that only include transformations between inertial frames with the same spatial handedness and exclude time reversal.

The defining condition (7) of Lorentz transformations gives

$$[\det(\Lambda^\mu{}_\nu)]^2 = 1 \,. \tag{12}$$

Moreover, setting $\mu = \nu = 0$ in Eq. (7) gives

$$\left(\Lambda^0{}_0\right)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i{}_0)^2 \geq 1 \,. \tag{13}$$

Mathematically, the subgroup of proper Lorentz transformations have

$$\boxed{\det(\Lambda^\mu{}_\nu) = 1 \quad \Lambda^0{}_0 \geq 1 \,,} \tag{14}$$

and these transformations are continuously connected to the identity.

From now on, we shall generally only consider such proper Lorentz transformations.

1.4 Cartesian basis vectors

Recall that on a general manifold, a coordinate system x^a provides a set of basis vectors $\partial/\partial x^a$ that span the tangent space at any point.

Since the basis vectors are differential operators corresponding to partial differentiation with respect to the coordinates, we often represent $\partial/\partial x^a$ in a diagram as an arrow tangent to the associated coordinate curves.

Recall also that the scalar product between two vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$, is $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v})$ or, in components, $g_{ab}u^a v^b$.

If we take $\boldsymbol{u}$ and $\boldsymbol{v}$ to be the basis vectors $\partial/\partial x^a$ and $\partial/\partial x^b$ of some coordinate system, then their scalar product is just the appropriate component of the metric in those coordinates, g_{ab}.

In Minkowski space, the global Cartesian coordinates x^μ associated with some inertial frame define a set of basis vectors $\partial/\partial x^\mu$ that we shall write as $\boldsymbol{e}_\mu$, i.e.,

$$\boldsymbol{e}_\mu \equiv \frac{\partial}{\partial x^\mu} \, . \tag{15}$$

These basis vectors are orthonormal since

$$\boldsymbol{g}(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu) = \eta_{\mu\nu} \, . \tag{16}$$

If we change coordinates, we generate a new set of basis vectors $\partial/\partial x'^a$ related to the basis vectors in the x^a coordinates by

$$\frac{\partial}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a}\frac{\partial}{\partial x^b} \, . \tag{17}$$

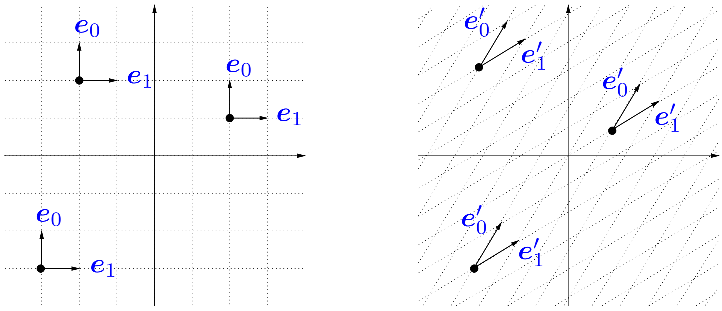


Figure 1: Coordinate curves for two systems of coordinates x^μ and x'^μ, corresponding to Cartesian inertial frames S and S' in standard configuration. The coordinate basis vectors for each system are also shown, indicated as arrows tangent to the coordinate curves. The 2- and 3-directions are suppressed and null vectors would lie at 45 degrees to the vertical.

Applying this to a Lorentz transformation in spacetime, we have

$$\boldsymbol{e}'_\mu = \Lambda_\mu{}^\nu \boldsymbol{e}_\nu \,, \tag{18}$$

where $\boldsymbol{e}'_\mu \equiv \partial/\partial x'^\mu$.

Note that the basis vectors transform with the inverse transformation matrix.

Since we are making a Lorentz transformation, the components of the metric in the transformed coordinates are still $\eta_{\mu\nu}$ and the new basis vectors are still orthonormal.

These ideas are illustrated in Fig. 1.

1.5 4-vectors and the lightcone

Vectors in 4D spacetime are usually referred to as *4-vectors*.

As usual, a vector at a point P can be decomposed into components relative to a basis there, for example,

$$\boldsymbol{v} = v^\mu \boldsymbol{e}_\mu \,, \tag{19}$$

where v^μ are the components of the vector.

Under a Lorentz transformation, the coordinate components of a vector (i.e., the components relative to the coordinate basis vectors) transform as

$$v'^\mu = \Lambda^\mu{}_\nu v^\nu \,. \tag{20}$$

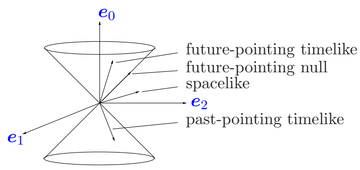
A vector $\boldsymbol{v}$ is timelike, spacelike, or null according to the character of $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v})$; in Cartesian coordinates

$$\boxed{\begin{array}{ll} \eta_{\mu\nu}v^\mu v^\nu > 0 & \quad \text{timelike}\,; \\ \eta_{\mu\nu}v^\mu v^\nu < 0 & \quad \text{spacelike}\,; \\ \eta_{\mu\nu}v^\mu v^\nu = 0 & \quad \text{null}\,. \end{array}}$$

For the basis vectors in an inertial frame, $\boldsymbol{e}_0$ is timelike, while $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ are spacelike.

A timelike or null vector is *future pointing* if $v^0 > 0$, and *past pointing* if $v^0 < 0$.

Note that the future- and past-pointing characterisations are invariant under proper Lorentz transformations (the proof is the same as the proof given in Topic I that the temporal ordering of causally-connected events is Lorentz invariant).



At any point P, the set of all null vectors there define the lightcone and this separates timelike and spacelike vectors (see figure to the right).

To every vector we can associate a dual vector by mapping with the metric.

In Cartesian coordinates, the components of the dual vector associated with the vector v^μ are

$$v_\mu = \eta_{\mu\nu} v^\nu \,, \tag{21}$$

which leaves the 0-component unchanged but reverses the spatial components.

Under a Lorentz transformation, the components of a dual vector transform with the inverse transformation matrix, i.e.,

$$X'_\mu = \Lambda_\mu{}^\nu X_\nu \,. \tag{22}$$

2 Particle dynamics

2.1 4-velocity of a massive particle

A massive particle follows a trajectory through spacetime that is usually called a *wordline*.

A convenient way to parameterise the wordline is with the *proper time* of the particle, τ.

Recall that proper time is the time measured by an ideal clock carried by the particle, and is related to the invariant path length by $ds^2 = c^2 d\tau^2$.

This means that τ is an affine parameter for the worldline.

The tangent vector to the wordline is the *4-velocity* of the particle, and has components

$$u^\mu = \frac{dx^\mu}{d\tau} \,. \tag{23}$$

For a massive particle, the 4-velocity is future-pointing and timelike.

Since proper time is an affine parameter, the length of the 4-velocity is constant:

$$\eta_{\mu\nu} u^\mu u^\nu = \left(\frac{ds}{d\tau}\right)^2 = c^2 \,. \tag{24}$$

Writing out the Cartesian components of u^μ, we have

$$\begin{aligned} u^\mu &= \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right) \\ &= \frac{dt}{d\tau} \left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) , \end{aligned} \tag{25}$$

which involves the components of the usual 3-velocity of particle, dx/dt, dy/dt and dz/dt.

With a slight abuse of notation[1], let us write the components of the 3-velocity as $\vec{u}^i = dx^i/dt$ and $\vec{u} = (\vec{u}^1, \vec{u}^2, \vec{u}^3)$, so that, compactly,

$$u^\mu = \frac{dt}{d\tau} (c, \vec{u}) \ . \tag{26}$$

The relation between coordinate and proper time is fixed by the normalisation of the 4-velocity:

$$\begin{aligned} c^2 &= \eta_{\mu\nu} u^\mu u^\nu \\ &= \left(\frac{dt}{d\tau} \right)^2 \left(c^2 - |\vec{u}|^2 \right) , \end{aligned} \tag{27}$$

so that

$$\frac{dt}{d\tau} = \left(1 - \frac{|\vec{u}|^2}{c^2} \right)^{-1/2} = \gamma_u \, , \tag{28}$$

where we have introduced the Lorentz factor γ_u.

2.1.1 Velocity transformation laws

The transformation laws for the 3-velocity of a particle (already derived in Topic I directly from the differentials of the Lorentz transformations) can now be derived simply from the transformation of the components of the 4-velocity:

$$u'^\mu = \Lambda^\mu{}_\nu u^\nu \, . \tag{29}$$

[1] This is not ideal, but at least it has the virtue of distinguishing, say, the 1-component of the 4-velocity, $u^1 = dx/d\tau$, from the x- or 1-component of the 3-velocity, $\vec{u}^1 = dx/dt$.

Let x^μ and x'^μ correspond to inertial frames S and S', respectively, related by a Lorentz boost with speed $v = c\beta$ along the x-direction, so that

$$\begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} \vec{u}'^1 \\ \gamma_{u'} \vec{u}'^2 \\ \gamma_{u'} \vec{u}'^3 \end{pmatrix} = \begin{pmatrix} \gamma_v & -\beta\gamma_v & 0 & 0 \\ -\beta\gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u c \\ \gamma_u \vec{u}^1 \\ \gamma_u \vec{u}^2 \\ \gamma_u \vec{u}^3 \end{pmatrix} . \tag{30}$$

The first component relates the particle's Lorentz factor in the two frames:

$$\frac{\gamma_u}{\gamma_{u'}} = \frac{1}{\gamma_v} \frac{1}{(1 - \vec{u}^1 v/c^2)} . \tag{31}$$

Combining this with the other components gives the usual results

$$\begin{aligned} \vec{u}'^1 &= \frac{\left(\vec{u}^1 - v\right)}{(1 - \vec{u}^1 v/c^2)} , \\ \vec{u}'^2 &= \frac{\vec{u}^2}{\gamma_v (1 - \vec{u}^1 v/c^2)} , \\ \vec{u}'^3 &= \frac{\vec{u}^3}{\gamma_v (1 - \vec{u}^1 v/c^2)} . \end{aligned} \tag{32}$$

2.2 4-acceleration

In an inertial frame, a free particle has $d^2x^i/dt^2 = 0$, so that $\vec{u} = \text{const.}$ and $\gamma_u = \text{const.}$

It follows that the components of the 4-velocity are also constant in Cartesian coordinates so

$$\frac{du^\mu}{d\tau} = 0 . \tag{33}$$

This equation is not a tensor equation but we can easily find a tensor equation (and so one that is valid in all coordinate systems) by replacing the derivative with the intrinsic derivative $D/D\tau$ along the particle's wordline since, in global Cartesian coordinates, the metric connection vanishes:

$$\frac{Du^\mu}{D\tau} = 0\,. \tag{34}$$

Since u^μ is the tangent vector to the wordline in an affine parameterisation, we see that

> free massive particles move on timelike geodesics in Minkowski space.

For a particle acted on by external forces (note that we are not considering gravity yet!), the particle will accelerate and so we define the *acceleration 4-vector* by

$$a^\mu = \frac{Du^\mu}{D\tau}\,. \tag{35}$$

In Cartesian coordinates, this reduces to $a^\mu = du^\mu/d\tau$.

The acceleration 4-vector is always orthogonal to the 4-velocity: in Cartesian inertial coordinates

$$\eta_{\mu\nu}a^\mu u^\nu = \eta_{\mu\nu}\frac{du^\mu}{d\tau}u^\nu = \frac{1}{2}\frac{d}{d\tau}\left(\eta_{\mu\nu}u^\mu u^\nu\right) = 0\,, \tag{36}$$

so, generally, $\boldsymbol{g}(\boldsymbol{a},\boldsymbol{u}) = 0$.

The components of $\boldsymbol{a}$ may be related to the usual 3-acceleration of the particle in an inertial frame as follows.

Writing $u^\mu = \gamma_u(c,\vec{u})$, we have

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma_u\frac{d}{dt}(\gamma_u c, \gamma_u\vec{u})\,. \tag{37}$$

The derivative of the Lorentz factor is

$$\frac{d\gamma_u}{dt} = \frac{d}{dt}\left(1-\frac{\vec{u}\cdot\vec{u}}{c^2}\right)^{-1/2} = \frac{\gamma_u^3}{c^2}\vec{u}\cdot\vec{a}\,, \tag{38}$$

where $\vec{a} = d\vec{u}/dt$ is the usual 3-acceleration in the inertial frame.

It follows that

$$a^\mu = \gamma_u^2 \left(\frac{\gamma_u^2}{c} \vec{u} \cdot \vec{a}, \vec{a} + \frac{\gamma_u^2}{c^2} (\vec{u} \cdot \vec{a}) \vec{u} \right) . \tag{39}$$

In the instantaneous rest frame of the particle, $\vec{u} = \vec{0}$, and the components of the 4-acceleration in that frame are simply $a^\mu = (0, \vec{a}_{\mathrm{IRF}})$, where $\vec{a}_{\mathrm{IRF}}$ is the 3-acceleration in the instantaneous rest frame.

Note that the magnitude of $\vec{a}_{\mathrm{IRF}}$ determines the (invariant) magnitude of the 4-acceleration:

$$|\boldsymbol{a}|^2 = -|\vec{a}_{\mathrm{IRF}}|^2 , \tag{40}$$

which shows that the 4-acceleration is a spacelike vector.

2.3 Relativistic mechanics of massive particles

The 4-momentum of a massive particle of rest mass m is the future-pointing, timelike 4-vector

$$\boldsymbol{p} = m\boldsymbol{u} . \tag{41}$$

At any point along the worldline of the particle, the (squared) magnitude of the 4-momentum is

$$|\boldsymbol{p}|^2 = m^2 c^2 . \tag{42}$$

In some inertial frame, the components of $\boldsymbol{p}$ are

$$p^\mu = (\gamma_u mc, \gamma_u m\vec{u}) . \tag{43}$$

In previous courses, you will have seen that the correct relativistic generalisation of the 3-momentum of a massive point particle is

$$\vec{p} = \gamma_u m\vec{u} , \tag{44}$$

so the spatial components of $\boldsymbol{p}$ are simply the 3-momentum.

Recall that this relativistic definition of the 3-momentum ensures the following are true.

1. The 3-momentum reduces to the usual non-relativistic limit $\vec{p} \approx m\vec{u}$ for $|\vec{u}| \ll c$.
2. For a free particle, $\vec{p}$ is constant since $\vec{u}$ is.
3. For a system of point particles interacting through short-range ("contact") interactions, the sum of the individual 3-momenta of all particles is conserved.
4. Newton's second law takes the form $\vec{f} = d\vec{p}/dt$, where $\vec{f}$ is the 3-force acting on the particle.

The time component of the 4-momentum is the total energy E of the particle (i.e., the sum of the rest-mass energy and kinetic energy):

$$E = \gamma_u mc^2 \,. \tag{45}$$

To see this, consider the rate of working $\vec{f} \cdot \vec{u}$ of the force accelerating a particle:

$$\begin{aligned}
\vec{u} \cdot \vec{f} &= \vec{u} \cdot \frac{d\vec{p}}{dt} \\
&= \vec{u} \cdot \frac{d}{dt}\left(\gamma_u m\vec{u}\right) \\
&= \gamma_u m \left(\vec{u} \cdot \vec{a} + \gamma_u^2 \vec{u} \cdot \vec{a} \frac{|\vec{u}|^2}{c^2}\right) \\
&= \gamma_u^3 m\vec{u} \cdot \vec{a} \\
&= mc^2 \frac{d\gamma_u}{dt} \,.
\end{aligned} \tag{46}$$

With $E = \gamma_u mc^2$, the rate of working by the force is therefore dE/dt as required.

We can now write the components of the 4-momentum in an inertial frame as

$$p^\mu = (E/c, \vec{p}) \,. \tag{47}$$

Forming the invariant $|\boldsymbol{p}|^2$ in an inertial frame, we find the *energy–momentum invariant*

$$E^2 - |\vec{p}|^2c^2 = m^2c^4 \,. \tag{48}$$

For a free particle, the total 4-momentum is constant, i.e., $dp^\mu/d\tau = 0$ in the coordinates of an inertial frame or, generally,

$$\frac{Dp^\mu}{D\tau} = 0 \,. \tag{49}$$

For an isolated system of particles undergoing collisional interactions, the total 4-momentum is the sum of the individual 4-momenta[2] and is constant; this combines *both* conservation of 3-momentum *and* energy into a Lorentz-invariant (i.e., 4-vector) law.

2.3.1 Force 4-vector

For a particle acted on by a force, the 4-momentum is not constant.

We can always introduce a 4-vector quantity called the *4-force* or force 4-vector, $\boldsymbol{f}$, by

$$\frac{Dp^\mu}{D\tau} = f^\mu \,. \tag{50}$$

Since $|\boldsymbol{p}|^2 = m^2c^2$ is constant, p^μ is orthogonal to $Dp^\mu/D\tau$ and so the 4-velocity and 4-force are necessarily orthogonal:

$$\boldsymbol{g}(\boldsymbol{f}, \boldsymbol{u}) = 0 \,. \tag{51}$$

In some inertial frame,

$$f^\mu = \gamma_u \frac{d}{dt}\left(\frac{E}{c}, \vec{p}\right) = \gamma_u \left(\frac{\vec{f}\cdot\vec{u}}{c}, \vec{f}\right) \,, \tag{52}$$

[2] As Minkowski space is pseudo-Euclidean, we can define addition of 4-vectors at different events by addition of the components in any set of global Cartesian coordinates.

where we have used $dE/dt = \vec{f} \cdot \vec{u}$.

Writing the components of the 4-force in the form on the right of Eq. (52) makes it clear that $\eta_{\mu\nu} f^\mu u^\nu = 0$.

Finally, note that the 4-force can be related to the 4-acceleration via $\boldsymbol{f} = m\boldsymbol{a}$.

2.4 4-momentum of a photon

For a particle with zero rest mass, such as a photon, the energy and 3-momentum still assemble into a 4-vector with components $p^\mu = (E/c, \vec{p})$.

This has to be the case if 4-momentum is to be conserved in scattering events involving photons and (charged) particles.

However, for zero rest mass the limit of Eq. (48) gives

$$E = |\vec{p}|c\,, \tag{53}$$

and so the 4-momentum is a (future-pointing) null vector:

$$\boldsymbol{g}(\boldsymbol{p}, \boldsymbol{p}) = 0\,. \tag{54}$$

For a free particle, the 4-momentum is conserved as in the massive case.

If we write the photon worldline as $x^\mu(\lambda)$ for some arbitrary parameter λ, then

$$\frac{Dp^\mu}{D\lambda} = 0\,. \tag{55}$$

The photon path is null, since photons travel at the speed of light, so we cannot use the proper time τ as a parameter ($d\tau = 0$).

However, we can always adopt a (dimensional) parameterisation such that

$$p^\mu = \frac{dx^\mu}{d\lambda}\,, \tag{56}$$

i.e., the tangent vector to the path is the 4-momentum.

Equation (55) then tells us that $x^\mu(\lambda)$ is an affinely-parameterised null geodesic.

Free massless particles move on null geodesics in Minkowski space, with $p^\mu = dx^\mu/d\lambda$ for some affine parameterisation.

To see why we can take $p^\mu = dx^\mu/d\lambda$, we note[3] that in an inertial frame

$$\begin{aligned} p^\mu &= \frac{E}{c}\left(1, \frac{\vec{p}}{|\vec{p}|}\right) \\ &= \frac{E}{c^2}\left(c\frac{dt}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \\ &= \frac{E}{c^2}\frac{dx^\mu}{dt}\,. \end{aligned} \tag{57}$$

Hence, p^μ is always parallel to the tangent vector $dx^\mu/d\lambda$ for any choice of parameterisation λ, and with a suitable choice of λ we can make $p^\mu = dx^\mu/d\lambda$.

2.4.1 Doppler effect revisited

For photons, we can introduce the *4-wavevector* $\boldsymbol{k}$ as $\boldsymbol{p} = \hbar\boldsymbol{k}$, with components in an inertial frame S of

$$k^\mu = \left(\frac{2\pi}{\lambda}, \vec{k}\right)\,. \tag{58}$$

Here, λ is the wavelength in S and $\vec{k}$ is the 3D wavevector, with $|\vec{k}| = 2\pi/\lambda$.

[3] This also shows that for neighbouring events separated by time dt on the worldline of a photon of energy E in some inertial frame, the ratio of E to dt is Lorentz invariant since $E dx^\mu/dt$ must be a 4-vector.

Consider an observer at rest in inertial frame S observing light with wavelength λ propagating at an angle θ to the x-axis; the components of the 4-wavevector in S are

$$k^\mu = \frac{2\pi}{\lambda}(1, \cos\theta, \sin\theta, 0)\,. \tag{59}$$

Suppose the light is emitted by a source that is moving at speed βc along the x-axis; in the rest-frame of the source (S'), the 4-wavevector has components

$$k'^\mu = \Lambda^\mu{}_\nu k^\nu\,, \tag{60}$$

where $\Lambda^\mu{}_\nu$ is the standard Lorentz boost (Eq. 8).

The emitted wavelength in the rest-frame, λ', follows from k'^0:

$$\begin{aligned} k'^0 = \frac{2\pi}{\lambda'} &= \frac{2\pi}{\lambda}\gamma\,(1-\beta\cos\theta) \\ \Rightarrow \frac{\lambda}{\lambda'} &= \gamma\,(1-\beta\cos\theta)\,. \end{aligned} \tag{61}$$

For the particular case $\theta = 0$, this reduces to the result derived kinematically in Topic I,

$$\frac{\lambda}{\lambda'} = \sqrt{\frac{1-\beta}{1+\beta}}\,. \tag{62}$$

2.5 Example of collisional relativistic mechanics: Compton scattering

Compton scattering describes scattering of a photon from a charged particle.

This can be considered as a collision between a photon with initial 4-momentum $\boldsymbol{p}$ and an electron, say, with initial 4-momentum $\boldsymbol{q}$.

In the final state, the photon has 4-momentum $\bar{\boldsymbol{p}}$ and

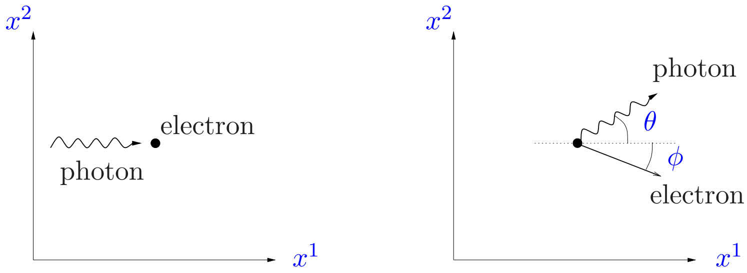


Figure 2: The Compton effect showing a photon initially propagating along the x-axis scattering off an electron at rest (left). After the collision (right), the photon propagates at an angle θ to the x-axis, and the electron recoils.

the electron has 4-momentum $\bar{\boldsymbol{q}}$.

We shall consider the collision in the inertial frame in which the electron is initially at rest, and the photon is propagating along the positive x-direction and has frequency ν.

Suppose the photon scatters through an angle θ, and its final frequency is $\bar{\nu}$, and in the process the electron recoils (see Fig. 2).

The components of the relevant 4-momenta are

$$\begin{aligned} p^\mu &= (h\nu/c, h\nu/c, 0, 0) \\ q^\mu &= (m_e c, 0, 0, 0) \\ \bar{p}^\mu &= (h\bar{\nu}/c,\ (h\bar{\nu}/c)\cos\theta,\ (h\bar{\nu}/c)\sin\theta, 0)\,, \end{aligned} \tag{63}$$

where h is Planck's constant and m_e is the electron rest mass.

(We shall not require the components of the final 4-momentum of the electron.)

The total 4-momentum is conserved, so

$$\boldsymbol{p} + \boldsymbol{q} - \bar{\boldsymbol{p}} = \bar{\boldsymbol{q}}\,. \tag{64}$$

We can also use the fact that the squared magnitude of the total 4-momentum is Lorentz invariant, and so equate the magnitude of the left-hand side of Eq. (64) evaluated in the initial rest-frame of the electron with the magnitude of the right-hand side evaluated in the final rest-frame.

Using $|\boldsymbol{p}|^2 = 0$, and similarly for $|\bar{\boldsymbol{p}}|^2$, and $|\boldsymbol{q}|^2 = |\bar{\boldsymbol{q}}|^2 = m_e^2 c^2$, we have

$$\eta_{\mu\nu} p^\mu q^\nu - \eta_{\mu\nu} \bar{p}^\mu q^\nu - \eta_{\mu\nu} p^\mu \bar{p}^\nu = 0 \,. \tag{65}$$

Substituting for the components from Eq. (63), we find

$$\begin{aligned} 0 &= h\nu m_e - h\bar{\nu} m_e - \left(\frac{h\nu}{c}\right)\left(\frac{h\bar{\nu}}{c}\right)(1-\cos\theta) \\ \Rightarrow \quad \bar{\nu} &= \frac{\nu}{1 + (h\nu/m_e c^2)(1-\cos\theta)} \,. \end{aligned} \tag{66}$$

We see that, generally, the photon frequency is reduced during the collision, with energy being transferred to kinetic energy of the recoiling electron.

This change in frequency follows only from a particle-like (i.e., quantum mechanical) description of light – in classical electromagnetism, the electron would be forced to oscillate at the frequency of the incident electromagnetic wave and so would also radiate at this frequency.

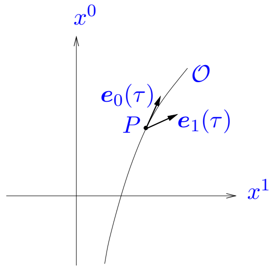
3 The local reference frame of a general observer

Consider a general observer $\mathcal{O}$ following a worldline $x^\mu(\tau)$.

Their 4-velocity has components $u^\mu = dx^\mu/d\tau$ and the 4-acceleration is $a^\mu = Du^\mu/D\tau$.

At any event on the wordline, we can define the instantaneous rest-frame of the particle as the inertial frame in which the particle is instantaneously at rest.

At proper time τ, the coordinate basis vectors of the instantaneous rest-frame at the observer's position constitute an orthonormal set of basis vectors $\boldsymbol{e}_\mu(\tau)$; see figure to the right.



By construction, the timelike basis vector $\boldsymbol{e}_0(\tau)$ is equal (up to a factor of c) to the instantaneous 4-velocity $\boldsymbol{u}(\tau)$.

The three spacelike vectors $\boldsymbol{e}_i(\tau)$, $i = 1, 2, 3$, are therefore orthogonal to the observer's 4-velocity.

At some later time τ', the basis vector $\boldsymbol{e}_0(\tau')$ is uniquely determined by the 4-velocity $\boldsymbol{u}(\tau')$, but the remaining three spacelike vectors $\boldsymbol{e}_i(\tau')$ are only determined up to a spatial rotation.

Additional information is required to specify the $\boldsymbol{e}_i$, such as demanding that they point along the directions specified by three orthogonal gyroscopes carried (with no torque applied) by the observer.

For the special case of a non-accelerating observer carrying three such gyroscopes, the $\boldsymbol{e}_\mu(\tau)$ undergo parallel transport along the particle's wordline.[4]

This leads us to the following idealisation of a local laboratory for an arbitrary observer: the observer (possibly accelerating) carries along four orthonormal vectors $\boldsymbol{e}_\mu(\tau)$ that satisfy

$$\boldsymbol{g}(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu) = \eta_{\mu\nu} \quad \text{and} \quad c\boldsymbol{e}_0(\tau) = \boldsymbol{u}(\tau) \,. \tag{67}$$

Such a frame of vectors is called an *orthonormal tetrad*.

[4]More generally, for an accelerated observer the $\boldsymbol{e}_i(\tau)$ *cannot* be parallel-transported since they have to remain orthogonal to $\boldsymbol{u}(\tau)$. If the orientation of the $\boldsymbol{e}_i(\tau)$ is determined by gyroscopes, the basis vectors at proper time $\tau+d\tau$ are obtained from those at τ by first parallel-transporting to the observer's new position, then applying the additional pure Lorentz boost required to boost the parallel-transported $\boldsymbol{e}_0$ onto $\boldsymbol{u}(\tau + d\tau)$. Such basis vectors are said to be Fermi–Walker transported and are the idealisation of a local *non-rotating* laboratory.

The results of any local measurement made by the observer at proper time τ can be represented as the components of tensor-valued quantities in this tetrad.

4 Minkowski space in other coordinate systems

In Minkowski space, it is usually most convenient to work in the Cartesian coordinates of an inertial frame.

The advantages of working in these coordinates are the following:

1. the coordinates have a simple physical interpretation in terms of distances and times measured by observers in some inertial frame; and
2. covariant differentiation of tensors reduces to partial differentiation of the components.

However, for some applications other coordinate systems are more appropriate.

A trivial example is to use spherical polar coordinates, say, rather than spatial Cartesian coordinates, in some inertial frame.

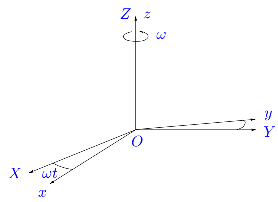
A less trivial example is to use a rotating coordinate system, an example that we shall now discuss.

4.1 Non-inertial coordinates: a rotating frame

Let $X^\mu = (cT, X, Y, Z)$ be Cartesian coordinates of an inertial frame S.

Introduce new coordinates $x^\mu = (ct, x, y, z)$ where

$$\begin{aligned} X &= x\cos\omega t - y\sin\omega t \\ Y &= x\sin\omega t + y\cos\omega t \\ Z &= z \\ T &= t\,; \end{aligned} \tag{68}$$



see figure to the right.

Points with fixed x, y and z coordinates rotate with angular speed ω about the Z axis in S.

Evaluating the differentials

$$dX = dx\cos\omega t - dy\sin\omega t - \omega dt\,(x\sin\omega t + y\cos\omega t)\ , \tag{69}$$

$$dY = dx\sin\omega t + dy\cos\omega t + \omega dt\,(x\cos\omega t - y\sin\omega t)\ , \tag{70}$$

we find (exercise!)

$$dX^2 + dY^2 = dx^2 + dy^2 + \omega^2\left(x^2+y^2\right)dt^2 + 2\omega dt\,(xdy - ydx)\ . \tag{71}$$

The line element,

$$ds^2 = c^2dT^2 - dX^2 - dY^2 - dZ^2\,, \tag{72}$$

becomes

$$ds^2 = \left[c^2 - \omega^2\left(x^2+y^2\right)\right]dt^2 + 2\omega y dt dx - 2\omega x dt dy - dx^2 - dy^2 - dz^2 \tag{73}$$

in terms of the x^μ coordinates.

Free particles move on timelike geodesics in Minkowski space, with equation of motion

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0\,. \tag{74}$$

Rather than calculating the metric connection directly, it is often quicker to follow the "Lagrangian" route, with $L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, where overdots denote differentiation with respect to proper time τ.

For the line element in the rotating coordinates, the Lagrangian is

$$L = \left[c^2 - \omega^2\left(x^2+y^2\right)\right]\dot{t}^2 + 2\omega y\dot{x}\dot{t} - 2\omega x\dot{y}\dot{t} \\ - \dot{x}^2 - \dot{y}^2 - \dot{z}^2\,. \quad (75)$$

The Euler–Lagrange equation for $t(\tau)$ gives

$$\frac{d}{d\tau}\left([c^2-\omega^2(x^2+y^2)]\dot{t} + \omega(y\dot{x}-x\dot{y})\right) = 0$$
$$\Rightarrow \quad [c^2-\omega^2(x^2+y^2)]\ddot{t} - 2\omega^2(x\dot{x}+y\dot{y})\dot{t} \\ +\omega(y\ddot{x}-x\ddot{y}) = 0\,. \quad (76)$$

For $x(\tau)$ and $y(\tau)$, we have

$$\ddot{x} = \omega y\ddot{t} + \omega^2 x\dot{t}^2 + 2\omega\dot{t}\dot{y}\,, \quad (77)$$
$$\ddot{y} = -\omega x\ddot{t} + \omega^2 y\dot{t}^2 - 2\omega\dot{t}\dot{x}\,, \quad (78)$$

while for z we find $\ddot{z} = 0$.

Substituting for $\ddot{x}$ and $\ddot{y}$ into Eq. (76), a large number of cancellations take place to leave

$$\ddot{t} = 0\,. \quad (79)$$

This is not unexpected: t is also the time coordinate in the Cartesian inertial coordinates X^μ, so $\ddot{t} = 0$ must hold for a free particle.

If we now parameterise x, y and z in terms of t rather than τ, using $dt/d\tau = \text{const.}$, we get

$$\frac{d^2x}{dt^2} = \omega^2 x + 2\omega\frac{dy}{dt}\,, \quad (80)$$
$$\frac{d^2y}{dt^2} = \omega^2 y - 2\omega\frac{dx}{dt}\,, \quad (81)$$
$$\frac{d^2z}{dt^2} = 0\,. \quad (82)$$

These are just the usual equations of motion for a free particle in a rotating frame, involving the centrifugal and Coriolis accelerations.