General Relativity

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Contents

Abstract

Abstract of this course

1 Particle Dynamics

1.1 Lorentz transformation

Definition homogeneous Lorentz transformation:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

...
Definition transformation matrix:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

This is the end

2 Electromagnetism

Definition Electromagnetic field tensor:

$$F\mu\nu = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$

$F\mu\nu$ is antisymmetric by construction and contains four independents fields

3 Spacetime Curvature

Definition Riemann Curvature Tensor:

$$R^{d}_{abc} = \nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c$$

The reason for doing so is that covariant derivative of dual vector field is not commutative

3.1 Subsection 1

4 Gravitation field equations

4.1 Energy Momentum Tensor

Definition Energy Momentum Tensor:

$$T^{\mu\nu}(x) = \rho_0(x) u^{\mu}(x) u^{\nu}(x)$$

T^{00}: energy density
T^{i0}: i-th component of 3-momentum density (times c)
T^{ij}: flux of i-component of 3-momentum in j-direction

4.1.1 Properties of Energy-momentum tensor

- Always symmetric

4.2 Energy-momentum Tensor in Ideal Fluid

Consider Ideal Fluid:
We can find a local inertial frame so that $T^{i0} = 0$; and the spatial components are isotropic: $T^{ij} \propto \delta^{ij}$. Hence, in **instantaneous rest frame**:

$$T^{\mu\nu} = \mathrm{diag}(\rho c^2, p, p, p)$$

where ρc^2 is the rest frame energy density and p is isotropic pressure. While in general:

$$T^{\mu\nu} = (\rho + \frac{p}{c^2})u^\mu u^\nu - pg^{\mu\nu} \tag{1}$$

Equation 1 is the energy momentum tensor, valid in any coordinate system.

4.2.1 Continuity Equation

The energy-momentum tensor satisfies continuity equation:

$$\nabla_\mu T^{\mu\nu} = 0$$

4.3 Einstein Field Equation

Definition Einstein Field Equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}$$

Constant proportionality and negative sign on the right is required for consistency with the weak field limit

5 The Schwarzchild Solution

Adopting a passive view point: change the coordinate system without changing the functional form of the fields on our coordinates.

5.1 Geodesics in Schwarzchild spacetime

In this section we study the equation of motion for 4 coordinates, t, r, θ, ϕ

1. Equation of motion for θ:
$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\psi}^2 = 0 \tag{2}$$
A possible solution is $\theta = \pi/2$, planar motion in the equatorial plane; given the spherical symmetry.

2. Equation of motion for t:
$$(1 - \frac{2\mu}{r})\dot{t} = k \tag{3}$$
$k = (1 - 2\mu/r)\dot{t}$ is related to the energy of the particle as measured by stationary observer.

3. Equation of motion for ϕ:

$$r^2\dot{\phi} = h \tag{4}$$

Here, we assume in the plane $\theta = \pi/2$. h arises from the symmetry of the spacetime under ratotion about z-axis, can be interpreted as *specific angular momentum*.

4. Equation of motion for r:

$$(1-\frac{2\mu}{r})c^2\dot{t}^2 - (1-\frac{2\mu}{r})^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = \begin{cases} c^2 & \text{massive} \\ 0 & \text{massless} \end{cases} \tag{5}$$

5.2 Effective potential energy

1. In spherical coordinates, the Newtonian effective potential energy is

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{h^2}{2r^2} \tag{6}$$

- It has a centrifugal barrier at small r, preventing particles reaching $r = 0$
- Bound orbits have $E_N < 0$, two turning points for r w.r.t V at $V = E_N$
- Effective potential have one turning point at $r = h^2/GM$ It is a minimum, corresponding to stable circular orbit.

2. massive particle in general relativity:

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{h^2}{2r^2}(1-\frac{2\mu}{r}) \tag{7}$$

- the centrifugal term is slightly modified by a factor of $(1 - 2\mu/r)$
- solving for the extrema of V gives:

$$r_{\pm} = \frac{h}{2\mu c^2}(h \pm \sqrt{h^2 - 12\mu^2c^2}) \tag{8}$$

- two stationary points for $h > \sqrt{\mu c}$, r_- is a maximum and $r_+$is minimum
- none for smaller h i.e V is increasing with r
-

6 Shapes of orbits for massive and massless particles

6.1 Shape of Orbit

1. Massive article gives

$$\frac{d^2u}{d\phi^2} + u - 3\mu u^2 = \frac{GM}{h^2} \tag{9}$$

2. Massless article gives

$$\frac{d^2u}{d\phi^2} + u - 3\mu u^2 = 0 \tag{10}$$