Astrofluid Dynamics

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February 15, 2023

Contents

Abstract

Abstract of this course

1 Some basic concepts

Collisional v collisionless fluids
Eulerian and Lagrangian framework
Concepts of streamlines, particle paths and streaklines:
They coinside if the flow is steady.i.e.

2 Formulation of the Fluid Equations

This chapter talked about the conservation of mass and momentum.

2.1 Conservation of mass

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{u} = 0 \tag{1}$$

2.2 Conservation of momentum

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla}) vbu = -\boldsymbol{\nabla} p + \rho \mathbf{g} \tag{2}$$

We consider 4 different which contribute to the change of momentum?

3 Gravitation

In this section, we used $\vec{\mathbf{g}}$ to denote gravitational acceleration; Ψ to denote gravitational potential; and Ω to denote the energy required to take the system of point masses to infinity

Example: Spherical distribution of mass
Example: Infinitely cylindrical symmetrical mass
Example: Infinite planar distribution of masses
Example: Finite axisymmetric disk

3.1 Potential of a Spherical Mass Distribution

Ψ is affected by any matter outside r through our choice of setting Psi at infinity. i.e. We **can't** say that $\Psi = -GM/r$

3.2 Gravitational Potential Energy

3.3 Virial Theorem

Virial Theorem: states that for a system in steady state, $I \equiv mr^2 = constant$, $2T + \Omega = 0$
Kinetic energy T has contribution from local flows and random/thermal motions.
A result of virial theorem is that the gravitational potential sets the temperature or velocity dispersion of the system.

4 Equation of state and the energy equation

So far we have 4 variables, density ρ, pressure,p, gravitational potential Ψ and velocity which is $\boldsymbol{v}$. In terms of equations to solve them, we have the scalar equation of mass conservation and the vector equation for conservation of momentum. We also have Poisson equation for the potential term. Now what we need is another equation: 'Equation of state' to determinate pressure (which is a thermodynamic property). While doing so we might Introduce another equation, the energy equation when the system is not barotropic, i.e. p is a function of T.

4.1 Equation of state

1. Astrophysical fluid are treated as ideal gas, and corresponding EoS is:

$$p = nk_BT = \frac{k_B}{\mu m_p}\rho T \tag{3}$$

   where ρ is the mean particle mass.

2. This EoS introduces another scalar field, temperature, T

4.1.1 Barotropic

Barotropic means that p independent of T. i.e. only a function of ρ. This comes in two cases: Isothermal and Adiabatic.

1. Isothermal: Constant T

2. Adiabatic: Ideal has undergoes reversible thermodynamic changes

$$p = K\rho^\gamma \tag{4}$$

3. Derivation of C_p and C_v for both cases.

4.1.2 Energy equation for non-Barotropic case

1. It starts from first law

$$\bar{d}Q = d\mathcal{E} + pdV \tag{5}$$

2. total energy per unit volume is:

$$E = \rho(\frac{1}{2}u^2 + \Psi + \mathcal{E}) \tag{6}$$

3. take material derivative, note that $\frac{DE}{Dt} = \frac{\partial E}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} E$, gives Energy Equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E+p)\mathbf{u}] = \rho\frac{\partial \Psi}{\partial t} - \rho\dot{Q}_{cool} \tag{7}$$

   In many settings $\partial\Psi/\partial\Psi = 0$, If there is no cooling, the equation expresses the conservation for energy in which the total energy density E is driven by the divergence of the enthalpy flux $(E+p)\mathbf{u}$

4.2 Heating and Cooling Processes

Combining cooling and heating effects, we can parametrise $\dot{Q}_{cool}$

$$\dot{Q}_{cool} = A\rho T^\alpha - H \tag{8}$$

where the first and second term on RHS means radiative cooling and cosmic ray heating

4.3 Energy Transport process

Transport processes move energy through the fluid, via Thermal conduction, convention and radiation transport.

5 A lot of different system

5.1 Full set of equations describing the dynamics of an ideal non-relativisitc fluid

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{u} = 0 \qquad \text{(Continuity Equation)}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla} \cdot)\mathbf{u} = -\boldsymbol{\nabla} \cdot p - \rho \boldsymbol{\nabla} \cdot \Psi \qquad \text{(Momentum Equation)}$$

$$\nabla^2 \Psi = 4\pi\rho \qquad \text{(Poisson's Equation)}$$

(Energy Equation)

(Definition fo Total energy)

(EoS of total energy)

(Internal Energy)

5.2 Hydrostatic Equilibrium

A System of hydrostatic equilibrium if

$$\mathbf{u} = \frac{\partial}{\partial t} = 0 \tag{9}$$

Continuity equation is trivially satisfied Sub into momentum equation gives:

$$\frac{1}{\rho} \boldsymbol{\nabla} p = -\boldsymbol{\nabla} \Psi \qquad \text{(Equation of Hydrostatic equilibirum)}$$

Example: Isothermal atmosphere
Isothermal atmosphere with constant $\mathbf{g} = -g\hat{\mathbf{z}}$

$$\rho = \rho_0 \exp\left\{-\frac{\mu g}{R_* T} z\right\} \tag{10}$$

i.e. exponential atmosphere **Example: Isothermal self-gravitating slab**
Isothermal atmosphere with constant $\mathbf{g} = -g\hat{\mathbf{z}}$

$$\nabla^2 \Psi = 4\pi G \rho \tag{11}$$

Is gives

$$\Psi - \Psi_0 = 2A \ln \cosh \sqrt{\frac{2\pi G \rho}{A}} z \tag{12}$$

$$\rho = \frac{\rho_0}{\cosh^2 \sqrt{\frac{2\pi G \rho}{A}} z} \tag{13}$$

5.3 Stars/Self-Gravitating Polytropes

1. In this section, we consider spherically-symmetric, self-gravitating system in hydrostatic equilibirum, which is called "star". Note that non-rotating stars are barotropes.

2. In this system, we have $p = p(\rho)$, and barotropic EoS can be written as $p = K\rho^{1+1/n}$ If the star is isentropic, i.e. constant entropy. $1 + 1/n = \gamma$.

3. When n is constant, the structure is called a Polytrope

4. Assuming a polytropics EoS, the equation of hydrostatic equilibrium is

$$-\nabla\Psi = \frac{1}{\rho}\nabla\left(K\rho^{1+1/n}\right) = (n+1)\nabla\left(K\rho^{1/n}\right)$$

Which has a solution for ρ:

$$\rho = \left(\frac{\Psi_T - \Psi}{(n+1)K}\right)^n$$

In which Ψ_T is $\Psi(\rho = 0)$ at surface.

Given boundary condition at for central density and central potential ρ_c and Ψ_c.

$$\rho = \rho_c\left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_C}\right)^n$$

Now feed into Poisson Equation gives a differential equation for Ψ

Define θ (which is equivalent to Ψ)and $\mathcal{E}$ (which is scaled radial coordinate), we get Lane-Emden Eqn of index n

$$\frac{1}{\mathcal{E}}\frac{d}{d\mathcal{E}^2}\left(\mathcal{E}^2\frac{\mathrm{d}\theta}{\mathrm{d}\mathcal{E}}\right) = -\theta^n$$

which is an differential equation for θ.

Analytical solution for n =0,1,5

5.4 Isothermal Sphere

1. Isothemal case means that $p = K\rho$,

From the momentum equation gives:

$$\Psi - \Psi_c = -K\ln(\rho/\rho_c) \tag{14}$$

From Possion's Equation gives differential equation for ρ

Take $\rho = \rho_c e^{-\Psi}$,we get a differential equation for Ψ

$$\frac{1}{\mathcal{E}}\frac{d}{d\mathcal{E}^2}\left(\mathcal{E}^2\frac{\mathrm{d}\Psi}{\mathrm{d}\mathcal{E}}\right) = e^{-\Psi} \tag{15}$$

which is equivalent to Lane-Equation in this isothermal case.

(16)

5.5 Scaling Relations

Scaling relation relates the mass and radius of a polytrope star.

6 Sound waves, supersonic flows and shock waves

6.1 Sound Waves

In this section, we find the sound wave a derivative of a $p(\rho)$ **Example: Isothermal**
$c_s^2 = \sqrt{\frac{R_* T}{\mu}}$ **Example: Adiabatic**
$c_s^2 = \sqrt{\mu \frac{R_* T}{\mu}}$

6.2 Sound Waves in Stratified Atmosphere

Now we consider sound waves traveling in a fluid with a background structure, i.e. constant gravitational field. We will end up with a dispersion relationship and a k that could be imaginary, meaning a decaying wave.

$$\omega^2 = c_u^2 \left(k^2 - \frac{\mathrm{i}k}{H} \right)$$

DISPERSION RELATION

6.3 Sound waves across interfaces

We then matches B.C. to get reflection coefficient and transmission radiation.

$$t = \frac{2k_1}{k_1 + k_2} \quad r = \frac{k_1 - k_2}{k_1 + k_2}$$

6.4 Supersonic Fluid and Shocks

Shock occurs when there is a disturbance in the fluid cause by compression by a large factor.
Definition Mach Number:

$$M_1 = u_1 / c_{s,1}$$

This number helps us to simplify the R-H relationships in the next section.

6.5 The Rankine-Hugoniot Relations

We discovered 3 R-H relations, from conservation of mass, momentum and energy.

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$
$$\frac{1}{2} u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2} u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2}$$

6.5.1 Adiabatic Case

The last R-H relationship becomes, in an adiabatic idea gas:

$$\frac{1}{2} u_1^2 + \frac{c_{s,1}^2}{\gamma - 1} = \frac{1}{2} u_2^2 + \frac{c_{s,2}^2}{\gamma - 1}$$

6.5.2 Isothermal Case

The last R-H relationship becomes $T_1 = T_2$ with this relationship, we have $c_s^2 = u_1 u_2$ and

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \left(\frac{u_1}{c_s}\right)^2 = M_1^2$$

Note that if $u_1 > c_s$ then $u_2 < c_s$, flow behind the shock is subsonic.

7 Bernuolli's Equation

This gives us Bernoulli's Principle: For steady barotropic flows, the quantity

$$H = \frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi$$

is constant along a streamline. The quantity H is called Bernoulli's constant.
If $p = 0, H =$ constant is the statement that kinetic + potential energy is constant along streamlines.

If $p \neq 0$, pressure differences accelerate or decelerate the flow as it flows along the streamline.

7.1 Rotational and Irrotational Flow

7.2 Laplacian

Laplacian Equation is derived if we have incompressible and irrotational flow. i.e $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and $\boldsymbol{\nabla} \times \mathbf{u} = 0$