TSP-2022/23 — Thermal and Statistical Physics (Part II)

Problem sheet II: questions 1-10

1. **Point defect**
A point defect in a solid may be occupied by 0, 1 (spin up or down) or 2 electrons, and the solid provides a reservoir of electrons at chemical potential μ. The energy for occupation by a single electron is ϵ, and that for 2 electrons is $2\epsilon+U$, where U is the Coulomb repulsion energy between the electrons. Obtain an expression for the average electron occupancy of the defect and sketch $\langle n\rangle(\mu-\epsilon)$ for low temperatures $T \ll U/k_B$.

2. **Plasma**
Show that the equilibrium constant K_N for the ionisation reaction He $\rightleftharpoons$ He$^+$ + e$^-$ is to a good approximation

$$K_N = \frac{1}{4V}\left(\frac{2\pi\hbar^2}{m_e k_B T}\right)^{3/2} \mathrm{e}^{e\phi/k_B T}$$

where ϕ is the first ionisation potential of He, which is 24.6 V.

Find the proportion of He that is ionised at 10^4 K (i) at atmospheric pressure, and (ii) at 10^{-2} Nm^{-2}. What is the cause of the change in the equilibrium constant? This effect is important for spectral lines from interstellar gases, one finds a surprisingly large intensity corresponding to spectral lines of ionised atoms.

3. **Trap**
Atoms can be held in a spherical trap with the potential energy potential $V(r) = ar$. Calculate the partition function Z_N of a gas of N indistinguishable non-interacting atoms in this trap at temperature T in the classical limit. Sketch the temperature dependence of the entropy of the classical gas in this trap for two different values of a and demonstrate that by decreasing a adiabatically, the gas can be cooled reversibly.

Find athe chemical potential of the system. Estimate the number of atoms required for quantum statistics to become important.

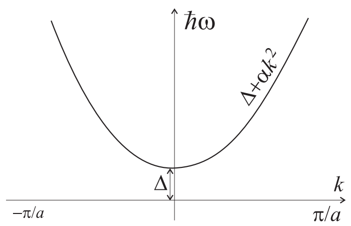
4. **Degenerate or non-degenerate**
The temperature at the centre of the sun is $T = 1.6 \times 10^7$ K, and plasma at the centre of the sun consists of hydrogen at a density of $\rho_{\mathrm{H}} = 6 \times 10^4$ kg m^{-3} and helium at a density of $\rho_{\mathrm{He}} = 1 \times 10^5$ kg m^{-3}.
(a) Calculate the thermal wavelengths of the electrons, protons and He nuclei.
(b) Determine whether the electrons, protons and He nuclei are degenerate or non-degenerate under these conditions.
(c) Estimate the pressure at the centre of the sun due to these particles and that due to the radiation pressure.
(d) Is it the pressure due to the particles or the radiation which prevents gravitational collapse of the sun?

5. **Helium-3**
At temperatures below 0.4 K, a dilute solution of ^{3}He in liquid ^{4}He behaves like a gas of ^{3}He atoms moving freely *in vacuo* except that the effective mass of each ^{3}He atom is enhanced by a factor of about 2.4. The concentration of ^{3}He is 5 atomic percent and the density of the solution is 140 kg/m^3. Sketch the temperature dependence of the heat capacity per ^{3}He atom at low temperatures. Calculate the Fermi temperature, T_F, and the coefficient γ of the specific heat at low temperatures, $c_V = \gamma T$.
[Answer: $T_F = \varepsilon_F/k_B = 0.33$ K, $\gamma = 2.0 \times 10^{-22}$ J atom^{-1}K^{-2}.]

6. **Repulsion of Fermi particles**
For Fermi particles at low temperature, examine the temperature change in (a) the Joule process: the gas expands in vacuum, and (b) the Joule-Kelvin process: the gas is forced through a valve while keeping it insulated so that no heat is exchanged with the environment. That is, find
$$\left(\frac{dT}{dV}\right)_U \quad \text{and} \quad \left(\frac{dT}{dp}\right)_H .$$
Argue how your result reflects the Fermi pressure at $T \to 0$, in both cases.

7. **Bose gas at low temperature**
Find the temperature dependence of chemical potential of the Bose gas at low temperature $T \to 0$ in 1- and in 2-dimensions. You know from the lectures that chemical potential μ_{Bose} is almost zero at large N, but here we want to specifically find its T-dependence, however small it might be.

8. **Spin waves**
Spin waves in many ferromagnets show a gap at low energy, due to coupling of the orbital moments to the crystalline lattice. The resulting dispersion relation curves are typically as shown below.



Find and sketch the temperature dependence of the specific heat at low temperatures, assuming that the dispersion relation is isotropic.

9. **Black body radiation**
A long air-filled coaxial transmission line, of length L and small diameter, is short circuited at each end. Show that, at room temperature and at a cyclic frequency $\nu = 10^9$Hz, the mean energy of black body radiation between the conductors in a small frequency range $d\omega$ will be approximately
$$k_B T \frac{L}{\pi c} d\omega \ .$$
If the outer diameter is 1cm and the inner diameter 2mm, explain why it would be reasonable at 10^{12}Hz to replace this expression by one proportional to
$$k_B T \frac{V}{\pi^2 c^3} \omega^2 d\omega$$
where V is the volume between the conductors.

10. **Chemical potential**
Write brief notes describing the chemical potential and examples of its use in thermodynamics and statistical mechanics. In your essay, include a sketch of μ as a function of (i) the number of particles per unit volume at constant temperature, and (ii) the temperature at constant number of particles per unit volume, in both the classical and quantum regimes.