I. RECAP OF SPECIAL RELATIVITY

1 Inertial frames

We specify an *event* in spacetime uniquely by assigning it three spatial coordinates and one time coordinate.

Define a frame S using Cartesian axes and a system of synchronised clocks at rest in the system, so that coordinates (ct, x, y, z) label events in space and time.

Inertial frames are defined by the requirement that a free particle is at rest or moves in a straight line with fixed speed (Newton's first law), i.e.,

$$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0\,. \tag{1}$$

Two inertial frames S and S' differ only by:

1. a translation;
2. a rotation;
3. relative motion at constant velocity.

Inertial frames are fundamental to the *principle of relativity*:

All laws of physics take same form in every inertial frame.

No exception to the principle of relativity has ever been found.

It applies equally well in both Newtonian theory and special relativity (SR).

Newtonian theory and SR differ in how to relate the coordinates of events in S and S'.

Consider S and S' in *standard configuration*: axes aligned, the same spacetime origin (i.e., the event with $ct = x = y = z = 0$ has $ct' = x' = y' = z' = 0$) and relative velocity along the x (and x') axis; see Fig. 1.

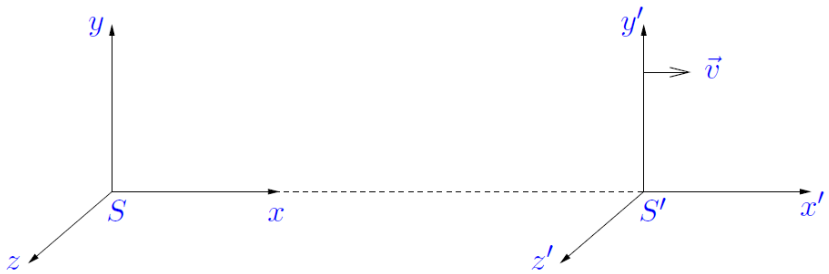


Figure 1: Inertial frames S and S' in standard configuration with relative velocity $\vec{v}$ along the x-axis.

Coordinates of events in S and S' are related via a linear transformation; this is restricted by symmetry and $x' = 0 \Rightarrow x = vt$ and $x = 0 \Rightarrow x' = -vt'$ to

$$\begin{aligned} t' &= At + Bx \qquad & x' &= A(x - vt) \\ y' &= y & z' &= z\,. \end{aligned} \tag{2}$$

Newtonian theory and SR differ in the constants A and B.

2 Newtonian geometry of space and time

Newtonian theory assumes an *absolute time* – the same for every observer.

It follows that $t' = t$, and we have the *Galilean transformation*:

$$\boxed{t' = t, \quad x' = x - vt, \quad y' = y, \quad z' = z\,.} \tag{3}$$

The inverse transformation simply replaces v with $-v$ (symmetry!).

3-velocities transform in the "common-sense" way:

$$u'_x = \frac{dx'}{dt'} = \frac{dx'}{dt} = \frac{dx}{dt} - v = u_x - v\,. \tag{4}$$

Acceleration is the same in both frames since $du'_x/dt' = du_x/dt$.

For two events A and B, the Gallilean transformation implies that

- the time difference $\Delta t = t_B - t_A$ is invariant; and
- $\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$ is invariant for simultaneous events (since Δx, Δy and Δz are).

Space and time are separate entities in Newtonian theory.

3 Spacetime geometry of special relativity

In special relativity, we abandon the notion of absolute time.

It is replaced by a new postulate:

The speed of light c is the same in all inertial frames.

Consider a photon emitted from the (coincident) origin of S and S' at $t = t' = 0$ in some arbitrary direction; then

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 = 0 \quad (5)$$

at any subsequent event on the photon's path.

The linear transformation in Eq. (2) implies the *Lorentz transformation*:

$$\boxed{ct' = \gamma(ct - \beta x), \quad x' = \gamma(x - \beta\, ct), \quad y' = y, \quad z' = z\,,} \quad (6)$$

where $\beta \equiv v/c$ and $\gamma \equiv (1-\beta^2)^{-1/2}$.

(Here, we have assumed that $A > 0$ so that we recover the identity transformation as $v \to 0$ and there is no time reversal.)

The inverse transformation is again obtained by setting $v \to -v$.

Note how time and space are mixed by the Lorentz transformation.

However, for two events A and B, the (squared) *interval*

$$\boxed{\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2} \quad (7)$$

is *invariant* under any Lorentz transformation.

In special relativity, space and time are united into a four-dimensional continuum called *spacetime* with invariant geometry characterised by Δs^2.

We call this *Minkowski* spacetime.

3.1 Lorentz transformations as 4D 'rotations'

Different Cartesian inertial frames S and S' simply relabel events in Minkowski spacetime, i.e., perform a *co-*

ordinate transformation

$$(ct, x, y, z) \to (ct', x', y', z') \,. \tag{8}$$

It is often convenient to define the *rapidity* parameter ψ (which runs from $-\infty$ to ∞) by $\beta = \tanh\psi$, so that

$$\gamma = \cosh\psi \qquad \text{and} \qquad \gamma\beta = \sinh\psi \,. \tag{9}$$

For S and S' in standard configuration, we have

$$\begin{aligned} ct' &= ct\cosh\psi - x\sinh\psi \,, \\ x' &= -ct\sinh\psi + x\cosh\psi \,, \\ y' &= y \,, \\ z' &= z \,. \end{aligned} \tag{10}$$

These are like a rotation in the ct-x plane, but with hyperbolic rather than trigonometric functions.

The hyperbolic functions are necessary to ensure the invariance of Δs^2 given the minus signs in its definition.

3.2 More complicated Lorentz transformations

More generally, the relation between two Cartesian inertial frames S and S' can differ from that for the standard configuration since[1]:

1. the 4D origins may not coincide, i.e., the event at $ct = x = y = z = 0$ may not be at $ct' = x' = y' = z' = 0$;
2. the relative velocity of the two frames may be in an arbitrary direction in S, rather than along the x-axis; and

[1] Lorentz transformations can be considered more formally as linear transformations that preserve the interval Δs^2. In this case, the definition admits transformations that are not continuously connected to the identity; i.e., parity transformations and/or time reversal. We shall not consider such transformations further.

3. the spatial axes in S and S' may not be aligned, e.g., the components of the relative velocity in S' may not be minus those in S.

We can always deal with the origins not coinciding (known as inhomogeneous Lorentz transformations or Poincaré transformations) by appropriate temporal and spatial displacements.

We can find the form of the remaining Lorentz transformation in the general case by decomposing as follows.

1. Apply a purely spatial rotation in the frame S to align the new x-axis with the relative velocity of the two frames.
2. Apply a standard Lorentz transformation as in Eq. (6).
3. Apply a spatial rotation in the transformed coordinates to align the axes with those of S'.

Given a reference frame S, the *Lorentz boost* of this frame for a general relative velocity $\vec{v}$ is obtained by rotating the spatial axes of S so that the relative velocity is along the new x-axis, applying the standard Lorentz transformation, and applying the inverse spatial rotation in the transformed frame.

If the relative velocity is along the original x-axis, this reduces to the standard Lorentz transformation.

More generally, reference frames connected by a Lorentz boost have their spatial axes as aligned as possible given the relative velocity of the frames, i.e., they are generated by hyperbolic "rotations" in the plane defined by the ct-axis and the relative velocity.

3.3 The interval and the lightcone structure

As we have seen, the interval is invariant under Lorentz transformations.

This is particularly transparent using the hyperbolic form of the standard transformation:

$$\begin{aligned}\Delta s^2 &= c^2(\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 \\ &= [(c\Delta t)\cosh\psi - (\Delta x)\sinh\psi]^2 \\ &\qquad - [-(c\Delta t)\sinh\psi + (\Delta x)\cosh\psi]^2 - \Delta y^2 - \Delta z^2 \\ &= c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2\,. \end{aligned} \tag{11}$$

The interval is also invariant under more general Lorentz transformations since a shift in origin does not alter the differences $(c\Delta t, \Delta x, \Delta y, \Delta z)$, and rotations preserve the spatial interval $\Delta x^2 + \Delta y^2 + \Delta z^2$.

The interval is an underlying geometrical property of the spacetime itself, i.e., an invariant “distance” between events in spacetime.

It follows that the sign of Δs^2 is also invariantly defined; it is used to classify separations between events as

$\Delta s^2 > 0$	the interval is timelike;
$\Delta s^2 = 0$	the interval is null or lightlike;
$\Delta s^2 < 0$	the interval is spacelike.

At any event A in spacetime, the *lightcone* separates timelike and spacelike events (see Fig. 2).

Since light travels in straight lines at speed c in all intertial frames, the lightcone of A is formed from all events that can be connected to A by light signals.

Events within the forward lightcone of event A can be influenced by A, i.e., causal signals (travelling at or below of the speed of light) from A can reach B.

Events within the past lightcone of A can causally influence A.

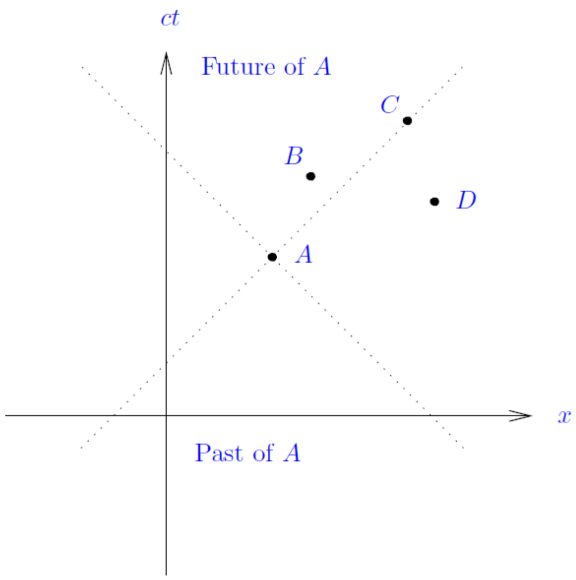


Figure 2: Lightcone structure around the event A. Events B and A are separated by a timelike interval, and B lies in the forward lightcone of A. The events could be causally connected. Events C and A are separated by a null (or lightlike) interval and could be connected by a light signal. Events D and A are separated by a spacelike interval and cannot be causally connected.

Events outside the lightcone of A cannot influence or be influenced by A.

For timelike-separated events, it is possible to find an inertial frame where the events occur at the same spatial coordinates.

For spacelike events, one can find an inertial frame where the events are simultaneous.

3.3.1 Relativity of simultaneity

Since under a standard Lorentz boost,

$$c\Delta t' = \gamma(c\Delta t - \beta\Delta x)\,, \tag{12}$$

events that are simultaneous in one inertial frame (e.g., $\Delta t = 0$ in S) will generally not be in another.

The concept of simultaneity is therefore not Lorentz invariant.

However, the temporal ordering of events is Lorentz invariant *provided they are timelike or null separated*; i.e., casually connected.

To see this, consider two events with coordinate separations $c\Delta t > 0$, Δx, Δy and Δz; if they are causally connected

$$c\Delta t \geq \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}\,. \tag{13}$$

It follows that $c\Delta t \geq |\Delta x|$ and so

$$\begin{aligned} c\Delta t' &= \gamma(c\Delta t - \beta\Delta x) \\ &\geq \gamma(c\Delta t - |\beta||\Delta x|) \\ &> 0\,, \end{aligned} \tag{14}$$

since $|\beta||\Delta x| < c\Delta t$ as $|\beta| < 1$.

4 Length contraction and time dilation

4.1 Length contraction

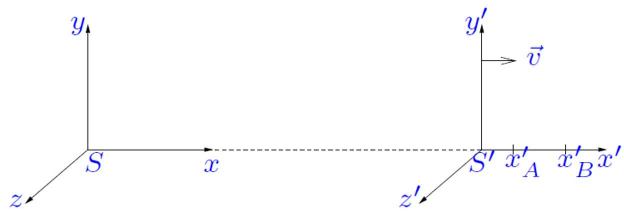


Figure 3: Two inertial frames S and S' in standard configuration, with a rod of proper length $\ell_0 = x'_B - x'_A$ at rest in S'.

Consider a rod of *proper length* ℓ_0 at rest in the inertial frame S' such that the x'-coordinates of all events at its endpoints satisfy $\ell_0 = x'_B - x'_A$ (Fig. 3).

In inertial frame S, the rod is moving at speed v along the x-axis and the x-coordinates of its endpoints at time t are $x_A(t)$ and $x_B(t)$.

The spatial coordinates of the ends of the rod in S' are related to coordinates in S' by the Lorentz transformations

$$x'_A = \gamma\left[x_A(t) - vt\right], \qquad x'_B = \gamma\left[x_B(t) - vt\right]. \tag{15}$$

An observer in S measures the length of the rod by comparing the coordinates of its endpoints, $x_A(t)$ and $x_B(t)$, at the *same* time t:

$$\ell = x_B(t) - x_A(t) = \frac{1}{\gamma}\left(x'_B - x'_A\right) = \frac{\ell_0}{\gamma}, \tag{16}$$

so they measure a *contracted length*

$$\boxed{\ell = \ell_0\left(1 - v^2/c^2\right)^{1/2}.} \tag{17}$$

The rod suffers no contraction in the y- and z-directions (i.e., perpendicular to its velocity).

It follows that the volume V of a moving object is related to proper volume V_0 by $V = V_0(1 - v^2/c^2)^{1/2}$.

Since the total number of objects in a system is Lorentz invariant, number densities thus transform from the rest rame as $n = \gamma n_0$.

4.2 Time dilation

Consider a clock at rest in the inertial frame S' with period T_0.

Let event A correspond to one tick, and event B to the next one; as the clock is at rest in S', $t'_B = t'_A + T_0$ and $x'_A = x'_B$ (see Fig. 4).

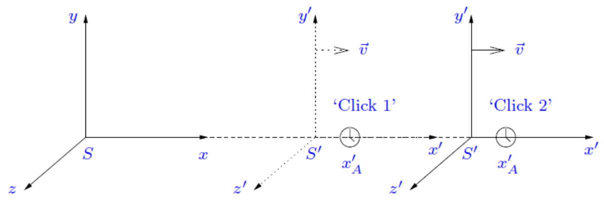


Figure 4: Two inertial S and S' in standard configuration, with a clock at rest in S'. The period between succesive ticks of the clock in its rest frame is T_0.

The times of these events in S are therefore

$$t_A = \gamma(t'_A + vx'_A/c^2) \quad \text{and} \quad t_B = \gamma(t'_A + T_0 + vx'_A/c^2) \tag{18}$$

(since $x'_A = x'_B$).

The time between ticks as measured in S is $T = t_B - t_A = \gamma T_0$ so that

$$\boxed{T = \frac{T_0}{(1 - v^2/c^2)^{1/2}}\,.} \tag{19}$$

We see that a moving clock appears to run *slower* by a factor of $(1 - v^2/c^2)^{1/2}$.

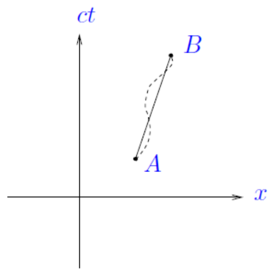
Note that, throughout this course, we shall consider only *ideal clocks* – clocks that are unaffected by acceleration – for example, the half-life of a decaying particle.

5 Paths in spacetime

5.1 Minkowski spacetime line element

The invariant interval $\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$ corresponds to the "distance" in spacetime between two events A and B measured along the straight line connecting them.

For a general (e.g., wiggly) path through spacetime, we must express the intrinsic geometry of Minkowski spacetime in infinitesimal form using the invariant *Minkowski line element* for infinitesimally-separated events:



$$\boxed{ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2\,.} \tag{20}$$

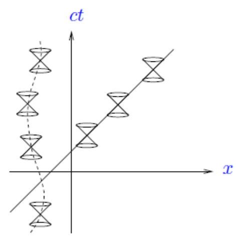
For a path connecting events A and B (see figure to the right), the invariant "distance" along the path is given by the line integral

$$\Delta s = \int_A^B ds\,. \tag{21}$$

5.2 Particle worldlines and proper time

A particle describes a *worldline* in spacetime.

For a massive particle passing through an event A, the particle's worldline must be inside the lightcone through A and each infinitesimal step must lie within the lightcone at each point (dashed line in the figure to the right).



For a photon or other massless particle, the worldline will be tangent to the lightcone (solid line).

We can write the spacetime path as $x(t), y(t), z(t)$ or, parametrically, as $t(\lambda), x(\lambda), y(\lambda), z(\lambda)$ for some parameter λ.

The most natural parameter for a massive particle is *proper time* – the time measured by an ideal clock carried by the observer.

The increment in proper time, $d\tau$, is just the increment in time in the *instantaneous rest frame* of the particle, where $dx' = dy' = dz' = 0$.

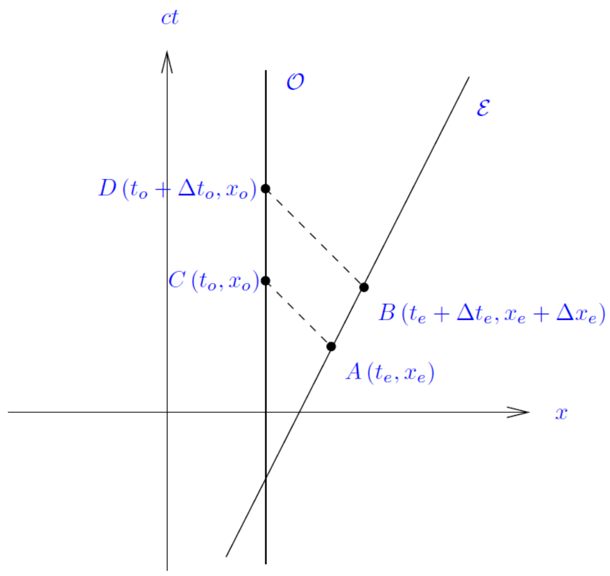


Figure 5: Spacetime diagram of the Doppler effect. An observer $\mathcal{E}$ moves at speed v along the x-axis of an intertial frame S in which an observer $\mathcal{O}$ is at rest at position x_o. A wavecrest is emitted by $\mathcal{E}$ at the event A with coordinates (t_e, x_e) in S and is received by $\mathcal{O}$ at the event C with coordinates (t_o, x_o). A second crest is emitted by $\mathcal{E}$ at the event B, which occurs at a time Δt_e later than A in S, and is received by $\mathcal{O}$ at the event D a time Δt_o later than C.

It follows that $c^2 d\tau^2 = ds^2$ and so, for two infinitesimally close events on the particle's worldline separated by dt, dx, dy, dz in some inertial frame,

$$\begin{aligned} c^2\,d\tau^2 &= c^2\,dt^2 - dx^2 - dy^2 - dz^2 \\ \Rightarrow \quad d\tau &= (1 - v^2/c^2)^{1/2}\,dt = dt/\gamma_v\,, \end{aligned} \tag{22}$$

where v is the speed of the particle in that frame and γ_v is the associated Lorentz factor.

We can integrate to get the time recorded on a clock moving with the particle:

$$\Delta\tau = \int_A^B d\tau = \int_A^B \left(1 - \frac{v^2(t)}{c^2}\right)^{1/2} dt\,. \tag{23}$$

5.3 Doppler effect

Consider an observer $\mathcal{E}$ who moves at speed v along the x-axis of an intertial frame S in which an observer $\mathcal{O}$ is at rest at position x_o (see Fig. 5).

Let successive wavecrests be emitted by $\mathcal{E}$ at events A and B, which are separated by proper time $\Delta\tau_{AB}$; this is the *proper period* of the source.

The relation between $\Delta\tau_{AB}$ and the time Δt_e between the emission events in S is

$$\Delta\tau_{AB} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \Delta t_e \, . \tag{24}$$

The wavecrests are received by $\mathcal{O}$ at the events C and D, which are separated by time Δt_o in S; since $\mathcal{O}$ is at rest in S, the proper time between C and D is $\Delta\tau_{CD} = \Delta t_o$.

In time Δt_e, the source $\mathcal{E}$ moves a distance $\Delta x_e = v\Delta t_e$ along the x-axis in S, and the second wavecrest has to travel Δx_e further than the first to be received by $\mathcal{O}$ at x_o.

It follows that

$$\Delta t_o = \left(1 + \frac{v}{c}\right) \Delta t_e \, , \tag{25}$$

so that the ratio $\Delta\tau_{AB}/\Delta\tau_{CD}$ of proper times is

$$\frac{\Delta\tau_{AB}}{\Delta\tau_{CD}} = \frac{(1-\beta^2)^{1/2}\Delta t_e}{(1+\beta)\Delta t_e} = \frac{(1-\beta)^{1/2}}{(1+\beta)^{1/2}} \, . \tag{26}$$

This ratio is also the ratio of the received frequency, as measured by $\mathcal{O}$, to the proper frequency (i.e., the frequency in the rest-frame of the source $\mathcal{E}$).

5.4 Addition of velocities

A particle on a wordline $x(t), y(t), z(t)$ has velocity in the inertial frame S with components $u_x = dx/dt$, $u_y = dy/dt$ and $u_z = dz/dt$.

The components of the velocity in a different inertial frame follow from the appropriate Lorentz transformation applied to infinitesimally-separated events on the wordline.

Transforming to S', related to S by a standard Lorentz boost, we have

$$\begin{aligned} dt' &= \gamma_v(dt - v\,dx/c^2)\,, \\ dx' &= \gamma_v(dx - v\,dt)\,, \\ dy' &= dy\,, \\ dz' &= dz\,, \end{aligned} \tag{27}$$

where γ_v is the Lorentz factor for speed v.

It follows that the velocity components in S' are given by

$$\begin{aligned} u'_x &= \frac{dx'}{dt'} = \frac{u_x - v}{1 - u_x v/c^2}\,, \\ u'_y &= \frac{dy'}{dt'} = \frac{u_y}{\gamma_v(1 - u_x v/c^2)}\,, \\ u'_z &= \frac{dz'}{dt'} = \frac{u_z}{\gamma_v(1 - u_x v/c^2)}\,. \end{aligned} \tag{28}$$

The inverse transformations are obtained by replacing v with $-v$ (and switching u'_i and u_i).

These results replace the "common-sense" addition of velocities in Newtonian mechanics; they reduce to the Newtonian results in the limit $v/c \to 0$.

Now consider three inertial frames S, S' and S'', where S' and S are related by a standard boost along the x-direction with speed v, and S'' and S' are related by a standard boost along the x'-direction with speed u'.

In terms of rapidities, $\tanh\psi_v = v/c$ and $\tanh\psi_{u'} = u'/c$, the composition of the two Lorentz transforms gives, for example,

$$\begin{aligned}
x'' &= \cosh\psi_{u'}x' - \sinh\psi_{u'}ct' \\
&= \cosh\psi_{u'}(\cosh\psi_v x - \sinh\psi_v ct) \\
&\quad - \sinh\psi_{u'}(\cosh\psi_v ct - \sinh\psi_v x) \\
&= \cosh(\psi_v + \psi_{u'})x - \sinh(\psi_v + \psi_{u'})ct\,. \qquad (29)
\end{aligned}$$

The equivalent result for t'' is

$$ct'' = \cosh(\psi_v + \psi_{u'})ct - \sinh(\psi_v + \psi_{u'})x\,, \qquad (30)$$

and $y'' = y$ and $z'' = z$.

It follows that S'' is related to S by a Lorentz boost in the x-direction with speed $u = c\tanh(\psi_v + \psi_{u'})$.

We see that the composition of two *colinear* boosts is another boost along the same direction and the rapidities add (like adding angles for rotations about a common axis).

Since a particle at rest in S'' is moving along the x'-axis in S' at speed u' and along the x-axis at speed u in S, we recover (the inverse of) Eq. (28):

$$\begin{aligned}
u &= c\tanh(\psi_v + \psi_{u'}) \\
&= c\frac{\tanh\psi_v + \tanh\psi_{u'}}{1 + \tanh\psi_v \tanh\psi_{u'}} \\
&= \frac{u' + v}{1 + u'v/c^2}\,. \qquad (31)
\end{aligned}$$

6 Acceleration in special relativity

The components of *acceleration* of a particle in an inertial frame S are the usual $a_x = du_x/dt$ etc.

The corresponding quantities in S', connected to S by a standard Lorentz boost, follow from considering the differentials of the velocity transformations (28):

$$\begin{aligned}
du'_x &= \frac{du_x}{\gamma_v^2(1-u_xv/c^2)^2}\,, \\
du'_y &= \frac{du_y}{\gamma_v(1-u_xv/c^2)} + \frac{u_y v du_x}{c^2\gamma_v(1-u_xv/c^2)^2}\,, \\
du'_z &= \frac{du_z}{\gamma_v(1-u_xv/c^2)} + \frac{u_z v du_x}{c^2\gamma_v(1-u_xv/c^2)^2}\,,
\end{aligned} \tag{32}$$

and

$$dt' = \gamma_v(dt - v\,dx/c^2) = \gamma_v(1-u_xv/c^2)\,dt\,. \tag{33}$$

The acceleration thus transforms as

$$\begin{aligned}
a'_x &= \frac{du'_x}{dt'} = \frac{1}{\gamma_v^3(1-u_xv/c^2)^3}a_x\,, \\
a'_y &= \frac{du'_y}{dt'} = \frac{1}{\gamma_v^2(1-u_xv/c^2)^2}a_y + \frac{u_y v}{c^2\gamma_v^2(1-u_xv/c^2)^3}a_x\,, \\
a'_z &= \frac{du'_z}{dt'} = \frac{1}{\gamma_v^2(1-u_xv/c^2)^2}a_z + \frac{u_z v}{c^2\gamma_v^2(1-u_xv/c^2)^3}a_x\,.
\end{aligned} \tag{34}$$

We see that acceleration is not invariant in special relativity but is, however, an *absolute* quantity in that all observers agree whether a particle is accelerating or not.

Example: Rectilinear acceleration

Consider a particle moving at a variable speed $u(\tau)$ along the x-axis in the inertial frame S, where τ is the particle's proper time.

Let the particle carry an accelerometer that reads $f(\tau)$ – this is the *proper acceleration*, the acceleration in the *instantaneous rest frame* of the particle at τ.

In the instantaneous rest frame at τ, $u'(\tau) = 0$ and

$du'/dt' = f(\tau)$; transforming back to the frame S using (the inverse of) Eq. (34), we have

$$\frac{du}{dt} = \left(1 - \frac{u^2}{c^2}\right)^{3/2} f(\tau)\,. \tag{35}$$

We can express things with respect to proper time using $d\tau = (1 - u^2/c^2)^{1/2}\,dt$ so that

$$\frac{du}{d\tau} = \left(1 - \frac{u^2}{c^2}\right) f(\tau)\,; \tag{36}$$

in terms of the rapidity $\psi(\tau)$, with $u(\tau) = c\tanh\psi(\tau)$, this is $cd\psi/d\tau = f(\tau)$ so

$$c\psi(\tau) = \int_0^\tau f(\tau')\,d\tau'\,, \tag{37}$$

taking $u = 0$ at $\tau = 0$.

To parameterise the worldline of the particle in S, we can use

$$\frac{dt}{d\tau} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} = \cosh\psi(\tau)\,, \tag{38}$$

$$\frac{dx}{d\tau} = u\left(1 - \frac{u^2}{c^2}\right)^{-1/2} = c\sinh\psi(\tau)\,. \tag{39}$$

Integrating these equations gives the coordinates in S of the wordline, $t(\tau)$ and $x(\tau)$.

Consider now the simple case of uniform or constant proper acceleration.

This does *not* mean that $du/dt =$ const., since u cannot exceed c.

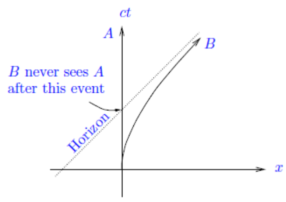
Rather, for $f =$ const. the rapidity rises linearly with τ, $\psi(\tau) = f\tau/c$, and the worldline is

$$t = t_0 + (c/f)\,\sinh f\tau/c\,, \tag{40}$$

$$x = x_0 + (c^2/f)(\cosh f\tau/c - 1)\,, \tag{41}$$

where t_0 and x_0 are integration constants.

Setting $ct_0 = x_0 = 0$, we have a hyperbolic trajectory through the origin, as shown to the right, with an oblique asymptote in the future of $ct = c^2/f + x$.



This means that there are regions of spacetime containing events that can never influence (i.e., communicate causally with) the accelerated particle (events to the left of the dotted line).

The boundary of this region defines an *event horizon* of the accelerated observer.

As an example, light emitted from an object at rest at $x = 0$ in S will only reach the accelerated observer if it is emitted before $t = c/f$.

Moreover, the accelerated observer sees the emitted light Doppler shifted to longer and longer wavelengths as the object approaches the event horizon and is observed as $\tau \to \infty$.