III. VECTOR AND TENSOR ALGEBRA

In general relativity, spacetime is described by a non-trivial (pseudo-)Riemannian manifold and this is the arena on which the rest of physics is enacted.

The equivalence principle tells us that, locally, the laws of physics reduce to those of special relativity when expressed in terms of locally-inertial coordinates defined by free-falling observers.

Our goal is therefore to formulate physical laws in such a way that they reduce to special relativity in locally-inertial coordinates.

The most efficient way to do this is to write down equations that are true in a general coordinate system (i.e., their form is the same in all coordinate systems) and then demand that they reduce to the usual form in special relativity when expressed in locally-inertial coordinates.

Such a coordinate-independent, or geometric, approach, naturally gives rise to vector-valued fields – at any point these are geometric objects that are independent of the choice of coordinate system (while their components are coordinate dependent).

In previous courses (e.g., electromagnetism) you have studied the calculus of vector fields in the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, and considered the components of vectors in simple coordinate systems such as Cartesian and spherical polar coordinates.

You have also met the notion of *tensors*, for example, the moment of inertia tensor that relates the angular velocity of a solid body to its angular momentum, and these are also essential geometric objects in general relativity.

In this handout, we shall see how to generalise familiar Euclidean ideas to define vectors and tensors in general (pseudo-)Riemannian manifolds and *arbitrary* coordinate systems.

1 Scalar and vector fields on manifolds

1.1 Scalar fields

A real (or complex) scalar field defined on (some subset of) a manifold $\mathcal{M}$ assigns a real (or complex) number to each point P in (the subset of) $\mathcal{M}$.

If we label the points in $\mathcal{M}$ using some coordinate system x^a, we can express the scalar field as a function $\phi(x^a)$ of the coordinates.

The value of a scalar field at a given point P is independent of the chosen coordinate system.

This means that if we change coordinates to x'^a, the scalar field is expressed as some different function of the new coordinates, $\phi'(x'^a)$, such that

$$\boxed{\phi'(x'^a) = \phi(x^a)\,.} \tag{1}$$

1.2 Vector fields and tangent spaces

When dealing with vectors in Euclidean space, you will have met two types of vector:

- *displacement vectors* connecting two points in the space;
- *local vectors* that are measured at a given observation point and refer solely to that point (e.g., the electric field).

Note that displacement vectors between infinitesimally-separated points are really local vectors, as are derivatives of displacement vectors (e.g., the velocity of a particle).

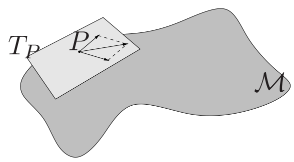
On a general manifold, we can only define local vectors – vectors defined at any given point P and that can be measured by a "bug" making local measurements in a small region around P.

In particular, we must abandon the idea of displacement vectors as these generally have no intrinsic meaning except in the infinitesimal limit.

Displacement vectors do make sense if we specify an embedding of $\mathcal{M}$ in some higher-dimensional Euclidean space, but we are interested only in intrinsic geometry here.

However, to gain some intuition, let us first consider the case where $\mathcal{M}$ is embedded in a Euclidean space but restrict attention to local vectors, such as the velocity of a particle confined to $\mathcal{M}$.

The usual velocity vector, defined by the derivative of the displacement in the Euclidean space, then lies tangent to the manifold at P.



For an ND manifold $\mathcal{M}$, the set of all possible local vectors at any point P lie in an ND subspace of the Euclidean embedding space.

This subspace is an ND vector space[1] $T_P(\mathcal{M})$, called the *tangent space* at P (see figure to the right).

The tangent spaces at different points are distinct so

[1] Recall that a vector space is, generally, a non-empty set of objects, called vectors, together with an associative and commutative operation of addition and an operation of scalar multiplication, which is distributive over addition. Moreover, the set must be closed under these operations, and must contain an additive zero vector, which leaves any vector unchanged under addition, and an additive inverse, which returns the zero vector when added to any vector.

we cannot add local vectors at different points, only at the same point.

These ideas can be generalised to remove any reference to embedding: at each point P of a general ND manifold $\mathcal{M}$, we can construct a ND vector space – the tangent space $T_P(\mathcal{M})$ – whose elements are (local) vectors.

1.2.1 Vectors as differential operators

We have not yet specified what we mean by a vector on a general manifold.

In older texts, one will often see vectors introduced as N-tuples, say $v^a = (v^1, v^2, \ldots, v^N)$, that transform in a specific way (see below) under changes of coordinates.

The v^a are the *coordinate components* of the vector and the operations of addition of vectors and multiplication by a scalar are defined by the corresponding operations on the components.

This approach is fine, but rather hides the geometric nature of vectors.

An alternative approach is to think of a vector at a point P as a differential operator there, which maps scalar fields on $\mathcal{M}$ to a number.

By extension, a vector *field* is associated with a differential operator at every point and maps scalar fields to scalar fields.

Intuitively, it is the directionality of the differential operator that captures the idea of vectors as having an associated direction.

Consider the operator

$$\boldsymbol{v} = v^a \frac{\partial}{\partial x^a} \tag{2}$$

at P, where x^a is some coordinate chart.

The sum of two such operators is also a differential operator, as is the result of multiplying by a scalar, so the space of all such operators at P is closed and forms a vector space.

In this way, we have explicitly constructed the tangent space $T_P(\mathcal{M})$.

Equation (2) expresses the vector $\boldsymbol{v}$ as a linear combination of the real-valued N-tuple v^a and the partial derivatives along the coordinate directions.

The N partial derivative operators $\{\partial/\partial x^1, \ldots, \partial/\partial x^N\}$ at P can therefore be considered a set of *basis vectors* for $T_P(\mathcal{M})$, and the v^a are the associated components.

If we change the coordinates to x'^a, the basis vectors will change since (by the chain rule)

$$\frac{\partial}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a}\frac{\partial}{\partial x^b}\,. \tag{3}$$

If the vector $\boldsymbol{v}$ is to remain invariant, its components must transform as

$$\boxed{v'^a = \frac{\partial x'^a}{\partial x^b} v^b} \tag{4}$$

since then

$$\begin{aligned}
\boldsymbol{v} \to v'^a &\frac{\partial}{\partial x'^a} \\
&= \underbrace{\frac{\partial x'^a}{\partial x^b}\frac{\partial x^c}{\partial x'^a}}_{\delta^c_b} v^b \frac{\partial}{\partial x^c} \\
&= v^b \frac{\partial}{\partial x^b} = \boldsymbol{v}\,.
\end{aligned} \tag{5}$$

Note that the components v^a and the basis vectors transform inversely under changes of coordinates.

Any N-tuple that transforms according to Eq. (4) forms the components of a vector.

For example, the coordinate differentials dx^a between two neighbouring points transform as

$$dx'^a = \frac{\partial x'^a}{\partial x^b} dx^b \,, \tag{6}$$

and so are the components of a vector (the infinitesimal "displacement" vector).

An important example of a vector is the *tangent vector* to a curve $x^a(u)$, which has components dx^a/du; the associated vector (i.e., differential operator) is

$$\frac{dx^a}{du}\frac{\partial}{\partial x^a} = \frac{d}{du} \,. \tag{7}$$

Finally, a word about notation: as most operations with vectors involve working with the components in some coordinate system, we shall often (rather sloppily!) write things like "the vector v^a" rather than the more correct "the vector with components v^a".

1.3 Dual vector fields

Another class of vector-like objects arises when we consider the gradient of a scalar field, i.e., N-tuples such as

$$X_a = \frac{\partial \phi}{\partial x^a} \,. \tag{8}$$

Under a change of coordinates, we have

$$X'_a = \frac{\partial \phi'}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a}\frac{\partial \phi}{\partial x^b} = \frac{\partial x^b}{\partial x'^a} X_b \,. \tag{9}$$

The X_a do not transform as the components of a vector (c.f. Eq. 4); rather, they transform, by construction, in the same way as the basis vectors $\partial/\partial x^a$.

Objects that transform as

$$\boxed{X'_a = \frac{\partial x^b}{\partial x'^a} X_b} \tag{10}$$

under a coordinate transformation are called the components of a *dual vector*.

(Again, this should be understood as the transformation law at the point P.)

Given the linearity of the transformation (10), it is clear that objects like X_a, with addition and multiplication by a scalar defined element-wise, form a vector space at P.

We shall see below that dual vectors at P should be considered as inhabiting a different vector space than $T_P(\mathcal{M})$, called the *dual vector space* $T_P^*(\mathcal{M})$.

Dual vectors are dual to vectors in the sense that the *contraction* of a dual vector X_a and vector v^a, defined by the summation $X_a v^a$, is invariant under coordinate transformations:

$$\begin{aligned} X'_a v'^a &= \underbrace{\frac{\partial x^b}{\partial x'^a} \frac{\partial x'^a}{\partial x^c}}_{\delta^b_c} X_b v^c \\ &= X_b v^b \, . \end{aligned} \tag{11}$$

We have so far defined dual vectors via the transformation law of their components.

However, as with vectors, we should think of dual vectors (as opposed to their components) as geometric objects that are invariant under changes of coordinates.

The way to formalise this is to regard dual vectors as linear maps that take vectors to real (or, more generally, complex) numbers.

Indeed, you may already be familiar (from courses in linear algebra) with the idea of a *dual vector space* to a vector space, defined as the set of linear maps of vectors to real (or, more generally, complex) numbers.

When expressed in terms of components, the result of the linear map between a dual vector X_a and vector v^a is just the contraction $X_a v^a$.

If we introduce a basis for the dual vector space, we can write down coordinate-independent expressions for dual vectors as linear maps on $T_P(\mathcal{M})$, but we shall not need such an approach here.

If all this seems unfamiliar and opaque, it might help to recall the bra-ket notation of quantum mechanics.

There, state vectors are written as $|\psi\rangle$ and are elements of a vector space.

The objects $\langle\phi|$ are elements of the dual vector space and are really linear maps of state vectors $|\psi\rangle$ to (complex) numbers as $\langle\phi|\psi\rangle$.

Finally, we note that there is, in general, no invariant way to relate vectors and dual vectors, i.e., given a vector v^a we cannot construct a dual vector.

An important exception is for (pseudo-)Riemannian manifolds, which are equipped with a metric.

We shall see shortly that the metric naturally associates vectors and dual vectors.

2 Tensor fields

Tensors are an extension of local vectors and dual vectors.

At a given point P, a tensor there can be formally introduced as a multi-linear map on tensor products of $T_p(\mathcal{M})$ and $T_P^*(\mathcal{M})$ that take k dual vectors and l vectors at P as input and returns a number.

Such a tensor is said to be of *type* (k, l) and to have *rank* $k + l$.

Here, we shall take the less formal route and define tensors via the tranformation laws of their components.

The components of a tensor of type (k, l) has k "upstairs" (sometimes called contravariant) indices and l "downstairs" (covariant) indices, e.g., T_{ab} is type $(0, 2)$ and T^{ab} is type $(2, 0)$.

Note that we can also have tensors with a mix of upstairs and downstairs indices, such as $T_a{}^b$.

(The reason for offsetting the indices, thus defining an order, will become clear later when we consider how the metric may be used to change the type of a tensor.)

The components of a type-(k, l) tensor transform under changes of coordinates like

$$\boxed{T'^{a\ldots b}{}_{c\ldots d} = \frac{\partial x'^a}{\partial x^p} \cdots \frac{\partial x'^b}{\partial x^q} \frac{\partial x^r}{\partial x'^c} \cdots \frac{\partial x^s}{\partial x'^d} T^{p\ldots q}{}_{r\ldots s} \, .} \tag{12}$$

We see that rank-0 tensors are scalar fields, while type-$(1, 0)$ tensors are vectors and type-$(0, 1)$ tensors are dual vectors.

As with vectors and dual vectors, we should think of tensors as geometric objects that are invariant under changes of coordinates (although the coordinate components do change, of course).

A tensor field assigns a tensor *of the same type* to every point in the manifold.

Finally, we shall sometimes want to write the tensor itself rather than its components; generally, we shall use the same bold symbol, for example, the tensor $\boldsymbol{T}$ with components T_{ab}.

2.1 Tensor equations

The reason that we are interested in working with tensor-valued objects is that they allow us to write down equations that are independent of any coordinate system.

In particular, suppose in some coordinate system one finds the components of two tensors, T_{ab} and S_{ab}, to be equal.

The tensor transformation law implies that their components are the same in *any* coordinate system, i.e., they are the same tensor.

In components, the *form* of the equation $T_{ab} = S_{ab}$ is the same in all coordinate systems.

Moreover, if the components of a tensor vanish in some coordinate system they vanish in all (the tensor itself vanishes).

2.2 Elementary operations with tensors

Addition and multiplication by a scalar

Tensors of the same type at the same point P can be added (subtracted) to give a tensor of the same type.

Addition is defined in the usual way for components, and the result is denoted by, e.g., $T_{ab} + S_{ab}$.

It is straightforward to check that the object with components $T_{ab} + S_{ab}$ is a tensor since under a coordinate transformation

$$\begin{aligned} T'_{ab} + S'_{ab} &= \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}T_{cd} + \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}S_{cd} \\ &= \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\left(T_{cd} + S_{cd}\right) . \end{aligned} \tag{13}$$

Tensors can also be multiplied by a real number c, which just multiplies each component by c, to return a tensor of the same type.

Outer (or tensor) product

The outer product of a type-(p, q) tensor $S^{a_1\ldots a_p}{}_{b_1\ldots b_q}$ and a type-(r, s) tensor $T^{c_1\ldots c_r}{}_{d_1\ldots d_s}$ is a type-$(p+r, q+s)$ tensor with components $S^{a_1\ldots a_p}{}_{b_1\ldots b_q}T^{c_1\ldots c_r}{}_{d_1\ldots d_s}$.

If we denote the tensors themselves (the coordinate-independent objects) as $\boldsymbol{S}$ and $\boldsymbol{T}$, the outer product is denoted by $\boldsymbol{S} \otimes \boldsymbol{T}$.

As an example, consider two vectors u^a and v^a and denote the outer product by T^{ab}, so that $T^{ab} = u^a v^b$.

Under a change of coordinates

$$\begin{aligned} T'^{ab} &= \frac{\partial x'^a}{\partial x^c}u^c\frac{\partial x'^b}{\partial x^d}v^d \\ &= \frac{\partial x'^a}{\partial x^c}\frac{\partial x'^b}{\partial x^d}T^{cd} , \end{aligned} \tag{14}$$

which shows that T^{ab} is indeed a type-$(2, 0)$ tensor.

Note that, in general, the outer product does not commute, $\boldsymbol{S} \otimes \boldsymbol{T} \neq \boldsymbol{T} \otimes \boldsymbol{S}$; for example, $u^a v^b$ does not equal $v^a u^b$ generally.

Contraction

In terms of components, the operation of contraction consists of setting an upstairs and downstairs index equal and summing.

For a type-(k,l) tensor, contraction returns a type-$(k-1, l-1)$ tensor.

For example, consider $T^{ab}{}_c$; contracting on the second and third indices gives a new object with just one upstairs index, say $S^a \equiv T^{ab}{}_b$ (summation convention!).

To show that S^a is indeed a vector, let us first tranform $T^{ab}{}_c$,

$$T'^{ab}{}_c = \frac{\partial x'^a}{\partial x^p} \frac{\partial x'^b}{\partial x^q} \frac{\partial x^r}{\partial x'^c} T^{pq}{}_r \,, \tag{15}$$

and then take the contraction in the new coordinates to find $S'^a \equiv T'^{ab}{}_b$ as

$$\begin{aligned} S'^a &= \frac{\partial x'^a}{\partial x^p} \underbrace{\frac{\partial x'^b}{\partial x^q} \frac{\partial x^r}{\partial x'^b}}_{\delta^r_q} T^{pq}{}_r \\ &= \frac{\partial x'^a}{\partial x^p} T^{pq}{}_q \\ &= \frac{\partial x'^a}{\partial x^p} S^p \,. \end{aligned} \tag{16}$$

This is just the expected transformation law for a vector showing that contraction does indeed return a tensor of appropriate type.

Note that the order of the indices matters when contracting – the vectors $T^{ab}{}_b$ and $T^{ba}{}_b$ are different in general.

We can combine the outer product and contraction to define a type of inner product.

For example, for tensors T^{ab} and S_{ab}, if we take the outer product to form $T^{ab}S_{cd}$ and then contract on, say, the second index of $\boldsymbol{T}$ and the first of $\boldsymbol{S}$, we have the type-$(1,1)$ tensor $T^{ab}S_{bc}$.

For the specific case of a vector v^a and a dual vector X_a, this composition reduces to what we previously called their contraction, i.e., the scalar v^aX_a.

Symmetrisation

A type-$(0,2)$ tensor S_{ab} is *symmetric* if $S_{ab} = S_{ba}$ and *antisymmetric* if $S_{ab} = -S_{ba}$.

Similarly, a type-$(2,0)$ tensor T^{ab} is symmetric if $T^{ab} = T^{ba}$ and antisymmetric if $T^{ab} = -T^{ba}$.

We can always decompose a type-$(0,2)$ [or type-$(2,0)$] tensor into a sum of symmetric and antisymmetric parts as

$$S_{ab} = \frac{1}{2}\left(S_{ab} + S_{ba}\right) + \frac{1}{2}\left(S_{ab} - S_{ba}\right) . \tag{17}$$

The operation of symmetrising (first term on the right) is usually denoted by putting round brackets around the enclosed indices:

$$S_{(ab)} \equiv \frac{1}{2}\left(S_{ab} + S_{ba}\right) . \tag{18}$$

Antisymmetrisation is usually denoted by square brackets:

$$S_{[ab]} \equiv \frac{1}{2}\left(S_{ab} - S_{ba}\right) . \tag{19}$$

These ideas extend to arbitrary numbers of indices; for $S_{ab\ldots c}$ we can construct totally-symmetric and totally-antisymmetric tensors as

$$S_{(ab\ldots c)} = \frac{1}{n!}\,(\text{sum over all perms of } a, b, \ldots, c)\ , \quad (20)$$

$$S_{[ab\ldots c]} = \frac{1}{n!}\,(\text{alternating sum over all perms})\ , \quad (21)$$

where n is the number of indices.

Here, the alternating sum denotes that a term enters with a positive sign if the permutation is even and a negative sign if it is odd.

For example, for S_{abc} we have

$$S_{[abc]} = \frac{1}{6}\,(S_{abc} - S_{acb} + S_{cab} - S_{cba} + S_{bca} - S_{bac})\ . \quad (22)$$

The normalisation $1/n!$ ensures that $S_{(ab\ldots c)} = S_{ab\ldots c}$ for a totally-symmetric tensor, and similarly for a totally-antisymmetric tensor.

We can also consider (anti)symmetrising on subsets of indices; for example

$$S_{(ab)c} = \frac{1}{2}\,(S_{abc} + S_{bac})\ . \quad (23)$$

It is straightforward to check that (anti)symmetry is a coordinate-independent notion, e.g., if the components of a tensor are symmetric in some coordinate system, they are symmetric in all.

Finally, we note that it only makes sense to discuss symmetry of pairs of upstairs or downstairs indices, but not a mix of up and downstairs.

2.3 Quotient theorem

Not all objects with indices are components of tensors, i.e., they may not transform correctly under changes of coordinates.

A useful way to test whether a set of quantities are the components of a tensor is provided by the *quotient theorem*:

> If a set of quantities when contracted with an arbitrary tensor produces another tensor, the original set of quantities form the components of a tensor.

To illustrate the proof of the quotient theorem, suppose v^a are the components of an arbitrary vector, and we have a set of quantities $T^a{}_{bc}$ that transform under a general change of coordinates in such a way that $T^a{}_{bc}v^c$ transforms as the components of a type-$(1,1)$ tensor.

This means that, however $T^a{}_{bc}$ transform (to $T'^a{}_{bc}$), they do so such that

$$T'^a{}_{bc}v'^c = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}T^d{}_{ef}v^f \,. \tag{24}$$

Since v^c is a vector, $v'^c = (\partial x'^c/\partial x^f)v^f$, and, since it is arbitrary, we must have

$$\begin{aligned} T'^a{}_{bc}\frac{\partial x'^c}{\partial x^f} &= \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}T^d{}_{ef} \\ \Rightarrow \quad T'^a{}_{bc} &= \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}\frac{\partial x^f}{\partial x'^c}T^d{}_{ef} \,. \end{aligned} \tag{25}$$

It follows that the transformation law for the quantities $T^a{}_{bc}$ must be the same as for the components of a type-$(1,2)$ tensor, and so $T^a{}_{bc}$ must be the components of such a tensor.

3 Metric tensor

We previously introduced the metric functions g_{ab} on a (pseudo-)Riemannian manifold via the line element

$$ds^2 = g_{ab}dx^a dx^b \,. \tag{26}$$

We argued that, at a given point, the metric functions must transform as

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} g_{cd} \tag{27}$$

to preserve ds^2.

This transformation law shows us that g_{ab} must be the coordinate components of a type-$(0,2)$ tensor, which we call the *metric tensor*.

In the geometric language of tensors, the metric defines a symmetric, bilinear map from pairs of vectors to real numbers.

It therefore defines a natural scalar (or inner) product between vectors, $\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v})$, where

$$\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v}) = g_{ab}u^a v^b \,. \tag{28}$$

The metric provides a map between vectors and dual vectors at a point, i.e., between the tangent space $T_P(\mathcal{M})$ and its dual $T_P^*(\mathcal{M})$.

To see this, consider the object $g_{ab}v^b$, where v^a is a vector.

This is contracting the outer product of the type-$(0,2)$ metric tensor and a type-$(1,0)$ tensor, which necessarily returns a type-$(0,1)$ tensor, i.e., a dual vector.

It is conventional to denote the dual vector $g_{ab}v^b$ with the same kernel symbol (v) as the vector from which it is derived, so we write

$$\boxed{v_a \equiv g_{ab}v^b \,.} \tag{29}$$

The operation of mapping vectors to dual vectors by the metric tensor is often referred to as "lowering an index".

The quantities v^a and v_a are the components of distinct mathematical objects (a vector and a dual vector, respectively) but, since we shall always be working with a manifold equipped with a metric, they should be regarded as just two ways of representing the same *physical* object.

Physics usually picks out the most convenient representation, e.g., a vector for the 4-velocity of a particle and a dual vector for the gradient of a scalar field, but the metric allows us to map between these freely.

More generally, we can change the type of tensors (lower their indices) by contracting with the metric; for example, given a type-$(1,1)$ tensor $T^a{}_b$,

$$T_{ab} \equiv g_{ac} T^c{}_b \tag{30}$$

is the associated type-$(0,2)$ tensor.

We can lower multiple indices with repeated application of the metric, e.g.,

$$T_{abc} \equiv g_{ap} g_{bq} T^{pq}{}_c \, . \tag{31}$$

3.1 Inverse metric

The matrix inverse of the metric functions transforms as a type-$(2,0)$ tensor under a change of coordinates.

To see this, let us denote the array formed from the inverse of the metric functions by $(g^{-1})^{ab}$, so that

$$(g^{-1})^{ab} g_{bc} = \delta^a_c \, . \tag{32}$$

If we transform g_{ab} and compute the inverse of these transformed components, we get a new matrix $(g'^{-1})^{ab}$ with

$$(g'^{-1})^{ab} = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} (g^{-1})^{cd} \, , \tag{33}$$

since then

$$
\begin{aligned}
(g'^{-1})^{ab} g'_{bc} &= \frac{\partial x'^a}{\partial x^p} (g^{-1})^{pq} \underbrace{\frac{\partial x'^b}{\partial x^q} \frac{\partial x^r}{\partial x'^b}}_{\delta^r_q} \frac{\partial x^s}{\partial x'^c} g_{rs} \\
&= \frac{\partial x'^a}{\partial x^p} \frac{\partial x^s}{\partial x'^c} \underbrace{(g^{-1})^{pq} g_{qs}}_{\delta^p_s} \\
&= \frac{\partial x'^a}{\partial x^p} \frac{\partial x^p}{\partial x'^c} \\
&= \delta^a_c
\end{aligned}
\tag{34}
$$

as required.

However, Eq. (33) is just the transformation law for a type-$(2,0)$ tensor, showing that $(g^{-1})^{ab}$ are indeed the components of a type-$(2,0)$ tensor.

It is cumbersome to write $(g^{-1})^{ab}$ for the inverse metric; instead it is usual to write it simply as g^{ab} so that $g^{ab} g_{bc} = \delta^a_c$.

Indeed, this is consistent with our earlier idea of lowering indices with the metric tensor since lowering those on g^{ab} gives

$$
g_{ac} g_{bd} g^{cd} = g_{ac} \delta^c_b = g_{ab} \,. \tag{35}
$$

The inverse metric provides a map ("raising the index") from dual vectors to vectors, e.g.,

$$
X^a \equiv g^{ab} X_b \,, \tag{36}
$$

given a dual vector X_a.

This is just the inverse of the map from vectors to dual vectors provided by the metric since lowering and then raising an index returns the original object[2]:

$$
v^a \xrightarrow{\boldsymbol{g}} g_{ab} v^b \xrightarrow{\boldsymbol{g}^{-1}} g^{ac} g_{cb} v^b = v^a \,. \tag{37}
$$

[2]This is why we can consistently use the same kernel letter after raising and lowering indices.

We can now use the metric and its inverse to lower and raise indices on general tensors, e.g., given $T^{ab}{}_c$, we define

$$T_a{}^{bc} \equiv g_{ad} g^{ce} T^{db}{}_e \,. \tag{38}$$

Note the careful positioning of the indices here: we raise and lower vertically with no horizontal shift of indices to keep track of which index was raised/lowered.

This is necessary to distinguish, for example, $g^{ac}T_{cb}$ from $g^{ac}T_{bc}$ – these are generally different (unless T_{ab} is symmetric).

Finally, if we raise only one index on the metric we get the components of a type-(1,1) tensor and these components are the kronecker delta: $g^a{}_b = g_b{}^a = \delta^a_b$.

This follows since g_{ab} and g^{ab} are inverses:

$$g^{ab} g_{bc} = \delta^a_c \,. \tag{39}$$

The tensor $g^a{}_b$ is a particularly special tensor as it is the only rank-2 tensor whose components are the same in all coordinate systems; indeed,

$$\begin{aligned} g'^a{}_b &= \frac{\partial x'^a}{\partial x^c} \frac{\partial x^d}{\partial x'^b} g^c{}_d \\ &= \frac{\partial x'^a}{\partial x^c} \frac{\partial x^c}{\partial x'^b} \\ &= \delta^a_b = g^a{}_b \end{aligned} \tag{40}$$

under a change of coordinates.

4 Scalar products of vectors revisited

We can now write the scalar product between two vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$, in terms of components in the equivalent forms:

$$g_{ab} u^a v^b = g^{ab} u_a v_b = u^a v_a = u_a v^a \,. \tag{41}$$

On a strictly Riemannian manifold, $g_{ab}v^a v^b \geq 0$ for any vector $\boldsymbol{v}$, with $g_{ab}v^a v^b = 0$ only if $\boldsymbol{v} = 0$.

On a pseudo-Riemannian manifold, these conditions are relaxed – we can have non-zero vectors (*null vectors*) v^a with $g_{ab}v^a v^b = 0$.

Generally, we can define the "length" of a vector $|\boldsymbol{v}|$ by

$$|\boldsymbol{v}| \equiv |g_{ab}v^a v^b|^{1/2}\,; \tag{42}$$

on a pseudo-Riemannian manifold the length of a non-zero vector can be zero.

We can also define a generalised "angle" θ between two non-null vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, with

$$\cos\theta \equiv \frac{u_a v^a}{|u_b u^b|^{1/2}|v_c v^c|^{1/2}}\,. \tag{43}$$

One should be aware that on a pseudo-Riemannian manifold it is possible to have $|\cos\theta| > 1$.

We say that two vectors are *orthogonal* if their scalar product vanishes.