Theoretical Physics I - Classical Field Theory

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Michaelmas Term

Logistics

Synopsis

Lagrangian and Hamiltonian mechanics: Generalised coordinates and constraints; Lagrangian and Lagrange's equations of motion; symmetry and conservation laws, canonical momenta, Hamiltonian; principle of least action; velocity-dependent potential for electromagnetic forces, gauge invariance; Hamiltonian mechanics and Hamilton's equations; Liouville's theorem; Poisson brackets and quantum mechanics; relativistic dynamics of a charged particle.
Classical fields: Waves in one dimension, Lagrangian density, canonical momentum and Hamiltonian density; multidimensional space, relativistic scalar field, Klein-Gordon equation; natural units; relativistic phase space, Fourier analysis of fields; complex scalar field, multicomponent fields; the electromagnetic field, field-strength tensor, electromagnetic Lagrangian and Hamiltonian density, Maxwell's equations.
Symmetries and conservation laws: Noether's theorem, symmetries and conserved currents; global phase symmetry, conserved charge; gauge symmetry of electromagnetism; local phase and gauge symmetry; stress-energy tensor, angular momentum tensor; quantum fields.
Broken symmetry: Self-interacting scalar field; spontaneously broken global phase symmetry, Goldstone's theorem; spontaneously broken local phase and gauge symmetry, Higgs mechanism.
Dirac field: [*not examinable*] Covariant form of Dirac equation and current; Dirac Lagrangian and Hamiltonian; global and local phase symmetry, electromagnetic interaction; stress-energy tensor, angular momentum and spin.
Phase transitions and critical phenomena: Mean field theory for the Ising and Heisenberg ferromagnets; Landau-Ginzburg theory; first order *vs.* continuous phase transitions; correlation functions; scaling laws and universality in simple continuous field theories.
Propagators and causality: Schrödinger propagator, Fourier representation, causality; Kramers-Kronig relations for propagators and linear response functions; propagator for the Klein-Gordon equation, antiparticle interpretation.

Resources

- 16 lectures, 4 examples classes, 1 exam (early Lent)
- these slides, lecture notes, examples (+solutions), past papers (+solutions)
- books

Books

- *The Feynman Lectures*, Vol. 2, Feynman. Good for inspiration/big picture.
- *Classical Mechanics*, Goldstein. Verbose, but definitive.
- *Classical Mechanics*, Kibble & Berkshire.
- *Classical Theory of Gauge Fields*, Rubakov.
- *Course of Theoretical Physics* Vol. 1 Mechanics and Vol. 2 Classical Theory of Fields, Landau & Lifshitz.
- *Quantum and Statistical Field Theory*, Le Bellac. For phase transitions and critical phenomena.

In a nutshell

A field theory has fields as degrees of freedom, *i.e.* maps from one set (the *source*) to another (the *target*).

In a nutshell

e.g. 1. Mechanics of a particle has maps $t \mapsto x_i(t) : \mathbb{R} \to \mathbb{R}^3$

In a nutshell

e.g. 1'. Mechanics of n particles has maps $t \mapsto (x_i, y_i, \dots) : \mathbb{R} \to \mathbb{R}^{3n}$

In a nutshell

e.g. 2. Transverse waves on a string are maps $(x,t) \mapsto (y,z) : \mathbb{R}^2 \to \mathbb{R}^2$

In a nutshell

e.g. 3. Electromagnetic fields are maps $(x_i, t) \mapsto (E_i, B_i) : \mathbb{R}^4 \to \mathbb{R}^6$

1st part of course: Classical dynamics of such systems

In a nutshell

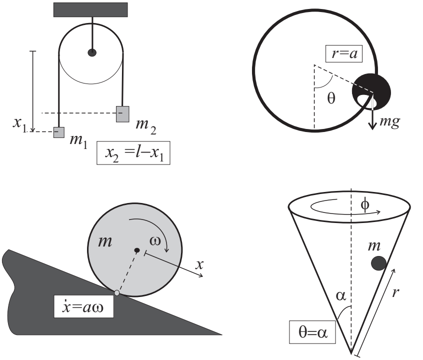
e.g. 4. Magnetization in a medium is a map $x_i \mapsto M_i : \mathbb{R}^3 \to \mathbb{R}^3$

2nd part of course: Phase transitions in such systems

Remark 1. As far as we are concerned, these 2 parts are related only in that the dofs are fields.

Remark 2. You might say ‘I already know all this’. cf. Newton’s laws, the wave equation, Maxwell’s equations.

If so, great! You can view this course merely as a tool for solving ever more complicated physical problems for the sole purpose of getting a degree.



But you could also view it as giving you a new worldview, in which you see that physics is not really about solving specific equations, but rather about seeing how they follow from very general principles, namely locality, symmetry, causality, ‘quantumness’, &c.

Lecture 1. Lagrangian Mechanics

Start with Newton's laws of motion. Let's say

1. $\exists\ (x_i, t)$, s.t.
2. $m\frac{d^2x_i}{dt^2} = F_i$
3. $\exists\ -F_i$

(Reminder: We have $t \mapsto x_i : \mathbb{R} \to \mathbb{R}^3$)

There are both features and bugs, *e.g.* features

- deterministic
- many-particle composites $M\frac{d^2X_i}{dt^2} = \sum_{\text{ext}} F_i$

There are both features and bugs, *e.g.* bugs

- whose watch measures t?
- what is F_i?
- what about constraints?
- there are many such x_i: $x_i \mapsto \rho_{ij} x_j + \alpha_i + \beta_i t$

Or should it be the other way round?
e.g. grains of sand
that a composite can be viewed as a point, should be an input, not an output

The last bug is (to me) a feature. It means you have a choice of frames. Moreover $(\rho_{ij}, \alpha^i, \beta^i)$ are elements of a symmetry group (the Galileo group).

But I'd rather be able to choose any frame. And the equations are not invariant (but the solutions are). And, again, the symmetry should be an input, not an output.

Hamilton’s principle of ‘least’ action

Suppose a particle moves along some curve between 2 fixed points in space in a fixed time interval. We’ll postulate that there exists a *lagrangian*, to wit a function of position, velocity, and time. We integrate this function along each curve to get the *action*, which is therefore a function on the set of possible curves. We declare that the classical trajectory is given by a curve which is an extremum of the action.

Remarks.

1. I didn't write it out in words to annoy you; this way shows that you can use any co-ordinates q^i you like on the target
2. 'least' $\neq$ 'extremal', *cf.* quantum mechanics
3. this is *not* equivalent to Newton's laws

Euler-Lagrange equations

In co-ordinates q^i, the extremal curves satisfy

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

As a check*, if we do a change of co-ordinates $q_i(s_j, t)$, the equations have the same form.

*An apparently mindless calculational check goes as follows.

Since $q = q(s,t)$, we have $\frac{\partial q}{\partial \dot{s}} = 0$ and $\dot{q} = \frac{\partial q}{\partial s}\dot{s} + \frac{\partial q}{\partial t}$, which in turn implies $\frac{\partial \dot{q}}{\partial \dot{s}} = \frac{\partial q}{\partial s}$. We also have $\frac{\partial L}{\partial s} = \frac{\partial L}{\partial q}\frac{\partial q}{\partial s} + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial s}$ and $\frac{\partial L}{\partial \dot{s}} = \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \dot{s}} = \frac{\partial L}{\partial \dot{q}}\frac{\partial q}{\partial s}$ (since we have already seen that $\frac{\partial q}{\partial \dot{s}} = 0$ and $\frac{\partial \dot{q}}{\partial \dot{s}} = \frac{\partial q}{\partial s}$.). Now consider $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\frac{\partial q}{\partial s}\right) - \frac{\partial L}{\partial q}\frac{\partial q}{\partial s} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial s} = \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right)\frac{\partial q}{\partial s} + \frac{\partial L}{\partial \dot{q}}\left(\frac{d}{dt}\frac{\partial q}{\partial s} - \frac{\partial \dot{q}}{\partial s}\right)$. The last term vanishes since derivatives commute and we get $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right)\frac{\partial q}{\partial s}$, implying that the Euler-Lagrange equation in the co-ordinates s can be written in the same form, *viz.* $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$.

Lagrange *vs.* Newton

To see the connection with Newton's laws, choose co-ordinates (x_i, t) and set $L = T - V = \frac{1}{2} m \dot{x}_i^2 - V(x_i, \dot{x}_i)$. The Euler-Langrange equation is

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i} + \frac{d}{dt}\frac{\partial V}{\partial \dot{x}_i}$$

cf. conservative forces, *cf.* magnetism.
So Newton $\not\subset$ Lagrange, *cf.* dissipative forces.
But Lagrange $\not\subset$ Newton either, *cf.* relativistic particles.

Advantages of the lagrangian approach.

- You can use any co-ordinates, *e.g.* Cartesian, spherical polars
- Constraints (at least holonomic ones, where $q_i(s_j, t) = \text{const}_i$) are easy: just choose the constraints to be a co-ordinates and ignore them. Or, add the constraint as a Lagrange multiplier.
- Once you have the lagrangian, the hard work is done
- The action should be invariant under any symmetry, meaning the lagrangian can shift by a most a total derivative; often this largely fixes L.
- *cf.* $L = \frac{1}{2}m\dot{x}_i^2$ under the Galileo group
- Conservation laws follow from symmetry too.

Symmetries and conservation laws

L doesn't depend explicitly on $q \implies L$ is invariant under shifts of q $\implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$. Thus, the *canonical momentum* $p_i := \frac{\partial L}{\partial \dot{q}_i}$ is conserved in time.

For a single particle in Cartesians we get $p_i = m\dot{x}_i$, the usual momentum. But for other co-ordinates, we get something quite different.

Suppose we have n particles and we shift them all by the same small amount δx. By Taylor's theorem $L \mapsto L + \sum_n \frac{\partial L}{\partial x_n} \delta x$. If L is in fact invariant under this shift (e.g. all potentials depend only on differences), then $\sum_n \frac{\partial L}{\partial x_n} = 0$ and so $\sum_n \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_n}\right) = 0$, by the E-L eq'ns. So $\sum \partial L / \partial \dot{x}_n$ is constant. But this is just $\sum m_n \dot{x}_n$. So total momentum is conserved in systems which are translation invariant.

Or suppose we have a system invariant under rotations about the z-axis. Use cylindrical polars (r, θ, z). The same argument yields that $\sum_n \frac{\partial L}{\partial \dot{\theta}_n} = \sum_n \frac{\partial}{\partial \dot{\theta}_n} \left(\frac{1}{2} m_n \dot{r}_n^2 + \frac{1}{2} m_n r_n^2 \dot{\theta}_n^2 + \frac{1}{2} m_n \dot{z}_n^2\right) = m_n r_n^2 \dot{\theta}_n$. So total angular momentum is conserved systems which are rotation invariant.

Finally, suppose $\frac{\partial L}{\partial t} = 0$. Then

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)$$

So the *hamiltonian* $H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$ is conserved. In Cartesians, with $L = T - V$, it's just $H = T + V$, the energy. So energy is conserved in systems which are time-translation invariant.

Lecture 2. Hamiltonian Mechanics

Last time we went from newtonian to lagrangian mechanics.
Advantages were

- holonomic constraints are easy
- we can use any co-ordinates q^i we like (on the target)
- locality is an input $S = \int L dt$
- causality has a meaning, *cf.* $\int dt = -\int d(-t)$
- symmetries govern L and result in conservation laws
- (quantumness is input too, *cf.* path integral)

- *cf.* $L = \frac{1}{2}m\dot{x}_i^2$ under $x_i \mapsto \rho_{ij}x_j + \alpha_i + \beta_i t, t \mapsto t + \gamma$.
- *cf.* $\dot{x}_i, \ddot{x}_i, x_i\dot{x}_i,$.

Today we'll study yet another formulation: *hamiltonian mechanics*

We already introduced the canonical momentum $p_i := \frac{\partial L}{\partial \dot{q}_i}$ and the hamiltonian $H = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$. These were introduced as conserved quantities corresponding to invariance under space and time translations, respectively. But they have a much more important role to play.

Suppose we want to trade $\dot{q}_i$ for p_i in our description. So $L = L(q_i, \dot{q}_i(p_i, q_i, t), t)$.
Let's make life easy and consider $f(x)$. So we want to trade x for $y := \frac{df}{dx}$.

Legendre transform

'Trade x for $y := \frac{df}{dx}$' means 'change variables from x to y'. This requires $\frac{df}{dx}$ to have an inverse! *i.e* convex (or concave) functions. *e.g.* $f(x) = x^n$ works only for even $n > 0$.

It is convenient to trade f as well, for $g(y) := xy - f$. Here $x = x(y)$ and $f = f(x(y))$.
Why? Because $\frac{dg}{dy} = \frac{dx}{dy}y + x - \frac{df}{dx}\frac{dx}{dy} = x$.
So we have $y = \frac{df}{dx}$ and $x = \frac{dg}{dy}$. *i.e.* The pairs $(y, f(x))$ and $(x, g(y))$ are conjugate.
We call g the *Legendre transform* of f.

An important fact: the *Legendre transform* of g is f!
Proof: $yx - g = (yx - (xy - f)) = f$.
This is important because it means that no information is lost in transforming. You can transform again and get back to where you started.

Now go back to mechanics. We want to trade $p := \frac{\partial L}{\partial \dot{q}_i}$ for $\dot{q}_i$. We take the Legendre transform of L, to get

$$H(p_i, q_i) = p_i \dot{q}_i(q_j, p_j) - L(q_i, \dot{q}_i(q_j, p_j))$$

We find

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + p_j \frac{\partial \dot{q}_j}{\partial p_i} - \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial p_i} = \dot{q}_i$$

and

$$\frac{\partial H}{\partial q_i} = p_j \frac{\partial \dot{q}_j}{\partial q_i} - \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = -\dot{p}_i$$

Thus we get *Hamilton's equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

and

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

We call $\{q_i\}$ the *configuration space* and $\{(q_i, p_i)\}$ the *phase space*. It's twice the size.

Motion in configuration space is determined by $q_i(0)$ and $\dot{q}_i(0)$ (2nd order ODE); motion in phase space is determined by $(q_i, p_i)(0)$ (1st order ODE).

Corollary: phase space trajectories are *flows*. They never cross!

e.g.

For the SHO, $L = \frac{1}{2}(\dot{x}^2 + x^2)$, we get $H = \frac{1}{2}(p^2 + x^2)$ and elliptical trajectories.

Conservation laws

Similar to before, if H doesn't depend on q_i, we immediately get

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0$$

so p_i is conserved.
If H has no explicit dependence on t, we get

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = -\dot{p}_i\dot{q}_i + \dot{q}_i\dot{p}_i = 0$$

so H is conserved.

Liouville's theorem

The flow of phase space trajectories is incompressible: their density in phase space is a constant. This is *Liouville's theorem*.

Proof: Suppose some points occupy phase space 'volume' V (in $2n$-dimensions) and has surface 'area' S. The rate of change of V is given by $\int_S \mathbf{v} \cdot d\mathbf{S}$ where

$$\mathbf{v} = (\dot{q}_1, \ldots, \dot{q}_n, \dot{p}_1, \ldots, \dot{p}_n) = \left(\frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q_1}, \ldots, -\frac{\partial H}{\partial q_n} \right)$$

is the velocity of the points. The divergence theorem (in $2n$-dimensions) says

$$\int_S \mathbf{v} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{v} dV.$$

But

$$\nabla \cdot \mathbf{v} = \sum_i \frac{\partial}{\partial q_i} \dot{q}_i + \frac{\partial}{\partial p_i} \dot{p}_i = \sum_i \frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} = 0.$$

Remarks

1. True only for hamiltonian systems. *e.g.* in a system which dissipation (e.g. damped SHO), all points head to $\dot{q}_i = 0$
2. *cf.* $\Delta p \Delta q \geq \hbar$ in QM

Liouville's equation

Since the density $\rho(p,q,t)$ is constant, we also have

$$0 = \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial p_i}\dot{p}_i + \frac{\partial \rho}{\partial q_i}\dot{q}_i$$

or

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial \rho}{\partial q_i}\frac{\partial H}{\partial p_i}$$

n.b. $\rho = \rho(H)$ is an important solution, *cf.* $\rho = e^{-H/kT}$

Remarks

1. Can use it to study uncertainty in 1 system
2. Can use it to study n systems in the mean
3. Can use it to study 1 big system in the mean
4. *cf.* (non-equilibrium) statistical mechanics

Lecture 3. Poisson Brackets

Last time we went from lagrangian to hamiltonian mechanics via Legendre transform $H = p_i\dot{q}_i - L$. Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

and

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Liouville's theorem says that the phase space density is constant.

Today we'll go further with hamiltonian mechanics and study *Poisson brackets.*

A general observable is $f(p,q,t)$. We have

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial f}{\partial t} \end{aligned} \tag{1}$$

We can write this as

$$\frac{df}{dt} = \{f,H\} + \frac{\partial f}{\partial t} \tag{2}$$

where

$$\{f,g\} := \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{3}$$

is the *Poisson Bracket* of the functions f and g.

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \tag{4}$$

e.g.

- For $f = q_i$, we get $\dot{q}_i = \frac{\partial H}{\partial p_i}$
- For $f = p_i$, we get $\dot{p}_i = -\frac{\partial H}{\partial q_i}$
- For $f = H$ get $\dot{H} = \frac{\partial H}{\partial t}$
- For $f = \rho$ get Liouville's equation (noting from Liouville's theorem that $\dot{\rho} = 0$)

Properties of Poisson brackets

$$\{f,g\} \equiv \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{5}$$

is

- Antisymmetric in f,g : $\{f,g\} = -\{g,f\}$
- Bi-linear f,g : $\{af+bh,g\} = a\{f,g\} + b\{h,g\}$ (for constant a,b)
- Leibniz: $\{fh,g\} = f\{h,g\} + \{f,g\}h$
- Jacobi: $\{\{f,g\},h\}+$ cyclic perms. $=0$.

cf. the commutator of matrices: $[A,B] := AB - BA$.

Conservation Laws

Suppose $f = f(p,q)$. Then f is conserved iff $\{f,H\} = 0$.
Proof.

$$\frac{df}{dt} = \{f,H\} + \frac{\partial f}{\partial t} \tag{6}$$

e.g. If q_i is ignorable, then $\{p_i,H\} = 0$.
e.g. $\{H,H\} = 0$, always, by anti-symmetry.
n.b. By Jacobi, if f,g are conserved, then so is $\{f,g\}$: conserved quantities close into an algebra.

From classical to quantum

The classical

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \tag{7}$$

is uncannily like the quantum

$$\frac{d}{dt}\langle \hat{O} \rangle = \frac{1}{\mathrm{i}\hbar}\langle [\hat{O}, \hat{H}] \rangle + \left\langle \frac{\partial \hat{O}}{\partial t} \right\rangle \tag{8}$$

viz. Ehrenfest's theorem.
So too are the properties of the Poisson brackets.

This suggests that we can go from classical to quantum by replacing

$$\{A, B\} \mapsto \frac{1}{\mathrm{i}\hbar}\left[\hat{A}, \hat{B}\right]$$

On the LHS we have functions on phase space; on the RHS we have linear Hermitian operators on Hilbert space!
n.b. $\{q, p\} = 1$, so we need $\hat{p} = -i\hbar\frac{d}{dq}$

$$\frac{S}{\hbar} = \frac{1}{\hbar}\int L\,dt = \int(\mathbf{p}\cdot\dot{\mathbf{q}}/\hbar - H/\hbar)dt = \int(\mathbf{k}\cdot\mathbf{dq} - \omega\,dt)$$

i.e. the wave-mechanical phase.
So Hamilton's principle says that the classical path is the one where the de Broglie phases interfere constructively.
cf. geometric *vs.* wave optics

Canonical transformations

In lagrangian mechanics, we can choose any co-ordinates on the target. Here we can do much more, *e.g.* $q \mapsto p, p \mapsto -q$.

Theorem: Suppose

$$Q_j = Q_j(q_i, p_i), P_j = P_j(q_i, p_i) \tag{9}$$

are s.t. Hamilton's equations are unchanged in form, *i.e.*

$$\dot{Q}_i = \frac{\partial H}{\partial P_i},\ \dot{P}_i = -\frac{\partial H}{\partial Q_i}.$$

Then $\{Q_i, P_j\} = \delta_{ij}$ (and all others vanish).

Proof (1 co-ordinate): We know

$$\dot{Q} = \{Q,H\} = \frac{\partial Q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H}{\partial q}\,. \tag{10}$$

Inserting

$$\begin{aligned} \frac{\partial H}{\partial p} &= \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p} \\ \frac{\partial H}{\partial q} &= \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\,. \end{aligned}$$

gives

$$\begin{aligned} \dot{Q} &= \frac{\partial H}{\partial P}\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) \\ &= \frac{\partial H}{\partial P}\{Q,P\}\,. \end{aligned}$$

and

$$\dot{P} = -\frac{\partial H}{\partial Q}\{Q,P\}\,.$$

Remark. A better proof would have set up Hamiltonian mechanics without mentioning co-ordinates at all. But this needs symplectic geometry.

Lecture 4. Charged Particles in Electromagnetic Fields

Last time we studied the Poisson bracket

$$\{f,g\} := \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{11}$$

in the hamiltonian formalism.

Today we'll study particle motion (both non-relativistic and relativistic) in electromagnetic fields. Mostly via lagrangians.

Non-relativistic charged particles

For electrostatics, it's obvious what to do. We know that the electric force is conservative, so comes from a potential. We set $L = T - V$ with $V = e\phi(x_i)$ and we are done: the force is $F_i = -\frac{\partial V}{\partial x_i} = -eE_i$ where $E_i = --\frac{\partial \phi}{\partial x_i}$ is the usual electric field.
Adding magnetic forces, we want the Lorentz force

$$\mathbf{F} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{12}$$

so will need the more general

$$F_i = -\frac{\partial V}{\partial x_i} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}_i}\right) \tag{13}$$

so $V = V(x_i, \dot{x}_i)$

Guess linear dependence: $V = e\left(\phi(x_j,t) - \dot{x}_i A_i(x_j,t)\right)$.
Theorem: This works, with $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$.
Proof:

$$\begin{aligned}
F_i &= -\frac{\partial V}{\partial x_i} + \frac{d}{dt}\left(\frac{\partial V}{\partial v_i}\right) \qquad (14)\\
&= -\frac{\partial}{\partial x_i} e(\phi - v_j A_j) + \frac{d}{dt}(-eA_i)\\
&= -e\frac{\partial \phi}{\partial x_i} + ev_j\frac{\partial A_j}{\partial x_i} - e\frac{\partial A_i}{\partial x_j}v_j - e\frac{\partial A_i}{\partial t}\\
&= e(\mathbf{E} + [\mathbf{v} \times (\nabla \times \mathbf{A})])_i
\end{aligned}$$

The canonical momentum is not the usual momentum:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial v_i} = mv_i + eA_i \tag{15}$$

(*cf.* Landau levels) and nor is the Hamiltonian:

$$\begin{aligned} H &= \mathbf{p} \cdot \dot{\mathbf{q}} - L & (16) \\ &= \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + e\phi & (17) \end{aligned}$$

Gauge symmetry

Let $f = f(x_i, t)$ and send $\phi \mapsto \phi - \frac{\partial f}{\partial t}$, $A_i \mapsto A_i - \frac{\partial f}{\partial x_i}$.
Then $L \supset e(\phi - \frac{dx_i}{dt}A_i) \mapsto L + e(\phi - \frac{dx_i}{dt}A_i - \frac{df}{dt})$.
A huge symmetry of the action: *gauge symmetry*.
cf. current theory of everything.
Gauge fixing: choose f to make (ϕ, A_i) nice, *e.g.*
$\phi = 0, \mathbf{n} \cdot \mathbf{A} = 0, \nabla \cdot \mathbf{A} = 0, \ldots$

Relativistic particles

First some notation. We define contravariant 4-vectors, *e.g.* x^μ with $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$ and covariant 4-vectors *e.g.* x_μ with $(x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$.

Only summations with indices up/downstairs are valid *e.g.* $x^\mu x_\mu = c^2t^2 - x^2 - y^2 - z^2$. They are Lorentz invariants.

Given a field $\phi(x^\mu)$, $d\phi = \frac{\partial\phi}{\partial x^\mu} dx^\mu$ is invariant, so $\frac{\partial\phi}{\partial x^\mu} := \partial_\mu \phi$ is covariant.

We can lower an index with the *Minkowski metric tensor* $g_{\mu\nu} := \mathrm{diag}(1, -1, -1, -1)$, *e.g.* $x_\mu = g_{\mu\nu} x^\nu$.

Relativistic particles

Lagrangian mechanics puts no structures on space/time (beyond orientation) so should handle relativity.
But what is the action, *e.g.* for a relativistic particle?
Symmetry: it should be Lorentz invariant. So (a function of) the proper time

$$S = -mc^2 \int d\tau = -mc \int \sqrt{dx^\mu dx_\mu} = -mc^2 \int \sqrt{1 - \dot{x}_i \dot{x}_i / c^2} dt$$

In other words

$$L = mc^2/\gamma = -mc^2 + \frac{1}{2}mv^2 + \dots$$

cf. $L = T - V$.

Remarks

- This L is not Lorentz invariant!
- The canonical momentum is $p_i = \gamma m v_i$.
- The Hamiltonian is $H = \gamma mc^2$.
- The equation of motion is $p_i = \text{const} \implies v_i = const$ (*n.b.* $v \leq c$).

Relativistic coupling to electromagnetism

We can package $A^\mu = (\phi/c, A^i)$, whence we see that
$L \supset e(\phi - v^i A^i) = e\frac{dx_\mu}{dt}A^\mu$.
The action is $S = \int e\frac{dx_\mu}{dt}A^\mu dt = e\int dx_\mu A^\mu$.
Already Lorentz (and gauge) invariant.

Reparameterization invariance

Recall that a field is just a map from a source to a target. Here we set things up with $\mathbb{R} \to \mathbb{R}^3 : t \mapsto x^i(t)$.
This is ugly, because Lorentz transformations mix t and x^i.
We could instead do $\mathbb{R} \to \mathbb{R}^4 : \lambda \mapsto x^\mu(\lambda)$, where λ parameterizes the worldline.
Any (oriented) parameterization will do, so now the action should be invariant under $\lambda \mapsto \lambda'(\lambda)$.
The action is unambiguously proportional to

$$\int \sqrt{\frac{dx_\mu}{d\lambda}\frac{dx^\mu}{d\lambda}}d\lambda$$

This is a gauge symmetry on the wordline!
We can gauge fix *e.g.* $\lambda = t$.

Remark. It is a very small step from here to string theory!

Lecture 5. Lagrangian and Hamiltonian Field Theory

Last time we studied charged particles moving in (background) electromagnetic fields.
Lorentz + gauge + reparameterization invariance led to the action

$$S = \int \left(-mc\sqrt{\frac{dx_\mu}{d\lambda}\frac{dx^\mu}{d\lambda}} + e\frac{dx_\mu}{d\lambda}A^\mu \right) d\lambda$$

Today we'll study (dynamical) fields.

Recall that ‘particle mechanics’ has dof maps $\mathbb{R} \to \mathbb{R}^n$, where the source is some kind of ‘time’.
In field theory, we’ll have dof maps $\mathbb{R}^m \to \mathbb{R}^n$, where the source is some kind of ‘spacetime’.
Clearly, this is what the Universe wants us to do.
(In quantum field theory, particles are the quanta of these fields)

We will need to generalize Hamilton's principle, *& c.*, but this is conceptually straightforward.
The main hurdle is to keep track of indices *& c.* in an efficient way.
Start with a simple non-relativistic example and build up.

Waves in an elastic rod

Consider longitudinal waves (‘sound waves/phonons’) in an elastic rod.
What are the degrees of freedom? We mark off points in the rod with a co-ordinate ϕ and mark off points in the lab with co-ordinate x. As the rod moves, we get $\phi(x,t)$. It’s convenient to choose $\phi(x,t=0)=x$.

What is the Lagrangian?
Pro: the symmetries are translations of both the lab and the rod, but this is broken to the common subgroup.

What is the Lagrangian?
Am: consider T and V for a bit of rod.

$$T = \int \frac{1}{2}\rho \left(\frac{\partial \varphi}{\partial t}\right)^2 dx, \tag{18}$$

where ρ is the mass per unit length, and the (elastic) potential energy as

$$V = \int \frac{1}{2}\kappa \left(\frac{\partial \varphi}{\partial x}\right)^2 dx, \tag{19}$$

Lagrangian density

Get

$$
\begin{aligned}
L &= T - V = \int \left[\frac{1}{2}\rho \left(\frac{\partial \varphi}{\partial t}\right)^2 - \frac{1}{2}\kappa \left(\frac{\partial \varphi}{\partial x}\right)^2 \right] dx \equiv \int \mathscr{L}\, dx\,, \\
\text{and } S &= \int L dt = \int \mathscr{L}\, dxdt \qquad (20)
\end{aligned}
$$

where the *lagrangian density* is

$$
\mathscr{L}(\varphi, \partial\varphi/\partial t, \partial\varphi/\partial x) = \frac{1}{2}\rho \left(\frac{\partial \varphi}{\partial t}\right)^2 - \frac{1}{2}\kappa \left(\frac{\partial \varphi}{\partial x}\right)^2 \qquad (21)
$$

Euler-Lagrange equations

We insist that the action be extremal with respect to maps $\mathbb{R}^2 \to \mathbb{R} : (x,t) \mapsto \phi(x,t)$. So

$$\delta S = \int \left(\frac{\partial \mathscr{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathscr{L}}{\partial \varphi'} \delta \varphi' + \frac{\partial \mathscr{L}}{\partial \dot{\varphi}} \delta \dot{\varphi} \right) dx\, dt\,. \tag{22}$$

But

$$\int \frac{\partial \mathscr{L}}{\partial \varphi'} \delta \varphi' \, dx = \int \frac{\partial \mathscr{L}}{\partial \varphi'} \frac{\partial}{\partial x} \delta \varphi \, dx = \left[\frac{\partial \mathscr{L}}{\partial \varphi'} \delta \varphi \right]_{-\infty}^{+\infty} - \int \frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial \varphi'} \right) \delta \varphi \, dx\,. \tag{23}$$

So

$$\delta S = \int \left[\frac{\partial \mathscr{L}}{\partial \varphi} - \frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial \varphi'} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \dot{\varphi}} \right) \right] \delta \varphi \, dx\, dt\,. \tag{24}$$

i.e.

$$\frac{\partial \mathscr{L}}{\partial \varphi} - \frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial \varphi'} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \dot{\varphi}} \right) = 0\,. \tag{25}$$

Wave equation

For

$$\mathscr{L} = \frac{1}{2}\rho\dot{\varphi}^2 - \frac{1}{2}\kappa\varphi'^2 \tag{26}$$

we get the 1-d wave equation

$$0 + \frac{\partial}{\partial x}\kappa\varphi' - \frac{\partial}{\partial t}\rho\dot{\varphi} = 0 \tag{27}$$

with solution

$$\varphi \propto e^{i\omega t - ikx}, \omega = \sqrt{\frac{\kappa}{\rho}}k\,.$$

Canonical momentum density

Analogously to $p_i := \partial L / \partial \dot{q}_i$, we define the *canonical momentum density*

$$\pi(x,t) := \frac{\partial \mathscr{L}}{\partial \dot{\varphi}} \tag{28}$$

Here $\pi = \rho \dot{\varphi}$.
We define the (total) canonical momentum

$$P(t) := \int dx \pi(x,t) \tag{29}$$

Hamiltonian density

Analogously to $H = \sum_i p_i \dot{q}_i - L$, we define the *Hamiltonian density* $\mathscr{H}$,

$$\mathscr{H}(\varphi, \varphi', \pi) = \pi\dot{\varphi} - \mathscr{L} \,, \tag{30}$$

Here, we get

$$\mathscr{H} = \frac{\pi^2}{2\rho} + \frac{1}{2}\kappa\varphi'^2 \,, \tag{31}$$

i.e. the total energy density. We define the (total) Hamiltonian

$$H := \int dx \mathscr{H} \tag{32}$$

i.e. the total energy.

Conservation Laws

If $\partial \mathscr{L} / \partial \varphi = 0$, the Euler-Lagrange equation reads

$$\frac{\partial}{\partial t}\pi(x,t) + \frac{\partial}{\partial x}J(x,t) = 0 \tag{33}$$

where $J(x,t) := \partial \mathscr{L} / \partial \varphi'$. So π is not conserved. But

$$\frac{dP}{dt} = \int dx \frac{\partial}{\partial t}\pi = -\int dx \frac{\partial}{\partial x}J = 0 \tag{34}$$

so P is conserved.
i.e. momentum can be exchanged between elements of the rod, but is conserved overall.

Multi-dimensional space

The next step is to go from 1 space dimension to d. So

$$S = \int \int .. \int \mathscr{L}(\varphi, \partial\varphi/\partial t, \nabla\varphi)\, dt dx_1 .. dx_d \tag{35}$$

and the Euler-Lagrange equation is

$$\frac{\partial \mathscr{L}}{\partial \varphi} = \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial(\partial\varphi/\partial t)}\right) + \nabla \cdot \left(\frac{\partial \mathscr{L}}{\partial(\nabla\varphi)}\right). \tag{36}$$

When $\partial\mathscr{L}/\partial\varphi = 0$, we now get the *continuity equation*

$$\dot{\pi}(x,t) + \operatorname{div} J(x,t) = 0, \quad \text{with the vector } \; J := \frac{\partial \mathscr{L}}{\partial(\nabla\varphi)}\,. \tag{37}$$

Again, canonical momentum is globally, conserved, but its local density satisfies a continuity equation.

Spacetime

There is no relativity here (yet), but let's put space and time on an equal footing. Letting $x^0 = (c)t$, we get

$$\frac{\partial \mathscr{L}}{\partial \varphi} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial \varphi/\partial x^\mu)}\right) \equiv \partial_\mu \frac{\partial \mathscr{L}}{\partial[\partial_\mu \varphi]}, \tag{38}$$

and the continuity equation becomes $\partial_\mu J^\mu = 0$, with $J^0 := (c)\pi$

Why did it work so nicely? Remember the worldview! The lagrangian formulation doesn't need any structure on the source (bar orientation). So 'space' and 'time' are meaningless (*a priori*).
Pro: the lagrangian *formalism* is invariant not just under Lorentz transformations, but under all (orientation-preserving) diffeomorphisms of the source. *Caveat*: J^μ is then not a vector but a *differential form of degree d*.

Lecture 6. Real Scalar fields

Last time we studied the formulation of classical field theory.
The local action

$$S = \int \mathscr{L}(\phi, \partial_\mu \phi, x^\mu) d^{d+1} x^\mu$$

yielded the Euler-Lagrange eq'n

$$\frac{\partial \mathscr{L}}{\partial \varphi} = \partial_\mu \frac{\partial \mathscr{L}}{\partial [\partial_\mu \varphi]}, \tag{39}$$

and $\frac{\partial \mathscr{L}}{\partial \varphi} = 0$ yielded the local conservation law

$$\partial_\mu J^\mu = 0$$

.

Today we'll study the simplest relativistic example: the real scalar field.

Real scalar field

We'll build it using the worldview. The degrees of freedom are maps $\mathbb{R}^4 \to \mathbb{R} : x^\mu \mapsto \phi(x^\mu)$.
The action S should be Lorentz invariant.
Because Lorentz transformations have determinant 1, $d^{d+1}x^\mu$ is Lorentz invariant, so $\mathscr{L}$ must also be Lorentz invariant.

$$\mathscr{L} = \alpha(\partial^\mu \varphi)(\partial_\mu \varphi) + \beta \partial^\mu \partial_\mu \varphi + \gamma \varphi \partial^\mu \partial_\mu \varphi + \delta \varphi + \varepsilon \varphi^2 \qquad (40)$$

can be written as

$$\mathscr{L} = (\alpha - \gamma)(\partial^\mu \varphi)(\partial_\mu \varphi) + \partial^\mu (\beta \partial_\mu \varphi + \gamma \varphi \partial_\mu \varphi) + \delta \varphi + \varepsilon \varphi^2 \ , \quad (41)$$

and we can choose $\alpha - \gamma = \frac{1}{2}$ w.l.o.g.

$$
\mathscr{L} = \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi) + \delta\varphi + \varepsilon\varphi^2 \,, \tag{42}
$$

leads to the equation of motion

$$
\partial^\mu \partial_\mu \varphi - \delta - 2\varepsilon\varphi = 0 \,. \tag{43}
$$

So $\delta = 0$ for BC $\phi \to 0$. Set $\varepsilon = -m^2/2$.

Klein-Gordon lagrangian

In toto, get

$$\mathscr{L} = \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi) - \frac{1}{2}m^2\varphi^2 \;, \tag{44}$$

with the *Klein-Gordon* eq'n of motion,

$$\partial^\mu \partial_\mu \varphi + m^2 \varphi = 0 \;. \tag{45}$$

Klein-Gordon hamiltonian

Since

$$\mathscr{L} = \frac{1}{2c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2 \,, \tag{46}$$

the momentum density is

$$\pi = \frac{\partial\mathscr{L}}{\partial\dot{\varphi}} = \frac{1}{c^2}\frac{\partial\varphi}{\partial t} \tag{47}$$

and the hamiltonian density is

$$\mathscr{H} = \frac{1}{2}c^2\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 \,. \tag{48}$$

So $m^2 > 0$.

Natural units

In the dark ages time and space were different. But now they are the same. So rather than measuring times in seconds and distance in metres in makes sense to use the same units. To not do so is the same (and is equally stupid) as the use of different units to measure lengths in the UK vs. in developed countries.
So $c = 1$ is a good choice. A length is measured by the time it takes light to travel that distance.
In the dark ages energy and frequency were different ...
So $\hbar = 1$ is a good choice. Times are measured by the inverse amount of energy.
You can choose what you want for the remaining unit. Particle physicists (currently) work in GeV.
To translate into the *vulgate*, use $\hbar c = 0.2 GeVfm$.
In this schema $\mathscr{L}$ has units of M^4 (in $d+1=4$) and ϕ has units of M.

Fourier analysis

In $d = 1 + 1$, the Klein-Gordon eq'n is

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + m^2 \varphi = 0\,. \tag{49}$$

Write $\varphi(x,t)$ as a Fourier integral:

$$\varphi(x,t) = \int dk\, N(k) \left[a(k) e^{ikx - i\omega t} + a^*(k) e^{-ikx + i\omega t}\right] \tag{50}$$

($N(k)$ is a normalisation factor). We get a solution if $\omega = +\sqrt{m^2 + k^2}$.

The Hamiltonian

$$H = \int \left(\frac{1}{2}\pi^2 + \frac{1}{2}\varphi'^2 + \frac{1}{2}m^2\varphi^2 \right) dx \tag{51}$$

then takes a simple form. Indeed, using

$$\int dx\, e^{i(k\pm k')x} = 2\pi\,\delta(k\pm k') \tag{52}$$

we get

$$\begin{aligned} \int \varphi^2\, dx = 2\pi \int dk\, dk'\, N(k)\, N(k') \\ \big[a(k)a(k')\delta(k+k')e^{-i(\omega+\omega')t} + a^*(k)a^*(k')\delta(k+k')e^{i(\omega+\omega')t} \\ + a(k)a^*(k')\delta(k-k')e^{-i(\omega-\omega')t} + a^*(k)a(k')\delta(k-k')e^{i(\omega-\omega')t}\big] \end{aligned} \tag{53}$$

But $\omega(-k) = \omega(k)$ and we can choose $N(-k) = N(k)$ s.t.

$$\begin{aligned} \int \varphi^2\, dx = 2\pi \int dk\, [N(k)]^2 \\ \big[a(k)a(-k)e^{-2i\omega t} + a^*(k)a^*(-k)e^{2i\omega t} + a(k)a^*(k) + a^*(k)a(k)\big] \end{aligned}$$

Claim: In all get

$$H = 2\pi \int dk \, [N(k)\omega(k)]^2 [a(k)a^*(k) + a^*(k)a(k)] \tag{54}$$

or with $N(k) := \frac{1}{2\pi \cdot 2\omega(k)}$,

$$H = \int dk \, N(k) \, \frac{1}{2} \omega(k) \, [a(k)a^*(k) + a^*(k)a(k)] \tag{55}$$

Proof: H is conserved and $\frac{1}{2}(\omega^2 + k^2 + m^2) = \omega^2$.

$$H = \int dk\, N(k)\, \omega(k)\, |a(k)|^2$$

i.e. add up density of modes $N(k)$ times energy per mode $\omega(k)|a(k)|^2$.
Modes behave like independent oscillators with amplitude $a(k)$.
cf. quantum field theory.

In $d = 3 + 1$, have

$$\varphi(r,t) = \int d^3k\, N(k) \left[a(k) e^{ik\cdot r - i\omega t} + a^*(k) e^{-ik\cdot r + i\omega t}\right] \tag{56}$$

and

$$\int d^3r\, e^{i(k\pm k')\cdot r} = (2\pi)^3\, \delta^3(k \pm k')\,. \tag{57}$$

So choose

$$N(k) = \frac{1}{(2\pi)^3\, 2\omega(k)} \tag{58}$$

to get

$$H = \int \frac{d^3k}{(2\pi)^3\, 2\omega(k)}\; \omega(k)\, |a(k)|^2\,. \tag{59}$$

Multiple fields

With multiple fields ϕ_i, we vary independently, yielding Euler-Lagrange equations

$$\frac{\partial \mathscr{L}}{\partial \varphi_j} = \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial(\partial \varphi_j/\partial t)}\right) + \nabla \cdot \left(\frac{\partial \mathscr{L}}{\partial(\nabla \varphi_j)}\right). \tag{60}$$

e.g. for transverse waves on a string

$$T = \frac{1}{2}\rho \int \left[\left(\frac{\partial \varphi_y}{\partial t}\right)^2 + \left(\frac{\partial \varphi_z}{\partial t}\right)^2\right] dx \tag{61}$$

and

$$\begin{aligned} V &= F \int \left[\sqrt{1 + \left(\frac{\partial \varphi_y}{\partial x}\right)^2 + \left(\frac{\partial \varphi_z}{\partial x}\right)^2} - 1\right] dx \\ &= \frac{1}{2} F \int \left[\left(\frac{\partial \varphi_y}{\partial x}\right)^2 + \left(\frac{\partial \varphi_z}{\partial x}\right)^2\right] dx \end{aligned} \tag{62}$$

Lecture 7. Complex Scalar Fields and Electromagnetic Fields

Last time we studied the (relativistic) real scalar field

$$\mathscr{L} = \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi) - \frac{1}{2}m^2\varphi^2 \, , \tag{63}$$

$$H = \int \frac{d^3k}{(2\pi)^3\, 2\omega(k)} \; \omega(k)\, |a(k)|^2 \, . \tag{64}$$

Today we'll study the (relativistic) complex scalar field and (the dynamics of) the electromagnetic field.
The first part is boring, but important: eventually we will couple electromagnetic fields to complex scalar fields (e.g. the Higgs field).

Suppose $\phi \in \mathbb{C}$. We can decompose

$$\varphi = \frac{1}{\sqrt{2}}\left(\varphi_1 + i\varphi_2\right) \tag{65}$$

and Fourier transform

$$\varphi(x,t) = \int dk\, N(k)\left[a(k)e^{ikx - i\omega t} + b^*(k)e^{-ikx + i\omega t}\right] \tag{66}$$

(where $a = \frac{1}{\sqrt{2}}(a_1 + ia_2)$, $\qquad b^* = \frac{1}{\sqrt{2}}(a_1^* + ia_2^*)$)

We can write $\mathscr{L}[\varphi_1]+\mathscr{L}[\varphi_2]$ as

$$\mathscr{L}=\frac{\partial\varphi^*}{\partial t}\frac{\partial\varphi}{\partial t}-\frac{\partial\varphi^*}{\partial x}\frac{\partial\varphi}{\partial x}-m^2\varphi^*\varphi\,. \tag{67}$$

with conjugates

$$\pi=\frac{\partial\mathscr{L}}{\partial\dot{\varphi}}=\frac{\partial\varphi^*}{\partial t}\,,\qquad \pi^*=\frac{\partial\mathscr{L}}{\partial\dot{\varphi}^*}=\frac{\partial\varphi}{\partial t}\,, \tag{68}$$

and hamiltonian density is

$$\mathscr{H}=\pi\dot{\varphi}+\pi^*\dot{\varphi}^*-\mathscr{L}=\pi^*\pi+\frac{\partial\varphi^*}{\partial x}\frac{\partial\varphi}{\partial x}+m^2\varphi^*\varphi\,. \tag{69}$$

In terms of Fourier components

$$H=\int dx\,\mathscr{H}\;=\;\int dk\,N(k)\,\omega(k)\left[|a(k)|^2+|b(k)|^2\right]\,. \tag{70}$$

Who cares? The complex scalar lagrangian density

$$\mathscr{L} = \frac{\partial \varphi^*}{\partial t}\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi^*}{\partial x}\frac{\partial \varphi}{\partial x} - m^2 \varphi^* \varphi\,. \tag{71}$$

is invariant under $\phi \mapsto e^{i\alpha}\phi$.

This can be promoted to a gauge symmetry by coupling to gauge fields.

Source of the basic interactions of Nature.

The electromagnetic field

We've already seen the *gauge potential* A_μ, with gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu f$.
It turns out that this (not E_i and B_i) is fundamental, *cf.* Aharonov-Bohm effect.
The latter are derived quantities. Indeed, the (*n.b.* gauge-invariant) *gauge field strength*

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

is s.t.

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Remark. Depending on how much of a pro you are, you can say F is

- an antisymmetric matrix
- a rank-2 antisymmetric tensor
- a differential form of degree 2
- the curvature of a connection on a principal $U(1)$ bundle over spacetime
- enough already. nobody likes a smart alec

Maxwell lagrangian

What is the lagrangian for electromagnetism?
Lorentz +Gauge invariance leaves only $\mathscr{L} = aF_{\alpha\beta}F^{\alpha\beta}$ With an external current J^μ we could have

$$\mathscr{L} = aF_{\alpha\beta}F^{\alpha\beta} - J^\mu A_\mu . \tag{72}$$

but only if $\partial_\mu J^\mu = 0$.

Maxwell's equations

The Euler-Lagrange equations for A_α are

$$\frac{\partial \mathscr{L}}{\partial A_\alpha} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\alpha)}\right) \tag{73}$$

The LHS gives the 'force' $-J^\alpha$. What about the RHS? Just need to use

$$\frac{\partial \partial_\mu A_\nu}{\partial \partial_\alpha A_\beta} = \delta_\mu^\alpha \delta_\nu^\beta$$

Elbow grease yields

$$J^\alpha + 4a\partial_\mu F^{\mu\alpha} = 0, \tag{74}$$

With $a = -1/4\mu_0$, these are just the inhomogeneous pair of Maxwell equations (recall here that $c^2 = 1/\varepsilon_0\mu_0$)

$$\mathrm{div}\mathbf{E} = \rho/\varepsilon_0\ , \qquad \mathrm{curl}\mathbf{B} = \varepsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t} + \mu_0\mathbf{J}\ .$$

The continuity equation comes from

$$\partial_\mu \partial_\nu F^{\mu\nu} = 0 = \partial_\nu J^\nu = \mu_0 \left(\frac{\partial \rho}{\partial t} + \mathrm{div}\mathbf{J} \right),$$

so charge is conserved, *cf.* gauge invariance.

What about the other 2 Maxwell equations? These are identities, following from $F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$. It follows that

$$\partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} = 0 \, ,$$

(the *Bianchi identity*). When written out explicitly, this gives $\mathrm{div}\mathbf{B} = \mathrm{curl}\mathbf{E} + \dot{\mathbf{B}} = 0$.

Remark. We learn that 2 of Maxwell's equations are not really equations at all. They are identities.

Gauge fixing

A nice gauge is the Lorenz gauge $\partial_\mu A^\mu = 0$. Not only is it Lorentz invariant, but Maxwell's eq'ns take the simple form $\partial_\nu \partial^\nu A_\mu = 0$.

Remark. If spacetime is topologically non-trivial (*e.g.* by deleting a worldline from $\mathbb{R}^4$, it is possible to find $F_{\alpha\beta}$ satisfying the Bianchi identity but not writable in the form $F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$.* We then have $\mathrm{div}\mathbf{B} = 0$ everywhere except on the line, where a source of magnetic charge is allowed. So there is a theoretical possibility of magnetic monopoles, first observed by Dirac. But none have actually been observed.

The pro says 'spacetime minus a line has non-trivial real cohomology in degree two.'

Lecture 8. Symmetries and Noether's Theorem

Last time we studied complex scalars and electromagnetic fields.

$$\mathscr{L} = \frac{\partial \varphi^*}{\partial t}\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi^*}{\partial x}\frac{\partial \varphi}{\partial x} - m^2 \varphi^* \varphi \,. \tag{75}$$

$$\mathscr{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - A_\alpha J^\alpha$$

Today we'll study symmetries and conservation laws, properly.
1st we'll prove a general theorem relating them; 2nd we'll apply it to $\phi \mapsto e^{-i\varepsilon}\phi$ (global and local).

Noether's Theorem

Suppose we have 1 field ϕ in $d = 1+1$.
Suppose $\mathscr{L}$ is invariant under $\phi \mapsto \phi + \delta\phi$. So

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathscr{L}}{\partial\varphi'}\delta\varphi' + \frac{\partial\mathscr{L}}{\partial\dot{\varphi}}\delta\dot{\varphi} = 0 \tag{76}$$

Using Euler-Lagrange

$$\frac{\partial\mathscr{L}}{\partial\varphi} - \frac{\partial}{\partial x}\left(\frac{\partial\mathscr{L}}{\partial\varphi'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\varphi}}\right) = 0 \tag{77}$$

get

$$\begin{aligned} 0 &= \frac{\partial}{\partial x}\left(\frac{\partial\mathscr{L}}{\partial\varphi'}\right)\delta\varphi + \frac{\partial\mathscr{L}}{\partial\varphi'}\frac{\partial}{\partial x}(\delta\varphi) + \frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\varphi}}\right)\delta\varphi + \frac{\partial\mathscr{L}}{\partial\dot{\varphi}}\frac{\partial}{\partial t}(\delta\varphi) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial\mathscr{L}}{\partial\varphi'}\delta\varphi\right) + \frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\varphi}}\delta\varphi\right) \end{aligned} \tag{78}$$

So

$$\frac{\partial}{\partial x}(J_x) + \frac{\partial \rho}{\partial t} = 0 \tag{79}$$

with

$$\rho = \frac{\partial \mathscr{L}}{\partial \dot{\varphi}} \delta \varphi\,, \qquad J_x = \frac{\partial \mathscr{L}}{\partial \varphi'} \delta \varphi\,. \tag{80}$$

A *continuity* or *current conservation* equation

With fields ϕ_j in multiple dimensions such that $\phi_j \mapsto \phi_j + \delta\phi_j$ sends $\mathscr{L}$ to itself, get

$$\partial_\mu J^\mu = 0$$

with

$$J^\mu = \sum_j \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi_j)} \delta\varphi_j \,. \tag{81}$$

Remarks.

- The am says the symmetry has to be continuous
- The pro says the symmetry has to be smooth (a *Lie group*).
- *cf.* $\phi \mapsto \phi + \varepsilon, \phi_i \mapsto R_{ij}\phi_j, \phi \mapsto e^{-i\varepsilon}\phi, \phi \mapsto -\phi$
- In general, $\mathscr{L}$ could shift by a total derivative
- Source *vs.* target symmetries
- *cf.* next lecture

Global phase invariance

$$\mathscr{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi$$

is invariant under

$$\begin{aligned} \varphi &\mapsto e^{-i\varepsilon}\varphi \simeq \varphi - i\varepsilon\varphi \\ \varphi^* &\mapsto e^{+i\varepsilon}\varphi^* \simeq \varphi^* + i\varepsilon\varphi^* \end{aligned} \tag{82}$$

global symmetry since ε is constant on source

The *Noether current* is

$$J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \delta\varphi + \delta\varphi^* \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi^*)} = -i(\partial^\mu \varphi^*)\varphi + i\varphi^*(\partial^\mu \varphi) \quad (83)$$

The *conserved charge* is

$$Q = \int \rho \, d^3 r \quad (84)$$

Indeed

$$\frac{dQ}{dt} = \int \frac{\partial \rho}{\partial t} d^3 r = -\int \nabla \cdot J \, d^3 r = -\int_{\infty\,\text{sphere}} J \cdot dS = 0\,. \quad (85)$$

Here

$$Q = -i \int \left(\frac{\partial \varphi^*}{\partial t} \varphi - \varphi^* \frac{\partial \varphi}{\partial t} \right) d^3 r\,. \quad (86)$$

In Fourier modes

$$Q = \int d^3k N(k) \left[|a(k)|^2 - |b(k)|^2 \right] \ . \tag{87}$$

cf. particles/antiparticles

Local phase invariance

With $\varepsilon = \varepsilon(x^\mu)$ we get

$$\partial^\mu \varphi \mapsto e^{-i\varepsilon(x)} \left[(\partial^\mu \varphi) - i(\partial^\mu \varepsilon)\varphi \right] \tag{88}$$

and $\mathscr{L} \mapsto$ a mess. But recall

$$A_\mu \to A_\mu + \partial_\mu \varepsilon / e \tag{89}$$

so the *covariant derivative*

$$D_\mu := \partial_\mu + ieA_\mu \tag{90}$$

is s.t.

$$D_\mu \varphi \mapsto [\partial_\mu + ieA_\mu + i(\partial_\mu \varepsilon)] e^{-i\varepsilon} \varphi = e^{-i\varepsilon} D_\mu \varphi \tag{91}$$

s.t.

$$\mathscr{L} = (D_\mu \varphi)^* (D^\mu \varphi) - m^2 \varphi^* \varphi \tag{92}$$

is *gauge invariant*.

Add dynamics for A_μ to get

$$\mathcal{L} = \mathcal{L}_{em} + \mathcal{L}_{KG} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^*(D^\mu\varphi) - m^2\varphi^*\varphi \tag{93}$$

Since

$$\varphi \to \varphi - ie\varepsilon\varphi\ , \qquad A_\mu \to A_\mu + \partial_\mu\varepsilon\ . \tag{94}$$

is a symmetry, it behoves us to find the Noether current. We have

$$J^\mu \propto -ie\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\varepsilon\varphi + ie\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\varepsilon\varphi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\partial_\nu\varepsilon\ . \tag{95}$$

1st 2 terms give

$$J^\mu_{KG} = ie[\varphi^*(D^\mu\varphi) - (D^\mu\varphi)^*\varphi] = ie[\varphi^*(\partial^\mu\varphi) - (\partial^\mu\varphi)^*\varphi] - 2e^2A^\mu\varphi^*\varphi\ . \tag{96}$$

3rd term gives

$$J^{\mu}_{em} \propto -F^{\mu\nu}\partial_\nu \varepsilon = -\partial_\nu (F^{\mu\nu}\varepsilon) + (\partial_\nu F^{\mu\nu})\varepsilon \tag{97}$$

Consider $\partial_\nu (F^{\mu\nu}\varepsilon)$. Its contribution to the charge is

$$\int d^3x^i \partial_0 (F^{00}\varepsilon) + \partial_i (F^{0i}\varepsilon) = 0 \tag{98}$$

So take

$$J^{\mu}_{em} = \partial_\nu F^{\mu\nu} \ , \tag{99}$$

Conserved on its own, so KG current is too.

Lecture 9. The Energy-Momentum Tensor

Last time we studied Noether's theorem

$$J^\mu = \sum_j \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi_j)} \delta\varphi_j \, . \tag{100}$$

and applied it to a scalar field coupled to electromagnetism

$$\mathscr{L} = \mathscr{L}_{em} + \mathscr{L}_{KG} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)^* (D^\mu \varphi) - m^2 \varphi^* \varphi \tag{101}$$

Mea Culpa

In deriving the conserved current for electromagnetism, I gave a tortuous argument why a term $J^\mu \supset \partial_\nu (F^{\mu\nu} \varepsilon)$ could be dropped.

A better and stronger argument is: $\partial_\mu \partial_\nu (F^{\mu\nu} \varepsilon) = 0$, by (anti)symmetry.

Sorry. I will try not to let it happen again.

Today we'll study the energy-momentum tensor: a set of conserved currents corresponding to symmetry under spacetime (*source*) translations $x^\mu \mapsto x^\mu + \varepsilon^\mu$.

The trick is to follow the effects through to the target
$x^\mu \mapsto x^\mu + \varepsilon^\mu \implies \varphi(x^\mu) \mapsto = \varphi + \varepsilon^\mu \partial_\mu \varphi$.
The resulting shift in $\mathscr{L}$ via ϕ is

$$\mathscr{L}(\varphi, \partial^\mu \varphi) \mapsto \mathscr{L}(\varphi, \partial^\mu \varphi) + \varepsilon_\mu \frac{\partial \mathscr{L}}{\partial \varphi} \partial^\mu \varphi + \varepsilon_\mu \frac{\partial \mathscr{L}}{\partial(\partial^\nu \varphi)} \partial^\mu \partial^\nu \varphi \quad (102)$$

so if $\mathscr{L}$ has no explicit dependence on x^μ we have

$$\frac{\partial \mathscr{L}}{\partial \varphi} \partial^\mu \varphi + \frac{\partial \mathscr{L}}{\partial(\partial^\nu \varphi)} \partial^\mu \partial^\nu \varphi = \partial^\mu \mathscr{L} \quad (103)$$

Using Euler-Lagrange, we get

$$\partial^\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial^\nu \varphi)} \partial^\mu \varphi \right) = \partial^\mu \mathscr{L} \quad (104)$$

or

$$\partial^\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial^\nu \varphi)} \partial^\mu \varphi - \delta^\mu_\nu \mathscr{L} \right) = 0. \quad (105)$$

Re-labelling, we find that the *energy-momentum* or *stress-energy tensor*

$$T^{\mu\nu} := \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi - g^{\mu\nu} \mathscr{L} \tag{106}$$

is conserved *i.e.*

$$\partial_\mu T^{\mu\nu} = 0 \tag{107}$$

e.g. 1

for waves in a 1-d rod

$$\mathscr{L} = \frac{1}{2}\rho(\dot{\varphi})^2 - \frac{1}{2}\kappa(\varphi')^2 \tag{108}$$

we get

$$\begin{aligned} T_{tt} &= \rho(\dot{\varphi})^2 - \mathscr{L} = \mathscr{H} \,, \qquad & T_{tx} &= -\rho\dot{\varphi}\varphi' \,, \\ T_{xx} &= \kappa(\varphi')^2 + \mathscr{L} = \mathscr{H} \,, \qquad & T_{xt} &= -\kappa\dot{\varphi}\varphi' \,. \end{aligned} \tag{109}$$

n.b. not symmetric

e.g. 2 for relativistic scalar

$$\mathscr{L} = \frac{1}{2}(\partial^\mu\varphi)(\partial_\mu\varphi) - \frac{1}{2}m^2\varphi^2 \,.$$

we get

$$T^{\mu\nu} = (\partial^\mu\varphi)(\partial^\nu\varphi) - g^{\mu\nu}\mathscr{L} \tag{110}$$

n.b. symmetric

e.g. 3

for electromagnetism

$$\mathscr{L} = -\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = -\tfrac{1}{4}g^{\alpha\gamma}g^{\beta\delta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\gamma A_\delta - \partial_\delta A_\gamma) \quad (111)$$

we get

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\lambda)}\partial^\nu A_\lambda - g^{\mu\nu}\mathscr{L} \\ &= -F^{\mu\lambda}\partial^\nu A_\lambda + \tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\ &= -F^{\mu}_{\lambda}\partial^\nu A^\lambda + \tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\ . \end{aligned} \quad (112)$$

n.b. neither symmetric nor gauge-invariant!

Fix 1: Add $\partial_\lambda \Omega^{\lambda\mu\nu}$ where $\Omega^{\lambda\mu\nu} = -\Omega^{\mu\lambda\nu}$ (for then $\partial_\mu \partial_\lambda \Omega^{\lambda\mu\nu} = 0$). Choosing $\Omega^{\lambda\mu\nu} = -F^{\lambda\mu} A^\nu$ gives

$$T^{\mu\nu} = -F^{\mu}{}_{\lambda} F^{\nu\lambda} + \tfrac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \ , \tag{113}$$

But why?!

Fix 2: Work out $T^{\mu\nu}$ for a combination of a translation and a gauge transformation f and pick a suitable f.
Better.

Fix 3. Couple to gravity *i.e.* general relativity which is co-ordinate invariant and has dynamical metric tensor $g^{\mu\nu}$. The action becomes

$$S = \int d^4x \sqrt{-g}\,\mathcal{L}\ . \tag{114}$$

and we define

$$\delta S = \frac{1}{2}\int d^4x \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu}\ , \tag{115}$$

i.e.

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} = 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + \frac{1}{g}\frac{\partial g}{\partial g^{\mu\nu}}\mathcal{L}\ . \tag{116}$$

or

$$T_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}\ . \tag{117}$$

This is manifestly symmetric.

Angular momentum

When $T_{\mu\nu}$ is conserved and symmetric, we get another conserved tensor for free! Indeed

$$M^{\lambda\mu\nu} = x^{\mu} T^{\lambda\nu} - x^{\nu} T^{\lambda\mu} \tag{118}$$

is s.t.

$$\partial_{\lambda} M^{\lambda\mu\nu} = T^{\mu\nu} - T^{\nu\mu} = 0 \tag{119}$$

We define the *total angular momentum tensor*

$$J^{\mu\nu} = \int d^3 r M^{0\mu\nu} \,. \tag{120}$$

A covariant definition of the spin is via the *Pauli-Lubanski vector*

$$S^{\mu} = -\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} U_{\nu} J_{\alpha\beta} \tag{121}$$

since $U^{\mu} = (1,0,0,0) \implies S^i = J^i$.

Lecture 10. Broken Symmetry

Last time we studied the stress energy tensor, corresponding to $x^\mu \mapsto x^\mu + \varepsilon^\mu$

Today we'll study spontaneous symmetry breaking, both global and local.
(It explains the origin of all particle masses bar one, so it's quite important really.)
Really we need quantum field theory, but we'll see that we can get quite far with a bit of 'winging it'.

So far we only allowed up to 2 fields or derivatives. But suppose we add a quartic term to the complex KG field

$$\mathscr{L} = (\partial^\mu \varphi^*)(\partial_\mu \varphi) - m^2 \varphi^* \varphi - \frac{1}{2}\lambda (\varphi^* \varphi)^2 \,. \tag{122}$$

We need $\lambda > 0$ for boundedness of the energy.
Indeed the Hamiltonian density is

$$\mathscr{H} = \pi^* \pi + \nabla \varphi^* \cdot \nabla \varphi + V(\varphi) \tag{123}$$

where the 'potential' is

$$V(\varphi) = m^2 \varphi^* \varphi + \frac{1}{2}\lambda (\varphi^* \varphi)^2 \,. \tag{124}$$

But now we could take $m^2 < 0$. So what happens if $m^2 \mapsto -m^2$?

$$V(\varphi) = -m^2 \varphi^* \varphi + \frac{1}{2}\lambda (\varphi^* \varphi)^2 \,. \tag{125}$$

There are many ground states (with $\varphi_0^* \varphi_0 = \frac{m^2}{\lambda}$)!

The phase symmetry carries the ground states into one another.
But the vacuum must pick one: the symmetry is broken in any ground state.
Pick one, say $\varphi_0 = m/\sqrt{\lambda}$ and expand

$$\varphi = \varphi_0 + \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2)$$

Get

$$\mathscr{L} = \frac{1}{2}(\partial^\mu \chi_1)(\partial_\mu \chi_1) + \frac{1}{2}(\partial^\mu \chi_2)(\partial_\mu \chi_2) - V(\varphi_0) - m^2\chi_1^2 - 0\chi_2^2 + \mathscr{O}(\chi^3) \tag{126}$$

$$\mathscr{L} = \frac{1}{2}(\partial^\mu \chi_1)(\partial_\mu \chi_1) + \frac{1}{2}(\partial^\mu \chi_2)(\partial_\mu \chi_2) - V(\varphi_0) - m^2 \chi_1^2 - 0\chi_2^2 + \mathscr{O}(\chi^3) \tag{127}$$

Remarks

- asymmetric in $\chi_{1,2}$
- χ_2 has 'right sign' quadratic term
- χ_2 has dispersion relation $\omega = \sqrt{k^2 + 2m^2}$, so its quanta are particles of mass $\sqrt{2}m$
- χ_1 has dispersion relation $\omega = \sqrt{k^2}$, so its quanta are massless particles

The appearance of a massless particle is no accident. It corresponds to fluctuations along the vacua. *Goldstone's theorem* says that there will be one such *Goldstone boson* for each* symmetry that is broken. (In fact this is not quite true for broken spacetime symmetries, e.g. phonons.)

*What does 'each' mean? Well a Lie group is smooth, which means that it can be described by a 'manifold'. A manifold is a smooth space which near any point looks like $\mathbb{R}^n$ for some n. If the Lie group G looking like $\mathbb{R}^n$ is broken to a subgroup $H \subset G$ looking like $\mathbb{R}^m$, there will be $n-m$ Goldstone bosons.

Gauge symmetry breaking

Recall that adding A^μ allows a local or gauge symmetry with

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu \varphi)^*(D^\mu \varphi) - V(\varphi) \tag{128}$$

where $D_\mu = \partial_\mu + ieA_\mu$.
V is unchanged, so we still get breaking with $\varphi = \varphi_0 + \ldots$.
But now there are extra terms from the covariant derivatives, *e.g.*

$$\mathscr{L} \supset (ie\varphi_0 A_\mu)^*(ie\varphi_0 A^\mu) = \frac{e^2 m^2}{\lambda} A_\mu A^\mu \tag{129}$$

In Lorenz gauge, $\partial_\mu A^\mu = 0$, the Euler-Lagrange equation for A_μ is

$$\partial_\nu \partial^\nu A_\mu + 2\frac{e^2 m^2}{\lambda} A_\mu = 0 \,. \tag{130}$$

so the dispersion relation for A^μ becomes

$$\omega = \sqrt{k^2 + 2e^2 m^2/\lambda} \tag{131}$$

with the quanta being particles with a mass $\mu = em\sqrt{2/\lambda}$!
These correspond to short-ranged forces $\sim e^{-\mu r}/r^2$.
cf. beta decays and the weak nuclear force.

But there is a mystery here: light has 2 polarizations (or a photon has 2 helicitystates), but a massive vector particle has 3 spin states. Surely it is not possible to discontinuously change the number of d.o.f. by continuously changing m^2?!

The resolution of the mystery is that the Goldstone boson χ_2 is no longer present! To see this note

$$D_\mu \varphi = \frac{1}{\sqrt{2}} \left(\partial_\mu \chi_1 + i \partial_\mu \chi_2 \right) + ie\varphi_0 A_\mu + \dots \qquad (132)$$

χ_2 can be removed from the theory by the gauge transformation

$$A_\mu \mapsto A_\mu - \frac{1}{\sqrt{2} e \varphi_0} \partial_\mu \chi_2 \qquad (133)$$

Note this trick doesn't work for $\varphi_0 = 0$ and nor does it work for χ_1. So $2+2=3+1$. Phew!

More generally, we get a massive gauge boson for ‘each’ broken gauge symmetry.

In the Standard Model of the electroweak interactions, there are 4 gauge bosons, corresponding to the Lie group $SU(2) \times U(1)$.* This is broken to a $U(1)$ subgroup corresponding to electromagnetism, so there are 3 massive ($W^{\pm}, Z$) and 1 massless (γ) gauge bosons. The photon was discovered some time ago, the $W^{\pm}, Z$ were discovered in the 80’s, and the Higgs boson (the analogue of χ_1) was discovered at the LHC in 2012. We’ve been partying ever since.

*$SU(2)$ is just the group of 2×2 unitary matrices with determinant one, and $U(1)$ is just the group of 1×1 unitary matrices, *i.e.* the complex numbers of modulus 1. So there is nothing too mysterious here. They are Lie groups, and as manifolds they correspond to the 3-dimensional sphere and the circle, respectively.

Lecture 11. The Ising Model and Mean Field Theory

Last time we studied spontaneous symmetry breaking in dynamical theories, *e.g.* electroweak interactions.

Today we'll study symmetry breaking in systems in (thermal) equilibrium and phase transitions more generally.

(The course has 2 parts, remember.)

Introduction

Most real world systems have many particles.
Studying them as such is nigh-on impossible and obviously wrong.
It's better to take a step back and describe them via long-range, effective (a.k.a. 'coarse-grained) descriptions.
e.g. hydrodynamics.
Fits with the worldview, and also leads us back to fields.

Though complicated microscopically, systems often exhibit simple but striking behaviours.

e.g. phases and phase transitions.

Singular nature of phase transitions a hallmark of the many-particle ($n \to \infty$)/thermodynamic limit, *cf.* $e^{-1/T}$.

Moreover, a degree of universality is apparent between systems.

We’ll start today with 2 microscopic models of ‘ferromagnets’: the Ising model and the Heisenberg model.
By means of a coarse approximation – mean field theory – we’ll get analytic solutions.

Ferromagnets

Model ferromagnet as a lattice of spins (with associated magnetic moment);

Below a critical temperature T_c these line up giving macroscopic spontaneous magnetization M.

The direction is arbitrary, *cf.* spontaneous symmetry breaking.

Second-order phase transition

We can also apply an external magnetic field B.
First-order phase transition

Ising Model

The 'source' is a spatial lattice (dimension d) and the 'fields' are spins taking values in ± 1. The hamiltonian is

$$H = -\frac{J}{2} \sum_{i,\delta} s_i s_{i+\delta} - \mu \sum_i s_i B. \tag{134}$$

The partition function is $Z = \sum_{\{s_i\}} e^{-\beta H(\{s_i\})}$.
We can't solve this for $d \geq 3$.

We can solve

$$H_0 = -\mu \sum_i s_i B. \tag{135}$$

Indeed

$$Z_0 = \left[e^{\beta b} + e^{-\beta b}\right]^N, \tag{136}$$

with $b = \mu B$, and

$$\langle s_i \rangle = \frac{e^{\beta b} - e^{-\beta b}}{e^{\beta b} + e^{-\beta b}} = \tanh(\beta b). \tag{137}$$

Mean field theory

We therefore expand $s_i = s + (s_i - s)$ and take $(s_i - s)$ to be 'small'. We get

$$H \supset -(Js/2)(\sum_{i,\delta} s_i + \sum_{i,\delta} s_{i+\delta}) = -2dJs \sum_i s_i.$$

So we must have

$$s = \tanh\left[\beta(b + 2dJs)\right] \tag{138}$$

Consider $b = 0$. Set $\varepsilon = 2d\beta Js$ s.t.

$$\varepsilon = 2d\beta J \tanh \varepsilon, \tag{139}$$

We get a 2nd order phase transition at $T_c = T_c = 2dJ/k_B$.
Just below T_c, we can expand in ε to get

$$s = \pm\sqrt{3}\,(-t)^{\frac{1}{2}} + \dots\ , \tag{140}$$

where the *reduced temperature* is $t := T/T_c - 1$.
(In general, $M(B \to 0) \propto \pm|t|^{\beta}$ and we have *critical exponent* β.)

The *magnetic susceptibility* $\chi := dM/dB(B \to 0) \propto |t|^{-\gamma}$. Here we get

$$\frac{ds}{db} = \operatorname{sech}^2 [\beta(b + 2Jds)] \left(\beta + \frac{T_c}{T} \frac{ds}{db} \right), \tag{141}$$

s.t. for $t > 0$

$$\chi = \frac{1}{k_B T_c} t^{-1}, \tag{142}$$

For $t < 0$, $s \neq 0$; get

$$\chi = \frac{1}{2 k_B T_c} (-t)^{-1} \tag{143}$$

(by expanding $s = \tanh \beta(b + s/\beta_c)$. Ugh!)

We also have the *critical isotherm* $M(T=T_c) \propto B^{1/\delta}$. Here we get

$$s(T=T_c, b) \simeq \left(\frac{3b}{k_B T_c}\right)^{1/3} \tag{144}$$

(from $s \simeq (\beta_c b + s) - \frac{1}{3}(\beta_c b + s)^3 + \ldots$ and neglecting b, b^2).

<rant>

Hey condensed matter people! Sort your notation and units out! (*e.g.* $k_B = 1$ would be a good start.) </rant>

Heisenberg ferromagnet

Now do it all over again, *mutatis mutandis*, for the (slightly) more realistic *Heisenberg ferromagnet*. Now the 'fields' are maps $\mathbb{Z}^d \to S^2$ (so $\mathbf{s}_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$) and

$$H = -\frac{J}{2}\sum_{i,\delta}\mathbf{s}_i \cdot \mathbf{s}_{i+\delta} - B\sum_i \hat{z}\cdot\mathbf{s}_i\,. \tag{145}$$

We can assume $\langle \mathbf{s}_i \rangle = s\hat{z}$ w.l.o.g.

The mean field approximation now yields

$$H = -\left(2dJs + B\right)\sum_i \hat{z}\cdot \mathbf{s}_i$$

so

$$\begin{aligned} Z = \sum_{\{s_i\}} e^{-\beta H(\{s_i\})} &\simeq \left[\int d\cos\theta \, d\phi \, e^{(2d\beta Js+\beta B)\cos\theta}\right]^N \\ &= (2\pi)^N \left[\int_{-1}^{1} dx\, e^{(2d\beta Js+\beta B)x}\right]^N . \end{aligned}$$

and ($\varepsilon := 2d\beta Js$ remember)

$$\begin{aligned} \langle s_i \cdot \hat{z}\rangle &= \frac{(2\pi)^N}{Z}\left[\int_{-1}^{1} dx\, e^{(2d\beta Js+\beta B)x}\right]^{N-1}\left[\int_{-1}^{1} dx\, x\, e^{(2d\beta Js+\beta B)x}\right] \\ &= \frac{\int_{-1}^{1} dx\, x\, e^{(2d\beta Js+\beta B)x}}{\int_{-1}^{1} dx\, e^{(2d\beta Js+\beta B)x}} = -\frac{1}{\varepsilon+\beta B} + \coth\left(\varepsilon+\beta B\right) , \end{aligned} \tag{146}$$

So our master equation is

$$\varepsilon = 2d\beta J\left[-\frac{1}{\varepsilon+\beta B}+\coth\left(\varepsilon+\beta B\right)\right]. \tag{147}$$

Again, we expand at $B = 0$

$$\varepsilon = \frac{T_c}{T}\left(\varepsilon - \frac{1}{15}\varepsilon^3 + \dots\right), \tag{148}$$

where $T_C := 2dJ/3k_B$, s.t.

$$s = \pm\sqrt{\frac{5}{3}}\left(-t\right)^{\frac{1}{2}} + \dots \tag{149}$$

so the critical exponent $\beta = \frac{1}{2}$, again.

For the susceptibility $\chi := \frac{ds}{dB}|_{B=0}$ we get

$$\chi = \begin{cases} \frac{1}{3k_B T_c} t^{-1} \\ \frac{1}{9k_B T_c} (-t)^{-1} \end{cases}$$

So the susceptibility exponent $\gamma = 1$.
(You get this by expanding $s = -\frac{1}{2d\beta Js + \beta B} + \coth(2d\beta Js + \beta B)$, differentiating w.r.t. B, setting $B = 0$, and plugging in either $s = 0$ or $s = \pm\sqrt{\frac{5}{3}}(-t)^{\frac{1}{2}}$.)

For the critical isotherm we set $t = 0$ and expand

$$s \simeq \frac{3s + \beta_c B}{3} - \frac{(3s + \beta_c B)^3}{45} + \cdots \implies s(T = T_c, B) \simeq \left(\frac{5B}{9k_B T_c}\right)^{1/3}, \tag{150}$$

so $\delta = 3$

Remarks

- DIY - it's as easy as a, b, c (or s, b, t)
- The estimated exponents come out the same, because of mean field theory
- But often the exponents really are the same, because of *universality*
- *e.g.* Ising *vs.* liquid-gas: magnetisation *vs.* density; susceptibility *vs.* compressibility
- At critical points, fluctuations on all lengths are important
- Conformal field theory: the worldview wins again!

Lecture 12. Landau-Ginzburg Theory

Last time we studied phase transitions, solving microscopic models (Ising and Heisenberg ferromagnets) using mean field theory.

Today we'll study a different approach: Landau-Ginzburg theory.
It'll still be mean field theory, but we'll start from macroscopics (and symmetries).
(It's also the starting point for going beyond mean field theory.)

This works because near a critical point, there are correlations on large length scales. So microscopic details are unimportant.

The basic idea is to describe systems via their symmetries. Whatever symmetries the microscopic dynamics has, the macroscopic dynamics should have too.
(There might even be more, *emergent* symmetries.)
And of course, some symmetries might be spontaneously broken in the preferred state.

Order Parameters

But to give this teeth, we need to decide on what the important long range degrees of freedom (the fields) are and how the symmetries act on them.
In systems with symmetry breaking, we have multiple, macroscopically distinguishable equilibrium states, connected by the symmetry.
We parameterise these with some *order parameter(s)*, which inherit an action of the symmetry.
Note that the order parameter measures both the direction and size of the breaking.

Examples

e.g. 1 In the Ising model, we have the average magnetization/spin, which takes values $m \in \mathbb{R}$. The symmetry which flips spins sends $m \mapsto -m$.

e.g. 2 In the Heisenberg model, we have the average magnetization/spin *vector*, which takes values $\mathbf{m} \in \mathbb{R}^3$. The symmetry which rotates spins rotates **m**.

e.g. 3 In a superconductor, the order parameter is $\phi \in \mathbb{C}$ and the electromagnetic gauge transformation rephases ϕ.

It's convenient to allow the order parameter to vary spatially (on macroscopic scales).
(Allows us to study responses to localized perturbations, transport phenomena, and thermal fluctuations.).
So we have fields, *i.e.* maps from a source (position space) to a target (the space of order parameters). But now we don't extremize the lagrangian; rather we minimize the *free energy.* F.
But the mathematics is the same!

But what is the free energy F?
The most general function of order parameters and derivatives compatible with the symmetries, of course!

Ising Model

For the Ising Model, we have symmetry $m \mapsto -m$ (as well as translation and rotation invariance in x_i), so

$$f(m,T) = f_0(T) + \alpha(T)m^2 + \frac{1}{2}\beta(T)m^4 + \gamma(T)\nabla m \cdot \nabla m + \dots \quad (151)$$

with $\beta, \gamma > 0$. Minimizing f, we get that $\nabla m = 0$, so $m(x_i) = \overline{m}$ and

$$\alpha\overline{m} + \beta\overline{m}^3 = 0. \quad (152)$$

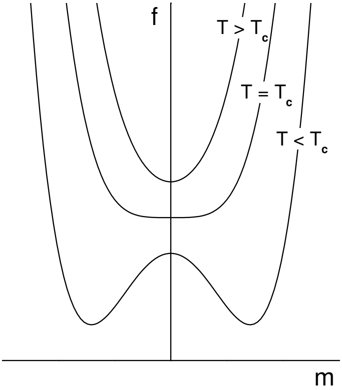
The minima are at

$$\overline{m} = \pm\sqrt{-\alpha/\beta} \quad \text{for} \quad \alpha < 0, \qquad \text{and} \qquad \overline{m} = 0 \quad \text{for} \quad \alpha > 0. \quad (153)$$

so we want $\alpha(T) \simeq a(T - T_c) + \dots, \beta(T) \simeq b + \dots, \gamma(T) \simeq c + \dots,$ and

$$\overline{m} \simeq \pm\left(\frac{a}{b}\right)^{\frac{1}{2}} \sqrt{(T_c - T)} \quad \text{for} \quad T < T_c. \quad (154)$$

$$f(m,T) \simeq f_0(T) + a(T-T_c)m^2 + \frac{1}{2}bm^4 \tag{155}$$



Can we reproduce our other critical exponents? With a magnetic field $m \mapsto -m$ is explicitly broken. At leading order

$$f(m,T,B) \simeq f_0(T) + a(T-T_c)m^2 + \frac{1}{2}bm^4 + c\nabla m \cdot \nabla m - mB. \quad (156)$$

For susceptibility, get (for $T > T_c$)

$$\chi = \left.\frac{d\overline{m}}{dB}\right|_{B=0} = \frac{1}{2a}(T-T_c)^{-1}. \quad (157)$$

For critical isotherm, get

$$\overline{m} = \left(\frac{1}{2b}\right)^{1/3} B^{1/3}. \quad (158)$$

So the various critical exponents are unchanged.

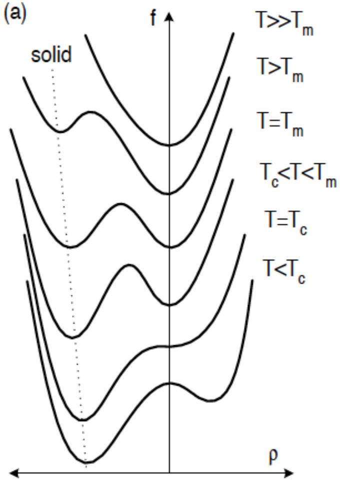
First-order transitions

We can even describe 1st order transitions in this framework.
e.g. solid-liquid transition.
The order parameter is the density, but there is no symmetry. So

$$f = f_0 + a(T - T_c)\rho^2 + c\rho^3 + \frac{b}{2}\rho^4\,,$$

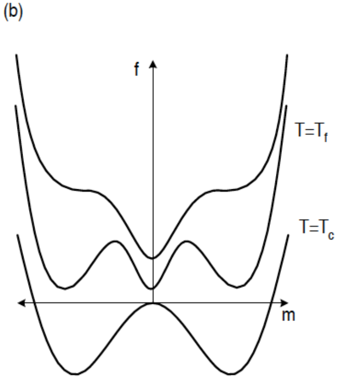
Solid-liquid transition

$$f = f_0 + a(T - T_c)\rho^2 + c\rho^3 + \frac{b}{2}\rho^4\,,$$



With symmetry

$$f = f_0 + a(T - T_c)m^2 - \frac{b}{2}m^4 + \frac{c}{3}m^6,$$



Remarks

- The Landau-Ginzburg picture is unlikely to be accurate for 1st order transitions
- Even for 2nd order, still get mean field theory
- Why include ∇m? $B(x)$
- Fluctuations: beyond mean field theory.

$$\frac{\partial}{\partial}$$

Lecture 13. Propagators for Particles

Last time we studied phase transitions using the Landau-Ginzburg theory.

Today we'll go back to studying dynamical fields. We'll look at the issue of *causality*. We'll start by going back to particle mechanics, i.e. 1-d field theories.

Causality is the notion that causes precede effects.
It certainly seems to be the case in Nature, and moreover it's hard to imagine what life would be like without it!

So far, the only thing we've done is to insist that time has a direction (so we can define $\int dt$.

This doesn't solve the problem of causality; rather it defines it!

Start with a simple system – a *lightly damped* harmonic oscillator – and a simple cause – a delta function impulse at time $t = t'$.

What evolution results? A Green function.

Damped SHO

(This example is a bit naughty, because there's no lagrangian).
The equation of motion is

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \phi(t). \tag{159}$$

The Green function is the solution with a δ-function source

$$\left[\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + \omega_0^2\right] G(t-t') = \delta(t-t')$$

In Fourier space

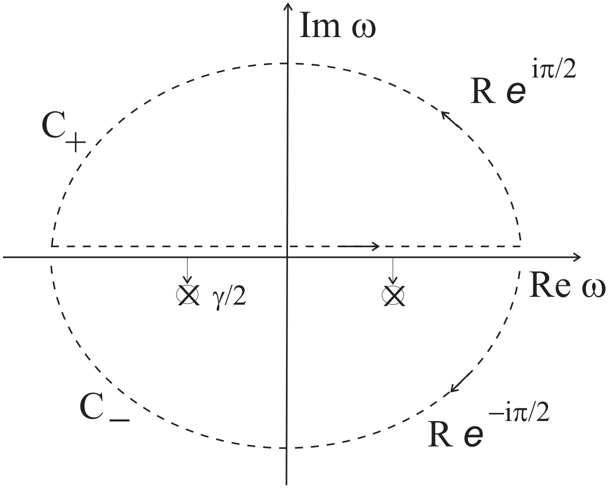
$$\left[-\omega^2 - i\gamma\omega + \omega_0^2\right] G(\omega) = 1\,, \quad \text{for } G(t-t') = \int G(\omega) e^{-i\omega(t-t')}\,\frac{d\omega}{2\pi}\,. \tag{160}$$

Transforming back, we get

$$G(t-t') = \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t')}}{\omega_0^2 - i\omega\gamma - \omega^2} \frac{d\omega}{2\pi} \tag{161}$$

We compute this via contour integration. There are simple poles in the lower half plane (since $\gamma^2/4 < \omega_0^2$) at

$$\omega_{1,2} = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}.$$



Im ω
R e^{iπ/2}
C+
Re ω
γ/2
C−
R e^{−iπ/2}

For $t < t'$, we close in the UHP, to get

$$G(t-t') = 0$$

For $t > t'$, we close in the LHP and use the residue theorem to get

$$\begin{aligned} G(t-t') &= -2\pi i \left[Res(\omega = \omega_1) + Res(\omega = \omega_2)\right] & (162) \\ &= \frac{i}{\omega_1 - \omega_2}\left[e^{-i\omega_1(t-t')} - e^{-i\omega_2(t-t')}\right] \\ &= \frac{1}{\sqrt{\omega_0^2 - \gamma^2/4}} e^{-(\gamma/2)|t-t'|} \sin\sqrt{\omega_0^2 - \gamma^2/4}\,(t-t') & (163) \end{aligned}$$

So causes precede effects (and causes have local influence).

Errrr, hang on a minute …

To see that there is a problem, consider the *un*damped oscillator, with $\gamma = 0$. The Fourier transformed Green function is

$$\left[-\omega^2 + \omega_0^2 \right] G(\omega) = 1 \qquad \text{for} \;\; G(t-t') = \int G(\omega) e^{-i\omega(t-t')} \, \frac{d\omega}{2\pi}. \tag{164}$$

This has simple poles on the real axis at $\omega = \pm\omega_0$. So the integral is undefined!

This is just as well, because a Green function is undefined as well. Or rather, it is not *uniquely defined*.
Here is the 'definition' again

$$\left[\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + \omega_0^2\right] G(t-t') = \delta(t-t')$$

So G is the solution of a 2nd order ODE, to which I haven't applied any boundary conditions. There are lots of such solutions.
In particular, I can add a solution to the homogeneous equation, and get another solution.

Now, we see how to enforce causality: we do it via the boundary conditions. In particular, our 'boundary condition' will be that $G(t-t') = 0$ for $t < t'$.

How to enforce this BC? It's just the SHO, so away from $t = t'$, we know that $G = A \sin \omega_0 (t - t') + B \cos \omega_0 (t - t')$. We want $G = 0$ at $t = t'$ and the derivative should jump by 1, so we have

$$G(t - t') = \begin{cases} 0, t < t' \\ \frac{1}{\omega_0} \sin \omega_0 (t - t'), t > t' \end{cases} \tag{165}$$

This coincides with the limit $\gamma \to 0$ of

$$\frac{1}{\sqrt{\omega_0^2 - \gamma^2/4}} e^{-(\gamma/2)|t-t'|} \sin \sqrt{\omega_0^2 - \gamma^2/4}\, |t - t'|.$$

which suggests another way to get the Green function. Just work out a prescription for going around the poles in the Fourier transform.

Here we clearly need to go above both poles. And indeed, we then get

$$
\begin{aligned}
G(t-t') &= -2\pi i\left[Res(\omega=\omega_1)+Res(\omega=\omega_2)\right] &(166)\\
&= \frac{i}{\omega_1-\omega_2}\left[e^{-i\omega_1(t-t')}-e^{-i\omega_2(t-t')}\right] \\
&= \frac{1}{\omega_0}\sin\omega_0|t-t'|. &(167)
\end{aligned}
$$

Next time we'll extend this to study causality in field theories.

But why the heck did I get a unique Green function for the damped harmonic oscillator?

See if you can figure it out for yourself. If not, I'll tell you in the next lecture.

Lecture 14. Propagators for Fields

Last time we studied propagators, a.k.a causal Green functions, for particles (*i.e.* x(t)).

Today we'll do the same for fields.

But first we need to clear up a mystery. We argued that Green f'ns are not unique, because we can add any sol'n to the homogeneous eq'n and get another Green f'n.
But for the damped SHO, we apparently found a unique Green f'n.
This is not good.

Going back to basics, the sol'ns of the homogeneous equation are

$$
\begin{aligned}
A\frac{1}{\sqrt{\omega_0^2-\gamma^2/4}}e^{-(\gamma/2)(t-t')}\sin\sqrt{\omega_0^2-\gamma^2/4}\,(t-t')& \\
+B\frac{1}{\sqrt{\omega_0^2-\gamma^2/4}}e^{-(\gamma/2)(t-t')|}\cos\sqrt{\omega_0^2-\gamma^2/4}\,(t-t')& \quad (168)
\end{aligned}
$$

These are not integrable or square integrable unless $A=B=0$.

As result there is no theorem that the inverse Fourier transform exists!
cf. the Legendre transform, which only exists for convex/concave functions.
These things actually matter!

The most exciting phrase to hear in science, the one that heralds new discoveries, is not “Eureka" but “That’s funny …”.
– Isaac Asimov (1920-1992)

As an exercise, see if anything “funny” happens when you increase γ from light to critical to heavy damping …

Let's get back to field theory propagators. *e.g.* the Schroedinger-like equation

$$i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi=F(x,t) \tag{169}$$

Here $F(x,t)$ is some kind of 'interaction'. As usual, we have

$$\psi(x,t)=\int_{t>t'}dt'\int dx' G(x,x';t,t')F(x',t'). \tag{170}$$

with

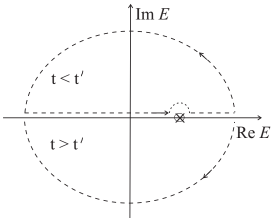
$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\right)G(x,x';t,t')=\delta(x-x')\delta(t-t').$$

Taking the Fourier transform (but with $\exp[-ipx/\hbar]$ and $\exp[-iEt/\hbar]$), we get

$$\left(i\hbar\frac{\partial}{\partial t}-\frac{p^2}{2m}\right)G(p;t,t')=\delta(t-t')$$
$$\left(E-\frac{p^2}{2m}\right)G(p;E)=1$$

$$G(p;t,t') = \int e^{-i\frac{E}{\hbar}(t-t')} \frac{1}{E - p^2/2m} \frac{dE}{2\pi\hbar}$$

with a single pole at $E = p^2/2m$.
(*n.b.* don't try to do the p integral first!)
For $t > t'$, we need to close in the LHP.
So for a causal BC, the contour should go above the pole.



Im E
$t < t'$
$t > t'$
Re E

$$G(p;t,t') = \int e^{-i\frac{E}{\hbar}(t-t')} \frac{1}{E-p^2/2m} \frac{dE}{2\pi\hbar}$$

so we get

$$G(p;t>t') = -2\pi i \left[\frac{1}{2\pi\hbar} e^{-i\frac{t-t'}{\hbar}(p^2/2m)} \right] = -\frac{i}{\hbar} e^{-i\frac{p^2}{2m\hbar}(t-t')}$$

and (in $d=1$)

$$G(x,x';t,t') \propto \frac{1}{\sqrt{t-t'}} e^{-\frac{im(x-x')^2}{2\hbar(t-t')}}$$

Note that this is invariant under the galilean group.

It's common to use the 'pole-moving trick' or '$i\varepsilon$ prescription to move the pole off the real line, s.t. $E \to E + i\varepsilon$ and we get

$$G(p;E) \to \frac{1}{E - p^2/2m + i\varepsilon} \,. \tag{171}$$

which gives the same result.

What about relativistic theories, *e.g.* the Klein-Gordon equation? Now we get

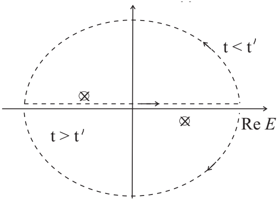
$$G(x^\mu) = \int d^4p^\nu \frac{e^{-ip_\nu x^\nu}}{p^\mu p_\mu - m^2}$$

As before , we get the causal propagator by sending $p^0 \to p^0 + i\varepsilon$. It vanishes for $x^0 < 0$ and by Lorentz invariance it vanishes for all spacelike x^μ.

Amazingly, it is not the right propagator for doing quantum field theory!

We need the Feynman propagator

$$G_F(x^\mu) = \int d^4p^\nu \frac{e^{-ip_\nu x^\nu}}{p^\mu p_\mu - m^2 + i\varepsilon}$$



The propagator decays exponentially outside the light cone.
This is not inconsistent with causality. The question is not whether fields can propagate, but whether quantum measurements can effect one another. They can't if the observables commute. So the relevant propagator corresponds to the commutator, which vanishes.
For a complex scalar theory, this corresponds to cancellation of contributions from particles and anti-particles.

Lecture 15. Linear Response

Last time we studied propagators.

Today we'll study the related topic of linear response and derive the Kramers-Kronig relations.

Consider first a harmonic oscillator with position $x(t)$, potential $V(x) = \frac{1}{2}\kappa x^2$ and equilibrium at $x = 0$. Apply a small force f, i.e. a potential $V \sim -fx$. The system responds by shifting its equilibrium by an amount $\propto f$. To wit $x = \frac{1}{\kappa} f$ and we define the *static linear response* to be $\frac{1}{\kappa}$.

More generally, we can ask for the response to a small time-dependent force. We know how to find this using the (causal!) Green function. We have

$$x(t) = \int_{-\infty}^{\infty} G(t-t')f(t')\,dt' \tag{172}$$

We can do this for all sorts of systems and observables, *e.g.* position/force, charge/voltage, polarization/electric field, magnetisation/magnetic field, pressure/volume, volume/pressure. For a general pair (u, f), we define the *linear response* or *generalised susceptibility* by

$$u(t) = \int_{-\infty}^{\infty} \alpha(t - t') f(t')\, dt' \tag{173}$$

In frequency space we have

$$u(\omega) = \alpha(\omega) f(\omega) \implies \alpha(\omega) = u(\omega)/f(\omega)$$

In words $\alpha(\omega)$ is the response to a sinusoidal perturbation of frequency ω.

Causality, *i.e.* $\alpha(t-t') = 0$ for $t < t'$ has consequences for $\alpha(\omega) \in \mathbb{C}$. Indeed, knowing either the real or the imaginary part determines the other.

Kramers-Kronig relations

Write $\alpha(\omega) = \alpha'(\omega) + i\alpha''(\omega)$ and $v(t)$, odd in t, s.t. $\alpha(t) = \Theta(t)v(t)$. Since this is a product, the Fourier transform is the convolution

$$\alpha(\omega) = \int_{-\infty}^{\infty} \Theta(\omega - \omega_1) v(\omega_1) \frac{d\omega_1}{2\pi}. \tag{174}$$

But what is the Fourier transform of a step function? Well,

$$\begin{aligned}
\Theta(\omega) &= \int_{-\infty}^{\infty} \Theta(t) e^{i\omega t} dt = \left(\int_0^{\infty} e^{i\omega t - \varepsilon t} dt \right) \bigg|_{\varepsilon \to 0} \\
&= \frac{1}{\varepsilon - i\omega} \bigg|_{\varepsilon \to 0} = \frac{\varepsilon}{\omega^2 + \varepsilon^2} \bigg|_{\varepsilon \to 0} + \frac{i\omega}{\omega^2 + \varepsilon^2} \bigg|_{\varepsilon \to 0} \\
&= \pi \delta(\omega) + \mathsf{P} \frac{i}{\omega}
\end{aligned}$$

So we get

$$\alpha(\omega) \equiv \alpha' + i\alpha'' = \int_{-\infty}^{\infty} \left(\pi\,\delta(\omega - \omega_1) + \mathsf{P}\frac{i}{\omega - \omega_1} \right) v(\omega_1) \frac{d\omega_1}{2\pi}\,. \tag{175}$$

But since $v(t)$ is antisymmetric, $v(\omega)$ is pure imaginary. So $v(\omega) = 2i\alpha''(\omega)$ and

$$\alpha'(\omega) = \mathsf{P}\int_{-\infty}^{\infty} \frac{\alpha''(\omega_1)}{\omega_1 - \omega}\,\frac{d\omega_1}{\pi}, \tag{176}$$

If $v(t)$ is instead even in t, we get

$$\alpha''(\omega) = -\mathsf{P}\int_{-\infty}^{\infty} \frac{\alpha'(\omega_1)}{\omega_1 - \omega}\,\frac{d\omega_1}{\pi}\,. \tag{177}$$

These are the *Kramers-Kronig* relations.

The Kramers-Kronig relations have applications all over physics: optics, spectroscopy, high energy physics, geophysics.
The imaginary part measures the tendency of a system to dissipate energy (it's in phase with the driving force). It's odd in ω so sensitive to the 'arrow of time'. The real part is associated with dispersive effects and is even under time reversal.

Lecture 16. The Dirac Field*

Last time we studied linear response and the Kramers-Kronig relations

Today we'll study the Dirac field. This is not examinable.

We'll do it in an *ad hoc* way, following Dirac.

A much better way goes like this.

The Klein-Gordon field is invariant under the Lorentz group.

What if we allow the target to carry a non-trivial *representation*[*]?

e.g. A^μ carries the *vector* representation (4-d, real).

The Dirac field ψ carries the *spinor* representation (4-d complex).

Then follow the worldview.

[*] Of the double cover of the Lorentz group.

Dirac equation

Dirac wanted a linear equation, with dispersion relation $E = +\sqrt{p^2 + m^2}$. Suppose

$$(i\gamma^\nu \partial_\nu - m)\psi = 0 \tag{178}$$

Acting with $(i\gamma^\mu \partial_\mu + m)$, we get

$$(-\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu - m^2)\psi = 0. \tag{179}$$

Since $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$

$$(-\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu - m^2)\psi = 0. \tag{180}$$

so need

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{181}$$

Numbers don't work, but 4×4 matrices do.

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{182}$$

e.g.

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},$$

$\psi \in \mathbb{C}^4$ is a 4-component *spinor*.
We have 1 order E-L eq'ns so 4 physical degrees of freedom: spin-$\frac{1}{2}$ particle and anti-particles.

Dirac lagrangian

A suitable lagrangian is

$$\mathscr{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi \,. \tag{183}$$

Here $\overline{\psi} := \psi^\dagger \gamma^0$.
Can add $-\frac{i}{2}\partial_\mu(\overline{\psi}\gamma^\mu \psi)$ to get

$$\mathscr{L} = \frac{i}{2}\left[\overline{\psi}\gamma^\mu \partial_\mu \psi - (\partial_\mu \overline{\psi})\gamma^\mu \psi\right] - m\overline{\psi}\psi \,. \tag{184}$$

For the equations of motion, get

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \overline{\psi})} = 0 = \frac{\partial \mathscr{L}}{\partial \overline{\psi}} = i\gamma^\mu \partial_\mu \psi - m\psi \tag{185}$$

and

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi)} = i(\partial_\mu \overline{\psi})\gamma^\mu = \frac{\partial \mathscr{L}}{\partial \psi} = -m\overline{\psi} \,. \tag{186}$$

Hamiltonian

The hamiltonian density is

$$\mathscr{H} = \pi \frac{\partial \psi}{\partial t} + \frac{\partial \overline{\psi}}{\partial t} \overline{\pi} - \mathscr{L} \tag{187}$$

So we have

$$\pi = \frac{\partial \mathscr{L}}{\partial(\partial \psi / \partial t)} = i \overline{\psi} \gamma^0 = i \psi^\dagger \,, \qquad \overline{\pi} = \frac{\partial \mathscr{L}}{\partial(\partial \overline{\psi} / \partial t)} = 0 \,, \tag{188}$$

and

$$\begin{aligned} \mathscr{H} &= i \psi^\dagger \frac{\partial \psi}{\partial t} - \mathscr{L} \\ &= \psi^\dagger \left(-i \alpha \cdot \nabla + \beta m \right) \psi \end{aligned} \tag{189}$$

$(\gamma^0 = \beta, \gamma^j = \beta \alpha_j)$

Energy-momentum tensor

$\mathscr{L} = 0$ when the E-L eq'ns hold, so

$$T^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi)} \partial^\nu \psi - g^{\mu\nu} \mathscr{L} = i\overline{\psi}\gamma^\mu \partial^\nu \psi \, , \tag{190}$$

This is not symmetric, so

$$M^{\lambda\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} = i\overline{\psi}\gamma^\lambda (x^\mu \partial^\nu - x^\nu \partial^\mu)\psi \, . \tag{191}$$

is not conserved. Adding

$$S^{\lambda\mu\nu} = \frac{i}{4}\overline{\psi}\gamma^\lambda (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)\psi \, , \tag{192}$$

gives a conserved quantity. The $0ij$ component yields that

$$J = \int d^3 r \;\; \psi^\dagger \left[-i(r \times \nabla) + \frac{1}{2}\Sigma \right] \psi \, . \tag{193}$$

is a conserved charge, *cf. total angular momentum*, with

$$\Sigma_l = \begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix} \tag{194}$$

Conserved current

The symmetry $\psi \to e^{i\alpha}\psi$ yields conserved current

$$J^\mu = \overline{\psi}\gamma^\mu \psi \tag{195}$$

We can gauge this to get

$$\mathscr{L} = i\overline{\psi}\gamma^\mu D_\mu \psi - m\overline{\psi}\psi \tag{196}$$

where $D_\mu = \partial_\mu + ieA_\mu$.
In QFT, this is the *QED* lagrangian.

$$\mathscr{L} = i\overline{\psi}\gamma^{\mu}D_{\mu}\psi - m\overline{\psi}\psi \tag{197}$$

In an external electromagnetic field, the Dirac equation becomes

$$\gamma^{\nu}\left(i\hbar\partial_{\nu} - eA_{\nu}\right)\psi - mc\psi = 0\,.$$

Acting with $\gamma^{\nu}\left(i\hbar\partial_{\nu} - eA_{\nu}\right) + mc$ yields

$$\left[\gamma^{\mu}\gamma^{\nu}\left(i\hbar\partial_{\mu} - eA_{\mu}\right)\left(i\hbar\partial_{\nu} - eA_{\nu}\right) - m^2c^2\right]\psi = 0\,.$$

Setting $\gamma^{\mu}\gamma^{\nu} := g^{\mu\nu} + \sigma^{\mu\nu}$ get

$$\left[\left(i\hbar\partial_{\mu} - eA_{\mu}\right)^2 - \frac{ie\hbar}{2}F_{\mu\nu}\sigma^{\mu\nu} - m^2c^2\right]\psi = 0\,,$$

The extra piece is $\left[e\hbar\Sigma\cdot B - ie\hbar\,\alpha\cdot E\right]\psi$

Closing words

Congratulations - you made it to the end of your first course of Theoretical Physics.

Has a bad rep, and indeed there are many symbols and formulæ.

But this obscures the simplicity of the rules of the game, *viz.*

Physics = Locality + symmetry + quantumness

We have done the first 2. Now comes the 3rd …