Astrofluid Dynamics

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Abstract

Abstract of this course

1 Some basic concepts

Collisional v collisionless fluids
Eulerian and Lagrangian framework
Concepts of streamlines, particle paths and streaklines:
They coinside if the flow is steady.i.e.

2 Formulation of the Fluid Equations

This chapter talked about the conservation of mass and momentum.

2.1 Conservation of mass

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{u} = 0 \tag{1}$$

2.2 Conservation of momentum

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla}) vbu = -\boldsymbol{\nabla} p + \rho \mathbf{g} \tag{2}$$

We consider 4 different which contribute to the change of momentum?

3 Gravitation

In this section, we used $\vec{\mathbf{g}}$ to denote gravitational acceleration; Ψ to denote gravitational potential; and Ω to denote the energy required to take the system of point masses to infinity

Example: Spherical distribution of mass
Example: Infinitely cylindrical symmetrical mass
Example: Infinite planar distribution of masses
Example: Finite axisymmetric disk

3.1 Potential of a Spherical Mass Distribution

Ψ is affected by any matter outside r through our choice of setting Psi at infinity. i.e. We **can't** say that $\Psi = -GM/r$

3.2 Gravitational Potential Energy

3.3 Virial Theorem

Virial Theorem: states that for a system in steady state, $I \equiv mr^2 = constant$, $2T + \Omega = 0$
Kinetic energy T has contribution from local flows and random/thermal motions.
A result of virial theorem is that the gravitational potential sets the temperature or velocity dispersion of the system.

4 Equation of state and the energy equation

So far we have 4 variables, density ρ, pressure,p, gravitational potential Ψ and velocity which is $\boldsymbol{v}$. In terms of equations to solve them, we have the scalar equation of mass conservation and the vector equation for conservation of momentum. We also have Poisson equation for the potential term. Now what we need is another equation: 'Equation of state' to determinate pressure (which is a thermodynamic property). While doing so we might Introduce another equation, the energy equation when the system is not barotropic, i.e. p is a function of T.

4.1 Equation of state

1. Astrophysical fluid are treated as ideal gas, and corresponding EoS is:

$$p = nk_BT = \frac{k_B}{\mu m_p}\rho T \tag{3}$$

   where ρ is the mean particle mass.

2. This EoS introduces another scalar field, temperature, T

4.1.1 Barotropic

Barotropic means that p independent of T. i.e. only a function of ρ. This comes in two cases: Isothermal and Adiabatic.

1. Isothermal: Constant T
2. Adiabatic: Ideal has undergoes reversible thermodynamic changes

$$p = K\rho^\gamma \tag{4}$$

3. Derivation of C_p and C_v for both cases.

4.1.2 Energy equation for non-Barotropic case

1. It starts from first law

$$\bar{d}Q = d\mathcal{E} + pdV \tag{5}$$

2. total energy per unit volume is:

$$E = \rho(\frac{1}{2}u^2 + \Psi + \mathcal{E}) \tag{6}$$

3. take material derivative, note that $\frac{DE}{Dt} = \frac{\partial E}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}E$, gives Energy Equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = \rho\frac{\partial \Psi}{\partial t} - \rho\dot{Q}_{cool} \tag{7}$$

   In many settings $\partial\Psi/\partial\Psi = 0$, If there is no cooling, the equation expresses the conservation for energy in which the total energy density E is driven by the divergence of the enthalpy flux $(E+p)\mathbf{u}$

4.2 Heating and Cooling Processes

Combining cooling and heating effects, we can parametrise $\dot{Q}_{cool}$

$$\dot{Q}_{cool} = A\rho T^\alpha - H \tag{8}$$

where the first and second term on RHS means radiative cooling and cosmic ray heating

4.3 Energy Transport process

Transport processes move energy through the fluid, via Thermal conduction, convention and radiation transport.

5 A lot of different system

5.1 Full set of equations describing the dynamics of an ideal non-relativisitc fluid

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{u} = 0 \tag{Continuity Equation}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla} \cdot)\mathbf{u} = -\boldsymbol{\nabla} \cdot p - \rho \boldsymbol{\nabla} \cdot \Psi \tag{Momentum Equation}$$

$$\nabla^2 \Psi = 4\pi\rho \tag{Poisson's Equation}$$

(Energy Equation)

(Definition fo Total energy)

(EoS of total energy)

(Internal Energy)

5.2 Hydrostatic Equilibrium

A System of hydrostatic equilibrium if

$$\mathbf{u} = \frac{\partial}{\partial t} = 0 \tag{9}$$

Continuity equation is trivially satisfied Sub into momentum equation gives:

$$\frac{1}{\rho}\boldsymbol{\nabla} p = -\boldsymbol{\nabla}\Psi \tag{Equation of Hydrostatic equilibirum}$$

Example: Isothermal atmosphere

Isothermal atmosphere with constant $\mathbf{g} = -g\hat{\mathbf{z}}$

$$\rho = \rho_0 \exp\left\{-\frac{\mu g}{R_* T} z\right\} \tag{10}$$

i.e. exponential atmosphere **Example: Isothermal self-gravitating slab**

Isothermal atmosphere with constant $\mathbf{g} = -g\hat{\mathbf{z}}$

$$\nabla^2 \Psi = 4\pi G\rho \tag{11}$$

Is gives

$$\Psi - \Psi_0 = 2A \ln\cosh\sqrt{\frac{2\pi G\rho}{A}} z \tag{12}$$

$$\rho = \frac{\rho_0}{\cosh^2\sqrt{\frac{2\pi G\rho}{A}} z} \tag{13}$$

5.3 Stars/Self-Gravitating Polytropes

1. In this section, we consider spherically-symmetric, self-gravitating system in hydrostatic equilibirum, which is called "star". Note that non-rotating stars are barotropes.
2. In this system, we have $p = p(\rho)$, and barotropic EoS can be written as $p = K\rho^{1+1/n}$ If the star is isentropic, i.e. constant entropy. $1 + 1/n = \gamma$.
3. When n is constant, the structure is called a Polytrope
4. Assuming a polytropics EoS, the equation of hydrostatic equilibrium is

$$-\nabla\Psi = \frac{1}{\rho}\nabla\left(K\rho^{1+1/n}\right) = (n+1)\nabla\left(K\rho^{1/n}\right)$$

Which has a solution for ρ:

$$\rho = \left(\frac{\Psi_T - \Psi}{(n+1)K}\right)^n$$

In which Ψ_T is $\Psi(\rho = 0)$ at surface.

Given boundary condition at for central density and central potential ρ_c and Ψ_c.

$$\rho = \rho_c\left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_C}\right)^n$$

Now feed into Poisson Equation gives a differential equation for Ψ

Define θ (which is equivalent to Ψ)and $\mathcal{E}$ (which is scaled radial coordinate), we get Lane-Emden Eqn of index n

$$\frac{1}{\mathcal{E}}\frac{d}{d\mathcal{E}^2}\left(\mathcal{E}^2\frac{\mathrm{d}\theta}{\mathrm{d}\mathcal{E}}\right) = -\theta^n$$

which is an differential equation for θ.

Analytical solution for n =0,1,5

5.4 Isothermal Sphere

1. Isothemal case means that $p = K\rho$,

From the momentum equation gives:

$$\Psi - \Psi_c = -K\ln(\rho/\rho_c) \tag{14}$$

From Possion's Equation gives differential equation for ρ

Take $\rho = \rho_c e^{-\Psi}$,we get a differential equation for Ψ

$$\frac{1}{\mathcal{E}}\frac{d}{d\mathcal{E}^2}\left(\mathcal{E}^2\frac{\mathrm{d}\Psi}{\mathrm{d}\mathcal{E}}\right) = e^{-\Psi} \tag{15}$$

which is equivalent to Lane-Equation in this isothermal case.

(16)

5.5 Scaling Relations

Scaling relation relates the mass and radius of a polytrope star.

6 Sound waves, supersonic flows and shock waves

6.1 Sound Waves

In this section, we find the sound wave a derivative of a $p(\rho)$

Example: Isothermal sound wave
$c_s^2 = \sqrt{\frac{R_* T}{\mu}}$

Example: Adiabatic sound wave
$c_s^2 = \sqrt{\gamma \frac{R_* T}{\mu}}$

6.2 Sound Waves in Stratified Atmosphere

Now we consider sound waves traveling in a fluid with a background structure, i.e. constant gravitational field. We will end up with a dispersion relationship and a k that could be imaginary, meaning a decaying wave.

$$\omega^2 = c_u^2 \left(k^2 - \frac{\mathrm{i}k}{H} \right)$$

DISPERSION RELATION

6.3 Sound waves across interfaces

We then matches B.C. to get reflection coefficient and transmission radiation.

$$t = \frac{2k_1}{k_1 + k_2} \qquad r = \frac{k_1 - k_2}{k_1 + k_2}$$

6.4 Supersonic Fluid and Shocks

Shock occurs when there is a disturbance in the fluid cause by compression by a large factor.
Definition Mach Number:

$$M_1 = u_1 / c_{s,1}$$

This number helps us to simplify the R-H relationships in the next section.

6.5 The Rankine-Hugoniot Relations

We discovered 3 R-H relations, from conservation of mass, momentum and energy.

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$
$$\frac{1}{2} u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2} u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2}$$

6.5.1 Adiabatic Case

The last R-H relationship becomes, in an adiabatic idea gas:

$$\frac{1}{2} u_1^2 + \frac{c_{s,1}^2}{\gamma - 1} = \frac{1}{2} u_2^2 + \frac{c_{s,2}^2}{\gamma - 1}$$

6.5.2 Isothermal Case

The last R-H relationship becomes $T_1 = T_2$ with this relationship, we have $c_s^2 = u_1 u_2$ and

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \left(\frac{u_1}{c_s}\right)^2 = M_1^2$$

Note that if $u_1 > c_s$ then $u_2 < c_s$, flow behind the shock is subsonic.

7 Bernoulli's Equation and transonic flows

1. In this section, we started with the Berbulli's equation which gives us Bernoulli Principle, ain nvaraint term connecting the velocity and p, ρ, ψ.
2. Then we briefly discuss the vorticities of the fluid.
3. In the third section, we researched on the De Laval Nozzle, the behaviour of a flow in a tube of area A:
$$\left(u^2 - c_s^2\right) \nabla \ln u = c_s^2 \nabla \ln A$$
before we applied Bernoulli's principle to the isotermal and adiabatic case to discuss the phyiscal meaning of the equations.
4. Lastly, we studied Spherical Accretion and wind, where we considered steady-state and spherically-symmertic accretion flow in the gravitational potential of a central body, i.e.a sun.

7.1 Bernoulli equation

This gives us Bernoulli's Principle: For steady barotropic flows, the quantity

$$H = \frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi$$

is constant along a streamline. The quantity H is called Bernoulli's constant.

If $p = 0, H =$ constant is the statement that kinetic + potential energy is constant along streamlines.

If $p \neq 0$, pressure differences accelerate or decelerate the flow as it flows along the streamline.

7.2 Rotational and irrotational Flow

Let's start with the momentum equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p - \nabla \Psi$$

7.3 The De Laval Nozzle

Laplacian Equation is derived if we have incompressible and irrotational flow. i.e $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and $\boldsymbol{\nabla} \times \mathbf{u} = 0$

7.4 Spherical Accretion

In this section, we are considering symmtric flow of mater in gravitational potential of point like central body, assuming:

1. rest at infinity
2. flow in steady state

3. barotropic equation of state

BYconsidering mass conservaiton and momentum conservation for steady states:

$$\left(u^2 - c_s^2\right)\frac{\mathrm{d}}{\mathrm{d}r}\ln u = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)$$

which means that there is a ctritical point where:

$$r = r_s = \frac{GM}{2c_s^2}$$

that we call a **Sonic pint**, where u has an extremum or $u = c_s$

Example: Isothermal accretion case, $c_s = \sqrt{\frac{\mathcal{R}_* T}{\mu}} = \text{const}$
We can find the mass accretion rate:

$$\dot{M} = \frac{\pi G^2 M^2 e^{3/2}\rho_\infty}{c_s^3}$$

Example: Polytropic accretion case, $p = K\rho^{1+1/n}$**:**
We can find the mass accretion rate:

$$\dot{M} = \frac{\pi (GM)^2 \rho_\infty}{c_{s,\infty}^3}\left(\frac{n}{n-\frac{3}{2}}\right)^{n-3/2}$$

we can recover the iso termal case by taking limit:

$$n \to \infty$$

7.4.1 Dependency of the accretion rate

1. Dependence of accretion rate on mass of object

$$\dot{M} = AM^2$$

- If initial mass is M_0 and it accretes like this for time t, then we can integrate to get

$$\int_{M_0}^{M}\frac{\mathrm{d}M}{M^2} = At \quad \Rightarrow \quad M = \frac{M_0}{1 - AM_0 t}$$

so $\quad M \to \infty$ as $t \to 1/AM_0$ - In reality, the accretion rate will become limited by fuel supply and/or the Eddington limit (which has $M\dot{\propto}M$, so exponential growth).

2. Dependence of accretion rate on reservoir properties:

$$\dot{M} \propto \frac{\rho_\infty}{c_\infty^3} \propto \frac{p_\infty}{c_\infty^5}$$

- Much higher accretion rates from colder material.

3. The $n = 3/2$ is a singular case: - Sonic point goes to origin, with infinite sound speed and density

$$c_s^2 = \left(\frac{n}{n-\frac{3}{2}}\right)c_{s,\infty}^2, \rho_s = \left(\frac{n}{n-\frac{3}{2}}\right)^n \rho_\infty$$
$$r_s = \frac{GM}{2c_s^2} \to 0 \text{ as } n \to \frac{3}{2}$$

- But accretion rate remains finite

$$\dot{M} = \frac{\pi (GM)^2\rho_\infty}{c_{s,\infty}^3}\left(\frac{n}{n-\frac{3}{2}}\right)^{n-3/2} \quad \to \frac{\pi (GM)^2\rho_\infty}{c_{s,\infty}^3} \quad \text{as } n \to \frac{3}{2}$$

4. Can extend (with less rigor) to the case of the mass moving at speed v_∞ through a uniform medium. Accretion rate

$$\dot{M} \sim \frac{(GM)^2\rho_\infty}{(c_\infty^2 + v_\infty^2)^{3/2}} \qquad \text{BONDI-HOYLE ACCRETION}$$

8 Fluid Instabilities

We say that for a fluid in a steady state, $(\partial t = 0)$, this is in a state of equilibrium. This results in:

1. small perturbation that grows with time and unstable
2. small perturbation that decays with time or osicalltes around the equilibirum configuration, the configuration is stable.

In this section, we explore the example of instabilities.

8.0.1 Convective Instability

For convective instabilites, we consider an equilibrium system that under an uniform field. Consider a system that has (p, ρ) that moves to a surrounding medium of (p', ρ').As its pressure changes to p', its denisity that undergoes adiabatic change, will become ρ^* instead of ρ'.

Now we have to compare ρ' and ρ^* to see if the density difference will grow or decay the instability. Hence, we have the Schwarzschild stability criterion which reads

$$\frac{\mathrm{d}T}{\mathrm{d}z} > \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}$$

Since hydrostatic equilibrium requires $\mathrm{d}p/\mathrm{d}z < 0$, we see that (since $\gamma > 1$) - Always stable to convection if $\mathrm{d}T/\mathrm{d}z > 0$; - Otherwise, can tolerate a negative temperature gradient provided

$$\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right| < \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\left|\frac{\mathrm{d}p}{\mathrm{d}z}\right|$$

So convective instability develops when T declines too steeply with increasing height.