NST Part II Physics Michaelmas Term 2022

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ADVANCED QUANTUM PHYSICS

Examples Sheet 1

1. **Operator methods and measurement**

   The Hamiltonian $\hat{H}$ has two normalized eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$, corresponding to different eigenvalues E_1 and E_2.

   (a) Show that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal.

   (b) An observable $\hat{A}$ has the properties $\hat{A}|\psi_1\rangle = |\psi_2\rangle$ and $\hat{A}|\psi_2\rangle = |\psi_1\rangle$; calculate its eigenvalues and eigenvectors (as combinations of $|\psi_1\rangle$ and $|\psi_2\rangle$).

   (c) At time $t = 0$, a measurement of $\hat{A}$ results in the measured value -1. Find the state of the system $|\psi(t)\rangle$ at times $t > 0$, and show that the probability that a measurement of $\hat{A}$ again gives the value -1 is given by $P(t) = \cos^2[(E_1 - E_2)t/2\hbar]$.

2. **Ladder operators**

   The potential energy of a one-dimensional harmonic oscillator of mass m and angular frequency ω is given by $V(\hat{x}) = m\omega^2\hat{x}^2/2$. Using the raising and lowering operators,

   $$\hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x}), \qquad \hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{x})\,,$$

   show that:

   (a) The expectation values of the position and momentum are zero for an energy eigenstate $|\psi_n\rangle$.

   (b) The expectation values of the potential and kinetic energies are each equal to $(n + 1/2)(\hbar\omega/2)$ where n is the quantum number of the state $|\psi_n\rangle$.

   (c) The uncertainties Δx and Δp in position and momentum are related by $\Delta x\,\Delta p = (n + 1/2)\hbar$.

   $\Big[$The ladder operators have the properties $\hat{a}^\dagger|\psi_n\rangle = \sqrt{n+1}|\psi_{n+1}\rangle$ and $\hat{a}|\psi_n\rangle = \sqrt{n}|\psi_{n-1}\rangle$.$\Big]$

3. **Heisenberg picture**

In the Heisenberg picture, time dependent operators $\hat{A}(t) \equiv e^{i\hat{H}t/\hbar}\hat{A}e^{-i\hat{H}t/\hbar}$ are introduced. For the one-dimensional harmonic oscillator of question 2, show that the ladder operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ in the Heisenberg representation satisfy

$$\begin{aligned}\hat{a}(t) &\equiv e^{i\hat{H}t/\hbar}\hat{a}(0)e^{-i\hat{H}t/\hbar} = e^{-i\omega t}\hat{a}(0)\,,\\ \hat{a}^\dagger(t) &\equiv e^{i\hat{H}t/\hbar}\hat{a}^\dagger(0)e^{-i\hat{H}t/\hbar} = e^{i\omega t}\hat{a}^\dagger(0)\,.\end{aligned}$$

Use this result to demonstrate that the position operator in the Heisenberg representation obeys the equation of motion

$$\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} = \frac{\hat{p}(t)}{m}\,.$$

Show that this last result holds also for the more general case $\hat{H} = \hat{p}^2/2m + V(\hat{x})$.

4. **Addition of angular momenta**

(a) Consider the addition of two angular momenta, $\ell_1 = 2$ and $\ell_2 = 1$. Tabulate the possible values of the corresponding quantum numbers m_1, m_2 and $M = m_1 + m_2$ (relating to $\hat{L}_z$), and show that the values of M correspond to the expected values $L = 3, 2, 1$ of the total angular momentum quantum number L. Repeat for the case $\ell_1 = 3$, $\ell_2 = 1$.

(b) For the case $\ell_1 = 2$ and $\ell_2 = 1$, the state $|L, M\rangle = |3, 3\rangle$ can be written down straightforwardly as $|\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle = |2, 2\rangle \otimes |1, 1\rangle$. Use ladder operators to construct the state $|L, M\rangle = |3, 2\rangle$ as a linear combination of the product states $|\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle$, and then orthogonality to construct the state $|L, M\rangle = |2, 2\rangle$.

[The angular momentum ladder operators $\hat{L}_\pm$ act as

$$\hat{L}_\pm|L, m_L\rangle = \hbar\sqrt{L(L+1) - m_L(m_L \pm 1)}|L, m_L \pm 1\rangle\,. \quad]$$

(c) Verify that the states obtained in (b) are the same as would be written down using the tables of Clebsch-Gordan coefficients appended to this examples sheet (see the table labelled $2 \otimes 1$).

(d) Using the $2 \otimes 1$ table, write down the state $|L, M\rangle = |1, -1\rangle$ as a linear combination of the $|\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle$ states.

(e) Show that the scalar product $\hat{\boldsymbol{L}}_1 \cdot \hat{\boldsymbol{L}}_2$ of two angular momentum operators can be expressed as

$$\hat{\boldsymbol{L}}_1 \cdot \hat{\boldsymbol{L}}_2 = \tfrac{1}{2}(\hat{L}_1)_+(\hat{L}_2)_- + \tfrac{1}{2}(\hat{L}_1)_-(\hat{L}_2)_+ + (\hat{L}_1)_z(\hat{L}_2)_z\ ,$$

where $(\hat{L}_{1,2})_\pm = (\hat{L}_{1,2})_x \pm i(\hat{L}_{1,2})_y$ are ladder operators. By operating directly with $(\hat{\boldsymbol{L}}_1 + \hat{\boldsymbol{L}}_2)^2$ and $(\hat{L}_1)_z + (\hat{L}_2)_z$, verify that the linear combination of product states written down in (d) does indeed have total angular momentum quantum numbers $L = 1$ and $M = -1$.

(f) Convince yourself that each table of Clebsch-Gordan coefficients corresponds to a unitary (in fact, orthogonal) matrix. For the cases $j_1 \otimes j_2 = (1/2) \otimes (1/2)$, $1 \otimes 1$, $(3/2) \otimes (3/2)$ and $2 \otimes 2$ (for which $j_1 = j_2$), what is the symmetry of the total angular momentum eigenstates $|j, m_j\rangle$ for each possible value of j under interchange of the labels 1 and 2 ?

5. **Matrix methods**

   Show that for a system with orbital angular momentum $\ell = 1$, for the basis of states $|\phi_1\rangle = |Y_{11}\rangle$, $|\phi_2\rangle = |Y_{10}\rangle$, $|\phi_3\rangle = |Y_{1-1}\rangle$, the angular momentum operators may be represented by the matrices

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad \hat{L}_y = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}, \quad \hat{L}_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

   A rotating body has the Hamiltonian

$$\hat{H} = \frac{\hat{L}_x^2}{2I_x} + \frac{\hat{L}_y^2}{2I_y} + \frac{\hat{L}_z^2}{2I_z}.$$

   Find the energy levels and corresponding eigenstates when $\ell = 1$.

6. **Spin**

   In terms of the Pauli matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, the operator corresponding to the component of spin along the axis (θ, ϕ) in spherical polar coordinates for a spin-half particle is $(\hbar/2)\boldsymbol{\sigma}\cdot\boldsymbol{n}$, where $\boldsymbol{n}$ is the unit vector $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Show that the eigenvalues of spin in this direction are $\pm\hbar/2$ (as expected), and deduce the corresponding wavefunctions. Hence, infer the wavefunctions for particles whose spins are aligned along the $+x$, $-x$, $+y$ and $-y$ directions.

7. **Perturbation theory**

   The energy levels of the hydrogen atom are influenced by the finite size of the proton. A simple model of this problem is to treat the proton as a uniformly charged hollow spherical shell of radius $b = 5 \times 10^{-16}$ m. Show that, for this model, the change in the electrostatic potential energy corresponds to introducing a perturbation

$$\hat{H}' = \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right); \qquad r < b$$

   into the normal Schrödinger equation for the hydrogen atom. Using first-order perturbation theory, estimate the energy shifts of the hydrogen 2s and 2p states and comment on your findings. Why is the energy shift the same for all three 2p states, and why can each of the 2s and 2p states be considered independently even though they are initially degenerate?

   [Hint: The integrals can be simplified considerably by noting that the size of the nucleus is much smaller than the atomic Bohr radius, i.e. $b \ll a_0$.]

   [The 2s and 2p hydrogen atom wavefunctions are

$$\psi_{2\mathrm{s}} = \sqrt{\frac{1}{8\pi a_0^3}}\left(1 - \frac{r}{2a_0}\right)e^{-r/2a_0}, \quad \psi_{2\mathrm{p}_0} = \frac{re^{-r/2a_0}}{\sqrt{32\pi a_0^5}}\cos\theta, \quad \psi_{2\mathrm{p}_{\pm 1}} = \mp\frac{re^{-r/2a_0}}{\sqrt{64\pi a_0^5}}e^{\pm i\phi}\sin\theta.]$$

8. **Perturbation Theory: Polarizability of Hydrogen**

The polarisibility of the hydrogen atom in its ground state may be estimated using perturbation theory. [The induced dipole moment in an applied electric field $\boldsymbol{E}$ is $\alpha\epsilon_0\boldsymbol{E}$ where α is the polarisibility.]

Working to second order in the electric field strength, show that the energy shift in the ground state $|0\rangle$ is

$$\Delta E = (eE)^2 \sum_{k\neq 0} \frac{|\langle k|z|0\rangle|^2}{E_0 - E_k}\,,$$

where E_k is the unperturbed energy of state $|k\rangle$. Hence show that the polarisability is

$$\alpha = \frac{2e^2}{\epsilon_0} \sum_{k\neq 0} \frac{|\langle k|z|0\rangle|^2}{E_k - E_0}\,.$$

Show that the same result may be obtained from the perturbed wavefunction to first-order in $\boldsymbol{E}$ by evaluating the expectation value of the induced electric dipole moment.

Evaluation of α is tedious, but a useful upper bound may be obtained by noting that $E_k \geq E_1$, where E_1 is the energy of the first excited state. Using this observation, show that $\alpha \leq (64/3)\pi a_0^3$. Compare this upper bound with the experimental value of $\alpha = 8.5 \times 10^{-30}\,\mathrm{m}^3$.

[The ground state of the hydrogen atom, $|0\rangle = (\pi a_0^3)^{-1/2} e^{-r/a_0}$, will be needed to compute the matrix element $\langle 0|z^2|0\rangle$.]

9. **Degenerate perturbation theory**

A particle of mass m is constrained to move in the xy–plane such that the Hamiltonian is given by

$$\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}m\omega^2(\hat{x}^2 + \hat{y}^2) + \lambda\hat{x}\hat{y}\,.$$

(a) Using raising and lowering operators show that for $\lambda = 0$ the (unperturbed) energy eigenvalues can be described by the equation $E_{n_x,n_y} = (n_x + n_y + 1)\hbar\omega$.

(b) For the ground state and first two excited states, describe the unperturbed eigenstates for the system in terms of one-dimensional harmonic oscillator eigenstates $|n_x\rangle$, $|n_y\rangle$. What are the degeneracies of each of these energy levels?

(c) For the case $\lambda \neq 0$, use degenerate perturbation theory to determine the energy splitting for the lowest energy degenerate level, as well as the first-order corrections to the wavefunctions.

10. **Variational method**

Use a trial wavefunction of the form

$$\psi(x) = \begin{cases} A(a^2 - x^2) & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

to place an upper bound on the ground state energy of the one-dimensional harmonic oscillator with potential $V(x) = m\omega^2x^2/2$, where m is the mass of the particle and ω is the oscillator frequency. Compare your answer with the exact result, and comment.

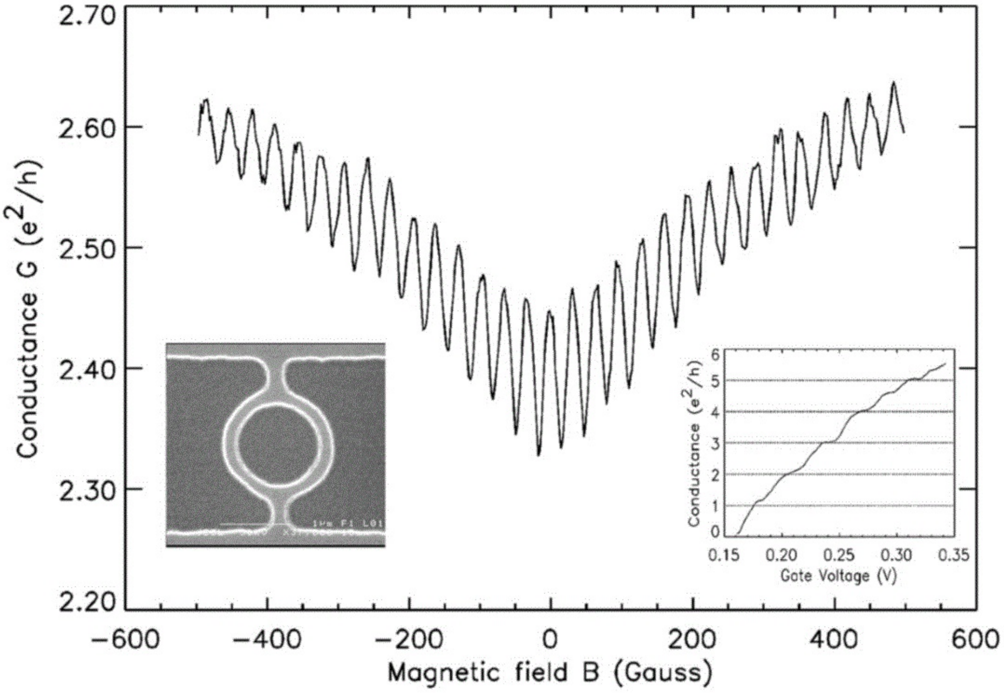

Figure 1: Aharonov-Bohm effect in a semiconductor quantum ring. [From S. Pedersen *et al.*, Phys. Rev. B **61** (2000) 5457.]

11. **Variational method**

    (a) E_1 and E_2 are the ground state energies of a particle moving in attractive potentials $V_1(\boldsymbol{r})$ and $V_2(\boldsymbol{r})$, respectively. Using the variational method, show that $E_1 \leq E_2$ if $V_1(\boldsymbol{r}) \leq V_2(\boldsymbol{r})$.

    [Hint: Use the wavefunction of a particle moving in $V_2(\boldsymbol{r})$ as a trial wavefunction for the potential $V_1(\boldsymbol{r})$.]

    (b) Consider a particle moving in a localized one-dimensional attractive potential $V(x)$, i.e. a potential such that $V(x) \leq 0$ for all x, and $V(x) \to 0$ as $|x| \to \infty$. Use the variational principle with trial function $A\exp(-\lambda x^2)$ to show that the upper bound on the ground state energy is negative, and hence that for any such potential at least one bound state must exist.

12. **Aharonov-Bohm effect**

    A ring-shaped semiconductor device is fabricated from a high mobility two-dimensional electron gas (see Fig.1; the lighter grey is the conducting region). The ring is cooled in a cryostat to 0.3 K. A voltage is applied across the ring (between points at the bottom and the top of the image) and the current flow is measured as a function of the magnetic field applied perpendicular to the plane of the ring. Explain why the oscillations in conductance occur, account for their periodicity, and obtain a value for the average diameter of the ring. (1 Tesla = 10^4 Gauss.)

13. **Spin precession (Tripos 1999)**

A spin 1/2 particle has gyromagnetic ratio γ, so that its magnetic moment is given by $\hat{\boldsymbol{\Gamma}} = \gamma\hat{\boldsymbol{S}}$ where $\hat{\boldsymbol{S}}$ is the spin operator. Show that the equation of motion for the spin state $|\psi(t)\rangle$ of such a particle in a magnetic field $\boldsymbol{B}$ is

$$-\frac{1}{2}\gamma\,(\boldsymbol{B}\cdot\hat{\boldsymbol{\sigma}})|\psi(t)\rangle = i\frac{\partial}{\partial t}|\psi(t)\rangle\,,$$

where $\hat{\boldsymbol{\sigma}}$ is a vector with the Pauli matrices $\hat{\sigma}_i$ as components.

If $\boldsymbol{B}$ is a constant field in the z-direction with magnitude B_0, and

$$|\psi(0)\rangle = \cos(\theta/2)|{\uparrow}\rangle + \sin(\theta/2)|{\downarrow}\rangle\,,$$

show that at time t,

$$|\psi(t)\rangle = \cos(\theta/2)\exp(i\omega_0 t/2)|{\uparrow}\rangle + \sin(\theta/2)\exp(-i\omega_0 t/2)|{\downarrow}\rangle\,,$$

where $\omega_0 = \gamma B_0$, and find the expectation values of the components of the magnetic moment $\hat{\boldsymbol{\Gamma}}$ at time t.

Using the general result

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{A}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle$$

for the time evolution of the expectation value of an operator $\hat{A}$, show that for an arbitrarily varying magnetic field $\boldsymbol{B}(t)$ the magnetic moment operator satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\boldsymbol{S}}\rangle = \langle\hat{\boldsymbol{\Gamma}}\times\boldsymbol{B}(t)\rangle\,,$$

and demonstrate explicitly that the expectation values found above for the constant field satisfy this relation. Interpret your results physically.

14. **Proton magnetic moment**

A beam of hydrogen molecules is moving in the y-direction. An x Stern-Gerlach apparatus is used as a filter which rejects para-H_2 (resultant nuclear spin zero) and passes ortho-H_2 (resultant nuclear spin one) with spin component $+\hbar$ in the x-direction. A magnetic field B in the z-direction acts over 20 mm of path between two such filters in series, and it is found that no molecules of kinetic energy 0.025 eV emerge when $B = 1.8(n + 1/2) \times 10^{-3}\,\mathrm{T}$, where n is an integer. Explain this phenomenon and deduce a value for the magnetic moment of the proton.

[The $\hat{S}_x$ operator and eigenstates in the basis of $\hat{S}_z$ eigenstates are:

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ; \qquad \begin{array}{ccc} m_x = +1 & m_x = 0 & m_x = -1 \\ \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} , & \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} , & \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \end{array}$$

These may be used to construct the time-dependent wavefunction.]

15. **Muon magnetic moment**

In the E821 experiment discussed in lectures, monoenergetic relativistic muons travel in a circular orbit in a plane perpendicular to a uniform magnetic field of strength $B = 1.4513\,\mathrm{T}$. The muons are initially polarised with their spin vector aligned along the momentum direction. Figure 2 shows the number of detected electrons from $\mu^- \to e^- + \overline{\nu}_e + \nu_\mu$ decays as a function of time.

(a) Explain briefly why the periodic structure seen in the plot reflects the frequency difference $\omega_\mathrm{a} \equiv \omega_\mathrm{s} - \omega_\mathrm{c}$ between the spin precession and cyclotron frequencies, and, from the observed period, estimate the muon g-factor, g_μ, and its uncertainty.

[The muon mass is $m_\mu = 206.77\, m_\mathrm{e}$. The non-relativistic expression for ω_a obtained in lectures applies also to relativistic particles.]

(b) In E821, rather than directly measuring the field strength B, improved precision is obtained by measuring instead the spin precession frequency ω_p of protons at rest in the same magnetic field. Show that the anomalous muon magnetic moment $a_\mu \equiv (g_\mu - 2)/2$ can be determined as

$$a_\mu = \frac{\mathcal{R}}{\lambda - \mathcal{R}} ,$$

where $\mathcal{R} \equiv \omega_\mathrm{a}/\omega_\mathrm{p} = 0.0037072064(20)$ is measured in E821, and $\lambda \equiv \mu_\mu/\mu_\mathrm{p} = 3.18334513(39)$ is the ratio of muon and proton magnetic moments, taken from another experiment. Obtain a_μ and its uncertainty, and compare with a recent QED prediction: $a_\mu(\mathrm{QED}) = 0.001165918204(356)$ [From A. Keshavarzi *et al.*, Phys. Rev. D **97** (2018) 114025.]

(c) (*Optional*) For a relativistic particle, the cyclotron frequency is given by $\omega_\mathrm{c} = qB/m\gamma$, where γ is the Lorentz boost factor. Use the data shown in Figure 2 to obtain estimates of γ, and of the number of cyclotron orbits, N_c, undertaken during a time interval $T_\mathrm{a} = 2\pi/\omega_\mathrm{a}$. Consult the E821 paper referenced in the figure caption to see whether or not it is a coincidence that $\gamma \approx N_\mathrm{c}$.

[The mean lifetime of the muon in its rest frame is $2.197\,\mu$s.]

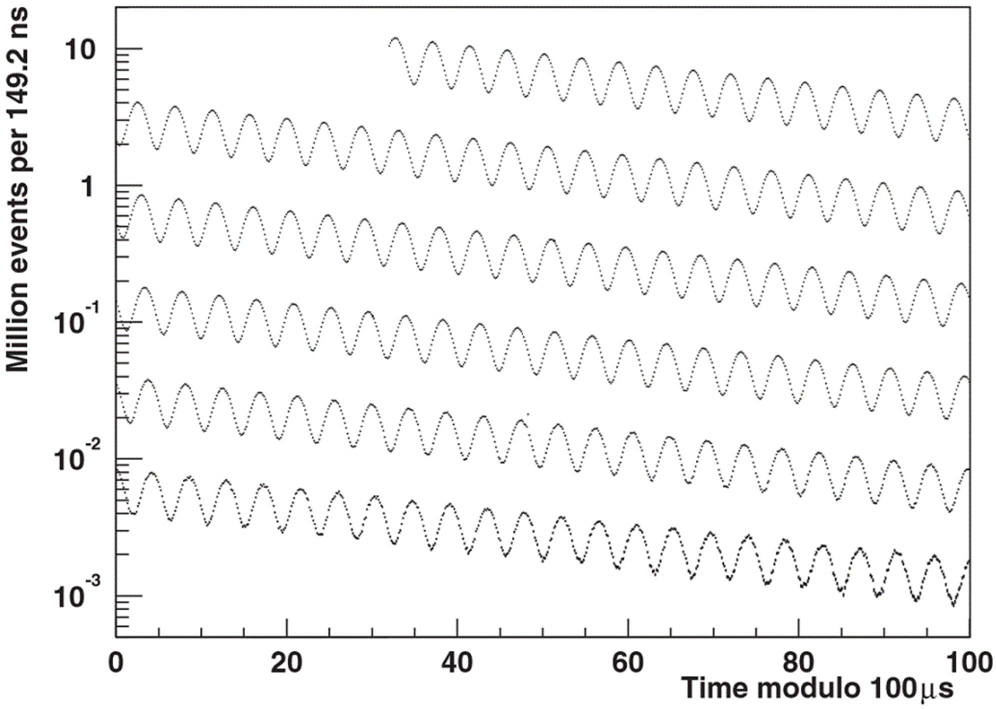

Figure 2: Distribution of electron counts versus time for 3.6×10^9 muon decays recorded by the E821 $g_\mu - 2$ experiment (corresponding to about 40% of the total data set). The data is wrapped around modulo $100\,\mu$s. [From G. W. Bennett *et al.*, Phys. Rev. D **73** (2006) 072003.]

16. **Landau levels**

An electron confined to the xy–plane is subjected to electric and magnetic fields $\boldsymbol{E} = E\boldsymbol{x}$, $\boldsymbol{B} = B\boldsymbol{z}$. In the Landau gauge, $\boldsymbol{A} = (0, Bx, 0)$, the Hamiltonian is

$$\hat{H} = \frac{\hat{p}_x^2 + (\hat{p}_y + eB\hat{x})^2}{2m} + eE\hat{x}\,.$$

Show that the Schrödinger equation $\hat{H}\psi = E\psi$ admits solutions of the form $\psi(x, y) = e^{ik_y y}\chi(x)$. Hence show that the electron energy spectrum for each k_y is given by

$$E_n = (n + 1/2)\hbar\omega_c - (mE^2/2B^2) - (E/B)\hbar k_y\,,$$

where $\omega_c = eB/m$. [Hint: Complete the square in the Hamiltonian $\hat{H}_\chi$ for the component $\chi(x)$.] Show that a wavepacket moves with a velocity $-E/B$ in the y-direction.

43. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			
.	.			
.	.			

1/2×1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\,e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\,e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\,e^{2i\phi}$$

2×1/2

		5/2
		+5/2
+2	+1/2	1

		5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

1×1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

3/2×1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1

2×1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

3/2×1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

1×1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\,Y_\ell^m\,e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle$$
$$= (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

3/2×3/2

		3
		+3
+3/2	+3/2	1

		3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

		3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

		3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

		3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

		3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

		3
		−3
−3/2	−3/2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

		7/2
		+7/2
+2	+3/2	1

		7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

		7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

		7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

		7/2	5/2	3/2	1/2
		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

		7/2	5/2	3/2
		−3/2	−3/2	−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

		7/2	5/2
		−5/2	−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

		7/2
		−7/2
−2	−3/2	1

2×2

		4
		+4
+2	+2	1

		4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

		4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

		4	3	2	1
		+1	+1	+1	+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

		4	3	2	1	0
		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

		4	3	2	1
		−1	−1	−1	−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

		4	3	2
		−2	−2	−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

		4	3
		−3	−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

		4
		−4
−2	−2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^{\,2}_{1,0} = -\sqrt[9]{\frac{3}{2}}\sin\theta\,\cos\theta$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2 \qquad d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1) \qquad d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

ANSWERS

1. (b) ± 1; $\big(|\psi_1\rangle \pm |\psi_2\rangle\big)/\sqrt{2}$;

4. (b) $|3,2\rangle = \sqrt{1/3}|2,2\rangle|1,0\rangle + \sqrt{2/3}|2,1\rangle|1,1\rangle$;
   $|2,2\rangle = \sqrt{2/3}|2,2\rangle|1,0\rangle - \sqrt{1/3}|2,1\rangle|1,1\rangle$;
   (d) $|1,-1\rangle = \sqrt{1/10}|2,0\rangle|1,-1\rangle - \sqrt{3/10}|2,-1\rangle|1,0\rangle + \sqrt{3/5}|2,-2\rangle|1,1\rangle$.

5. (b) $(\hbar^2/2)(I_x^{-1} + I_y^{-1})$, $(\hbar^2/2)(I_x^{-1} + I_z^{-1})$, $(\hbar^2/2)(I_y^{-1} + I_z^{-1})$,
   $|Y_{10}\rangle$, $(|Y_{11}\rangle + |Y_{1,-1}\rangle)/\sqrt{2}$, $(|Y_{11}\rangle - |Y_{1,-1}\rangle)/\sqrt{2}$.

6. $+\hbar/2$: $|\psi\rangle = \cos(\theta/2)|\uparrow\rangle + e^{i\phi}\sin(\theta/2)|\downarrow\rangle$;
   $-\hbar/2$: $|\psi\rangle = \sin(\theta/2)|\uparrow\rangle - e^{i\phi}\cos(\theta/2)|\downarrow\rangle$

7. $\Delta E(2s) = (b^2/6a_0^2)R_\infty$; $\quad \Delta E(2p_0) = (b^4/240a_0^4)R_\infty$.

9. (b) $g = 1, 2, 3$;
   (c) $\Delta E = \pm\lambda\hbar/2m\omega$;

12. (a) $d = 1.28\,\mu\text{m}$.

14. $\mu_{\rm p} = 1.42 \times 10^{-26}\,\text{J}\,\text{T}^{-1}$.

15. (a) $g_\mu \approx 2.002338(3)$;
    (b) $a_\mu = 0.00116592093(65)$;
    (c) $\gamma \approx N_c \approx 29$.