II. MANIFOLDS AND COORDINATES

Special relativity tells us to think of space and time as a 4-dimensional continuum called *spacetime*.

The spacetime of special relativity – *Minkowski spacetime* – is an example of what mathematicians call a *manifold*.

In general relativity, spacetime is described by a more complicated manifold that responds dynamically to the distribution of mass and energy.

In this handout, we shall briefly introduce the concept of a manifold, and then discuss some of the geometric properties of the types of manifold (called Riemannian manifolds) relevant for general relativity.

1 Concept of a manifold

Informally, an N-dimensional manifold is a set of objects that locally resembles ND Euclidean space $\mathbb{R}^N$.

In relativity, the object are events and the set of events is spacetime.

What "locally resembles" means is that there exists a map ϕ from the ND manifold $\mathcal{M}$ to an *open subset* of $\mathbb{R}^N$ that is one-to-one and onto.[1]

Under the map ϕ, a point $P \in \mathcal{M}$ maps to a point in the open subset U of $\mathbb{R}^N$ with *coordinates* x^a, $a = 1, \ldots, N$.

Generally, we cannot cover the entire manifold with a

[1] An open subset U of $\mathbb{R}^N$ is such that for any point one can construct a sphere centred on the point whose interior lies entirely inside U. A map from $\mathcal{M}$ to U is one-to-one and onto if every element of U is mapped to by exactly one element of $\mathcal{M}$.

single map ϕ (or, equivalently, set of coordinates), but it is sufficient if we can subdivide $\mathcal{M}$ and map each piece separately onto open subsets of $\mathbb{R}^N$.

The manifold is *differentiable* if these subdivisions join up smoothly so that we can define scalar fields on the manifold that are differentiable everywhere.

We can generally think of manifolds as surfaces embedded in some higher-dimensional Euclidean space, and we shall often do so, but it is important to appreciate that a given manifold exists independent of any embedding.

A non-trivial example of a manifold is the set of rotations in 3D; these can be parameterised by three Euler angles, which form a coordinate system for the 3D manifold.

2 Coordinates

As we have just seen, points in an ND manifold can be labelled by N real-valued coordinates $(x^1, x^2, \ldots, x^N)$.

We shall denote these collectively by x^a with $a = 1, \ldots, N$.

The coordinates are not unique: think of them as labels of points in the manifold that can change under a coordinate transformation (i.e., a change of map ϕ) while the point itself does not.

We have also noted that, generally, it will not be possible to cover a manifold with a single *non-degenerate* coordinate system, i.e., one where the correspondence between points and coordinate labels is one-to-one.

In such cases, multiple coordinate systems are required to cover the whole manifold.

Example 1: coordinates (ρ, ϕ) in the plane $\mathbb{R}^2$

The Euclidean plane $\mathbb{R}^2$ is a 2D manifold that can be covered globally with the usual Cartesian coordinates.

However, we could instead use plane-polar coordinates, (ρ, ϕ) with $0 \leq \rho \leq \infty$ and $0 \leq \phi < 2\pi$.

Plane-polar coordinates are degenerate at $\rho = 0$ since ϕ is indeterminate there.

Example 2: coordinates (θ, ϕ) on the 2-sphere S^2

The 2-sphere is the set of points in $\mathbb{R}^3$ with $x^2+y^2+z^2 = 1$.

It is an example of a 2D manifold.

The spherical polar coordinates (θ, ϕ), with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$, are degenerate at the poles $\theta = 0$ and $\theta = \pi$, where ϕ is indeterminate.

For S^2, there is no single coordinate system that covers the whole manifold without degeneracy: at least two coordinate patches are required.

2.1 Curves and surfaces

Subsets of points in a manifold define *curves* and *surfaces.*

These are usually defined parametrically for some coordinate system, e.g., for a curve with parameter u:

$$x^a = x^a(u) \qquad (a = 1, 2, \ldots, N)\,. \tag{1}$$

For a *submanifold* (or surface) of M ($M < N$) dimensions, we need M parameters:

$$x^a = x^a(u^1, u^2, \ldots, u^M) \qquad (a = 1, 2, \ldots, N)\,. \qquad (2)$$

The special case $M = N - 1$ is called a *hypersurface*.

In this case, we can eliminate the $N - 1$ parameters from the N equations (2) to give

$$f(x^1, x^2, \ldots, x^N) = 0\,, \qquad (3)$$

for some function f.

Similarly, points in an M-dimensional surface can be specified by $N - M$ (independent) constraints

$$f_1(x^1, x^2, \ldots, x^N) = 0\,, \ldots, f_{N-M}(x^1, x^2, \ldots, x^M) = 0\,, \qquad (4)$$

i.e., by the intersection of $N - M$ hypersurfaces, as an alternative to the parametric representation of Eq. (2).

2.2 Coordinate transformations

Coordinates are used to *label* points in a manifold, but the labelling is arbitrary.

Later, we shall learn how to construct geometric objects that are independent of the way we assign coordinates, and that express the true physical content of the theory (think vectors in $\mathbb{R}^N$).

We can relabel points by performing a coordinate transformation given by N equations

$$x'^a = x'^a(x^1, x^2, \ldots, x^N) \quad (a = 1, 2, \ldots, N)\,. \qquad (5)$$

We shall view coordinate transformations as *passive*, i.e., assigning new coordinates x'^a to a given point in terms of the original coordinates x^a.

We shall further assume that the functions $x'^a(x^1, \ldots, x^N)$ are single-valued, continuous and differentiable.

Consider two neighbouring points P and Q with coordinates x^a and $x^a + dx^a$.

In the new (primed) coordinates

$$dx'^a = \sum_{b=1}^{N} \frac{\partial x'^a}{\partial x^b} \, dx^b \,, \tag{6}$$

where the partial derivatives are evaluated at the point P.

This defines an $N \times N$ *transformation matrix* at the point P with elements

$$J^a{}_b = \frac{\partial x'^a}{\partial x^b} = \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \cdots & \frac{\partial x'^1}{\partial x^N} \\ \vdots & & \vdots \\ \frac{\partial x'^N}{\partial x^1} & \cdots & \frac{\partial x'^N}{\partial x^N} \end{pmatrix} , \tag{7}$$

where the numerator (index a) labels the rows and the denominator (index b) the columns.

The determinant of $J \equiv \det(J^a{}_b)$ is the *Jacobian* of the transformation.

If $J \neq 0$ for some range of the coordinates, the coordinate transformation can be inverted locally to give x^a as a function of the x'^a.

The transformation matrix for the inverse

$$x^a = x^a(x'^1, x'^2, \ldots, x'^N) \tag{8}$$

is the inverse of $J^a{}_b$; this follows from the chain rule for partial derivatives

$$\sum_{b=1}^{N} \frac{\partial x'^a}{\partial x^b} \frac{\partial x^b}{\partial x'^c} = \frac{\partial x'^a}{\partial x'^c} = \delta^a_c \,. \tag{9}$$

It also follows that the determinant of the inverse transformation is $1/J$.

2.3 Einstein summation convention

It will rapidly get cumbersome to include the summation over indices explicitly, as in Eq. (6).

We therefore introduce the *Einstein summation convention*:

Whenever an index occurs twice in an expression, once as a subscript and once as a superscript, summation over the index from 1 to N is implied.

For example, for an infinitesimal displacement

$$dx'^a = \frac{\partial x'^a}{\partial x^b} dx^b \, . \tag{10}$$

Here, the index a is a free index and may take any value from 1 to N, while the index b is summed over 1 to N.

Note the following points about the summation convention.

- A superscript in the denominator of a partial derivative is considered a subscript, which is why the index b in Eq. (10) is summed over.
- Indices that are summed over are called *dummy* indices because can be replaced by any other index not already in use, e.g.,

$$\frac{\partial x'^a}{\partial x^b} dx^b = \frac{\partial x'^a}{\partial x^c} dx^c \, . \tag{11}$$

- In any term, an index should *not* occur more than twice, and any repeated index must occur once as a subscript and once as a superscript (and is summed over).

3 Local geometry of Riemannian manifolds

The general definition of a differentiable manifold does not define its *geometry*.

To do so requires introducing additional structure to the manifold.

Consider two neighbouring points P and Q in a manifold, i.e., points with coordinates x^a and $x^a + dx^a$, in some coordinate system, which differ infinitesimally.

The *local geometry* near P is specified by giving the invariant "distance" or "interval" between the points.

In a *Riemannian manifold*, the interval takes the form (summation convention!)

$$\boxed{ds^2 = g_{ab}(x)dx^a dx^b\,,} \tag{12}$$

i.e., the interval is quadratic in the coordinate differentials.

The coefficients $g_{ab}(x)$ contain information about the local geometry but also depend on the particular coordinate system.

Strictly, the geometry is Riemannian if $ds^2 > 0$ and pseudo-Riemannian otherwise (the latter being the relevant case for spacetime).

It is also possible to consider more general intervals, but these are not relevant for general relativity because of the equivalence principle.

3.1 The metric

The *metric functions* relate infinitesimal changes in the coordinates to invariantly-defined "distances" in the manifold.

In general relativity, these will be proper distances and times.

The metric functions $g_{ab}(x)$ can always be chosen symmetric, $g_{ab}(x) = g_{ba}(x)$.

To see this, note that we can write a general g_{ab} as the sum of a symmetric and antisymmetric part:

$$g_{ab}(x) = \frac{1}{2}[g_{ab}(x) + g_{ba}(x)] + \frac{1}{2}[g_{ab}(x) - g_{ba}(x)] \, . \quad (13)$$

The contribution of the antisymmetric part to ds^2 vanishes since

$$\begin{aligned}(g_{ab} - g_{ba})dx^a dx^b &= g_{ab}dx^a dx^b - g_{ab}dx^b dx^a \\ &= (g_{ab} - g_{ab})dx^a dx^b = 0 \, , \end{aligned} \quad (14)$$

where we have relabelled the dummy indices $a \leftrightarrow b$ in the first line on the right.

It follows that in an N-dimensional Riemannian manifold there are $N(N+1)/2$ independent metric functions at each point.

Given two neighbouring points, the interval between them is independent of the coordinate system used.

Since the coordinate differentials change under a change of coordinates, so must the metric functions, i.e.,

$$\begin{aligned} ds^2 &= g_{ab}(x) \, dx^a \, dx^b \\ &= g_{ab}(x) \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d} \, dx'^c \, dx'^d \\ &= g'_{cd}(x') \, dx'^c \, dx'^d \, , \end{aligned} \quad (15)$$

where the metric functions in the new coordinates *at the same physical point* are $g'_{cd}(x')$.

We can read off from Eq. (15) that the metric functions must transform as

$$\boxed{g'_{cd}(x') = g_{ab}(x(x'))\frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d}\,.} \tag{16}$$

Since there are N arbitrary coordinate transformations that we can make, there are really only $N(N-1)/2$ independent functional degrees of freedom associated with $g_{ab}(x)$.

3.2 Intrinsic and extrinsic geometry

The interval (or *line element*) ds^2 characterises the local geometry (or curvature), which is an *intrinsic* property of the manifold independent of any possible embedding in some higher-dimensional space.

Intrinsic properties are those that can be determined by a "bug" *confined* to the manifold – the bug can set up a coordinate system, measure physical distances and hence determine the metric functions.

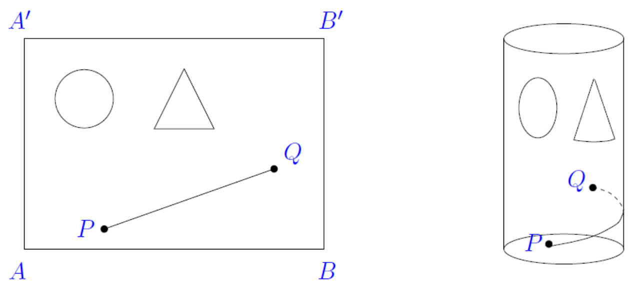

Figure 1: The Euclidean plane $\mathbb{R}^2$ can be rolled up into a cylindrical surface without distortion. The intrinsic geometry of the cylindrical surface is therefore the same as the plane. In particular, a bug confined to the surface would measure the sum of the angles of a triangle to be $180°$ and the circumference of a circle to be 2π times its radius.

As an example of the distinction between intrinsic and extrinsic geometry, consider the surface of a cylinder of radius a embedded in $\mathbb{R}^3$ (see Fig. 1).

In a cylindrical polar coordinate system, (z, ϕ), the interval is

$$ds^2 = dz^2 + a^2 d\phi^2 \,. \tag{17}$$

The intrinsic geometry is locally identical to the 2D Euclidean plane $\mathbb{R}^2$ since the coordinate transformation $y' = a\phi$ and $z' = z$ gives $ds^2 = dy'^2 + dz'^2$ everywhere.

This makes physical sense since the cylinder can be unrolled to give the plane without buckling, tearing or otherwise distorting.

However, the *extrinsic geometry* as seen within the embedding space $\mathbb{R}^3$ is clearly curved (non-Euclidean).

We can contrast the cylinder to a 2-sphere of radius a embedded in $\mathbb{R}^3$.

The intrinsic geometry, based on measurements made within the surface, is now not identical to the Euclidean plane since the surface of a sphere cannot be formed from the flat plane without deformation (this is why gift-wrapping a ball is hard!).

If we use polar coordinates (θ, ϕ), the interval on the 2-sphere is

$$ds^2 = a^2(d\theta^2 + \sin^2\theta \, d\phi^2) \,. \tag{18}$$

This cannot be transformed to Euclidean form $ds^2 = dx^2 + dy^2$ over the *entire* surface by any coordinate transformation, which shows that the intrinsic geometry is non-Euclidean – the space is intrinsically curved.

Note that at any point A, we can find coordinates (see example below) such that $ds^2 = dx^2 + dy^2$ in the local neighbourhood of A, but *not* over the entire surface.

General relativity is a theory involving the (local) intrinsic geometry of the spacetime manifold – no embedding in some higher-dimensional space is required.

Example 1: the 2-sphere in $\mathbb{R}^3$

For a surface embedded in a higher-dimensional space, the induced line element in the surface is determined by the line element in the embedding space and the "shape" of the surface.

Consider the 2-sphere embedded in $\mathbb{R}^3$; the embedding space has the Euclidean line element $ds^2 = dx^2 + dy^2 + dz^2$ in Cartesian coordinates.

If the sphere has radius a, points on its surface satisfy $x^2 + y^2 + z^2 = a^2$, so that

$$\begin{aligned} & 0 = 2xdx + 2ydy + 2zdz \\ \Rightarrow \quad & dz = -\frac{(xdx + ydy)}{z} = -\frac{(xdx + ydy)}{\sqrt{a^2 - x^2 - y^2}} \,. \end{aligned} \tag{19}$$

This is the constraint on dz that keeps us on the spherical surface for a displacement dx and dy in the x and y coordinates.

We obtain the induced line element by substituting dz in the line element of the embedding space ($\mathbb{R}^3$ here) to find

$$ds^2 = dx^2 + dy^2 + \frac{(xdx + ydy)^2}{a^2 - (x^2 + y^2)} \,. \tag{20}$$

Near the north or south poles, where $x^2 + y^2 \ll a^2$, the induced line element is approximately the Euclidean form, $ds^2 = dx^2 + dy^2$.

The induced metric looks neater if we use plane polar coordinates $x = \rho \cos\phi$ and $y = \rho \sin\phi$; then

$$\begin{aligned} dx &= \cos\phi d\rho - \rho \sin\phi d\phi \\ dy &= \sin\phi d\rho + \rho \cos\phi d\phi \,, \end{aligned} \tag{21}$$

and so $xdx + ydy = \rho d\rho$ and $dx^2 + dy^2 = d\rho^2 + \rho^2 d\phi^2$.

Putting these pieces together gives

$$ds^2 = \frac{a^2 d\rho^2}{(a^2 - \rho^2)} + \rho^2 d\phi^2 \,. \tag{22}$$

Example 2: the 3-sphere in $\mathbb{R}^4$

Now consider the 3-sphere, defined by $x^2 + y^2 + z^2 + w^2 = a^2$, embedded in 4D Euclidean space $\mathbb{R}^4$ with line element

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 \,. \tag{23}$$

Differentiating gives

$$\begin{aligned} 0 &= 2xdx + 2ydy + 2zdz + 2wdw \\ \Rightarrow \quad dw &= -\frac{(xdx + ydy + zdz)}{w} \\ &= -\frac{(xdx + ydy + zdz)}{\sqrt{a^2 - (x^2 + y^2 + z^2)}} \,, \end{aligned} \tag{24}$$

and so the induced line element is

$$ds^2 = dx^2 + dy^2 + dz^2 + \frac{(xdx + ydy + zdz)^2}{a^2 - (x^2 + y^2 + z^2)} \,. \tag{25}$$

As for the 2-sphere, the line element looks neater in polar coordinates; this time we use spherical-polar coordinates

$$\begin{aligned} x &= r \sin\theta \cos\phi \,, \\ y &= r \sin\theta \sin\phi \,, \\ z &= r \cos\theta \,. \end{aligned} \tag{26}$$

We find $xdx + ydy + zdz = rdr$ so that

$$ds^2 = \frac{a^2}{(a^2 - r^2)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \,. \tag{27}$$

This line element describes a non-Euclidean 3D space.

We shall meet this space again towards the end of the course, where we shall see that it describes the spatial part of a cosmological model with compact spatial sections (a closed universe).

In the limit $a \to \infty$ we recover 3D Euclidean space in spherical-polar coordinates.

More generally, for $r \ll a$ we recover $\mathbb{R}^3$ locally.

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4 Lengths and volumes

The metric functions determine an invariant distance measure on the manifold, and so also determine invariant "lengths" of curves and "volumes" of subregions.

4.1 Lengths along curves

Consider a curve $x^a(u)$ between points A and B on some manifold.

Since $ds^2 = g_{ab}(x)dx^a dx^b$ is the invariant distance between neighbouring points with coordinates separated by dx^a, the invariant length along the curve is

$$\boxed{L_{AB} = \int_{u_A}^{u_B} \left| g_{ab} \frac{dx^a}{du} \frac{dx^b}{du} \right|^{1/2} du \,.} \tag{28}$$

(The modulus sign is not required for a Riemannian manifold, where $ds^2 > 0$, but is generally required for application to spacetime.)

4.2 Volumes of regions

To calculate the volume of some region, we shall initially consider the simple case where the metric is diagonal, i.e., $g_{ab}(x) = 0$ for $a \neq b$.

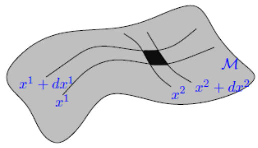
In this case,

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \cdots + g_{NN}(dx^N)^2 \,. \quad (29)$$

A coordinate system with a diagonal metric is called *orthogonal* since, as we shall discuss later when considering tangent vectors to curves, the coordinate curves (i.e., the curves obtained by allowing a single coordinate to vary in turn) are orthogonal to each other.

To be concrete, consider the 2D manifold $\mathcal{M}$ illustrated by the curved surface in the figure to the right.

The coordinates x^1 and x^2 form an orthogonal coordinate system in $\mathcal{M}$.



The volume element (infinitesimal rectangle for an orthogonal coordinate system in 2D) defined by coordinate increments dx^1 and dx^2 has sides of invariant length $\sqrt{g_{11}}dx^1$ and $\sqrt{g_{22}}dx^2$.

It follows that the invariant volume element is

$$dV = \sqrt{|g_{11}g_{22}|}dx^1dx^2 \,. \quad (30)$$

This generalises to the volume element of an ND manifold,

$$\boxed{dV = \sqrt{|g_{11}g_{22}\dots g_{NN}|}dx^1dx^2\dots dx^N \,.} \quad (31)$$

Similarly, one can define “area”-like elements on surfaces within manifolds by using the induced line element on the surface.

4.2.1 Invariance of the volume element

The result (31) for the volume element involves the determinant of the metric, since for a diagonal metric $g \equiv \det(g_{ab}) = g_{11}g_{22}\dots g_{NN}$.

This suggests that the generalisation to an arbitrary coordinate system is

$$dV = \sqrt{|g|}dx^1dx^2\dots dx^N\,. \tag{32}$$

Let us check that this is indeed an invariant volume element.

Consider a coordinate transformation $x^a \to x'^a$; under this, $dx^1dx^2\dots dx^N$ transforms with the Jacobian of the transformation matrix:

$$dx'^1dx'^2\dots dx'^N = Jdx^1dx^2\dots dx^N\,, \tag{33}$$

where, recall, $J = \det(\partial x'^a/\partial x^b)$.

Since the metric transforms as

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}g_{cd}\,, \tag{34}$$

the determinant of the metric transforms as

$$g' = g/J^2\,. \tag{35}$$

Here, we have used that $1/J = \det(\partial x^a/\partial x'^b)$, which follows since $\partial x^a/\partial x'^b$ is the inverse of the transformation matrix.

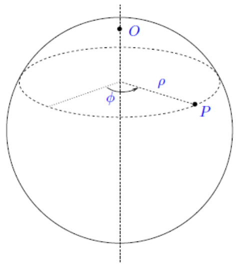
It follows that

$$\sqrt{|g'|}dx'^1dx'^2\dots dx'^N = \frac{\sqrt{|g|}}{J}Jdx^1dx^2\dots dx^N\,, \tag{36}$$

and so $dV = \sqrt{|g|}dx^1dx^2\dots dx^N$ is indeed invariant.

Example: surface of the 2-sphere in $\mathbb{R}^3$

Consider again the 2-sphere of radius a embedded in $\mathbb{R}^3$.



We write the line element as

$$ds^2 = \frac{a^2 d\rho^2}{(a^2 - \rho^2)} + \rho^2 d\phi^2 \,, \tag{37}$$

so the metric is diagonal with components

$$g_{11} = \frac{a^2}{(a^2 - \rho^2)} \quad \text{and} \quad g_{22} = \rho^2 \,. \tag{38}$$

Consider the circle $\rho = R$ (dashed circle in the figure to the right); we shall compute its length, the distance from its centre O to its perimeter, and the area enclosed.

The distance from the centre O to the perimeter along the curve $\phi =$ const. is given by

$$D = \int_0^R \frac{a}{(a^2 - \rho^2)^{1/2}} \, d\rho = a \sin^{-1}\left(\frac{R}{a}\right) \,. \tag{39}$$

For the circumference of the circle, we have

$$C = \int_0^{2\pi} R \, d\phi = 2\pi R \,. \tag{40}$$

For the area enclosed, we use Eq. (31) noting that in 2D the enclosed area is the "volume":

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^R \frac{a}{(a^2 - \rho^2)^{1/2}} \rho \, d\rho \, d\phi \\ &= 2\pi a^2 \left[1 - \left(1 - \frac{R^2}{a^2}\right)^{1/2}\right] \,. \end{aligned} \tag{41}$$

We can rewrite these results for C and A in terms of the (radius) distance D as follows:

$$C = 2\pi a \sin\left(\frac{D}{a}\right) \,, \tag{42}$$

$$A = 2\pi a^2 \left[1 - \cos\left(\frac{D}{a}\right)\right] \,. \tag{43}$$

We note the following points about these results:

- For $D \ll a$, we recover the Euclidean results $C = 2\pi D$ and $A = \pi D^2$.
- As D increases, both C and A increase until $D = \pi a/2$, after which C decreases.
- The coordinates (ρ, ϕ) are degenerate beyond the equator (the metric coefficient g_{11} makes it clear that the coordinates are poorly behaved at $\rho = a$).

  However, if we switch to coordinates (D, ϕ), this system is well defined beyond the equator, becoming degenerate only at $D = \pi a$ (the south pole).

  The metric in these coordinates is
  $$ds^2 = dD^2 + a^2 \sin^2\left(\frac{D}{a}\right) d\phi^2 \,. \tag{44}$$

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5 Local Cartesian coordinates

On a Riemannian manifold (assume $ds^2 > 0$ for now) it is generally *not* possible to choose coordinates such the line element takes the Euclidean form at *every* point.

This follows since $g_{ab}(x)$ has $N(N+1)/2$ independent functions, but there are only N functions involved in coordinate transformations.

However, it *is* always possible to adopt coordinates such that in the neighbourhood of some point P, the line element takes the Euclidean form.

More precisely, we can always find coordinates such that at P
$$g_{ab}(P) = \delta_{ab} \quad \text{and} \quad \left.\frac{\partial g_{ab}}{\partial x^c}\right|_P = 0 \,. \tag{45}$$

This means that, in the neighbourhood of P, we have

$$g_{ab}(x) = \delta_{ab} + O\left[(x - x_P)^2\right] \tag{46}$$

in these special coordinates.

Such coordinates are called *local Cartesian coordinates* at P.

In general relativity, we shall see that the generalisation of such coordinates to spacetime corresponds to coordinates defined by locally-inertial (i.e., free-falling) observers.

5.1 Proof of existence of local Cartesian coordinates

We shall prove the existence of local Cartesian coordinates by showing that a coordinate transformation $x^a \to x'^a$ has enough degrees of freedom to bring the metric to the form in Eq. (46).

Under the coordinate transformation, the metric and its derivatives transform as

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} g_{cd}\,, \tag{47}$$

$$\frac{\partial g'_{ab}}{\partial x'^e} = \frac{\partial}{\partial x'^e}\left(\frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\right) g_{cd} + \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\frac{\partial x^f}{\partial x'^e}\frac{\partial g_{cd}}{\partial x^f}\,. \tag{48}$$

We now try to construct the (as-yet) unknown relation $x^a(x')$ such that in the primed coordinates $g'_{ab} = \delta_{ab}$ and $\partial g'_{ab}/\partial x'^c = 0$ at P.

Consider the transformation matrices and their derivatives that appear in Eqs (47) and (48); at the point P, the numbers of independent degrees of freedom in these are

$$\begin{aligned} &\left.\frac{\partial x^a}{\partial x'^b}\right|_P && N^2 \text{ values,} \\ &\left.\frac{\partial^2 x^a}{\partial x'^b \partial x'^c}\right|_P && N^2(N+1)/2 \text{ values.} \end{aligned} \tag{49}$$

Contrast these with the number of degrees of freedom in the metric and its derivatives at P:

$$\begin{aligned} &g'_{ab}(P) && N(N+1)/2 \text{ values,} \\ &\left.\frac{\partial g'_{ab}}{\partial x'^c}\right|_P && N^2(N+1)/2 \text{ values.} \end{aligned} \tag{50}$$

If we try and set $g'_{ab}(P) = \delta_{ab}$ (which are $N(N+1)/2$ equations), we have more than enough degrees of freedom in the $\partial x^a/\partial x'^b$ to do so.

Indeed, we are left with $N(N-1)/2$ "unused" degrees of freedom in the $\partial x^a/\partial x'^b$.

For $N = 4$ in spacetime, these correspond to the six degrees of freedom (three boosts, three rotations) associated with homogeneous Lorentz transformations that preserve the Minkowski form of the metric.

Now consider trying to enforce further that $\partial g'_{ab}/\partial x'^c = 0$ at P.

These are $N^2(N+1)/2$ equations, which consume all of the second derivatives $\partial^2 x^a/\partial x'^b \partial x'^c$.

This proves that it is always possible to construct local Cartesian coordinates at a point.

Can we go further, i.e., can we also set the second derivatives of the metric to zero?

The answer is no: $\partial^2 g'_{ab}/\partial x'^c \partial x'^d = 0$ gives $N^2(N+1)^2/4$ equations, but the number of degrees of freedom in the third derivatives of the coordinates, $\partial^3 x^a/\partial x'^b \partial x'^c \partial x'^d$ is only $N^2(N+1)(N+2)/6$.

We see that there are generally

$N^2(N+1)^2/4 - N^2(N+1)(N+2)/6 = N^2(N^2-1)/12$

independent degrees of freedom in the second derivatives of the metric that cannot be eliminated by coordinate transformations.

It is these (20 for $N = 4$) that describe the *curvature* of the manifold and, in general relativity, the physical degrees of freedom associated with gravity.

6 Pseudo-Riemannian manifolds

In a Riemannian manifold, $ds^2 = g_{ab}\, dx^a\, dx^b$ is always positive for all dx^a.

Considered as a matrix, g_{ab} has to be positive definite at every point and so have all eigenvalues positive.

In a *pseudo-Riemannian* manifold, ds^2 can be positive, negative or zero depending on dx^a, which implies that some of the eigenvalues of g_{ab} are negative.

In a pseudo-Riemannian manifold one can always find coordinates such that at a point P

$$g_{ab}(P) = \eta_{ab}\,, \tag{51}$$

and the first derivatives of the metric vanish at P.

Here,

$$\eta_{ab} = \mathrm{diag}(\pm 1, \pm 1, \ldots, \pm 1)\,, \tag{52}$$

where the number of positive entries in η_{ab} minus the number of negative is the *signature* of the manifold.

(We shall always assume that the metric is sufficiently regular that the signature is the same at all points in the manifold.)

In the Minkowski spacetime of special relativity, we have the line element

$$ds^2 = d(ct)^2 - dx^2 - dy^2 - dz^2 \,. \tag{53}$$

This is an example of a pseudo-Riemannian manifold with $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$ taking the coordinates to be (ct, x, y, z).

7 Topology of manifolds

So far, we have discussed only the *local* geometry of manifolds, defined at any point by the line element.

In addition, a manifold also has a *global* geometry or *topology*, defined (crudely) by identification of points with different coordinates as being coincident.

For example, the surface of cylinder in $\mathbb{R}^3$ has same local intrinsic geometry as the Euclidean plane $\mathbb{R}^2$, but a different topology.

Indeed, the compact dimension on the surface of the cylinder could be detected by a "bug" confined to the surface since by continuing in a straight line (we shall define what we mean by a "straight line" in a general manifold later in the course) in a certain direction the bug would return to the same physical point.

Topology is an *intrinsic*, but non-local, property of a manifold.

General relativity is a local theory, in which the local intrinsic geometry is determined by energy density of matter/radiation at that point.

The field equations of general relativity do *not* constrain the global topology of the spacetime manifold.