Part-II RELATIVITY

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These lecture notes accompany the *Relativity* course in NST Part-II Physics and Part-II Astrophysics.

These notes are adopted from Prof. Anthony Challinor, the previous lecturer of this course, and are based on the slides of Prof. Mike Hobson.

RECOMMENDED BOOKS

Relativity, both special and general, are covered in numerous textbooks, many now considered classics.

Some suggestions, based on books I found particularly useful in preparing this course, are as follows.

- *General Relativity: An Introduction for Physicists*, Hobson M.P., Efstathiou G.P. and Lasenby A.N. (CUP 2006)
- *Spacetime and Geometry: An Introduction to General Relativity*, Carroll S.M. (Addison-Wesley 2003) – based on lecture notes that can be found at https://arxiv.org/abs/gr-qc/9712019
- *Introducing Einstein's Relativity*, d'Inverno R. (OUP 1992)
- *General Relativity*, Kenyon I.R. (OUP 1990)
- *Gravitation and Cosmology*, Weinberg S. (Wiley 1972)
- *Gravitation*, Misner C.W., Thorne K.S. and Wheeler J.A. (Freeman 1973)
- *General Theory of Relativity*, Dirac P.A.M. (Wiley 1975)

There are also many excellent lecture notes available freely online; along with Carroll's, I particularly recommend the less technical chapters of Matthias Blau's notes (`http://www.blau.itp.unibe.ch/newlecturesGR.pdf`).

0. INTRODUCTION

Much of this course is concerned with general relativity, a theory of space, time and gravity that naturally incorporates the ideas of special relativity.

We shall begin by seeing why Newtonian gravity is inconsistent with special relativity, and hence in need of an overhaul.

We shall uncover a striking feature that separates gravity apart from other fundamental forces – the universality of free-fall.

This observation leads to the radical idea that gravity is an inherent property of the geometry of spacetime itself, rather than a field defined over the fixed spacetime of special relativity, paving the way for the development of general relativity.

1 Newtonian gravity

Newtonian gravity is described by a space and time-dependent potential Φ, such that the force exerted on a test particle of mass m_G is

$$\vec{f} = -m_G \vec{\nabla}\Phi \,. \tag{1}$$

The quantity m_G is the *passive gravitational mass*, and it determines the gravitational force on the particle.

The potential is determined by all the matter that is present via Poisson's equation: for a mass density ρ,

$$\vec{\nabla}^2\Phi = 4\pi G\rho \,, \tag{2}$$

where G is the gravitational constant.

We immediately see that Newtonian gravity is inconsistent with special relativity: Poisson's equation implies that Φ (and so its observable gradient) responds instantaneously throughout space to changes in ρ, but we know in special relativity that no causal influence can travel faster than the speed of light.

Fixing this incompatibility will ultimately require a radical overhaul of how we think about gravity and, indeed, spacetime itself.

Sticking with Newtonian gravity for the moment, it is not immediately obvious that the mass density appearing in Poisson's equation should refer to the density of the passive gravitational mass.

Rather, let us also introduce the *active gravitational mass* m_A, so that the relevant mass density for a point particle at position $\vec{y}(t)$ at time t is

$$\rho(\vec{x},t) = m_A \delta^{(3)}(\vec{x} - \vec{y}(t)) \,. \tag{3}$$

For the point particle, the relevant solution of Poisson's equation is

$$\Phi(\vec{x},t) = -\frac{Gm_A}{|\vec{x} - \vec{y}(t)|} \,. \tag{4}$$

It follows that the force on a test particle of passive gravitational mass $m_{G,1}$ at position $\vec{x}_1$ at time t due to a particle of active gravitational mass $m_{A,2}$ at position $\vec{x}_2$ *at the same time* t is

$$\vec{f}_{2 \text{ on } 1} = -Gm_{G,1}m_{A,2}\frac{(\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|^3} \,. \tag{5}$$

Similarly, the force on the second particle due to the first is

$$\vec{f}_{1 \text{ on } 2} = -Gm_{G,2}m_{A,1}\frac{(\vec{x}_2 - \vec{x}_1)}{|\vec{x}_1 - \vec{x}_2|^3} \,. \tag{6}$$

If momentum is to be conserved, i.e., $\vec{f}_{2\text{ on }1} = -\vec{f}_{1\text{ on }2}$, we must have

$$m_{G,1}m_{A,2} = m_{G,2}m_{A,1} \,. \tag{7}$$

Since this must hold for arbitrary masses, we must have that the ratio of passive to active gravitational mass is the same for all particles.

With a suitable choice of the gravitational constant, we can take these masses to be equal, $m_G = m_A$, for all matter.

This sort of universality is not unusual in physics – a similar thing happens in electromagnetism, for example, where the passive and active electric charges are equal.

However, there is a further equality of masses in Newtonian gravity that is rather more surprising: the equality of gravitational and inertial masses.

A particle acted on by a force $\vec{f}$ experiences an acceleration such that

$$\vec{f} = m_I \frac{d^2\vec{x}}{dt^2} \,, \tag{8}$$

where m_I is the *inertial mass.*

For the gravitational force, the acceleration is

$$\frac{d^2\vec{x}}{dt^2} = -\frac{m_G}{m_I}\vec{\nabla}\Phi \,. \tag{9}$$

It is an experimental fact[1] (known since Galileo's time) that the ratio m_G/m_I is the same for all particles, so we can always take $m_G = m_I$ (further absorbing their universal ratio in the gravitational constant).

This means that if two particles of different composition

[1]The equality of gravitational and inertial masses is now verified to the level of one part in 10^{13}.

fall freely in a gravitational field, they have the same acceleration.

This is often rephrased as the *weak equivalence principle*:

> Freely-falling particles with negligible gravitational self-interaction follow the same path through space and time if they have the same initial position and velocity, independent of their composition.

This property of gravity is in striking contrast to other forces; for example, in electromagnetism the acceleration of a point particle in a given electric field depends on the ratio of the electric charge to inertial mass, which is definitely not universal.

2 Implications of the equivalence principle

Consider an observer in a free-falling, non-rotating elevator in a uniform gravitational field.

Relative to this observer, free-falling particles move on straight lines at constant velocity – the effects of the uniform gravitational field have been removed and the observer perceives that the usual laws of special relativistic kinematics hold.

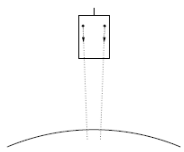
This idea motivates an extension of the weak equivalence principle to what is known as the *strong equivalence principle*:

> In an arbitrary gravitational field, *all* the laws of physics in a free-falling, non-rotating laboratory occupying a sufficiently small region of spacetime look locally like special relativity (with no gravity).

Note how the strong equivalence principle is supposed to apply to all laws of physics, not just the dynamics of free-falling particles.

Why the qualification of observations over a sufficiently small region of spacetime?

Consider the same elevator falling freely in the non-uniform gravitational field of the earth (see figure to the right).



Free particles initially at rest in the elevator will move together over time as they follow radial trajectories towards the centre of the earth.

It is these *tidal effects* that are the physical manifestation of the gravitational field, and that cannot be removed by passing to the free-falling frame.

However, for sufficiently local measurements in space and time, these tidal effects are undetectable, and physics relative to the free-falling elevator looks just like special relativistic physics in an inertial frame of reference in the absence of gravity.

The strong equivalence principle implies the *local* equivalence of a gravitational field and acceleration.

In particular, it implies that a constant gravitational field is unobservable – observations in a reference frame at rest in such a field would be indistinguishable from those in a uniformly-accelerating reference frame in the absence of gravity.

In special relativity, physics looks simple when referred to an inertial frame, one defined by comoving, unaccelerated observers with synchronised clocks.

However, with gravity, the equivalence principle tells us that physics looks equally simple *locally* in a free-falling reference frame, suggesting that we should *define* inertial reference frames locally by free-falling observers.

Acceleration should be defined relative to such local inertial frames, so that a particle acted on by no other force (and so free-falling) should be regarded as unaccelerated.

2.1 Gravity as spacetime curvature

The universality of free fall suggested to Einstein that the trajectories of free-falling particles should be determined by the local structure of spacetime, rather than by the action of a gravitational force with a mysterious universal coupling to matter.

Local inertial reference frames correspond to local systems of coordinates over spacetime so that the geometry over a small region looks like that of the spacetime of special relativity.

Gravity manifests itself through our inability to extend such coordinates globally, reflecting the *curvature of spacetime.*

General relativity abandons the idea of gravity as a force defined on the fixed spacetime of special relativity, replacing it with a geometric theory in which the geometry of spacetime determines the trajectories of free-falling particles, the geometry itself being curved by the presence of matter.

3 Further motivation: extreme gravity

Newtonian gravity is recovered from general relativity in the limit of low relative speeds of particles, $v \ll c$, and weak gravitational fields, typically $|\Phi| \ll c^2$.

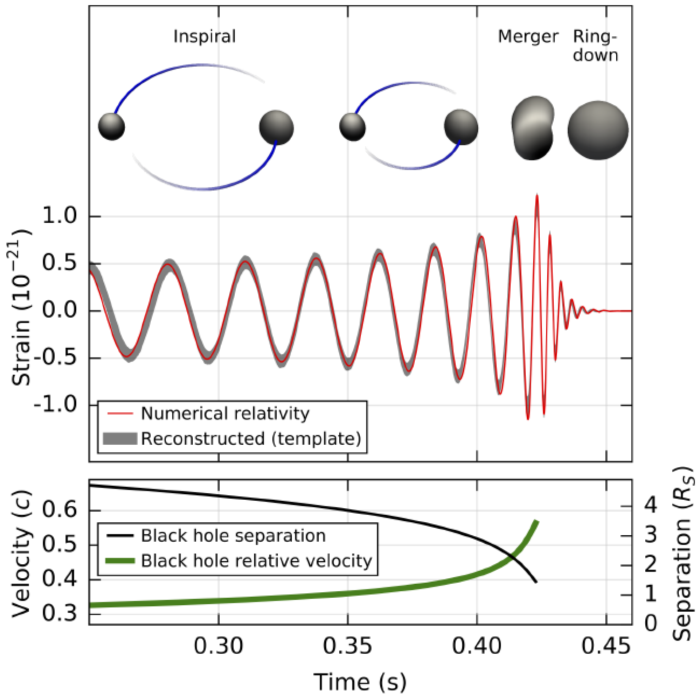


Figure 1: *Top*: Estimated gravitational wave strain amplitude inferred from the LIGO data for their discovery event. The signal is generated from the inspiral, merger and ring-down of two massive black holes. The properties of the source can be estimated by comparing the measured waveform with detailed calculations in general relativity. *Bottom*: the relative speed and separation (in units of the Schwarzschild radius, $R_S = 2GM/c^2$) of the blackholes during the event. For reference, the Newtonian potential at R_S away from a mass M is $|\Phi|/c^2 = 1/2$. Figure taken from Abbot et al., Phys. Rev. Lett. **116**, 061102 (2016).

Note that in situations where speeds are determined by gravity, these two regimes are generally equivalent.

To see this, consider a particle in a circular orbit of radius R around a mass M in Newtonian gravity: the speed is determined by

$$\frac{v^2}{R} = \frac{GM}{R^2}\,, \tag{10}$$

and so

$$\frac{v^2}{c^2} = \frac{GM}{Rc^2} = \frac{|\Phi|}{c^2}\,. \tag{11}$$

However, increasingly we are observing phenomena where

Newtonian gravity is a very poor approximation.

A striking example is the recent first detection of gravitational waves by the LIGO interferometer; see Fig. 1.

Gravitational waves are wavelike disturbances in the geometry of spacetime, which can be detected by looking for their characteristic quadrupole distortion (i.e., a shortening in one direction and stretching in an orthogonal direction) of the two arms of a laser interforemeter.

Gravitational waves propagate at the speed of light and are a natural prediction of general relativity; they do not arise in Newtonian gravity where the potential responds instantly to distant rearrangements of mass.

The first LIGO signal was generated by a truly extreme astrophysical source: two merging black holes each with a mass around 30 times that of the Sun at a distance from us of around 2 Gly.

As the blackholes orbited their common centre of mass, the system radiated gravitational waves causing the blackholes to spiral inwards and increase their speed until they merged to form a single black hole.

Such sources probe the strong-field regime of general relativity during the merger phase and involve highly relativistic speeds (see Fig. 1).

At its peak, the source was losing energy to gravitational waves at a rate of 3.6×10^{49} W, which is equivalent to 200 times the rest mass energy of the Sun per second!