II. Wiener process & Martingale

Def) Symmetric Random Walk (술 취한 사람이 비틀비틀 걸어가는 것을 비유.)

↳ We repeatedly toss a fair coin (prob. $= \frac{1}{2}$)

Let $X_j = \begin{cases} 1 & \text{if Head} \\ -1 & \text{if Tail} \end{cases}$ and $E[X_j] = 0$, $Var(X_j) = 1$

define $M_0 = 0$, $M_k = \sum_{j=1}^{k} X_j$, $k = 1, 2, \cdots$

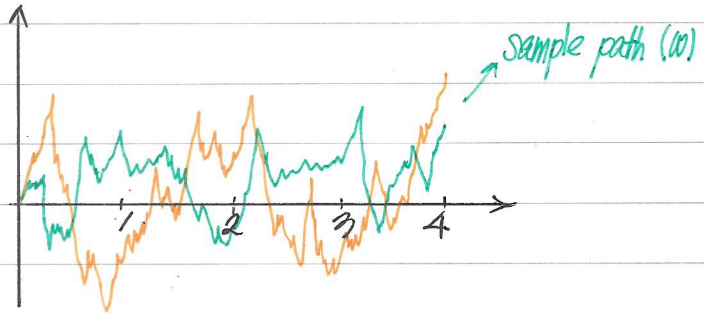
The process M_k, $k = 1, 2, \cdots$ is a symmetric random walk.

Def) Scaled Symmetric Random Walk.

↳ $W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$ $\left(\sim N(0, t) \text{ as } n \to \infty \right)$

Ex) $W^{(100)}$ up to $t = 4$

⇒ this was generated by 400 coin tosses with a step up or down of size $\frac{1}{10}$ on each coin toss.



* **Properties of Symmetric Random Walk**

↳ ① If we choose nonnegative integers $0 = k_0 < k_1 < \cdots < k_m$, the random variables $M_{k_1} = (M_{k_1} - M_{k_0}), (M_{k_2} - M_{k_1}), \cdots, (M_{k_m} - M_{k_{m-1}})$ are independent.

② Each of these random variables $M_{k_{i+1}} - M_{k_i}$ is called an increment of the random walk.

③ $E[M_{k_{i+1}} - M_{k_i}] = E\left[\sum_{j=k_i+1}^{k_{i+1}} X_j\right] = \sum_{j=k_i+1}^{k_{i+1}} E[X_j] = 0$

④ $Var(M_{k_{i+1}} - M_{k_i}) = \sum_{j=k_i+1}^{k_{i+1}} Var(X_j) = \sum_{j=k_i+1}^{k_{i+1}} 1 = k_{i+1} - k_i$

⑤ For $k < l$,

$$\begin{aligned} E[M_l \mid \mathcal{F}_k] &= E[(M_l - M_k) + M_k \mid \mathcal{F}_k] \\ &= E[M_l - M_k \mid \mathcal{F}_k] + E[M_k \mid \mathcal{F}_k] \\ &\overset{\text{independent}}{=} E[M_l - M_k] + M_k \\ &= M_k \end{aligned}$$

∴ martingale

⑥ Quadratic Variation

$[M, M]_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2 = \sum_{j=1}^{k} X_j^2 = k$

* **Properties of Scaled Symmetric Random Walk.**

↳ ① $E[W^{(n)}(t) - W^{(n)}(s)] = 0$

② $Var(W^{(n)}(t) - W^{(n)}(s)) = \frac{1}{n}(nt - ns) = t - s$

③ $E[W^{(n)}(t) \mid \mathcal{F}(s)] = W^{(n)}(s)$

④ $$\begin{aligned} [W^{(n)}, W^{(n)}](t) &= \sum_{j=1}^{nt} \left[W^{(n)}\left(\tfrac{j}{n}\right) - W^{(n)}\left(\tfrac{j-1}{n}\right) \right]^2 \\ &= \sum_{j=1}^{nt} \frac{1}{n} X_j^2 \\ &= \frac{1}{n} \times nt \\ &= t \end{aligned}$$

THM. Central Limit Theorem.

↳ Fix $t \geq 0$. As $n \to \infty$, the distribution of the scaled random walk $W^{(n)}(t)$ evaluated at time t converges to the normal distribution with mean zero and variance t.

$\Rightarrow W^{(n)}(t) \sim N(0,t)$ as $n \to \infty$

pf) * 같은 mgf를 가지면 distribution이 같다.

$$\varphi^{(n)}(u) = \mathbb{E}\left[e^{uW^{(n)}(t)}\right]$$

$$= \mathbb{E}\left[e^{u \cdot \frac{1}{\sqrt{n}} M_{nt}}\right]$$

$$= \mathbb{E}\left[e^{u \cdot \frac{1}{\sqrt{n}} \sum_{j=1}^{nt} X_j}\right]$$

$$= \mathbb{E}\left[\prod_{j=1}^{nt} e^{\frac{u}{\sqrt{n}} X_j}\right]$$

$$= \prod_{j=1}^{nt} \mathbb{E}\left[e^{\frac{u}{\sqrt{n}} X_j}\right]$$

if X, Y : independent, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

$$= \prod_{j=1}^{nt} \left(e^{\frac{u}{\sqrt{n}}} \times \frac{1}{2} + e^{-\frac{u}{\sqrt{n}}} \times \frac{1}{2}\right)$$

$$= \left(\frac{1}{2} e^{\frac{u}{\sqrt{n}}} + \frac{1}{2} e^{-\frac{u}{\sqrt{n}}}\right)^{nt}$$

$$\ln \varphi^{(n)}(u) = nt \cdot \ln\left(\frac{1}{2} e^{\frac{u}{\sqrt{n}}} + \frac{1}{2} e^{-\frac{u}{\sqrt{n}}}\right)$$

(Let $x = \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$)

$$= t \cdot \frac{\ln\left(\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}\right)}{x^2}$$

L'Hopital

$$\Rightarrow t \cdot \frac{\frac{u}{2}e^{ux} - \frac{u}{2}e^{ux}}{2x \cdot \left(\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}\right)}$$

$$= \frac{t}{\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}} \cdot \frac{\frac{u}{2}e^{ux} - \frac{u}{2}e^{-ux}}{2x}$$

$$\Rightarrow t \cdot \frac{\frac{u^2}{2}e^{ux} + \frac{u^2}{2}e^{-ux}}{2} \Rightarrow \frac{u^2 t}{2}$$

Note that if $X \sim N(\mu, \sigma^2)$, mgf of $X = \mathbb{E}[e^{uX}] = e^{\mu u + \frac{1}{2}\sigma^2 u^2}$

$\therefore X \sim N(0,t) \Rightarrow$ mgf of $X = e^{\frac{1}{2}u^2 t}$

$\therefore \lim_{n\to\infty} \varphi^{(n)}(u) = e^{\frac{1}{2}u^2 t}$

$\therefore W^{(n)}(t) \sim N(0,t)$ as $n \to \infty$

cf) $\exists$: there exists (존재한다.)

Def) Brownian Motion

↳ Limit of the scaled random walk $W^{(n)}(t)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, Suppose $\exists$ a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and depends on ω. Then, $W(t)$, $t \geq 0$, is a Brownian Motion if for all $0 = t_0 < t_1 < \cdots < t_m$ increments $W(t_1) = W(t_1) - W(t_0)$, $W(t_2) - W(t_1)$, $\cdots$, $W(t_m) - W(t_{m-1})$ are independent and each of these increments is normally distributed with $\mathbb{E}[W(t_{j+1}) - W(t_j)] = 0$, $Var(W(t_{j+1}) - W(t_j)) = t_{j+1} - t_j$

$W(t_{j+1}) - W(t_j) \sim N(0, t_{j+1} - t_j)$

정리 Wiener process.

↳ $W(t) \sim N(0,t)$ with independent increment

↳ $(t_1, t_2) \cap (s_1, s_2) = \emptyset$ 일 때, $W(t_2) - W(t_1)$ 과 $W(s_2) - W(s_1)$은 서로 independent

$W(0) = 0$

$\mathbb{E}[W(t)] = 0$

$Var(W(t) - W(s)) = t - s \quad (t \geq s)$

cf) $Cov(W(t), W(s)) = s \wedge t \quad (= \min(s,t))$

Def) Quadratic Variation

↳ Let $f(t)$ be a function defined for $0 \le t \le T$.
The quadratic variation of $f(t)$ up to time T is

$$[f,f](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2$$

where $\Pi = \{t_0, \cdots, t_n\}$ and $0 = t_0 < t_1 < \cdots < t_n = T$

$\|\Pi\| = \max_j |t_{j+1} - t_j|$

Remark)

↳ Let $f \in C^1_{[0,T]}$

$$QV(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|^2$$

$$= \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f'(t_j^*)|^2 \cdot |t_{j+1} - t_j|^2 \quad (\because \text{Mean Value THM})$$

중간값 정리

$$\le \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \|\Pi\| \cdot |f'(t_j^*)|^2 |t_{j+1} - t_j|$$

$$= \lim_{\|\Pi\| \to 0} \|\Pi\| \cdot \int_0^T f'(t_j^*)^2 \, dt = 0$$

$\therefore QV(f) = 0$

cf) $E[X] = m,\ Var(X) = 0 \Rightarrow X = m$

THM.

↳ $[W,W](T) = T,\quad \forall T \ge 0$ (fixed T)

즉, $\lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2 = T$ on $[0,T]$

pf) Let $Q_\pi = \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2$

$$E[(W(t_{j+1}) - W(t_j))^2]$$
$$= Var(W(t_{j+1}) - W(t_j)) = t_{j+1} - t_j$$

$$\therefore E[Q_\pi] = E\left[\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\right]$$
$$= \sum_{j=0}^{n-1} E[(W(t_{j+1}) - W(t_j))^2]$$
$$= \sum_{j=0}^{n-1} (t_{j+1} - t_j)$$
$$= T$$

Lemma. $E[W(t)^4] = 3t^2$

$$Var((W(t_{j+1}) - W(t_j))^2)$$
$$= E\Big[\big((W(t_{j+1}) - W(t_j))^2 - \underbrace{E[(W(t_{j+1}) - W(t_j))^2]}_{= t_{j+1} - t_j}\big)^2\Big]$$
$$= E[(W(t_{j+1}) - W(t_j))^4 - 2(t_{j+1} - t_j)(W(t_{j+1}) - W(t_j))^2 + (t_{j+1} - t_j)^2]$$
$$= E[(W(t_{j+1}) - W(t_j))^4] - 2(t_{j+1} - t_j)\underbrace{E[(W(t_{j+1}) - W(t_j))^2]}_{= t_{j+1} - t_j} + (t_{j+1} - t_j)^2$$
$$= 3(t_{j+1} - t_j)^2 - (t_{j+1} - t_j)^2$$
$$= 2(t_{j+1} - t_j)^2$$

$$\therefore Var(Q_\pi) = Var\left(\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\right)$$
$$= \sum_{j=0}^{n-1} Var((W(t_{j+1}) - W(t_j))^2)$$
$$= \sum_{j=0}^{n-1} 2(t_{j+1} - t_j)^2$$
$$\le 2 \cdot \|\Pi\| \cdot \sum_{j=0}^{n-1} (t_{j+1} - t_j)$$
$$= 2T \cdot \|\Pi\| \to 0 \text{ as } n \to \infty$$

$$\therefore \begin{cases} \lim_{n \to \infty} E[Q_\pi] = T \\ \lim_{n \to \infty} Var(Q_\pi) = 0 \end{cases}$$

$\therefore [W,W](T) = \lim_{\|\Pi\| \to 0} Q_\pi = E[Q_\pi] = T$

cf) $d[W,W] = dWdW$

Remark.

$\mathbb{E}[(W(t_{j+1})-W(t_j))^2] = t_{j+1}-t_j$

$Var[(W(t_{j+1})-W(t_j))^2] = 2(t_{j+1}-t_j)^2 \ll 1 \to 0$

$\therefore \underbrace{(W(t_{j+1})-W(t_j))^2}_{\Delta W \Delta W} \approx \underbrace{t_{j+1}-t_j}_{\Delta t}$

$\Rightarrow dWdW = dt$

$\lim_{\|\pi\|\to 0} \sum_{j=0}^{n-1} (W(t_{j+1})-W(t_j))(t_{j+1}-t_j) = 0$

pf) $\sum_{j=0}^{n-1} |(W(t_{j+1})-W(t_j))(t_{j+1}-t_j)|$

$= \sum_{j=0}^{n-1} |W(t_{j+1})-W(t_j)| \cdot (t_{j+1}-t_j)$

$\le \max_j |W(t_{j+1})-W(t_j)| \sum_{j=0}^{n-1} (t_{j+1}-t_j)$

$= T \cdot \underbrace{\max_j |W(t_{j+1})-W(t_j)|}_{} \to 0$ as $\|\pi\| \to 0$.

($\because W(t)$ is continuous)

$\Rightarrow dWdt = 0$

$\lim_{\|\pi\|\to 0} \sum_{j=0}^{n-1} (t_{j+1}-t_j)^2 = 0$

pf) $\sum_{j=0}^{n-1} (t_{j+1}-t_j)^2 \le \|\pi\| \cdot \sum_{j=0}^{n-1} (t_{j+1}-t_j)$

$= \|\pi\| \cdot T \to 0$ as $\|\pi\| \to 0$.

$\Rightarrow dtdt = 0$

cf) General properties of Expectation.

① $\mathbb{E}[aX+bY|G] = a\mathbb{E}[X|G] + b\mathbb{E}[Y|G]$

② $\mathbb{E}[\mathbb{E}[X|G]] = \mathbb{E}[X]$ Law of iterated expectation,

③ $\mathbb{E}[XY|G] = X\mathbb{E}[Y|G]$ if X is G-measurable

④ $\mathbb{E}[X|G] = \mathbb{E}[X]$ if X is independent of G

⑤ $\mathbb{E}[\mathbb{E}[X|G]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$ if $\mathcal{H} \subset G$

cf) convex

A function $f: \mathbb{R} \to \mathbb{R}$ is convex

if for any $x, y \in \mathbb{R}$ & $\lambda \in [0,1]$,

$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$

Ex) $\lambda = \frac{1}{2} \Rightarrow f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$

cf) Jensen's Inequality

$f: \mathbb{R} \to \mathbb{R}$, convex

X: integrable random variable.

$f(X)$ is also integrable.

Then, $\begin{cases} f(\mathbb{E}[X|G]) \le \mathbb{E}[f(X)|G], & \forall G \subset \mathcal{F} \\ f(\mathbb{E}[X]) \le \mathbb{E}[f(X)] \end{cases}$

Def) Filtration

↳ A sequence of σ-fields $\mathcal{F}_1, \mathcal{F}_2, \cdots$ s.t. $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ is called a filtration

Def) adapted

↳ We say that a sequence of random variables $\xi_1, \xi_2, \cdots$ is "adapted" to a filtration $\mathcal{F}_1, \mathcal{F}_2 \cdots$ if ξ_n is $\mathcal{F}_n$-measurable for each $n = 1, 2, \cdots$

Def) Martingale

↳ A sequence $\xi_1, \xi_2, \cdots$ of random variables is called a martingale with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \cdots$ if

① ξ_n is integrable for all n. $\mathbb{E}[|\xi_n|] < \infty$

② $\xi_1, \xi_2, \cdots$ is adapted to $\mathcal{F}_1, \mathcal{F}_2, \cdots$

③ "$\mathbb{E}[\xi_{n+1} | \mathcal{F}_n] = \xi_n$" for all n.

↳ $\mathbb{E}[\mathbb{E}[\xi_{n+1} | \mathcal{F}_n]] = \mathbb{E}[\xi_n]$

$\therefore$ $\mathbb{E}[\xi_{n+1}] = \mathbb{E}[\xi_n]$ $\Rightarrow$ fair game.

For $k < l$,

$\mathbb{E}[\xi_l | \mathcal{F}_k] = \xi_k$ $\Rightarrow$ martingale

Ex) Brownian Motion is martingale

↳ pf)
$$
\begin{aligned}
&\mathbb{E}[W(t) | \mathcal{F}(s)] \\
&= \mathbb{E}[W(t) - W(s) + W(s) | \mathcal{F}(s)] \\
&= \mathbb{E}[W(t) - W(s) | \mathcal{F}(s)] + \mathbb{E}[W(s) | \mathcal{F}(s)] \\
&= \mathbb{E}[W(t) - W(s)] + W(s) \\
&= W(s)
\end{aligned}
$$

Def) supermartingale / submartingale

↳ A sequence $\xi_1, \cdots, \xi_n$ is a supermartingale / submartingale with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \cdots$ if

① ξ_n is integrable for all n.

② $\xi_1, \xi_2, \cdots$ is adapted to $\mathcal{F}_1, \mathcal{F}_2, \cdots$

③ $\mathbb{E}[\xi_{n+1} | \mathcal{F}_n]$ $\leq$ ξ_n supermartingale

$\geq$ submartingale

↓

supermartingale	submartingale
$\mathbb{E}[\xi_{n+1}] \leq \mathbb{E}[\xi_n]$	$\mathbb{E}[\xi_{n+1}] \geq \mathbb{E}[\xi_n]$
unfavourable	favourable