CHAPTER 12 BINOMIAL TREES

Derivatives Securities
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Chapter Outline

- A simple binomial model.
  - Risk-neutral probabilities.
  - Deltas.
  - Volatilities.
- Multiple-step binomial trees.

Pricing of Options

조건부

- Since options are contingent claims, they are difficult to price fairly.
- There are two popular methods:
  - A binomial model (discrete, numerical). 이산적, 직관적
  - Black-Scholes-Merton model (continuous, analytic). 연속적
- Theoretically, two methods are identical. 이항모형 (n을 무한대로 보내면) 블랙숄즈머튼이 됨
- In this chapter, we discuss binomial models.

옵션의 만기일까지 주가가 변할수 있는 과정(path)를 tree모형으로 보여줌 (주가는 랜덤워크 따른다고 가정)

무차익논리

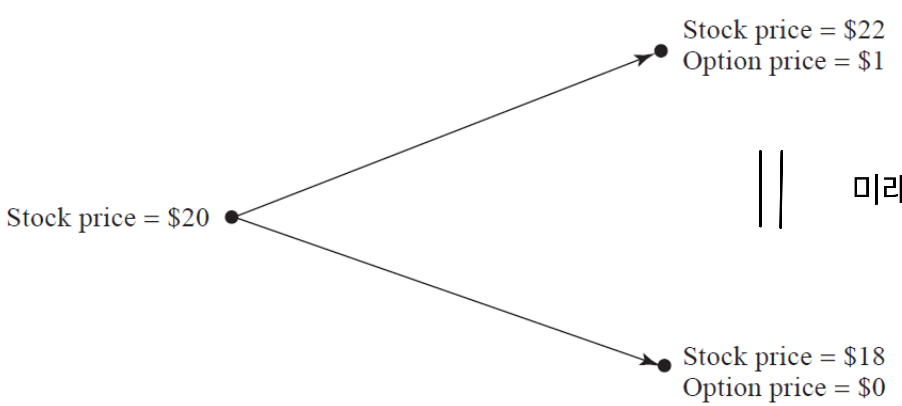
이항과정 수치화절차

위험중립 가치평가

Example: Binomial Models

- Suppose that:
  - The current price of a stock is \$20. $S_0=20$
  - In 3 months, the price will be either \$22 or \$18. $u: 1.1$ $d: 0.9$
  - There is a call option to buy the stock for \$21. $k=21$
  - The risk-free rate for 3 months is 12% per annum in continuous compounding.

Figure 12.1 Stock price movements for numerical example in Section 12.1.



미래 결과 상관없이 가치가 같아야 한다

현재 미래

무위험 포트폴리오 구성해서 옵션의 현재 가치를 추정

Example: Binomial Models

- Consider a portfolio consisted of:
  - A long position in Δ shares of the stock.
  - A short position in one call option.
- Then, the payoff from the portfolio in 3 months is either:
  - $22\Delta - 1$.
  - 18Δ
- Hence, the portfolio is risk-free if

$$22\Delta - 1 = 18\Delta$$

Example: Binomial Models

- Solving for Δ gives,

$$\Delta = 0.25$$

- If $\Delta = 0.25$, the portfolio is an risk-free asset, which yields

$$22\Delta - 1 = \$4.5$$

- The current price of the portfolio should be

$$4.5e^{-0.25\times 0.12} = \$4.367$$

Example: Binomial Models

- Therefore,

$$20 \times 0.25 - c = 4.367$$

- Solving for c gives

$$c = \$0.633$$

0.무위험포트 구성했다고 가정 (콜1개 매도, 주식 델타개 매수)
1. 미래시점 두 경우의 결과가 같게 델타 구하기
2. 델타 대입해서 미래시점에서 포트폴리오 가치 구하기
3. 무위험이자율만큼 현재시점으로 할인
4. 옵션의 현재가치 구하기

Example: Binomial Models

다른 방식으로도 증명가능(replicate)

- Here is an alternative way… Consider a replicating portfolio of the call option, which is consisted of:
  - Δ shares of the stock.
  - A zero-rate bond which yields x in 3 months.
- The payoff from the portfolio in 3 months is either:
  - $22\Delta + x$
  - $18\Delta + x$

각 상황모두 델타만큼 주식사고, 미래 x를 지급하는 채권 포트폴리오
에 넣어서 Replicate가능 (이 때 채권 매수인지 매도인지 모름)

- By the definition of a replicating portfolio,

$$\begin{cases} 22\Delta + x = 1 \\ 18\Delta + x = 0 \end{cases}$$

Example: Binomial Models

- Solving for Δ and x gives

연립시 델타 0.25개, 채권은 음수이므로 위에서 채권은 매도 한 것으로 파악가능

$$\begin{cases} \Delta = 0.25 \\ x = -\$4.5 \end{cases}$$

- The current price of one call option should be the same as the current price of the replicating portfolio, which is

$$c = 20 \times 0.25 - 4.5e^{-0.25\times 0.12} = \$0.633$$

√ 원서 pg 275, 한국어 292

A Simple Binomial Model

미래 포트폴리오 기댓값을 할인한 것에 대한 일반화

- The current price of a derivative is

상승시 옵션의 가치

$$f = e^{-rT}[pf_u + (1-p)f_d]$$

상승확률

Where

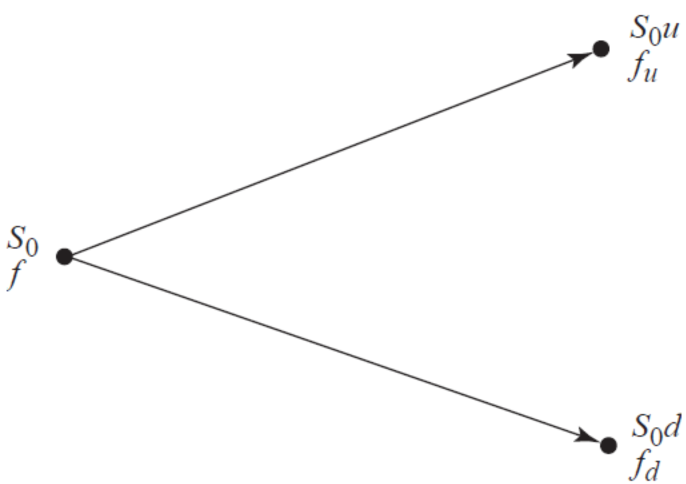
√
$$p = \frac{e^{rT} - d}{u - d}$$

u와 d는 각각 상승폭 하락폭

- f_u is the payoff from the derivative if the price of the underlying asset moves up with $u > 1$ rate of change.
- f_d is the payoff from the option if the price of the underlying asset moves down with $d < 1$ rate of change.

A Simple Binomial Model

Figure 12.2 Stock and option prices in a general one-step tree.



Proof: Simple Binomial Model

- Consider a portfolio consisted of:
  - A long position in Δ units of the underlying asset.
  - A short position in one unit of the derivative.
- If there is an upward movement in the price, the payoff from the portfolio is

$$S_0 u\Delta - f_u$$

- If there is an downward movement, the payoff from the portfolio is

$$S_0 d\Delta - f_d$$

Proof: Simple Binomial Model

- Then, the portfolio is risk-free if

$$S_0 u\Delta - f_u = S_0 d\Delta - f_d$$

- Solving for Δ gives

$$\Delta = \frac{f_u - f_d}{S_0(u - d)}$$

- The present value of the portfolio should be the same as the current price of the portfolio. That is,

$$(S_0 u\Delta - f_u)e^{-rT} = S_0\Delta - f$$

↓ 식정리

$$f = S_0\Delta(1 - ue^{-rT}) + f_u e^{-rT}$$

↓ $\Delta = \dfrac{f_u - f_d}{S_0 u - S_0 d}$ 대입

$$f = \cancel{S}_0\left(\frac{f_u - f_d}{\cancel{S}_0 u - \cancel{S}_0 d}\right)(1 - ue^{-rT}) + f_u e^{-rT}$$

↓ 식정리

$$f = \frac{f_u(1 - de^{-rT}) + f_d(ue^{-rT} - 1)}{u - d}$$

⇓ e^{-rT} 따로 빼주기

$$f = \frac{f_u(e^{rT} - d) + f_d(u - e^{rT})}{u - d} \times e^{-rT}$$

: p

: $1-p$

Proof: Simple Binomial Model

- Solving for f gives

$$f = e^{-rT} \times \frac{f_u(e^{rT} - d) + f_d(u - e^{rT})}{u - d}$$

- Let $p = \frac{e^{rT} - d}{u - d}$. Then,

$$f = e^{-rT}[pf_u + (1 - p)f_d]$$

단, $d < e^{rT} < u$ else 차익거래기회 발생

Example: Simple Binomial Model

- From the previous example,

$$\begin{cases} u = 1.1 \\ d = 0.9 \end{cases}$$

- Then,

$$p = \frac{e^{0.12\times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Example: Simple Binomial Model

- Also,

$$\begin{cases} f_u = \$1 \\ f_d = \$0 \end{cases}$$

- Therefore,

$$\begin{aligned} f &= e^{-0.25\times 0.12}(0.6523 \times 1 + 0.3477 \times 0) \\ &= \$0.633 \end{aligned}$$

Risk-Neutral Probabilities

- In the simple binomial model, p and $1 - p$ can be interpreted as probabilities. 전에 보았듯이 P는 확률로 해석 가능
  - The payoff $pf_u + (1 - p)f_d$ can be interpreted as the expected payoff under these probabilities.
- The **risk-neutral probabilities** are *ideological* probabilities under the assumption that all agents are risk-neutral. 확률인데 무슨확률? 위험중립확률! (이상적)
  - The discounting factor, or required rate of return, is always as same as the risk-free rate. 할인률과 기대이자율을 모두 무위험이자율로 만드는 그런 확률
- Risk-neutral probabilities are different from the real probabilities. 이런 이상적인(계산용이) 위험중립확률은 당연히 실제 확률과 다름

실제세계에서는 자산마다 위험도가 다르고 사람들이 위험회피 성향을 가지고 있기때문에 기대이자율과 할인율이 다 다름
그래서 이 세계를 위험중립이라고 가정 -> 할인율 기대이자율을 무위험이자율로 통침
그럼 실제세계와 위험중립세계에서 주가상승 확률이 같나? -> 다르다
다른게 문제가 되지않나 그럼? -> 옵션의 가치를 구할때는 실제 주가 상승확률을 고려하지 않는다

$$E(S_T) = pS_0u + (1-p)S_0d$$

$$E(S_T) = pS_0(u-d) + S_0d$$

실제로 위험중립 가정하에 구한 P를 주가의 기대값 식에 대입하면

$$P = \frac{e^{rT}-d}{u-d}$$

$$\Downarrow$$

$$E(S_T) = S_0 \cdot e^{rT} - \cancel{d \cdot S_0} + \cancel{S_0 \cdot d}$$

⇒ 주가가 무위험 이자율 만큼 성장함을 알수 있다.

Risk-Neutral Probabilities

- More formal definition of the risk-neutral probability is derived from the following equation:

$$pS_0u + (1-p)S_0d = S_0a$$

  - S_0 is initial price of the underlying asset.
  - u is the change rate in upward movement.
  - d is the change rate in downward movement.
  - a is the growing factor of the underlying asset.
- E.g., The values of a are different:
  - If the underlying asset gives dividend, $a = e^{(r-d)T}$.
  - If the underlying asset is currency, $a = e^{(r-r_f)T}$.

Deltas

파생상품을 헷지하기 위해 필요한 주식의 개수

- The **delta** of a stock option is the ratio of the change in the price of the stock option to the change in the price of the underlying stock. That is,

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

  - By holding Δ shares of the stock and one unit of the option, it is possible to create a riskless portfolio.

- **Delta hedging** is the hedging of options using Δ shares of the underlying stock.

Volatilities

위험중립에서 주가상승 확률을 결정하는 것은 주식 기대수익률이 아닌 u와 d
어떻게 하면 이 u와 d가 실제세계의 자산가격의 움직임을 잘 포착할 수 있을까?

- The parameters u and d should be chosen so that the binomial tree reflects the real movements in the price of the underlying assets.
- One popular way to determining u and d is

$$\begin{cases} u = e^{\sigma\sqrt{t}} \\ d = e^{-\sigma\sqrt{t}} \end{cases}$$

  - t is the length of one period.
  - $\sigma\sqrt{t}$ is the standard deviation of the price during t.
    ($\sigma^2 t$ is the variance of the price during t.)

위험중립에서 p를 구할때 실제 주가의 상승확률은 영향을 끼치지 않는 다는 것을 알았음
대신 u와 d의 영향을 받는다 이 u와 d를 어떻게 결정을 할거냐?
-> 이론 중 하나:기초자산의 변동성을 고려

중립세계

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

실제세계

$$p^* = \frac{e^{\mu\Delta t} - d}{u - d}$$

=> 분명히 p와 p*는 다름

pg 302 참고

⇓

$$u = e^{\sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{-\sigma\sqrt{\Delta t}}$$

변동성은 같음 (실제 주가움직임 잘 반영한다고 할수 있음)

(cf) 수학 5장 참고
P 측도 (실제) → Q 측도 (위험중립) 기대수익률은 변하나, 변동성 변하지 않음.

✓ 위험중립세계 의의: 그 자체로는 특별한 의미가 없지만 실제세계를 단순화하고 또 그 결과가 실제세계를 잘 반영하고 있음

지금까지는 single step -> 다소 비현실적, 더 많은 경우 고려해야

Multiple-Step Binomial Trees

- By extending the simple binomial model into multiple steps, a derivative can be priced more precisely.
- A multiple-step binomial tree is solved by applying the principles of the simple binomial model repeatedly.

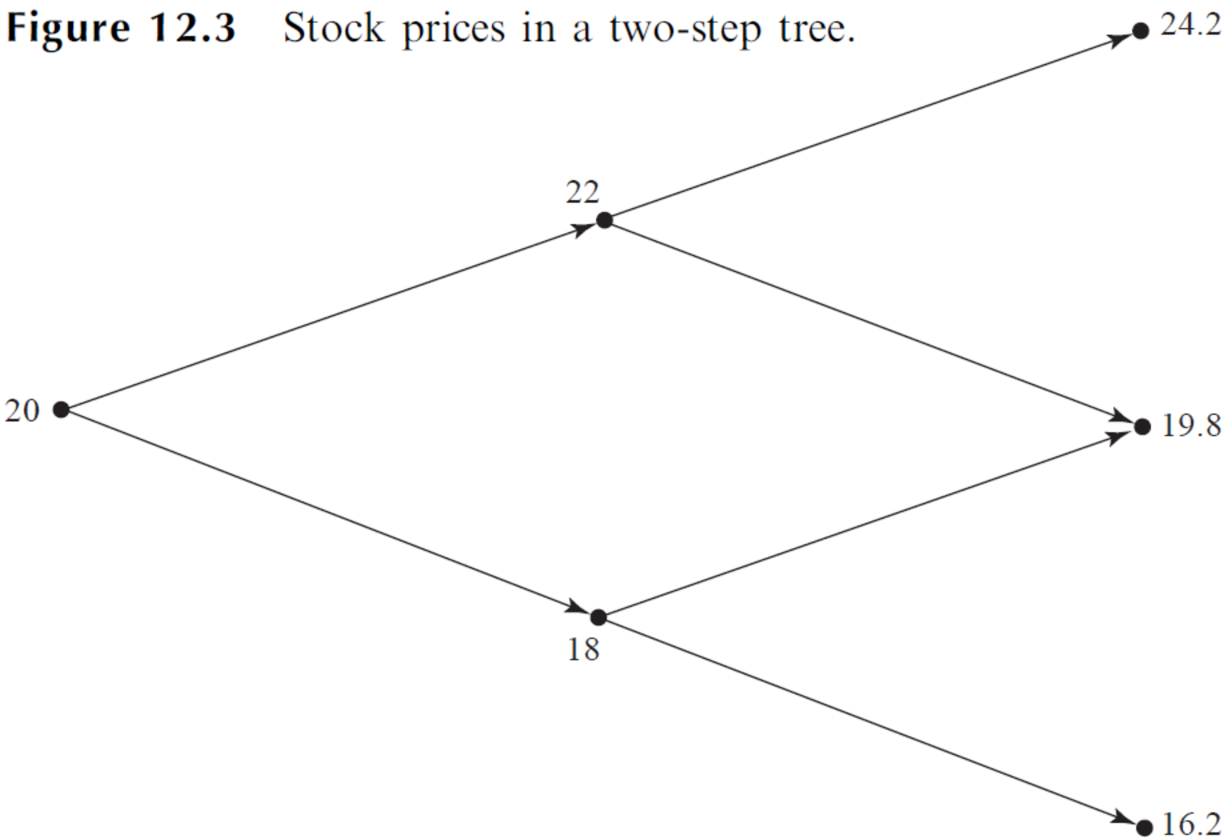
Example: European Calls

- Suppose that:
  - There are two periods, $t = 0$ from $t = 2$.
  - A period is 3-month long.
  - The initial stock price is $20. $S_0 = 20$
  - The stock price may go up by 10% or down by 10% in each period. $u: 1.1$ $d: 0.9$
  - The risk-free rate in each period is 12% per annum with continuous compounding.
  - There is an European call option which matures at $t = 2$.
  - The strike price of the call option is $21.

$K = 21$

Example: European Calls

Figure 12.3 Stock prices in a two-step tree.



Example: European Calls

두번 모두 오른 것

- In state uu, the payoff from the option is

$$24.2 - 21 = \$3.2$$

- In state ud and dd, the payoff from the option is $0 because the option-holder does not exercise it.
- The risk-neutral probability of moving up in each period is

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

$$p = \frac{e^{0.25\times 0.12} - 0.9}{1.1 - 0.9} = 0.6523$$

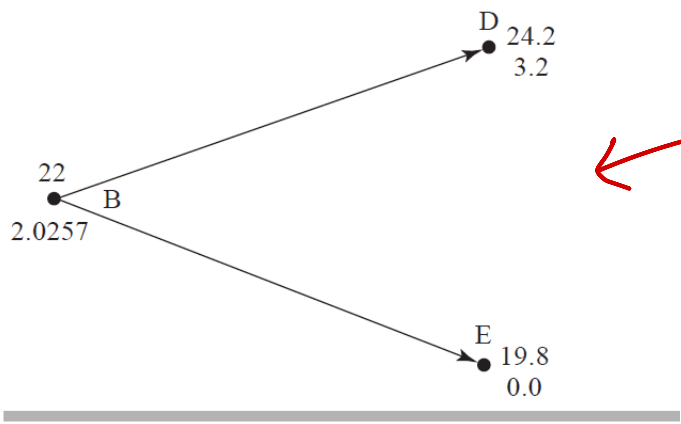
p는 스탭사이에 기간과 u,d에 의해 결정되므로 시점이 바껴도 일정함

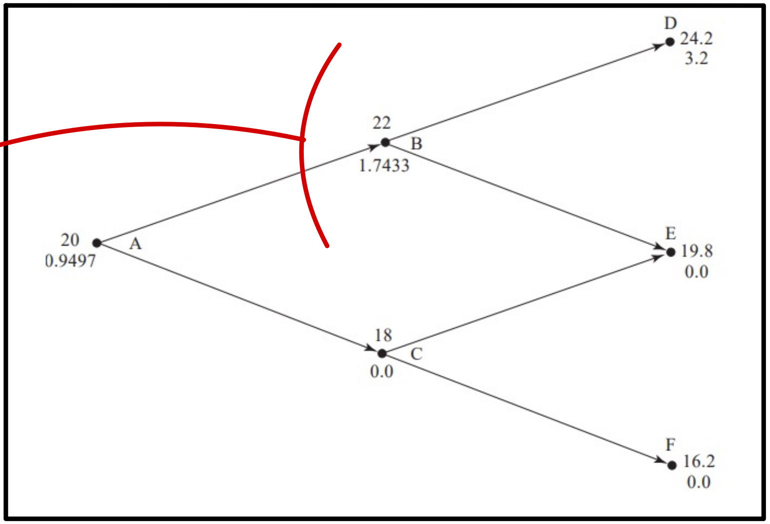
Example: European Calls

- By considering state uu and state ud, the value of the option in state u is

$$e^{-0.25\times 0.12}(0.6523\times 3.2+0.3477\times 0)$$
$$=\$2.0257$$

Figure 12.5 Evaluation of option price at node B of Figure 12.4.





Example: European Calls

- In the same way, the value of the option in state d is

  $$e^{-0.25\times 0.12}(0.6523 \times 0 + 0.3477 \times 0) = \$0$$

- Therefore, by considering state u and state d, the price of the option at $t = 0$ is

  $$e^{-0.25\times 0.12}(0.6523 \times 2.0257 + 0.3477 \times 0) = \$1.2823$$

지금까지는 유로피언 콜옵션의 경우
이제는 풋옵션을 고려해보자

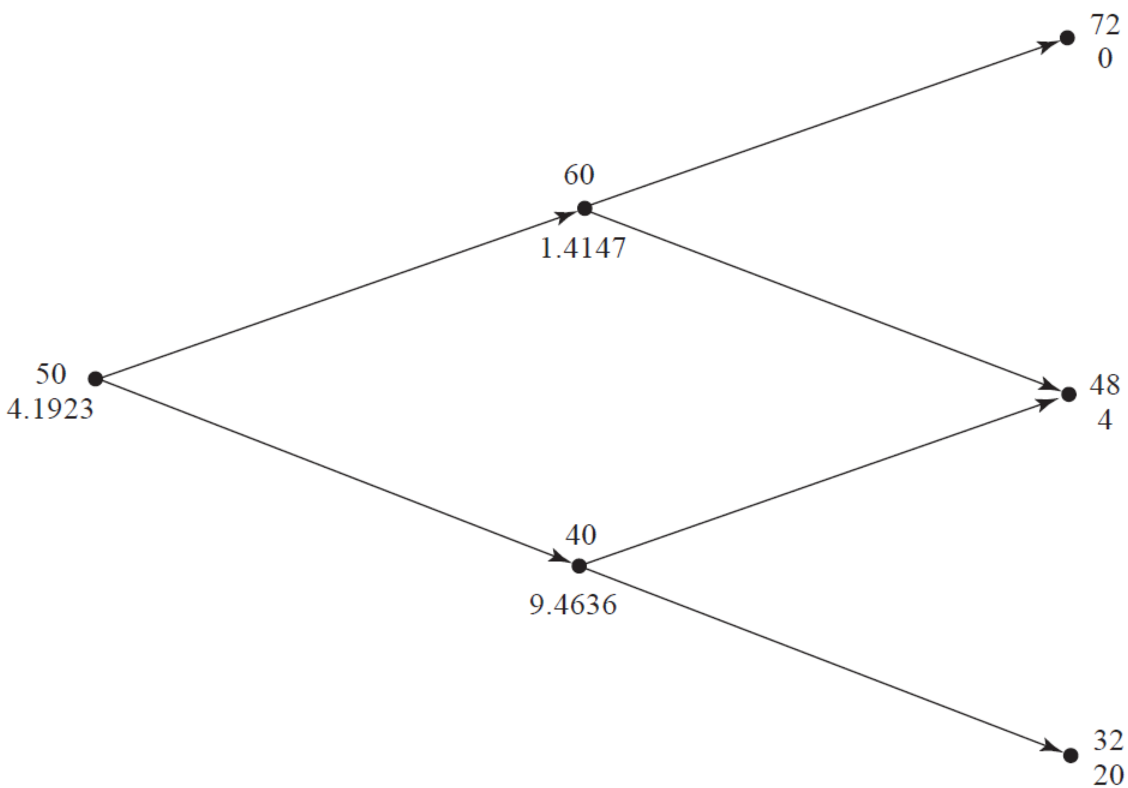
Example: European Puts

- Suppose that:
  - There are two periods, $t = 0$ from $t = 2$.
  - A period is 1-year long.
  - The initial stock price is \$50. $S_0 = 50$
  - The stock price may go up by 20% or down by 20% in each period. $u: 1.2$ $d: 0.8$
  - The risk-free rate in each period is 5% per annum with continuous compounding.
  - There is an European put option which matures at $t = 2$.
  - The strike price of the put option is \$52.

$K = 52$

Example: European Puts

Figure 12.7 Using a two-step tree to value a European put option. At each node, the upper number is the stock price and the lower number is the option price.



Example: European Puts

- Here, the risk-neutral probability of moving up is

$$p = \frac{e^{0.05} - 0.8}{1.2 - 0.8} = 0.6282$$

- The payoff from the option in state uu is \$0 since the option-holder does not exercise it.
- The payoff from the option in state ud is

$$52 - 48 = \$4$$

- The payoff from the option in state dd is

$$52 - 32 = \$20$$

Example: European Puts

- Then, the value of the option in state u is

  $e^{-0.05}(0.6282 \times 0 + 0.3718 \times 4) = \$1.4147$
- The value of the option in state d is

  $e^{-0.05}(0.6282 \times 4 + 0.3718 \times 20) = \$9.4636$
- Therefore, the price of the option at $t = 0$ is

  $e^{-0.05}(0.6282 \times 1.4147 + 0.3718 \times 9.4636)$
  $= \$4.1923$

아메리칸 옵션의 경으도 생각해보자 -> 유러피언 옵션처럼 + 조기행사의 경우도 고려

Example: American Puts

전 유로피언 예시랑 기본 가정은 같음 (최후의 옵션가치 어떻게 달라지는지 주시하자)

- Suppose that:
  - There are two periods, $t = 0$ from $t = 2$.
  - A period is 1-year long.
  - The initial stock price is $50. $S_0 = 50$
  - The stock price may go up by 20% or down by 20% in each period. $u: 1.2$ $d: 0.8$
  - The risk-free rate in each period is 5% per annum with continuous compounding.
  - There is an American put option which matures at $t = 2$.
  - The strike price of the put option is $52.

$K=52$

Example: American Puts

- Again, the risk-neutral probability of moving up is

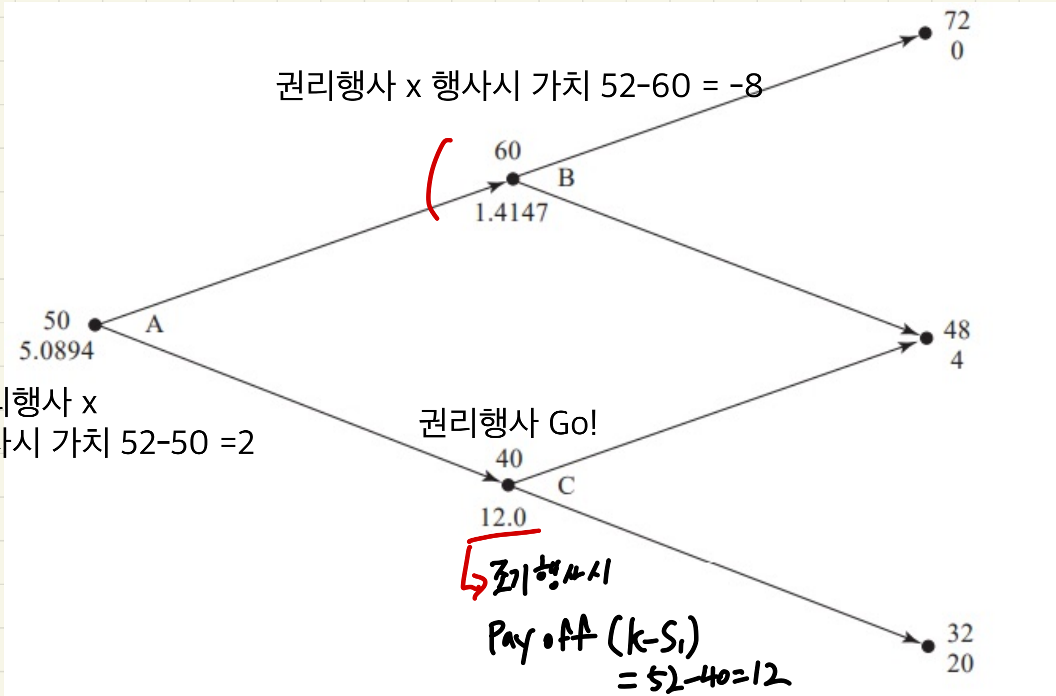
$$p = \frac{e^{0.05} - 0.8}{1.2 - 0.8} = 0.6282$$

- The payoff from the option in state uu is \$0 since the option-holder does not exercise it.
- The payoff from the option in state ud is

$$52 - 48 = \$4$$

- The payoff from the option in state dd is

$$52 - 32 = \$20$$



권리행사 x 행사시 가치 52-60 = -8

권리행사 x
행사시 가치 52-50 =2

권리행사 Go!

→ 조기행사시
Payoff $(K-S_1)$
$= 52-40=12$

VS

$$e^{-0.05}(0.6282 \times 4 + 0.3718 \times 20) = \$9.4636$$

→ 조기 행사 안 할시 f_d 기대값

조기 행사가
더 큰 이득
→ 조기 행사!

Example: American Puts

- The payoff from the option exercised at $t = 1$ in state u is $\$0$ since the option-holder does not exercise it.
- Considering the time value, the value of the option in state u is

$$e^{-0.05}(0.6282 \times 0 + 0.3718 \times 4) = \$1.4147$$

- The payoff from the option exercised at $t = 1$ in state d is

$$52 - 40 = \$12$$

Example: American Puts

- If the option-holder waits until $t = 1$, the expected payoff from the option in state d is

$$e^{-0.05}(0.6282 \times 4 + 0.3718 \times 20) = \$9.4636$$

- Therefore, the value of the option in state d is

$$\max(12, 9.4636) = \$12$$

- Then, the price of the option at $t = 0$ is

$$e^{-0.05}(0.6282 \times 1.4147 + 0.3718 \times 12) = \$5.0894$$

유로피안 옵션의 현재가치는 4.1923이었음을 기억하자