1. $d(XY) = Y\,dX + X\,dY + dX\,dY$

$f(t, X, Y) = XY$

$$d(XY) = df = \cancel{\frac{\partial f}{\partial t}dt} + \frac{\partial f}{\partial x}dX + \frac{\partial f}{\partial y}dY$$

$$+ \frac{1}{2}\cancel{\frac{\partial^2 f}{\partial x^2}dX\,dX} + \frac{1}{2}\cancel{\frac{\partial^2 f}{\partial y^2}dY\,dY}$$

$$+ \cancel{\frac{1}{2} \times 2 \times} \frac{\partial^2 f}{\partial x \partial y}dX\,dY$$

$$= Y\,dX + X\,dY + dX\,dY$$

2. $d(tW(t))$를 이용.

$$\int_0^t (t-s)\,dW(s) = \int_0^t W(s)\,ds$$

$\int_0^t W(s)\,ds$ 의 평균과 분산.

$$d(XY) = Y\,dX + X\,dY + dX\,dY$$

$$d(tW(t)) = W(t)\,dt + t\,dW(t) + \cancel{dt\,dW(t)}$$

$$tW(t) = \int_0^t W(s)\,ds + \int_0^t s\,dW(s)$$

W(t)를 적분

$$\int_0^t t\,dW(s) = \int_0^t W(s)\,ds + \int_0^t s\,dW(s)$$

$$\int_0^t (t-s)\,dW(s) = \int_0^t W(s)\,ds.$$

$$I(t) = \int_0^t (t-s)\,dW(s)$$

$$\mathbb{E}[I(t)] = \mathbb{E}[I(0)] = 0$$

$$\mathrm{Var}[I(t)] = \mathbb{E}[I(t)^2] - \mathbb{E}[I(t)]^2$$

$$= \mathbb{E}\left[\left(\int_0^t (t-s)\,dW(s)\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t (t-s)^2 ds \right]$$

$$= \int_0^t \mathbb{E}\left[(t-s)^2\right] ds$$

$$= \int_0^t (t-s)^2 ds$$

$$= \left[-\frac{1}{3}(t-s)^3 \right]_0^t$$

$$= \frac{1}{3} t^3$$

3. $dX(t) = -aX(t)dt + dW(t)$

$X(t) = ?$

$$d(e^{at}X(t)) = d(e^{at})X(t) + e^{at}dX(t) + d(e^{at})dX(t)$$

$$= a e^{at}\cancel{X(t)dt} + e^{at}(\cancel{-aX(t)dt} + dW(t))$$

$$+ ae^{at}\underline{dt}(\cancelto{0}{-aX(t)dt} + \cancelto{0}{dW(t)})$$

$$= e^{at}dW(t)$$

$$\int_0^t d(e^{as}X(s)) = \int_0^t e^{as}dW(s)$$

$$= e^{at}X(t) - e^{0}X(0) = \int_0^t e^{as}dW(s)$$

$$\frac{1}{e^{at}} \times e^{at}X(t) = \frac{1}{e^{at}}\left(\int_0^t e^{as}dW(s) + X(0)\right)$$

$$X(t) = e^{-at}\left(X(0) + \int_0^t e^{as}dW(s)\right)$$

4. $X(t) = ?$

1). $dX(t) = dt + 2\sqrt{X(t)}\, dW(t)$

$$= \left(1 - \frac{1}{2} \times 2\sqrt{X(t)} \frac{1}{\sqrt{X(t)}}\right) dt + 2\sqrt{X(t)} \circ dW(t)$$

$$= 2\sqrt{X(t)} \circ dW(t)$$

$$\frac{1}{2\sqrt{X(t)}} dX(t) = 1 \circ dW(t)$$

$$\int_0^t \frac{1}{2\sqrt{X(s)}} dX(s) = W(t)$$

$$= \sqrt{X(t)} - \sqrt{X(0)} = W(t) \qquad X(t) = \left(W(t) + \sqrt{X(0)}\right)^2$$

2) $dX(t) = \frac{1}{3} X(t)^{\frac{1}{3}} dt + X(t)^{\frac{2}{3}} dW(t)$

$$= \left(\frac{1}{3} X(t)^{\frac{1}{3}} - \frac{1}{2} X(t)^{\frac{1}{3}} \times \frac{2}{3} X(t)^{-\frac{2}{3}}\right) dt$$

$$+ X(t)^{\frac{2}{3}} \circ dW(t)$$

$$= X(t)^{\frac{2}{3}} \circ dW(t)$$

$$\frac{1}{X(t)^{\frac{2}{3}}} dX(t) = 1 \circ dW(t)$$

$$\int_0^t X(s)^{-\frac{2}{3}} dX(s) = W(t)$$

$$3X(t)^{\frac{1}{3}} - 3X(0)^{\frac{1}{3}} = W(t)$$

$$X(t) = \left(\frac{1}{3} W(t) + X(0)^{\frac{1}{3}}\right)^3$$

3. $$dX(t) = \frac{1}{2} a^2 m X(t)^{2m-1} dt + aX(t)^m dW(t)$$

$$= \left(\underbrace{\frac{1}{2} a^2 m X(t)^{2m-1} - \frac{1}{2} aX(t)^m \cdot amX(t)^{m-1}}_{=0}\right) dt$$

$$+ aX(t)^m \circ dW(t)$$

$$= aX(t)^m \circ dW(t)$$

$$\frac{1}{aX(t)^m} dX(t) = 1 \circ dW(t)$$

$2x$

$\frac{1}{2} x$

$x^2 \quad 2x''$

$$\int_0^t \frac{1}{a} X(s)^{-m} \, dX(s) = W(t)$$

$$\frac{1}{a}\int_0^t X(s)^{-m} \, dX(s) = \frac{1}{a}\left[\frac{1}{1-m} X(s)^{1-m}\right]_0^t$$

$$= \frac{1}{a(1-m)} X(t)^{1-m} - \frac{1}{a(1-m)} X(0)^{1-m}$$

$$X(t) = \left(a(1-m)\left(W(t) + \frac{1}{a(1-m)} X(0)^{1-m} \right)\right)^{m-1}$$

$$= \left(a(1-m) W(t) + X(0)^{1-m}\right)^{m-1}$$

5. $\int_0^t W(s)^2 \circ dW(s) = \frac{1}{3}W(t)^3$

$$d(W(t)^3) = d(W(t)^2 \cdot W(t))$$

$$= W(t)d(W(t)^2) + W(t)^2 dW(t) + dW(t)^2 dW(t)$$

$$\rightarrow d(W(t)^2) = (2W(t)dW(t) + dW(t)dW(t))$$

$$= W(t)(2W(t)dW(t) + dt) + W(t)^2 dW(t)$$

$$+ (2W(t)dW(t) + \cancel{dt})dW(t)$$

$$= 2W(t)^2 dW(t) + W(t)dt + W(t^2)dW(t)$$

$$+ 2W(t)dt$$

$$= 3W(t)^2 dW(t) + 3W(t)dt$$

$$= 3(W(t)^2 dW(t) + W(t)dt)$$

$$W(t)^2 \circ dW(t) = W(t)^2 dW(t) + \frac{1}{2}dW(t)^2 dW(t)$$

$$= W(t)^2 dW(t) + \frac{1}{2}(2W(t)dW(t) + dW(t)dW(t)) \times dW(t)$$

$$= W(t)^2 dW(t) + W(t)dW(t)$$

$$d(W(t)^3) = 3W(t)^2 \circ dW(t)$$

$$\int_0^t d(W(s)^3) = 3\int_0^t W(s)^2 \circ dW(s)$$

$$W(t)^3 = 3\int_0^t W(s)^2 \circ dW(s)$$

$$\int_0^t W(s)^2 \circ dW(s) = \frac{1}{3} W(t)^3$$

6.

1) $d(t^2 e^{W(t)}) \rightsquigarrow$

$f(t,x) = t^2 e^x$

$f_t = 2t e^x$, $f_x = t^2 e^x$, $f_{xx} = t^2 e^x$

f_{tx}, $f_{tt} = 0$

$$d(t^2 e^{W(t)}) = f_t dt + f_x dW(t) + \frac{1}{2} f_{xx} dW(t) dW(t)$$

$$= 2t e^{W(t)} dt + t^2 e^{W(t)} dW(t) + \frac{1}{2} t^2 e^{W(t)} dW(t) dW(t)$$

$$= t e^{W(t)} \left(2 + \frac{1}{2} t\right) dt + t^2 e^{W(t)} dW(t)$$

2) $d(e^{W(t)^2}) = \rightsquigarrow$

$f(t,x) = e^{x^2}$

$f_t = 0$ $f_x = 2x e^{x^2}$ $f_{xx} = 2e^{x^2} + 4x^2 e^{x^2}$

$$d\left(e^{W(t)^2}\right) = \cancel{f_t\,dt} + f_x\,dW(t) + \frac{1}{2} f_{xx}\,dW(t)\,dW(t)$$

$$= 2W(t)e^{W(t)^2}\,dW(t) + \frac{1}{2}\left(2e^{W(t)^2} + 4W(t)^2 e^{W(t)^2}\right)dt$$

$$d\left(e^{W(t)^2}\right) = \left(1 + 2W(t^2)\right)e^{W(t)^2}\,dt + 2W(t)\,e^{W(t)^2}\,dW(t)$$