3. BSM solution

$$\begin{cases} C_t + \frac{1}{2}\sigma^2 S^2 C_{ss} + rSC_s - rC = 0 & \text{: BSM} \\ C(S_T, T) = \max[S_T - E, 0] & \text{terminal condition.} \end{cases}$$

3. BSM solution
⇒ 해를 구하는 과정. " $C(s, t)$ "

sol) Let
$$\begin{cases} S = Ee^x, & (-\infty < x < \infty) \quad (0 < S < \infty) \\ t = T - \dfrac{\tau}{\frac{1}{2}\sigma^2} \\ C = EV(x, \tau) & (\because C = C(S, t)) \end{cases}$$

$t = T$ (만기) 시점에서 콜옵션 long 포지션의 가치를 t에 대해 표현; $C(S, T)$ - terminal condition

" " τ에 대해 표현; $EV(x, 0)$ - initial condition

$$\frac{\partial C}{\partial t} = E \cdot \frac{\partial V}{\partial t} = E \cdot \frac{\partial V}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = EV_\tau \cdot \left(-\frac{1}{2}\sigma^2\right)$$

$$V \begin{cases} x - S \\ \tau - t \end{cases}$$

$$\frac{\partial C}{\partial S} = E \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial S} = \cancel{E} V_x \cdot \frac{1}{\cancel{E}e^x} = \frac{V_x}{e^x}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{V_x}{e^x}\right) = \frac{\partial}{\partial S}\left(\frac{1}{e^x}\right) \cdot V_x + \frac{\partial}{\partial S}(V_x) \cdot \frac{1}{e^x}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{e^x}\right)\frac{\partial x}{\partial S} V_x + \frac{\partial}{\partial x}(V_x)\frac{\partial x}{\partial S} \cdot \frac{1}{e^x}$$

$$= -e^{-x}\frac{\partial x}{\partial S} V_x + e^{-x} V_{xx} \cdot \frac{\partial x}{\partial S}$$

$$= \frac{1}{Ee^{2x}}(V_{xx} - V_x)$$

대입

$$\therefore EV_\tau\left(-\frac{1}{2}\sigma^2\right) + \frac{1}{2}\sigma^2 (Ee^x)^2 \cdot \frac{1}{Ee^{2x}}(V_{xx} - V_x) + r(Ee^x)\frac{V_x}{e^x} - rEV = 0$$

$$\therefore V_\tau\left(-\frac{1}{2}\sigma^2\right) + \frac{1}{2}\sigma^2(V_{xx} - V_x) + rV_x - rV = 0$$

let $k = \dfrac{r}{\frac{1}{2}\sigma^2}$, $\quad V_\tau = V_{xx} + (k-1)V_x - kV$

V ; x와 τ를 변수로 하는 임의의 함수

$$C(S, T) = \max(S_T - E, 0)$$
$$EV(x, 0) = \max(Ee^x - E, 0)$$
$$\therefore V(x, 0) = \max(e^x - 1, 0) \quad \text{initial condition}$$

$$\text{Let } v = e^{\alpha x+\beta\tau}u(x,\tau)$$

$$v_\tau = \beta e^{\alpha x+\beta\tau}u(x,\tau) + e^{\alpha x+\beta\tau}u_\tau$$

$$v_x = \alpha e^{\alpha x+\beta\tau}u(x,\tau) + e^{\alpha x+\beta\tau}u_x$$

$$v_{xx} = \alpha^2 e^{\alpha x+\beta\tau}u(x,\tau) + 2\alpha e^{\alpha x+\beta\tau}u_x + e^{\alpha x+\beta\tau}u_{xx}$$

$$e^{\alpha x+\beta\tau} \times u(x,\tau)$$

곱미분

$$\therefore \beta u + u_\tau = (\alpha^2 u + 2\alpha u_x + u_{xx}) + (k-1)(\alpha u + u_x) - ku$$

$$\therefore \begin{cases} \beta = \alpha^2 + \alpha(k-1) - k \\ 0 = 2\alpha + (k-1) \end{cases} \Rightarrow u_\tau = u_{xx}$$

"2. PDE" 의 열방정식 "$u_t = u_{xx}$" 에서 가져옴

$$\therefore \alpha = -\frac{1}{2}(k-1), \quad \beta = -\frac{1}{4}(k+1)^2$$

$$\therefore v(x,0) = e^{-\frac{1}{2}(k-1)x}\cdot u(x,0) \Rightarrow u(x,0) = \max\left(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right)$$

$$v(x,0) = \max(e^x - 1, 0)$$

정리

$$\begin{cases} C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} + rSC_S - rC = 0 \\ C(S,T) = \max[S_T - E, 0] \end{cases} \rightarrow \begin{cases} u_\tau = u_{xx} \\ u(x,0) = \max\left(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right) \overset{\text{let}}{=} u_0(x) \end{cases}$$

$$\therefore u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(s)e^{-(x-s)^2/4\tau}\,ds$$

$$\text{Let } \hat{x} = \frac{s-x}{\sqrt{2\tau}} \Rightarrow d\hat{x} = \frac{1}{\sqrt{2\tau}}ds$$

"2. PDE"의 마지막 부분에서 가져옴.

$$\Rightarrow u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(s)e^{-(x-s)^2/4t}\,ds$$

$$\therefore u(x,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_0(x+\sqrt{2\tau}\hat{x})e^{-\frac{\hat{x}^2}{2}}\,d\hat{x}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}\left(e^{\frac{1}{2}(k+1)(x+\sqrt{2\tau}\hat{x})} - e^{\frac{1}{2}(k-1)(x+\sqrt{2\tau}\hat{x})}\right)e^{-\frac{1}{2}\hat{x}^2}\,d\hat{x} \quad (\because s>0)$$

s≤0 이라면 더이상 식을 전개할 의미·필요 없음.

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)(x+\sqrt{2\tau}\hat{x})}\cdot e^{-\frac{1}{2}\hat{x}^2}\,d\hat{x}}_{=I} - \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k-1)(x+\sqrt{2\tau}\hat{x})}\cdot e^{-\frac{1}{2}\hat{x}^2}\,d\hat{x}}_{=II}$$

$$= I - II$$

$$e^{\frac{1}{2}(k+1)s} - e^{\frac{1}{2}(k-1)s} > 0$$

s>0 이므로 $x+\sqrt{2\tau}\hat{x} > 0$ 이다.

$$\therefore \hat{x} > -\frac{x}{\sqrt{2\tau}}$$

⇒ $e^{\frac{ks}{2}}$ 를 각각 나눠주면 $e^{\frac{s}{2}} - e^{-\frac{s}{2}} > 0$

⇒ $e^{\frac{s}{2}} > e^{-\frac{s}{2}}$ ⇒ $s > -s$ ⇒ $s > 0$

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)(x+\sqrt{2\tau}\hat{x})} \cdot e^{-\frac{1}{2}\hat{x}^2} \, d\hat{x}$$

$$= \frac{e^{\frac{1}{2}(k+1)x}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)\sqrt{2\tau}\hat{x} - \frac{1}{2}\hat{x}^2} \, d\hat{x}$$

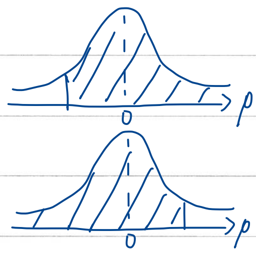
표준정규분포~(0, 1) 의 pdf : $\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

cdf : $\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt$

$$= \frac{e^{\frac{1}{2}(k+1)x}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(\underbrace{\hat{x} - \frac{1}{2}(k+1)\sqrt{2\tau}}_{\text{let } \rho})^2} \cdot e^{\frac{1}{4}(k+1)^2\tau} \, d\hat{x}$$

$$= \frac{e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}} - \frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\rho^2} \, d\rho$$

$-\rho$ (로 두)



(뒤집기)

$$= \frac{e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}} e^{-\frac{1}{2}\rho^2} \, d\rho$$

$$= e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau} N(d_1) \quad \text{where } N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}s^2} \, ds, \quad d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}$$

Similarly, $II = e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau} N(d_2)$ where $d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}$

$$\therefore u(x,\tau) = e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau} N(d_1) - e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau} N(d_2)$$

$$\therefore v(x,\tau) = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} \cdot u(x,\tau) = e^{x} N(d_1) - e^{-k\tau} N(d_2)$$

$k = \frac{r}{\frac{1}{2}\sigma^2}$, $e^x = \frac{S}{E}$, $\tau = (T-t)\cdot\frac{1}{2}\sigma^2$ 대입.

$$\therefore v(x,\tau) = \frac{S}{E} N(d_1) - e^{-r(T-t)} N(d_2)$$

$$\therefore C(S,t) = E \cdot v(x,\tau) = S \cdot N(d_1) - E e^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}$$

$$\begin{cases} C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} + rSC_S - rC = 0 \\ C(S_T, T) = \max[S_T - E, 0] \end{cases} \Rightarrow C(S,t) = S \cdot N(d_1) - E e^{-r(T-t)} \cdot N(d_2)$$

대입!

$x = \ln(S/E)$

$\tau = (T-t)\frac{1}{2}\sigma^2$

$\sqrt{2\tau} = \sigma\sqrt{T-t}$

$k = r/\frac{1}{2}\sigma^2$

where $\begin{cases} d_1 = \dfrac{\ln(S/E) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = \dfrac{\ln(S/E) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{cases}$

$$* \; N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}s^2} \, ds$$