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$X \sim N(0,t)$일 때 $\mathbb{E}[X^4] = 3t^2$

$$\mathbb{E}[e^{ux}] = \mathbb{E}\left[f(0) + \frac{f'(0)}{1!}X + \frac{f''(0)}{2!}X^2 + \cdots\right]$$

$$= \mathbb{E}\left[1 + uX + \frac{u^2}{2!}X^2 + \cdots\right]$$

$$= 1 + \mathbb{E}[X]\cdot u + \frac{\mathbb{E}[X^2]}{2!}u^2 + \cdots$$

$$\mathbb{E}[e^{ux}] = e^{\frac{1}{2}u^2 t}$$

$$= 1 + \frac{\frac{1}{2}u^2}{1!}t + \frac{\left(\frac{1}{2}u^2\right)^2}{2!}t^2 + \cdots$$

$$= 1 + \frac{1}{2}u^2 t + \frac{1}{8}u^4 t^2 + \cdots$$

$$\rightarrow 1 + \mathbb{E}[X]\cdot u + \frac{\mathbb{E}[X^2]}{2!}u^2 + \frac{\mathbb{E}[X^3]}{3!}u^3 + \frac{\mathbb{E}[X^4]}{4!}u^4 + \cdots$$

$$= 1 + \qquad \frac{1}{2}u^2 t + \qquad \frac{1}{8}u^4 t^2 + \cdots$$

$$\frac{E[X^4]}{4!}u^4 = \frac{1}{8}u^4 t^2 \quad , \quad \frac{E[X^4]}{4!} = \frac{1}{8}t^4$$

$$\rightarrow \boxed{E[X^4] = 3t^4}$$

2. $\mathbb{E}[W(t)W(s)] = ?$ 단$(t \geq s)$

$$\mathbb{E}[(W(t) - W(s) + W(s)) \cdot W(s)]$$

$$= \mathbb{E}[(W(t) - W(s)) \cdot W(s)] + \underbrace{\mathbb{E}[W(s)^2]}_{= Var[W(s)] + \mathbb{E}[W(s)]}$$

$$= \mathbb{E}[(W(t) - W(s))] \cdot \cancel{\mathbb{E}[W(s)]}^{\,0} + Var[W(s)] + \cancel{\mathbb{E}[W(s)]}^{\,0}$$

$$= Var[W(s)]$$

$$\boxed{\mathbb{E}[W(t)W(s)] = Var[W(s)]}$$

3. $Z(t) = e^{\sigma W(t) - \frac{1}{2}\sigma^2 t}$ is martingale.

$$\mathbb{E}[Z(t) \mid \mathcal{F}(s)] = \mathbb{E}\left[\frac{Z(t)}{Z(s)} \cdot Z(s) \mid \mathcal{F}(s)\right]$$

$$= \mathbb{E}\left[e^{\sigma W(t) - \frac{1}{2}\sigma^2 t - \sigma W(s) + \frac{1}{2}\sigma^2 s} \cdot Z(s) \mid \mathcal{F}(s)\right]$$

$$= \mathbb{E}\left[e^{\sigma(W(t) - W(s)) - \frac{1}{2}\sigma^2 (t-s)} \cdot Z(s) \mid \mathcal{F}(s)\right]$$

$$= e^{-\frac{1}{2}\sigma^2(t-s)} \cdot \mathbb{E}\left[e^{\sigma(W(t)-W(s))}\right] \cdot Z(s)$$

$$\rightarrow \mathbb{E}\left[e^{\sigma(W(t)-W(s))}\right] = e^{0 + \frac{1}{2}\sigma^2(t-s)}$$

$$\rightarrow e^{-\frac{1}{2}\sigma^2(t-s)} \cdot \mathbb{E}\left[e^{\sigma(W(t)-W(s))}\right] \cdot Z(s)$$

$$= e^{-\frac{1}{2}\sigma^2(t-s)} \cdot e^{\frac{1}{2}\sigma^2(t-s)} \cdot Z(s)$$

$$= e^0 \cdot Z(s) = \boxed{Z(s)}$$

4. $\mathbb{E}[\ln S(t)] = ?$ $\quad$ $\mathrm{Var}[\ln S(t)] = ?$

$$\ln S(t) = \ln S(0) + \ln e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$

$$= \ln S(0) + (\alpha - \tfrac{1}{2}\sigma^2)t + \sigma W(t)$$

$$\mathbb{E}[\ln S(t)] = \ln S(0) + (\alpha - \tfrac{1}{2}\sigma^2)t + \mathbb{E}[\overset{0}{\cancel{\sigma W(t)}}]$$

$$= \ln S(0) + \alpha - \tfrac{1}{2}\sigma^2 t$$

$$\mathrm{Var}[\ln S(t)] = \mathrm{Var}[\ln S(0) + (\alpha - \tfrac{1}{2}\sigma^2)t + \sigma W(t)]$$

$$= \mathrm{Var}[\sigma W(t)]$$

$$= \sigma^2 \mathrm{Var}[W(t)]$$

$$= \sigma^2 t$$