1. $X \sim N(0, t)$

Show $\mathbb{E}[X^4] = 3t^2$.

① $\mathbb{E}[e^{uX}] = \mathbb{E}\left[1 + uX + \frac{u^2}{2!}X^2 + \frac{u^3}{3!}X^3 + \frac{u^4}{4!}X^4 + \cdots\right]$

u⁴의 계수

∵ Taylor's Expansion $\quad = 1 + \mathbb{E}[X]u + \frac{\mathbb{E}[X^2]}{2!}u^2 + \frac{\mathbb{E}[X^3]}{3!}u^3 + \frac{\mathbb{E}[X^4]}{4!}u^4 + \cdots$

u⁴의 계수

② $\mathbb{E}[e^{uX}] = e^{\frac{1}{2}u^2 t} = 1 + \left(\frac{1}{2}u^2 t\right) + \frac{\left(\frac{1}{2}u^2 t\right)^2}{2!} + \cdots = 1 + \frac{1}{2}tu^2 + \frac{1}{8}t^2u^4 + \cdots$

∵ mgf. $\quad$ Taylor's Expansion

$\therefore \frac{\mathbb{E}[X^4]}{4!} = \frac{1}{8}t^2$

$\therefore \mathbb{E}[X^4] = 3t^2$. and if n is odd, $\mathbb{E}[X^n] = 0$.

2. $\mathbb{E}[W(t)W(s)]$

$= \mathbb{E}[(W(t)\overset{0}{\underline{-W(s)+W(s)}})W(s)]$

$= \mathbb{E}[(W(t)-W(s))W(s)] + \mathbb{E}[W(s)^2]$

$= \underline{\mathbb{E}[W(t)-W(s)]\cdot\mathbb{E}[W(s)]} + \underline{Var(W(s))}$

$\left(\begin{array}{l} \because \text{if } X, Y : \text{independent}, \\ \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] \end{array}\right)$ $\quad \because VarW(s) = \mathbb{E}[W(s)^2] - (\cancelto{0}{\mathbb{E}[W(s)]})^2$

$= s$.

※ $Var(W(t)-W(s))$

$= VarW(t) + VarW(s) - 2Cov(W(t), W(s))$

$= t + s - 2s$

$= t - s \quad (\because Cov(W(t), W(s)) = \mathbb{E}[W(t)W(s)])$

3. $Z(t) = e^{\sigma W(t) - \frac{1}{2}\sigma^2 t}$

$\mathbb{E}[Z(t)|\mathcal{F}(s)] = \mathbb{E}\left[\frac{Z(t)}{Z(s)}\cdot Z(s) \middle| \mathcal{F}(s)\right]$

$= \mathbb{E}\left[e^{\underbrace{\sigma(W(t)-W(s))}_{\text{independent}} - \underbrace{\frac{1}{2}\sigma^2(t-s)}_{\text{non-random}}}\cdot Z(s) \middle| \mathcal{F}(s)\right]$

$= e^{-\frac{1}{2}\sigma^2(t-s)}\cdot\mathbb{E}[e^{\sigma(W(t)-W(s))}]\cdot Z(s)$

$= e^{-\frac{1}{2}\sigma^2(t-s)}\cdot e^{\frac{1}{2}\sigma^2(t-s)}\cdot Z(s) = Z(s)$

3. pf)

① $Z(t)$ is integrable?

즉, $\mathbb{E}[|Z(t)|] < \infty$?

pf) $\mathbb{E}[|Z(t)|]$

$= \mathbb{E}[|e^{\sigma W(t) - \frac{1}{2}\sigma^2 t}|]$

$= \mathbb{E}[e^{\sigma W(t) - \frac{1}{2}\sigma^2 t}]$ (∵ 지수함수 정의)

$= e^{-\frac{1}{2}\sigma^2 t} \mathbb{E}[e^{\sigma W(t)}]$

$= e^{-\frac{1}{2}\sigma^2 t} \cdot e^{\frac{1}{2}\sigma^2 t}$ (∵ mgf)

$= 1 < \infty$

∴ $Z(t)$ is integrable

② $Z(t)$ is adapted to $\mathcal{F}$?

$Z(t)$는 $W(t)$의 함수 ⇒ ∴ $Z(t)$ is adapted.

4. Lognormal Process.

↳ $S(t) = S(0)e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)}$

∴ $\ln S(t) = \underbrace{\ln S(0) + (\alpha - \frac{1}{2}\sigma^2)t}_{\text{상수}} + \underbrace{\sigma W(t)}_{\text{상수}}$

∴ $\mathbb{E}[\ln S(t)] = \ln S(0) + (\alpha - \frac{1}{2}\sigma^2)t$ (∵ $\mathbb{E}[aX+b] = a\mathbb{E}[X] + b$)

$Var[\ln S(t)] = \sigma^2 t$ (∵ $Var[aX+b] = a^2 Var X$)