I. 기초수학

1. 미분법 -지수/로그/삼각 함수, chain rule
2. 테일러전개
3. Distribution -적률생성함수/이항분포/정규분포

1) 자연상수 e

"1원 / 단위기간 1년 / 이자율 100%"

$1(1+\frac{100}{100})^1 = 2$

1원 / 단위기간 6개월 / 이자율 50% 복리

$1(1+\frac{50}{100})^2 = 2.25$

1원 / 단위기간 4개월 / 이자율 33.3% 복리

$1(1+\frac{33.3}{100})^3 = 2.3703\cdots$

1원 / 단위기간 1년의 $\frac{1}{n}$ / 이자율 $\frac{1}{n}$ 비율

$1(1+\frac{1}{n})^n$

n을 무한대로 보내서 단위기간을 무수히 짧게 하면

$\lim_{n\to\infty} 1\cdot(1+\frac{1}{n})^n = 2.7182818\cdots = e$

2) ln

$\ln a = \log_e a = \frac{1}{\log_a e}$

$\ln e = \log_e e = 1$

1. 미분법

1) 지수함수 미분

$\log_e a$ ॥

① $(a^x)' = a^x \cdot \ln a$

② $(e^x)' = e^x$

③ $(a^{f(x)})' = a^{f(x)} \cdot \ln a \cdot f'(x)$

④ $(e^{f(x)})' = e^{f(x)} \cdot f'(x)$

1)-① 증명 // $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ 이므로

$$(a^x)' = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h}$$

$$= a^x \cdot \lim_{h\to 0} \frac{a^h - 1}{h}$$

여기서, $a^h - 1 = t$로 치환, $a^h = t+1$

$h = \log_a(1+t)$

$$= a^x \cdot \lim_{t\to 0} \frac{t}{\log_a(1+t)}$$

$$= a^x \cdot \lim_{t\to 0} \frac{1}{\frac{1}{t}\log_a(1+t)}$$

$$= a^x \cdot \lim_{t\to 0} \frac{1}{\log_a(1+t)^{\frac{1}{t}}} \quad \because e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= a^x \cdot \frac{1}{\log_a e} \qquad \frac{1}{\log_a e} = \log_e a = \ln a$$

$$= a^x \cdot \ln a$$

여기서 $a \to e$로 바꾸면 ②번 증명 가능

$(e^x)' = e^x \cdot \underbrace{\ln e}_{\log_e e = 1} = e^x$

2) 로그함수 미분

① $(\ln x)' = \frac{1}{x}$

② $(\log_a x)' = \frac{1}{x} \cdot \frac{1}{\ln a}$

③ $(\log_a f(x))' = \frac{1}{f(x)} \cdot \frac{1}{\ln a} \cdot f'(x)$

2-② 증명 //

$$(\log_a x)' = \lim_{h\to 0} \frac{\log_a(x+h) - \log_a x}{h}$$

로그의 뺄셈은 나누기로!

$$= \lim_{h\to 0} \frac{1}{h} \cdot \log_a\left(1+\frac{h}{x}\right)$$

$e = \lim_{h\to\infty}\left(1+\frac{1}{n}\right)^n$ 꼴을 만들어가는 과정

$$= \lim_{h\to 0} \log_a\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

$$= \log_a \lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}\cdot\frac{1}{x}}$$

$$= \frac{1}{x}\log_a \lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \frac{1}{x}\log_a e$$

$$= \frac{1}{x} \cdot \frac{1}{\log_e a}$$

$$= \frac{1}{x} \cdot \frac{1}{\ln a}$$

마찬가지로, a 대신 e 넣으면 ①번 증명가능

3) 삼각함수 미분

① $(\sin x)' = \cos x$

② $(\cos x)' = -\sin x$

4) Chain Rule

① 합성함수 미분

$\Rightarrow y = f(g(x))$

$y' = f'(g(x)) \cdot g'(x)$

$g(x)$를 t로 치환 $\rightarrow g'(x) = \frac{dt}{dx}$

$f'(g(x)) = f'(t) = \frac{dy}{dt}$

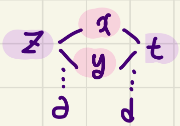
즉, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$y \rightarrow t \rightarrow x$

f　g

② 다변수 함수 chain rule

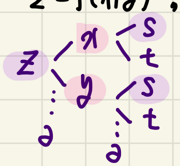
$\Rightarrow z = f(x, y)$, $x = g(t)$, $y = h(t)$



라운드(∂)와 d 구분!

$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$\Rightarrow z = f(x, y)$, $x = g(s, t)$, $y = h(s, t)$



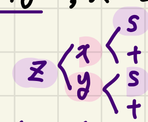
$\therefore \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$

교안 3p 예제 ①

$\Rightarrow z = x^2 + xy + y^2$, $x = t + s$, $y = t - s$ 일때,

$\frac{\partial z}{\partial s} = ?$



sol) $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$

$= (2x + y) \cdot 1 + (2y + x) \cdot -1$

$= x - y$

$= (t + s) - (t - s)$

$= 2s$

3p 예제 2번

⇒ $z = f(x, y)$, $x = r^2 + s^2$, $y = 3rs$ 일때

$\dfrac{\partial^2 z}{\partial r^2} = A \cdot \dfrac{\partial z}{\partial x} + B \cdot \dfrac{\partial z}{\partial y} + C \cdot \dfrac{\partial^2 z}{\partial x^2} + D \cdot \dfrac{\partial^2 z}{\partial x \partial y} + E \cdot \dfrac{\partial^2 z}{\partial y^2}$ 이다

각각의 계수 A, B, C, D, E 의 값은?

$z \left\langle \begin{array}{l} x \left\langle \begin{array}{l} r \\ s \end{array}\right. \\ y \left\langle \begin{array}{l} r \\ s \end{array}\right. \end{array}\right.$

$\underset{②}{\dfrac{\partial}{\partial r}} \times \overset{①}{\left(\dfrac{\partial z}{\partial r}\right)} = \dfrac{\partial^2 z}{\partial r^2}$

sol) 먼저, $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial r}$

$= \dfrac{\partial z}{\partial x} \cdot (2r) + \dfrac{\partial z}{\partial y} \cdot (3s)$

$\dfrac{\partial^2 z}{\partial r^2} = \dfrac{\partial}{\partial r} \cdot \dfrac{\partial z}{\partial r}$ 이므로,

$= \dfrac{\partial}{\partial r} \cdot \left\{ \dfrac{\partial z}{\partial x} \cdot (2r) + \dfrac{\partial z}{\partial y} \cdot (3s) \right\}$ ⋯ $\dfrac{\partial z}{\partial x} \cdot 2r$ 과 $\dfrac{\partial z}{\partial y} \cdot 3s$ 를 r로 미분

참고) 곱미분

$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$= \underbrace{\dfrac{\partial}{\partial r} \cdot \dfrac{\partial z}{\partial x}}_{①} \cdot 2r + \dfrac{\partial z}{\partial x} \cdot 2 + \underbrace{\dfrac{\partial}{\partial r} \cdot \dfrac{\partial z}{\partial y}}_{②} \cdot 3s + \dfrac{\partial z}{\partial y} \cdot 0$

$\dfrac{\partial z}{\partial x} \left\langle \begin{array}{l} x \left\langle \begin{array}{l} r \\ s \end{array}\right. \\ y \left\langle \begin{array}{l} r \\ s \end{array}\right. \end{array}\right.$

① $\dfrac{\partial}{\partial r}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) \cdot \dfrac{\partial x}{\partial r} + \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right) \cdot \dfrac{\partial y}{\partial r}$

$= \dfrac{\partial^2 z}{\partial x^2} \cdot 2r + \dfrac{\partial^2 z}{\partial x \partial y} \cdot 3s$

② $\dfrac{\partial}{\partial r}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) \cdot \dfrac{\partial x}{\partial r} + \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) \cdot \dfrac{\partial y}{\partial r}$

$= \dfrac{\partial^2 z}{\partial x \partial y} \cdot 2r + \dfrac{\partial^2 z}{\partial y^2} \cdot 3s$

대입하면

$= \left\{ \dfrac{\partial^2 z}{\partial x^2} \cdot 2r + \dfrac{\partial^2 z}{\partial x \partial y} \cdot 3s \right\} \cdot 2r + \dfrac{\partial z}{\partial x} \cdot 2 + \left\{ \dfrac{\partial^2 z}{\partial x \partial y} \cdot 2r + \dfrac{\partial^2 z}{\partial y^2} \cdot 3s \right\} \cdot 3s$

$= 2 \cdot \dfrac{\partial z}{\partial x} + 4r^2 \cdot \dfrac{\partial^2 z}{\partial x^2} + 12rs \cdot \dfrac{\partial^2 z}{\partial x \partial y} + 9s^2 \dfrac{\partial^2 z}{\partial y^2}$

$A = 2$ $B = 0$ $C = 4r^2$ $D = 12rs$ $E = 9s^2$

2. Taylor expansion (교안 4p) (맥클로린급수)

$f(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots \rightarrow f(0) = C_0$

$f'(x) = C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + 5C_5 x^4 + \cdots \quad f'(0) = C_1 \rightarrow C_1 = \frac{f'(0)}{1!}$

$f''(x) = 2C_2 + 3\cdot 2\cdot C_3 x + 4\cdot 3\cdot C_4 x^2 + 5\cdot 4\cdot C_5 x^3 + \cdots \quad f''(0) = 2\cdot 1\cdot C_2 \rightarrow C_2 = \frac{f''(0)}{2!}$

$f'''(x) = 3\cdot 2\cdot 1\cdot C_3 + 4\cdot 3\cdot 2\cdot C_4 x + 5\cdot 4\cdot 3\cdot C_5 x^2 + \cdots \quad f'''(0) = 3\cdot 2\cdot 1\cdot C_3 \rightarrow C_3 = \frac{f'''(0)}{3!}$

$f''''(x) = 4\cdot 3\cdot 2\cdot 1\cdot C_4 + 5\cdot 4\cdot 3\cdot 2\cdot C_5 x + \cdots \quad f''''(0) = 4\cdot 3\cdot 2\cdot 1\cdot C_4 \rightarrow C_4 = \frac{f''''(0)}{4!}$

f(x)가 무한히 미분 가능하다면,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots$$

로 표현할 수 있다 $\Rightarrow \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$

But, n번 미분가능하다면 약간의 오차가 발생

$T_n f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{n-1}(0)}{(n-1)!}x^{n-1} + \frac{f^n(0)}{n!}x^n$ - 테일러전개 항

$R_n f(x) = f(x) - T_n$ 오차, 나머지항

$\Rightarrow f(x) = T_n f(x) + R_n f(x)$ 으로 표현가능!

$\therefore f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{n-1}(0)}{(n-1)!}x^{n-1} + \frac{f^n(x^*)}{n!}x^n$ 을 만족시키는 x^*가 $0 < x^* < x$ 사이에 존재

n+1번 미분 가능하다면?

$$\Rightarrow f(x) = T_n f(x) + \frac{f^{n+1}(x^*)}{(n+1)!}x^{n+1}$$

예제 1번

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ 증명

sol) $f(x) = e^x$

$f^n(x) = e^x$ for $^\forall n$

$f^n(0) = e^0 = 1$

$\therefore f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

예제 2번

$e^{jx} = \cos x + j\sin x$ 증명

sol) $e^{jx} = \sum_{n=0}^{\infty} \frac{(jx)^n}{n!} = 1 + \frac{jx}{1} - \frac{x^2}{2!} - \frac{jx^3}{3!} + \frac{x^4}{4!} + \cdots$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$ 임을 이용해 전개해보기

3. Distribution

- Moment Generating Function
- Binomial Distribution
- Normal Distribution

1) Moment generating function (mgf)

⇒ 적률 : X^n의 기댓값 $= E[X^n]$

* 같은 Mgf를 가지면 distribution이 같다!

moment			
1차	$E[X]$		⋯ 평균 (μ)
2차	$E[X^2]$	$E[(X-\mu)^2]$	⋯ 분산
3차	$E[X^3]$	$E[(X-\mu)^3]/\sigma^3$	⋯ 왜도 (기울어진 정도)
4차	$E[X^4]$	$E[(X-\mu)^4]/\sigma^4$	⋯ 첨도 (뾰족한 정도)

⇒ 적률 생성함수 : $E[X^n]$을 생성하는 함수

$$M(t) = E[e^{tx}]$$

X는 확률변수, e^{tx}의 기댓값 $E[e^{tx}]$는 $-h<t<h$ 범위에 존재 (h는 양수)

왜 $E[e^{tx}]$가 $E[X^n]$을 생성하는 함수인가?

⇒ $E[e^{tx}]$를 t로 n번 미분 후 t=0 대입하면 $\underbrace{\text{n차적률}}_{E[X^n]}$ 구할 수 있기 때문!

여기서 f(x)는 확률함수
- 연속적: 확률 밀도 함수(pdf)
- 이산적: 확률 질량 함수(pmf)

증명) $M'(t) = \frac{dM(t)}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} e^{tx}\cdot f(x)\,dx$

$= \int_{-\infty}^{\infty} x\cdot e^{tx}\cdot f(x)\,dx$

$e^{tx}f(x)$를 t로 미분 $= x\cdot e^{tx}\cdot f(x) + e^{tx}\cdot 0$

$M'(0) = \int_{-\infty}^{\infty} xf(x)\,dx = E[X]$ ⋯ 1차적률

마찬가지로 $M''(0) = E[X^2]$ ⋯ 2차적률

$\therefore M^{(n)}(0) = E[X^n]$ ⋯ n차적률

3. Distribution

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2) Binomial Distribution (이항분포)

① 베르누이 시행 : 결과가 성공 or 실패 中 1가지인 시행 (확률 상관 x)

ex) 동전던지기

Success(S) or Failure(F)

$\therefore P(S) = p, \quad P(F) = 1-p$

② 베르누이 분포 (Bernoulli Distribution)

확률변수 X $\begin{cases} 1 & \text{성공} \\ 0 & \text{실패} \end{cases}$

↓ 확률질량함수(pmf)로 표현

· pmf : $f_X(x) = p^x(1-p)^{1-x}$ for $x = 1, 0$

$\Rightarrow f_X(1) = p, \quad f_X(0) = 1-p$

· $E[X] = p$

$= p \times 1 + (1-p) \times 0$

$= p$

· $Var(X) = p(1-p)$

$= E[X^2] - E[X]^2$

$= p - p^2$

$= p(1-p)$

③ 이항분포 (Binomial Distribution)

: n번의 독립적인 시행에서,

각 시행의 확률이 p로 같은 이산확률 분포

$X \sim B(n, p)$

ex) 주사위를 10번 던져서 숫자 6이 나오는 횟수 x

$X \sim B(10, \frac{1}{6})$

- 베르누이 시행을 n번 합하면 → 이항분포

6이 나옴/안나옴 10번시행

· pmf : $f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, \cdots, n$

$\binom{n}{x} = {}_nC_x = \frac{n!}{x!(n-x)!}$

ex) 주사위 10번 던져서 6이 2번 나올 확률

$= {}_{10}C_2 \times (\frac{1}{6})^2 \times (\frac{5}{6})^8$

· $E[X] = np, \quad Var = np(1-p)$

⇒ mgf로 증명

· mgf : $M(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx} \cdot p(x)$

$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$

e^{tx}와 p^x 묶기

$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$

이항정리

$= [1-p+pe^t]^n$

$M'(t) = n(1-p+pe^t) \cdot pe^t$

$\mu = M'(0) = np \cdots E[X]$

$\sigma^2 = E[X^2] - E[X]^2$

$= M''(0) - (np)^2$

$= np(1-p)$

3. Distribution

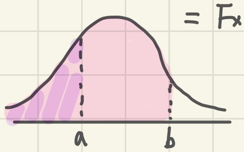
- Moment Generating Function
- Binomial Distribution
- Normal Distribution

참고) cdf (cumulative density function)

$F_X(x) = P_X(X \le x)$

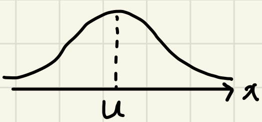
$\therefore P(a < X \le b) = P_X(X \le b) - P_X(X \le a)$

$= F_X(b) - F_X(a)$



3) Normal distribution (정규분포)

$\Rightarrow X \sim N(u, \sigma^2)$



· pdf : $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-u)^2}{2\sigma^2}}$ ··· 암기X

$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-u)^2}{2\sigma^2}} dx = 1$

표준화

$\therefore X \sim N(u, \sigma^2)$를 $Z = \frac{X-u}{\sigma} \sim N(0,1)$

3-2) Standard normal distribution (표준정규분포)

$\Rightarrow X \sim N(0, 1)$

· pdf $= \phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

· cdf $= \Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \phi(t)\, dt$

· mgf : $M(t) = E[e^{tZ}]$

$= \int_{-\infty}^{\infty} e^{tz} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}} \cdot e^{\frac{1}{2}t^2} dz$

$= e^{\frac{1}{2}t^2}$ ··· mgf of Z

· mgf of X 구하기

$Z = \frac{X-u}{\sigma}$ 에서, $X = Z\sigma + u$

$M(t) = E[e^{tX}] = E[e^{t(Z\sigma+u)}]$

$= e^{ut} \cdot E[e^{t\sigma Z}]$

$= e^{ut} \cdot e^{\frac{1}{2}\sigma^2 t^2}$

$= e^{ut + \frac{1}{2}\sigma^2 t^2}$