1. e^x를 Taylor's expansion을 이용하여 전개

$f(x) = e^x$

$f'(x) = e^x$

$f^{(n)}(x) = e^x$ for $\forall n$

$f^{(n)}(0) = 1$

$\therefore f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. $w = f(x-y, y-x)$일 때, $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$ 임을 보여라

Let $\begin{cases} x-y=r \\ y-x=s \end{cases} \longrightarrow w = f(r,s)$

i) $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial w}{\partial s} \cdot \frac{\partial s}{\partial x}$

$= \frac{\partial w}{\partial r} \cdot 1 + \frac{\partial w}{\partial s} \cdot (-1) = \frac{\partial w}{\partial r} - \frac{\partial w}{\partial s}$

ii) $\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial w}{\partial s} \cdot \frac{\partial s}{\partial y}$

$$= \frac{\partial w}{\partial r}\cdot(-1) + \frac{\partial w}{\partial s}(1) = \frac{\partial w}{\partial s} - \frac{\partial w}{\partial r}$$

$$\therefore \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = \left(\frac{\partial w}{\partial r} - \frac{\partial w}{\partial s}\right) + \left(\frac{\partial w}{\partial s} - \frac{\partial w}{\partial r}\right)$$

$$= \underline{0}$$

3. $X \sim N(0,t)$ 일 때, mgf of X = ?

$$Z = \frac{X}{\sqrt{t}} \sim N(0,1)$$

표준 정규분포 이용

$$\text{mgf of } Z = M_Z(u) = E[e^{uZ}]$$

$$= \int_{-\infty}^{\infty} e^{uz}\cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-u)^2}{2}} \cdot e^{\frac{u^2}{2}}\, dz$$

$$= e^{\frac{1}{2}u^2}$$

$$\text{mgf of } X = M_X(u) = E[e^{u\sqrt{t}Z}]$$

$$= e^{\frac{1}{2}tu^2}$$

$X \sim N(0, t)$

뜬도와 X

$$M_X(u) = \mathbb{E}[e^{uX}]$$

$$= \int_{-\infty}^{\infty} e^{ux} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\left(\frac{x^2 - 2tux}{2t}\right)} dx$$

$$= \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-tu)^2}{2t}}}_{\to N(t,\ \sqrt{t})} \times e^{\frac{(tu)^2}{2t}}$$

$$= e^{\frac{1}{2}tu^2}$$

4. $w = x^3 f(y/x, z/x)$ 일 때, $x \cdot \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = 3w$

Let $\begin{cases} x^3 = r \\ y/x = s \\ z/x = t \end{cases} \rightarrow w = r f(s, t)$

i) $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial x}$

$= \frac{\partial w}{\partial r} \cdot 3x^2 + \frac{\partial w}{\partial s} \cdot \left(-\frac{y}{x^2}\right) + \frac{\partial w}{\partial t} \cdot \left(-\frac{z}{x^2}\right)$

ii) $\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial y}$

$= \frac{\partial w}{\partial r} \cdot (0) + \frac{\partial w}{\partial s} \cdot \left(\frac{1}{x}\right) + \frac{\partial w}{\partial t} \cdot (0)$

iii) $\frac{\partial w}{\partial z} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial z}$

$$= \frac{\partial w}{\partial r}\cdot(0) + \frac{\partial w}{\partial s}\cdot(0) + \frac{\partial w}{\partial t}\cdot\left(\frac{1}{x}\right)$$

$\rightarrow x\cdot\frac{\partial w}{\partial x} + y\cdot\frac{\partial w}{\partial y} + z\cdot\frac{\partial w}{\partial z}$

$$= \frac{\partial w}{\partial r}\cdot 3x^3 + \frac{\partial w}{\partial s}\cdot\left(-\frac{y}{x}\right) + \frac{\partial w}{\partial t}\left(-\frac{z}{x}\right)$$

$$+ \frac{\partial w}{\partial s}\cdot\frac{y}{x}$$

$$+ \frac{\partial w}{\partial t}\cdot\frac{z}{x}$$

$$= \frac{\partial w}{\partial r}\cdot 3x^3$$

$$= \frac{\partial w}{\partial r}\cdot 3r$$

$$= \boxed{3w}$$