5. Risk Neutral.

THM. Lévy THM.

↳ Let $M(t)$, $t \geq 0$ be a martingale with respect to a filtration $\mathcal{F}(t)$. Assume $M(0)=0$. $M(t)$ has continuous paths, $\underbrace{[M,M](t)=t}$ ∵ $dMdM=dt$.
Then, $M(t)$ is a Brownian Motion (Wiener process)

pf) Show $M(t) \sim N(0,t)$

Let $\forall f \in C^2$. "$f(t, M(t))$"

$$df = f_t\,dt + f_x\,dM(t) + \frac{1}{2} f_{xx} \underbrace{dM(t)dM(t)}_{=dt}$$

$$f(t,M(t)) - f(0,M(0))$$

$$= \int_0^t f_t(s,M(s))\,ds + \int_0^t f_x(s,M(s))\,dM(s) + \frac{1}{2}\int_0^t f_{xx}(s,M(s))\,ds$$

$$= \int_0^t f_t + \frac{1}{2} f_{xx}\,ds + \int_0^t f_x(s,M(s))\,dM(s)$$

set $f_t + \frac{1}{2} f_{xx} = 0$. $\Rightarrow f(t,x) = e^{ux - \frac{1}{2}u^2 t}$

$$\left(\begin{array}{l} \because f_t = -\frac{1}{2}u^2 e^{ux-\frac{1}{2}u^2t} \\ f_{xx} = u^2 e^{ux - \frac{1}{2}u^2 t} \end{array} \Rightarrow f_t + \frac{1}{2} f_{xx} = 0. \right)$$

$\therefore f(0, M(0)) = 1$

$$e^{uM(t) - \frac{1}{2}u^2 t} - 1 = \text{"}\int_0^t f_x(s, M(s))\,dM(s)\text{"}$$

Itô-integral.

$$\therefore \mathbb{E}\left[e^{uM(t) - \frac{1}{2}u^2 t}\right] = 1$$

$$\underbrace{\mathbb{E}\left[e^{uM(t)}\right]}_{mgf} = e^{\frac{1}{2}u^2 t} \Rightarrow \therefore M(t) \sim N(0,t)$$

THM. Girsanov THM.

↳ Let $W(t)$, $0 \leq t \leq T$, be a Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}(t)$ be a filtration for this Brownian Motion.
Let $\theta(t)$, $0 \leq t \leq T$, be an adapted process.

Define
$$Z(t) = e^{-\int_0^t \theta(u)\,dW(u) - \frac{1}{2}\int_0^t \theta^2(u)\,du} > 0.$$

$$\widetilde{W}(t) = W(t) + \int_0^t \theta(u)\,du$$

↱ $\widetilde{W}(t)$ is a Brownian Motion.

and assume

$$\mathbb{E}\left[\int_0^T \theta^2(u) Z^2(u)\,du\right] < \infty$$

set $Z = Z(t)$

Then, $\mathbb{E}[Z] = 1$

and under $\widetilde{\mathbb{P}}$ given by $\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\,d\mathbb{P}(\omega)$, $\forall A \in \mathcal{F}$
the process $\widetilde{W}(t)$ is a Brownian Motion.

$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z$, $\widetilde{\mathbb{E}}[X] = \mathbb{E}[XZ$

cf) $dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)$

$$= rS(t)\,dt + (\alpha - r)S(t)\,dt + \sigma S(t)\,dW(t)$$

$$= \boxed{r} S(t)\,dt + \sigma \cdot \text{"}\underbrace{\frac{\alpha - r}{\sigma}}\text{"} S(t)\,dt + \sigma S(t)\,dW(t)$$

risk-free interest rate. $\stackrel{let}{=} \theta$: market price of risk

$$= rS(t)\,dt + \sigma S(t) \underbrace{(\theta\,dt + dW(t))}_{= d\widetilde{W}(t)}$$

$$= rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t).$$

$\widetilde{\mathbb{P}}$: risk-neutral measure.

↳ ★ 실제 세계에서 위험 중립 세계로 전환될 때 기대수익률은 변하나 변동성은 변하지 않는다.

◦ actual measure

↳ $P \Leftrightarrow E \Rightarrow \int dP$.

◦ risk-neutral measure.

↳ $\tilde{P} \Leftrightarrow \tilde{E} \Rightarrow \int d\tilde{P}$.

⇒ 위험 중립 세계에서의 확률과 실제 세계에서의 확률은 서로 다름. 실제 세계에서의 기대수익률과 확률을 구할 수 없기 때문에 계산을 위해서 위험중립세계로 전환해야 할 필요가 있다.

$\therefore \tilde{E}[\tilde{W}(t)] = 0 \Leftrightarrow E[W(t)] = 0$.

$*\ dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$

$\quad = rS(t)dt + \sigma S(t)d\tilde{W}(t)$

$\therefore S(t) = S(0)\cdot e^{(r-\frac{1}{2}\sigma^2)t + \sigma\tilde{W}(t)}$

$\therefore \tilde{E}[S(T)] = \tilde{E}[S(0)\cdot e^{(r-\frac{1}{2}\sigma^2)T + \sigma\tilde{W}(T)}]$

$\quad = S(0)\,e^{(r-\frac{1}{2}\sigma^2)T}\,\tilde{E}[e^{\sigma\tilde{W}(T)}]$

$\quad = S(0)\cdot e^{(r-\cancel{\frac{1}{2}\sigma^2})T}\cdot e^{\cancel{\frac{1}{2}\sigma^2}T}$

$\quad = S(0)e^{rT}$.

* Risk-Neutral을 이용한 call option 가치 평가.

↳ $C_0 = e^{-rT}\tilde{E}[\max(S(T)-K, 0)]$

$\cdot S(T) = S(0)\,e^{(r-\frac{1}{2}\sigma^2)T + \sigma\tilde{W}(T)}$

$\cdot \ln S(T) = \ln S(0) + (r-\frac{1}{2}\sigma^2)T + \sigma\tilde{W}(T)$

$\therefore$ " $\ln S(T) \sim N\big(\underbrace{\ln S(0) + (r-\frac{1}{2}\sigma^2)T}_{m},\ \underbrace{\sigma^2 T}_{Var}\big)$ "

log normal distribution.

$\tilde{E}[\ln S(T)] = \ln S(0) + (r-\frac{1}{2}\sigma^2)T$.

$\tilde{Var}[\ln S(T)] = \sigma^2 T$.

$\tilde{E}[S(T)] = S(0)e^{rT}$. $\tilde{Var}[S(T)] = ?$

$\therefore \tilde{E}[\max(S(T)-K, 0)]$

$= \int_{-\infty}^{\infty} \max(S(T)-K, 0)\cdot$ " $g(S(T))$ " $dS(T)$.

→ S(T)에 대한 pdf. (뒷장 remark 참고)

$= \int_{K}^{\infty} (S(T)-K)\,g(S(T))\,dS(T)$

$\left(\text{Let } Q = \frac{\ln S(T) - m}{s} \quad \cdot \begin{cases} m = \ln S(0) + (r-\frac{1}{2}\sigma^2)T. \\ s = \sigma\sqrt{T} \end{cases} \quad \therefore S(T) = e^{m+sQ}\ \&\ Q \sim N(0,1). \right)$

$= \int_{\frac{\ln K - m}{s}}^{\infty} (e^{m+sQ} - K)$ " $h(Q)$ " dQ

↳ $h(Q) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}Q^2}$.

$= \underbrace{\text{"}\int_{\frac{\ln K - m}{s}}^{\infty} e^{m+sQ}\,h(Q)\,dQ\text{"}}_{(i)} - \underbrace{\text{"}\int_{\frac{\ln K - m}{s}}^{\infty} K\cdot h(Q)\,dQ\text{"}}_{(ii)}$

(I) $$e^{m+sQ}\cdot h(Q) = e^{\ln S(0)+(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Q}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}Q^2}$$

$$= \frac{1}{\sqrt{2\pi}}e^{\ln S(0)+(r-\frac{1}{2}\sigma^2)T}\cdot e^{-\frac{1}{2}(Q^2-2\sigma\sqrt{T}Q+\sigma^2T)+\frac{1}{2}\sigma^2T}$$

$$= \frac{1}{\sqrt{2\pi}}e^{\ln S(0)+rT}\cdot e^{-\frac{1}{2}(Q-\sigma\sqrt{T})^2}$$

$$\therefore \int_{\frac{\ln K-m}{s}}^{\infty} e^{m+sQ}h(Q)\,dQ = \frac{1}{\sqrt{2\pi}}e^{\ln S(0)+rT}\int_{\frac{\ln K-m}{s}}^{\infty} e^{-\frac{1}{2}\overset{\text{let} = u.}{\underline{(Q-\sigma\sqrt{T})}}^2}dQ$$

$$= e^{\ln S(0)+rT}\cdot\frac{1}{\sqrt{2\pi}}\int_{-d_1}^{\infty} e^{-\frac{1}{2}u^2}du.$$

$$\boxed{= S(0)e^{rT}N(d_1).}$$

where $N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}p^2}dp$, $d_1=\frac{\ln(\frac{S(0)}{K})+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$.

(II) $$-K\int_{\frac{\ln K-m}{s}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}Q^2}dQ = -K\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}Q^2}dQ$$

$$= -KN(d_2).$$

where $d_2=\frac{\ln(\frac{S(0)}{K})+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$.

$$\therefore \hat{E}[\max(S(T)-K,0)] = S(0)e^{rT}N(d_1)-KN(d_2)$$

$$\therefore C_0 = e^{-rT}\hat{E}[\max(S(T)-K,0)]$$

$$= S(0)N(d_1)-Ke^{-rT}N(d_2).$$ closed form.

Remark) $-\infty < Q < \infty$, $Q \sim N(0,1)$.

$$\hookrightarrow h(Q)\,dQ = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Q^2} dQ \ , \ Q = \frac{\ln S(T) - m}{s} .$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-m}{s}\right)^2} \cdot \frac{du}{s} \ , \ u = \ln S(T) .$$

$$= \frac{1}{\sqrt{2\pi}\,s} e^{-\frac{(u-m)^2}{2s^2}} du \quad \left(= f(u)\,du \right)$$

$$= \boxed{\frac{1}{\sqrt{2\pi}\,s} e^{-\frac{(\ln S(T)-m)^2}{2s^2}} \cdot \frac{1}{S(T)}} dS(T)$$

$$= g(S(T))\,dS(T) .$$

* Black-Scholes Equation 유도. $(dS = \alpha S\,dt + \sigma S\,dW)$.

$$\hookrightarrow \underset{\text{포트폴리오 가치.}}{\Pi} = \underset{\hookrightarrow \text{기초자산 } S\text{를 } \Delta \text{ 단위만큼 매수}}{\Delta \cdot S} - \underset{\hookrightarrow \text{가격이 } f\text{인 option 매도}}{f} .$$

$$d\Pi = \underset{\hookrightarrow \text{짧은 순간에 사고 팔지 않으므로 변화 X.}}{\Delta} \cdot dS - df .$$

Under risk-neutral measure, $d\Pi = r\Pi\,dt$.

$$\therefore \Delta \cdot dS - \left(\frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dS + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \underbrace{dS\,dS}_{\sigma^2 S^2 dt .} \right) = r \cdot (\Delta S - f)\,dt .$$

$$\underset{\underset{0.}{\shortparallel}}{\left(\Delta - \frac{\partial f}{\partial x}\right)} dS = \left(\frac{\partial f}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \underset{\underset{0}{\shortparallel}}{\frac{\partial^2 f}{\partial x^2}} + r\Delta S - rf \right) dt .$$

Let $\Delta = \frac{\partial f}{\partial x}$.

$$\therefore \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial x^2} + r\frac{\partial f}{\partial x} S - rf = 0 . \quad \Rightarrow \quad C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} + rSC_S - rC = 0 .$$

$$C(T, S(T)) = \max(S(T) - K, 0)$$

Def) Arbitrage.

↳ An arbitrage is a portfolio value process $X(t)$ satisfying $X(0)=0$ and satisfying for some time $T>0$.

$$\begin{cases} \mathbb{P}\{X(T)\geq 0\} = 1 \\ \mathbb{P}\{X(T)>0\} > 0. \end{cases}$$

THM) First fundamental THM of Asset Pricing.

↳ If a market model has a risk-neutral probability measure, then it does not admit arbitrage. i.e. $\mathbb{P}\{X(T)>0\}=0$.

즉, Risk-Neutral ⇒ No arbitrage.

(실제로는 양방향).

pf) $\underline{X(0)=0}$ (by assumption) $\therefore \underline{\hat{\mathbb{E}}[X(T)]=\hat{\mathbb{E}}[X(0)]=0}$

Suppose that $\mathbb{P}\{X(T)<0\}=0$ $(\because \mathbb{P}\{X(T)\geq 0\}=1)$

$\Rightarrow \widetilde{\mathbb{P}}\{X(T)<0\}=0$. $(\because \mathbb{P}, \hat{\mathbb{P}}$: equivalent$)$.

And, $\hat{\mathbb{P}}\{X(T)>0\}=0$

Otherwise. $\hat{\mathbb{P}}\{X(T)>0\}>0$.

$\Rightarrow \hat{\mathbb{E}}[X(T)]>0$. Contradiction.

$\therefore \hat{\mathbb{P}}\{X(T)>0\}=0$ &

$\therefore \mathbb{P}\{X(T)>0\}=0$. ⇒ No Arbitrage.