Efficient Computation of Marginal Likelihood

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First Version: April 3, 2024

Abstract

A marginal likelihood is the likelihood of a model that integrates out all model parameters and is the essential quantity for Bayesian model comparison. Though conceptually simple, computing the marginal likelihood in practice is often difficult when the number of model parameters is large, particularly for models with latent variables. In this paper, I propose an approach that constructs an importance sampler to estimate the marginal likelihood based on the principle of minimizing its Monte Carlo sampling variance. The same approach can be used for other Monte Carlo estimators of marginal likelihood such as the Gelfand-Dey method. The proposed approach is simple yet efficient. Performance of the new approach is illustrated in empirical examples including the probit model and linear regressions with stochastic volatility.

Keywords: Model Comparison, Marginal Likelihood, Model Evidence
JEL Codes: C11, C22, C25, E37

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1 Introduction

The marginal likelihood integrates the likelihood function of a model with respect to the prior distribution of model parameters and lies at the heart of model comparison in the Bayesian framework (e.g. Aitkin (1991), Carlin and Chib (1995) and Kass and Raftery (1995)). Specifically, the marginal likelihood of the data $\boldsymbol{y} = \{\boldsymbol{y}_t\}_{t=1}^T$ for a given model is:

$$p(\boldsymbol{y}) = \int p(\boldsymbol{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta} \tag{1}$$

where $\boldsymbol{\theta}$ is the model parameters, $p(\boldsymbol{\theta})$ is the prior distribution of $\boldsymbol{\theta}$ (assumed to be proper) and $p(\boldsymbol{y}|\boldsymbol{\theta})$ is the model likelihood function. Computing $p(\boldsymbol{y})$ in practice is often difficult. Except in rare cases, the integral in Equation (1) is not analytically available and requires numerical integration that can be very challenging when the dimension of $\boldsymbol{\theta}$ is large. Various methods have been proposed in the literature to compute $p(\boldsymbol{y})$ efficiently. Examples include the importance sampling method (Hammersley and Handscomb (1964), Kloek and Dijk (1978), Geweke (1989)), the harmonic mean method (Newton and Raftery (1994), Gelfand and Dey (1994)), the Chib method (Chib (1995), Chib and Jeliazkov (2001)), the bridge sampling method (Meng and Wong (1996), Meng and Schilling (2002)) and the path sampling method (Gelman and Meng (1998), Friel and Pettitt (2008)), among many others. Although substantial progress has been made over the years, calculating marginal likelihoods for model comparison remains a hurdle in many applied studies, especially for models with latent variables. Many of the existing methods for marginal likelihood computation are conceptually simple but can be difficult to implement in practice for various reasons. This paper contributes to the literature by proposing a new approach to implement the importance sampling and harmonic mean methods that is both simple and effective.

The main idea of the proposed approach is that an auxiliary quadratic function of the model parameters $g(\boldsymbol{\theta})$ can be introduced to approximate the log model likelihood function $\log(p(\boldsymbol{y}|\boldsymbol{\theta}))$ such that the pseudo posterior proportional to $p(\boldsymbol{\theta})\exp(g(\boldsymbol{\theta}))$ has an analyti-

cally closed form while being close to the true posterior. Thus the sampling variations of the Monte Carlo estimate of marginal likelihood are limited to the approximation errors between $\exp(g(\boldsymbol{\theta}))$ and $p(\boldsymbol{y}|\boldsymbol{\theta})$ and can be greatly reduced. Relative to the approaches that construct importance samplers to directly approximate the posterior $p(\boldsymbol{\theta}|\boldsymbol{y})$ (e.g. the cross entropy approach of Chan and Eisenstat (2015) and Chan (2023)), the proposed approach is more flexible to allow complex posterior dependence between elements of $\boldsymbol{\theta}$ (e.g. the posterior dependence between latent variables and fixed model parameters in latent variable models). The computational cost of the proposed approach is very low as the parameters of the auxiliary function $g(\boldsymbol{\theta})$ can be determined by a sequence of low-dimensional linear regressions. Both the importance sampler of the importance sampling method and the tuning function of the harmonic mean method can be calibrated by the proposed approach. Encouraging results are found when the new approach is applied to a number of empirical examples including the probit model and the linear regression models with stochastic volatility.

In the remainder of the paper, Section 2 describes the proposed approach in detail. The approaches for comparison are described in Section 3. Empirical examples are provided in Section 4. Section 5 concludes. Additional details are provided in the appendices.

2 The Approach

The proposed approach is motivated by searching for an appropriate importance sampler for computing the marginal likelihood. Hence the discussion will first focus on the importance sampling method. Later in Section 2.3, I will discuss the use of the same approach to construct the tuning function for the harmonic mean method of Gelfand and Dey (1994) (i.e. the Gelfand-Dey method).

Without loss of generality, I assume that the model parameters take values on the whole

real line for convenience.[1] Let $\boldsymbol{\theta}$ denote a K-by-1 vector of the model parameters. The importance sampling (IS) method introduces an auxiliary distribution $q(\boldsymbol{\theta})$ to compute the marginal likelihood via the following equation:

$$p(\boldsymbol{y}) = \int \frac{p(\boldsymbol{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{q(\boldsymbol{\theta})} q(\boldsymbol{\theta}) d\boldsymbol{\theta} \tag{2}$$

An Monte Carlo estimate of the marginal likelihood is:

$$\hat{p}_{\text{IS}}(\boldsymbol{y}) = \frac{1}{m} \sum_{i=1}^{m} \frac{p(\boldsymbol{y}|\boldsymbol{\theta}^{(i)})p(\boldsymbol{\theta}^{(i)})}{q(\boldsymbol{\theta}^{(i)})} \tag{3}$$

where $\{\boldsymbol{\theta}^{(i)}\}_{i=1}^{m}$ are independent draws from the importance sampler $q(\boldsymbol{\theta})$. When $q(\boldsymbol{\theta})$ is close to the posterior $p(\boldsymbol{\theta}|\boldsymbol{y}) \propto p(\boldsymbol{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})$, the IS estimator of Equation (3) enjoys small sampling variance and hence greater accuracy. An obvious choice of $q(\boldsymbol{\theta})$, when $\boldsymbol{\theta}$ is of modest dimension, is a Gaussian distribution whose mean and covariance matrix equal the posterior mean and covariance matrix of $\boldsymbol{\theta}$. However, when the dimension of $\boldsymbol{\theta}$ increases (e.g. in models with latent variables), constructing an appropriate $q(\boldsymbol{\theta})$ in a practical way has proved extremely challenging.

2.1 The Simple Case

In this paper, I focus on the Gaussian distribution as the importance sampler. I begin with the simple case where the prior $p(\boldsymbol{\theta})$ is Gaussian and will discuss the more general case in Section 2.2. Calibrating the parameters of the Gaussian importance sampler is based on the principle of minimizing the variance of the IS estimator:

$$\begin{aligned} V(\hat{p}(\boldsymbol{y})) &= \frac{1}{m} \int \left(\frac{p(\boldsymbol{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{q(\boldsymbol{\theta})} - p(\boldsymbol{y}) \right)^2 q(\boldsymbol{\theta}) d\boldsymbol{\theta} \\ &= \frac{p(\boldsymbol{y})^2}{m} \int \left(\frac{p(\boldsymbol{\theta}|\boldsymbol{y})}{q(\boldsymbol{\theta})} + \frac{q(\boldsymbol{\theta})}{p(\boldsymbol{\theta}|\boldsymbol{y})} - 2 \right) p(\boldsymbol{\theta}|\boldsymbol{y}) d\boldsymbol{\theta} \end{aligned} \tag{4}$$

[1] If a parameter takes values on a subset of the real line (e.g. $\mathbb{R}_+$ or $(0, 1)$), one can always re-parameterize the parameter such that its support becomes $\mathbb{R}$ and adjust its prior accordingly.

where the second line of the equation is derived by using the Bayes theorem $p(\boldsymbol{\theta}|\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\boldsymbol{y})}$. For the task of minimizing $V(\hat{p}(\boldsymbol{y}))$, one can in theory calibrate $q(\boldsymbol{\theta})$ to directly minimize the Monte Carlo counterpart of the integral in Equation (4) based on draws from the posterior $p(\boldsymbol{\theta}|\boldsymbol{y})$. But such a direct optimization scheme is usually infeasible as the posterior $p(\boldsymbol{\theta}|\boldsymbol{y})$ appears inside the integrand and is rarely analytically available.

To simplify the problem, let $f(\boldsymbol{\theta}) = \log(p(\boldsymbol{\theta}|\boldsymbol{y})) - \log(q(\boldsymbol{\theta}))$. A second-order Taylor expansion at $f(\boldsymbol{\theta}) = 0$ gives $\exp(f(\boldsymbol{\theta})) + \exp(-f(\boldsymbol{\theta})) - 2 \approx f(\boldsymbol{\theta})^2$. Inserting this approximation into Equation (4) returns:

$$\begin{aligned} V(\hat{p}(\boldsymbol{y})) &\approx \frac{p(\boldsymbol{y})^2}{m} \int \left(\log(p(\boldsymbol{\theta}|\boldsymbol{y})) - \log(q(\boldsymbol{\theta}))\right)^2 p(\boldsymbol{\theta}|\boldsymbol{y}) d\boldsymbol{\theta} \\ &= \frac{p(\boldsymbol{y})^2}{m} \int \left(\log(p(\boldsymbol{\theta})) + \log(p(\boldsymbol{y}|\boldsymbol{\theta})) - \log(p(\boldsymbol{y})) - \log(q(\boldsymbol{\theta}))\right)^2 p(\boldsymbol{\theta}|\boldsymbol{y}) d\boldsymbol{\theta} \end{aligned} \tag{5}$$

Thus the task of minimizing $V(\hat{p}(\boldsymbol{y}))$ can be reduced to a least-squares problem based on posterior draws of $\boldsymbol{\theta}$. However, directly minimizing the Mote Carlo counterpart of the integral in Equation (5) is still difficult as one needs to estimate the mean and in particular the covariance matrix for $q(\boldsymbol{\theta})$.

Our solution is to introduce a series of auxiliary functions $g_t(\boldsymbol{\theta}; a_t) = a_{1,t} + \boldsymbol{a}'_{2,t}\boldsymbol{\theta} + \boldsymbol{\theta}'\boldsymbol{A}_t\boldsymbol{\theta}$ that is quadratic in $\boldsymbol{\theta}$ to approximate the log likelihood function $\log(p(\boldsymbol{y}_t|\boldsymbol{\theta}))$ for each $t = 1, ..., T$, where a_t denotes the union of the free parameter $a_{1,t}$, $a_{2,t}$ and A_t in each auxiliary function. As will be shown in the example of Section 4.1, the matrix $\boldsymbol{A}_t$ in the auxiliary function $g_t(\boldsymbol{\theta}; a_t)$ can often be written as $a_{3,t}\boldsymbol{A}^*_t$ where $a_{3,t}$ is a scalar and $\boldsymbol{A}^*_t$ is a known positive semi-definite matrix.[2] Thus $a_t = \{a_{1,t}, a_{2,t}, a_{3,t}\}$. Define the function $g(\boldsymbol{\theta}; a)$ as:

$$\begin{aligned} g(\boldsymbol{\theta}; a) &= \sum_{t=1}^{T} g_t(\boldsymbol{\theta}; a_t) \\ &= a_1 + \boldsymbol{a}'_2\boldsymbol{\theta} + \boldsymbol{\theta}'\boldsymbol{A}\boldsymbol{\theta} \end{aligned} \tag{6}$$

[2]In cases where such a simplification of $\boldsymbol{A}_t$ is unavailable, one can simply drop the quadratic term in $g_t(\boldsymbol{\theta})$ (i.e. $\boldsymbol{A}_t = 0$) to keep the approximation tractable at the cost of losing some approximation accuracy.

where $a_1 = \sum_{t=1}^{T} a_{1,t}$, $\boldsymbol{a}_2 = \sum_{t=1}^{T} \boldsymbol{a}_{2,t}$, $\boldsymbol{A} = \sum_{t=1}^{T} a_{3,t}\boldsymbol{A}_t^*$ and $a = \{a_t\}_{t=1}^{T}$. Since the prior $p(\theta)$ is Gaussian, it can be shown that $p(\theta)\exp\left(g(\theta; a)\right)$ is proportional to a Gaussian density function of θ (see Lemma 2.1). Denote the normalizing constant $\int p(\theta)\exp\left(g(\theta; a)\right) d\theta$ as $c(a)$. We construct the importance sampler as $q(\theta; a) = p(\theta)\exp\left(g(\theta; a)\right)/c(a)$. Inserting $q(\theta; a)$ into Equation (5) leads to:

$$V(\hat{p}(\boldsymbol{y})) \approx \frac{p(\boldsymbol{y})^2}{m} \int \left(\log(p(\boldsymbol{y}|\boldsymbol{\theta})) - g(\boldsymbol{\theta}; a) - \log(p(\boldsymbol{y})) + \log(c(a))\right)^2 p(\boldsymbol{\theta}|\boldsymbol{y}) d\boldsymbol{\theta} \tag{7}$$

For practicality, we ignore the cross product terms and approximate the square function inside the integral in Equation (7) as:

$$\sum_{t=1}^{T} \left(\log(p(\boldsymbol{y}_t|\boldsymbol{\theta})) - g_t(\boldsymbol{\theta}; a_t)\right)^2$$

Thus, minimizing $V(\hat{p}(\boldsymbol{y})$ of Equation (7) is simplified to minimize the components:

$$\begin{aligned} &\int \left(\log(p(\boldsymbol{y}_t|\boldsymbol{\theta})) - g_t(\boldsymbol{\theta}; a_t)\right)^2 p(\boldsymbol{\theta}|\boldsymbol{y}) d\boldsymbol{\theta} \\ =&\int \left(\log(p(\boldsymbol{y}_t|\boldsymbol{\theta})) - a_{1,t} + a_{2,t}'\theta - a_{3,t}\theta' A_t^*\theta\right)^2 p(\theta|y) d\theta \end{aligned} \tag{8}$$

for each $t = 1, ..., T$. One can simply run a regression of the log likelihood $\log(p(\boldsymbol{y}_t|\boldsymbol{\theta}))$ on a constant, $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'\boldsymbol{A}_t^*\boldsymbol{\theta}$ based on posterior draws of $\boldsymbol{\theta}$ and determine the optimal value of the free parameters $a_t^* = \{a_{1,t}^*, \boldsymbol{a}_{2,t}^*, a_{3,t}^*\}$ for each auxiliary function $g_t(\boldsymbol{\theta}; a_t)$, $t = 1, ..., T$. For $g_t(\boldsymbol{\theta}; a_t^*)$ to be useful, we require that $a_{3,t}^* \leq 0$ for all $t = 1, ..., T$ (see Lemma 2.1 and Remark 1 below).[3] Inserting $a^* = \{a_t^*\}_{t=1}^{T}$ into $q(\theta; a)$ pins down the optimized importance sampler $q(\theta; a^*)$.

Further intuition of the proposed approach can be gained by going back to the IS representation of the marginal likelihood in Equation (2). Recall that the goal is to find an importance sampler that is close to the posterior $p(\boldsymbol{\theta}|\boldsymbol{y}) \propto p(\boldsymbol{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})$. By introducing

[3]If the regression estimate of $a_{3,t}$ is positive, one can set $a_{3,t}^* = 0$ and re-run the regression by keeping only the constant and $\boldsymbol{\theta}$ as regressors.

the auxiliary functions $\{g_t(\boldsymbol{\theta}; a_t)\}_{t=1}^T$, one can re-write the marginal likelihood as:

$$p(\boldsymbol{y}) = \int \left(\frac{p(\boldsymbol{y}|\boldsymbol{\theta})}{\exp(g(\boldsymbol{\theta}; a))}\right) \left(\frac{p(\boldsymbol{\theta})\exp(g(\boldsymbol{\theta}; a))}{q(\boldsymbol{\theta}; a)}\right) q(\boldsymbol{\theta}; a) d\boldsymbol{\theta} \tag{9}$$

For the first term $\frac{p(\boldsymbol{y}|\boldsymbol{\theta})}{\exp(g(\boldsymbol{\theta};a))}$ of the integrand, the denominator $\exp(g(\boldsymbol{\theta}; a))$ is made "close" to the numerator $p(\boldsymbol{y}|\boldsymbol{\theta})$ through a series of simple linear regressions as discussed in the preceding paragraph, while the second term $\frac{p(\boldsymbol{\theta})\exp(g(\boldsymbol{\theta};a))}{q(\boldsymbol{\theta};a)}$ of the integrand is a constant as per the construction of $q(\theta)$. Thus the new approach achieves the goal of making the importance sampler $q(\boldsymbol{\theta})$ close to $p(\boldsymbol{\theta})p(\boldsymbol{y}|\boldsymbol{\theta})$ by using the auxiliary functions $\{g_t(\boldsymbol{\theta})\}_{t=1}^T$ as a series of "bridges". Instead of directly jumping from $q(\boldsymbol{\theta})$ to the target $p(\boldsymbol{\theta})p(\boldsymbol{y}|\boldsymbol{\theta})$, the new approach takes a shortcut via the constructed bridges, which facilitates easy construction of a useful importance sampler with minimal computation cost.

Lemma 2.1. *Suppose $\boldsymbol{\theta}$ is a K-by-1 vector of random variables taking values on the whole real line. If $p(\boldsymbol{\theta})$ is a Gaussian density function for $N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ and $g(\boldsymbol{\theta}) = a_1 + \boldsymbol{a}_2'\boldsymbol{\theta} + \boldsymbol{\theta}'\boldsymbol{A}\boldsymbol{\theta}$ is a quadratic function of $\boldsymbol{\theta}$ where a_1 is a scalar, $\boldsymbol{a}_2$ is a K-by-1 vector and $\boldsymbol{A}$ is a K-by-K negative semi-definite matrix, then the product $p(\boldsymbol{\theta})exp(g(\boldsymbol{\theta}))$ is proportional to the Gaussian density function of $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ satisfy $\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}_0^{-1} - 2\boldsymbol{A}$ and $\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0 + \boldsymbol{a}_2$.*

Proof. The proof can be found in Appendix A. □

Remark 1. In the simple case of this section, $\boldsymbol{A} = \sum_{t=1}^T a_{3,t}\boldsymbol{A}_t^*$ where each $\boldsymbol{A}_t^*$ is a positive semi-definite matrix (Equation (6)). Given that $\boldsymbol{\Sigma}_0$ is positive definite by definition, a sufficient condition for the matrix $\boldsymbol{\Sigma}$ satisfying $\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}_0^{-1} - 2\boldsymbol{A}$ to be positive definite is that $a_{3,t} \leq 0$ for all t .

2.2 The General Case

In the general case, consider a model with latent variables $\boldsymbol{\theta} = \{\boldsymbol{\theta}_t\}_{t=1}^T$ and fixed model parameters $\boldsymbol{\delta}$, where the prior $p(\theta, \delta) = p(\delta)p(\theta|\delta)$ is such that $p(\boldsymbol{\delta})$ could be non-Gaussian

while $p(\boldsymbol{\theta}|\boldsymbol{\delta})$ is Gaussian. Given an importance sampler $q(\boldsymbol{\theta}, \boldsymbol{\delta}) = q(\theta|\delta)q(\delta)$, the marginal likelihood can be written as:

$$p(\boldsymbol{y}) = \int \frac{p(\boldsymbol{y}|\boldsymbol{\theta}, \boldsymbol{\delta})p(\boldsymbol{\theta}, \boldsymbol{\delta})}{q(\boldsymbol{\theta}, \boldsymbol{\delta})} q(\boldsymbol{\theta}, \boldsymbol{\delta}) d\boldsymbol{\theta} d\boldsymbol{\delta} \tag{10}$$

with the Monte Carlo estimate:

$$\hat{p}(\boldsymbol{y}) = \frac{1}{m} \sum_{i=1}^{m} \frac{p(\boldsymbol{y}|\boldsymbol{\theta}^{(i)}, \boldsymbol{\delta}^{(i)})p(\boldsymbol{\theta}^{(i)}, \boldsymbol{\delta}^{(i)})}{q(\boldsymbol{\theta}^{(i)}, \boldsymbol{\delta}^{(i)})} \tag{11}$$

where $\{\boldsymbol{\theta}^{(i)}, \boldsymbol{\delta}^{(i)}\}_{i=1}^{m}$ are independent draws from the importance sampler $q(\boldsymbol{\theta}, \boldsymbol{\delta})$.

Constructing an appropriate importance sampler $q(\boldsymbol{\theta}, \boldsymbol{\delta})$ here is trickier than in the simple case of Section 2.1 due to the presence of $\boldsymbol{\delta}$. We focus on the case where the likelihood $p(y|\theta, \delta)$ can be decomposed as $\prod_{t=1}^{T} p(y_t|y^{t-1}, \theta_t, \delta)$ where $y^t = \{y_1, ..., y_t\}$ and y^0 is an empty set. To simplify notations, we will write $p(y_t|y^{t-1}, \theta_t, \delta)$ as $p(y_t|\theta_t, \delta)$ hereafter.

Let $\boldsymbol{\delta}$ be an L-by-1 vector, $\boldsymbol{\theta}_t$ a K-by-1 vector and $\boldsymbol{\theta} = [\boldsymbol{\theta}_1' \ ... \ \boldsymbol{\theta}_T']'$. We aim to construct an auxiliary function $g_t(\theta_t; \delta, a_t)$ that is quadratic in θ_t to approximate $\log(p(y_t|\theta_t, \delta))$ for each $t = 1, ..., T$. The details of constructing $g_t(\theta_t; \delta, a_t)$ are provided in Section 2.2.1. The resulting auxiliary function is $g_t(\theta_t; \delta, a_t) = a_{1,t} + a_{2,t}'\theta_t + a_{3,t}h_t(\delta)'\theta_t + a_{4,t}\theta_t' A_t^*(\delta)\theta_t$ for each $t = 1, ..., T$, where $h_t(\delta)$ is a known function mapping δ to a K-by-1 vector, $A_t^*(\delta)$ is a known function mapping δ to a K-by-K positive semi-definite matrix, and $a_t = \{a_{1,t}, a_{2,t}, a_{3,t}, a_{4,t}\}$ collects the free parameters of the function.

Now let $g(\theta; \delta, a) = \sum_{t=1}^{T} g_t(\theta_t; \delta, a_t)$ where $a = \{a_t\}_{t=1}^{T}$ denotes the free parameters of the auxiliary functions. One can write $g(\theta; \delta, a) = a_1 + a_2(\delta)'\theta + \theta' A(\delta)\theta$, where $a_1 = \sum_{t=1}^{T} a_{1,t}$, $a_2(\delta) = [a_{2,1}' + a_{3,1}h_1(\delta)' \ ... \ a_{2,T}' + a_{3,T}h_T(\delta)']'$ and $A(\delta)$ is a KT-by-KT block-diagonal matrix with the diagonal block $a_{4,t}A_t^*(\delta)$ for $t = 1, ...,$ T. Since $p(\theta|\delta)$ is Gaussian, one can apply Lemma 2.1 to show that $p(\theta|\delta)\exp\left(g(\theta; \delta, a)\right)$ is proportional to a Gaussian density function of θ and thus construct a Gaussian importance sampler $q(\theta|\delta; a) = p(\theta|\delta)\exp\left(g(\theta; \delta, a)\right)/c(\delta, a)$ where $c(\delta, a) = \int p(\theta|\delta)\exp\left(g(\theta; \delta, a)\right) d\theta$ is the normalizing constant and is analytically available. Similar to the derivation of Equation (5),

the Monte Carlo variance of the resulting IS estimate of the marginal likelihood can be approximated as:

$$V(\hat{p}(\boldsymbol{y})) \approx \frac{p(\boldsymbol{y})^2}{m} \int \left(\log(p(\boldsymbol{\theta}, \delta|\boldsymbol{y})) - \log(q(\boldsymbol{\theta}|\delta; a)) - \log(q(\delta))\right)^2 p(\boldsymbol{\theta}, \delta|\boldsymbol{y}) d\boldsymbol{\theta} d\delta \qquad (12)$$

By inserting $q(\theta|\delta; a) = p(\theta|\delta)\exp\left(g(\theta; \delta, a)\right)/c(\delta, a)$ and applying the Bayes law, the squared function in the integrand of Equation (12) can be re-written as:

$$\left(f_1(\theta, \delta; a) + f_2(\delta; a)\right)^2$$

where $f_1(\theta, \delta; a) = \log(p(y|\theta, \delta)) - g(\theta|\delta; a) = \sum_{t=1}^{T} \log(p(y_t|\theta_t, \delta)) - g_t(\theta_t|\delta; a_t)$, and $f_2(\delta; a) = \log(c(\delta; a)) + \log(p(\delta)) - \log(p(y)) - \log(q(\delta))$. By ignoring the cross product terms to simplify the work, minimizing the Monte Carlo variance $V(\hat{p}(\boldsymbol{y}))$ is reduced to minimize the following items:

$$\begin{aligned} &\int \left(\log(p(y_t|\theta_t, \delta)) - g_t(\theta_t|\delta; a_t)\right)^2 p(\boldsymbol{\theta}, \delta|\boldsymbol{y}) d\boldsymbol{\theta} d\delta \\ =&\int \left(\log(p(y_t|\theta_t, \delta)) - a_{1,t} - a_{2,t}'\theta_t - a_{3,t} h_t(\delta)'\theta_t - a_{4,t}\theta_t' A_t^*(\delta)\theta_t\right)^2 p(\boldsymbol{\theta}, \delta|\boldsymbol{y}) d\boldsymbol{\theta} d\delta \end{aligned} \qquad (13)$$

for each $t = 1, ..., T$ and:

$$\begin{aligned} &\int \left(f_2(\delta; a)\right)^2 p(\boldsymbol{\theta}, \delta|\boldsymbol{y}) d\boldsymbol{\theta} d\delta \\ =&\int \left(\log(c(\delta; a)) + \log(p(\delta)) - \log(p(y)) - \log(q(\delta))\right)^2 p(\delta|\boldsymbol{y}) d\delta \end{aligned} \qquad (14)$$

For the first item in Equation (13), one can simply run a regression of the log likelihood $\log(p(y_t|\theta_t, \delta))$ on a constant, θ_t, $h_t(\delta)'\theta_t$ and $\theta_t' A_t^*(\delta)\theta_t$ based on posterior draws of θ_t and δ to obtain the optimal value $a_t^* = \{a_{1,t}^*, a_{2,t}^*, a_{3,t}^*, a_{4,t}^*\}$ for each $t = 1, ..., T$. Inserting $a^* = \{a_t^*\}_{t=1}^T$ gives the optimized importance sampler $q(\theta|\delta; a^*)$.

The part $q(\delta)$ of the importance sampler is specified as a Gaussian distribution for simplicity. In principle, one can apply a non-linear least squares procedure to calibrate the mean and covariance matrix of $q(\delta)$ based on Equation (14), subject to the constraint that

the covariance matrix must be positive definite. In this paper, we opt for a simpler heuristic approach. Given that $g(\theta|\delta; a^*)$ approximates $\log(p(y|\theta, \delta))$, we have $c(\delta; a^*) \approx p(y|\delta)$ by the definition of $c(\delta; a)$. Therefore the item of Equation (14) is usefully approximated by $\int (\log(p(\delta|y)) - \log(q(\delta)))^2 p(\delta|y) d\delta$, which is minimized when $q(\delta) = p(\delta|y)$. Since δ takes value on the whole real line, we approximate the posterior $p(\delta|y)$ as a Gaussian distribution. Therefore, we construct $q(\delta)$ as a Gaussian distribution whose mean and covariance matrix equal the posterior mean and covariance matrix of δ.

To directly apply Lemma 2.1 in this general case, one needs to work with a full covariance matrix of the size TK-by-TK, which could be computationally cumbersome. Since the conditional prior $p(\boldsymbol{\theta}|\boldsymbol{\delta})$ often admits a Markovian structure $p(\boldsymbol{\theta}|\boldsymbol{\delta}) = p(\boldsymbol{\theta}_1|\boldsymbol{\delta}) \prod_{t=2}^{T} p(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1}, \boldsymbol{\delta})$ in practice, the calibration of $q(\boldsymbol{\theta}|\boldsymbol{\delta})$ can be simplified by a backward sequential matching procedure as in Lemma 2.2.

Lemma 2.2. *Suppose $\boldsymbol{\theta}_t$ is a K-by-1 vector of random variables taking values on the whole real line and $\boldsymbol{\theta} = \{\boldsymbol{\theta}_t\}_{t=1}^{T}$. If $p(\boldsymbol{\theta}) = p(\boldsymbol{\theta}_1) \prod_{t=2}^{T} p(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1})$, where $p(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1}) = N(\boldsymbol{\mu}_0 + \boldsymbol{\Phi}_0\boldsymbol{\theta}_{t-1}, \boldsymbol{\Sigma}_0)$ for $t > 1$ and $p(\boldsymbol{\theta}_1) = N(\tilde{\boldsymbol{\mu}}_0, \tilde{\boldsymbol{\Sigma}}_0)$, and $g(\boldsymbol{\theta}) = \sum_{t=1}^{T} a_{1,t} + \boldsymbol{a}_{2,t}'\boldsymbol{\theta}_t + \boldsymbol{\theta}_t'\boldsymbol{A}_t\boldsymbol{\theta}_t$ is a quadratic function of $\boldsymbol{\theta}$ where $a_{1,t}$ is a scalar, $\boldsymbol{a}_{2,t}$ is a K-by-1 vector and $\boldsymbol{A}_t$ is a K-by-K negative semi-definite matrix, then the product $p(\boldsymbol{\theta})exp(g(\boldsymbol{\theta}))$ is proportional to the function $q(\boldsymbol{\theta}) = q(\boldsymbol{\theta}_1) \prod_{t=2}^{T} q(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1})$ where $q(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1}) = N(\boldsymbol{\mu}_t + \boldsymbol{\Phi}_t\boldsymbol{\theta}_{t-1}, \boldsymbol{\Sigma}_t)$ for $t > 1$ and $p(\boldsymbol{\theta}_1) = N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$. The parameters $\boldsymbol{\mu}_t$, $\boldsymbol{\Phi}_t$ and $\boldsymbol{\Sigma}_t$ follow a recursive law of motion:*

- *For $t = T$, $\boldsymbol{\Sigma}_t^{-1} = \boldsymbol{\Sigma}_0^{-1} - 2\boldsymbol{A}_t$, $\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t = \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0 + \boldsymbol{a}_{2,t}$, $\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\Phi}_t = \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Phi}_0$.*
- *For $1 < t < T$, $\boldsymbol{\Sigma}_t^{-1} + \boldsymbol{\Phi}_{t+1}'\boldsymbol{\Sigma}_{t+1}^{-1}\boldsymbol{\Phi}_{t+1} = \boldsymbol{\Sigma}_0^{-1} - 2\boldsymbol{A}_t + \boldsymbol{\Phi}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Phi}_0$, $\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t - \boldsymbol{\Phi}_{t+1}'\boldsymbol{\Sigma}_{t+1}^{-1}\boldsymbol{\mu}_{t+1} = \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0 + \boldsymbol{a}_{2,t} - \boldsymbol{\Phi}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0$, $\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\Phi}_t = \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Phi}_0$.*
- *For $t = 1$, $\boldsymbol{\Sigma}_t^{-1} + \boldsymbol{\Phi}_{t+1}'\boldsymbol{\Sigma}_{t+1}^{-1}\boldsymbol{\Phi}_{t+1} = \tilde{\boldsymbol{\Sigma}}_0^{-1} - 2\boldsymbol{A}_t + \boldsymbol{\Phi}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Phi}_0$, $\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t - \boldsymbol{\Phi}_{t+1}'\boldsymbol{\Sigma}_{t+1}^{-1}\boldsymbol{\mu}_{t+1} = \tilde{\boldsymbol{\Sigma}}_0^{-1}\tilde{\boldsymbol{\mu}}_0 + \boldsymbol{a}_{2,t} - \boldsymbol{\Phi}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0$.*

Proof. The proof can be found in Appendix B. □

Remark 2. The computation cost in Lemma 2.2 is limited to working with matrices of size K-by-K, which is much cheaper than in Lemma 2.1 where one would directly work with a full covariance matrix of size TK-by-TK.

Remark 3. The posterior dependence between the latent variables $\boldsymbol{\theta}$ and the fixed model parameters $\boldsymbol{\delta}$ can be complicated. It is clear from the preceding discussions and Lemma 2.2 that the component $q(\boldsymbol{\theta}|\boldsymbol{\delta})$ of the importance sampler thus constructed allows non-linear dependence between $\boldsymbol{\theta}$ and $\boldsymbol{\delta}$ and hence can be more flexible for capturing the joint posterior distribution of $\boldsymbol{\theta}$ and $\boldsymbol{\delta}$ than alternatives that directly approximate $p(\boldsymbol{\theta}, \boldsymbol{\delta}|\boldsymbol{y})$ (e.g. using a joint Gaussian distribution to approximate $p(\boldsymbol{\theta}, \boldsymbol{\delta}|\boldsymbol{y})$).

2.2.1 Approximating the Log Likelihood Function

To approximate the point-t log likelihood function $\log(p(\boldsymbol{y}_t|\boldsymbol{\theta_t}, \boldsymbol{\delta}))$, a straightforward form of the auxiliary function, similar to the simple case of Section 2.1, would be $g(\theta_t; \delta, a_t) = a_{1,t} + a'_{2,t}\delta + a'_{3,t}\theta_t + a_{4,t}\theta'_t A^*_t \theta_t$, where $a_t = \{a_{1,t}, a_{2,t}, a_{3,t}, a_{4,t}\}$ are the free parameters and A^*_t is a known K-by-K positive definite matrix. However, we have found that, by exploiting the specific feature of the log likelihood function of a model under study, one can often construct a more accurate auxiliary function and thus improve the marginal likelihood estimate.

To that end, we note that one can always write the point-t log likelihood function in the following form:

$$\log(p(\boldsymbol{y}_t|\boldsymbol{\theta_t}, \boldsymbol{\delta})) = f_{1,t}(\boldsymbol{\delta}) + \boldsymbol{f}_{2,t}(\boldsymbol{\delta})'\boldsymbol{\theta}_t + f_{3,t}(\boldsymbol{\delta}) h_t(\boldsymbol{\theta}_t, f_{4,t}(\boldsymbol{\delta})) \tag{15}$$

where $f_{1,t} : \mathbb{R}^L \to \mathbb{R}$, $\boldsymbol{f}_{2,t} : \mathbb{R}^L \to \mathbb{R}^K$ and $f_{3,t} : \mathbb{R}^L \to \{\mathbb{R}_+, 0\}$ are known functions of $\boldsymbol{\delta}$, $\boldsymbol{f}_{4,t} : \mathbb{R}^L \to \mathbb{R}^H$ is a known function mapping δ to an H-by-1 vector (H could be different from K or L), while $h_t : \mathbb{R}^{K+H} \to \mathbb{R}$ is a known function of $\boldsymbol{\theta}_t$ and $f_{4,t}(\boldsymbol{\delta})$. Such a formulation of the log likelihood function is always valid. For example, one can trivially

let $f_{1,t}(\delta) = 0$, $f_{2,t}(\delta) = 0$, $f_{3,t}(\delta) = 1$, $f_{4,t}(\delta) = \delta$ and $h_t(\theta_t, f_{4,t}(\delta)) = \log(p(\boldsymbol{y}_t|\boldsymbol{\theta_t}, \boldsymbol{\delta}))$. However, depending on the specific model under study, other non-trivial forms of the functions $f_{1,t}(\delta)$, $f_{2,t}(\delta)$, $f_{3,t}(\delta)$, $f_{4,t}(\delta)$ and $h_t(\theta_t, f_{4,t}(\delta))$ can be derived. Examples are illustrated for the empirical models in Section 4. By re-writing $\log(p(\boldsymbol{y}_t|\boldsymbol{\theta_t}, \boldsymbol{\delta}))$ in the form of Equation (15), one only needs to find a quadratic function of θ_t to approximate the component $h_t(\theta_t, f_{4,t}(\delta))$ and hence reduce the complexity of approximating $\log(p(\boldsymbol{y}_t|\boldsymbol{\theta_t}, \boldsymbol{\delta}))$.

Specifically, denote $\xi_t = f_{4,t}(\delta)$ to simplify notations. Guided by a second-order Taylor expansion of $h_t(\theta_t, \xi_t)$, we introduce the auxiliary function:

$$g_t^*(\boldsymbol{\theta}; \boldsymbol{\delta}) = a_{1,t} + a_{2,t}'\xi_t + a_{3,t}'\text{vech}(\xi_t\xi_t') + a_{4,t}'\text{vec}(\xi_t\theta_t') + a_{5,t}'\theta_t + a_{6,t}\theta_t' A_t^*\theta_t \tag{16}$$

to approximate $h_t(\boldsymbol{\theta}_t, \boldsymbol{\xi_t})$, where $a_t = \{a_{1,t}, a_{2,t}, a_{3,t}, a_{4,t}, a_{5,t}, a_{6,t}\}$ are free parameters to be determined and $\boldsymbol{A}_t^*$ is a known K-by-K positive semi-definite matrix. Adding the other terms of the log likelihood in Equation (15) leads to the auxiliary function

$$g_t(\boldsymbol{\theta}; \boldsymbol{\delta}) = f_{1,t}(\boldsymbol{\delta}) + \boldsymbol{f}_{2,t}(\boldsymbol{\delta})'\boldsymbol{\theta}_t + f_{3,t}(\boldsymbol{\delta})g_t^*(\boldsymbol{\theta}; \boldsymbol{\delta}) \tag{17}$$

that approximates $\log(p(\boldsymbol{y}_t|\boldsymbol{\theta}, \boldsymbol{\delta}))$ for each $t = 1, ..., T$. Using similar arguments as in Section 2.1, the free parameters $a_{1,t}$, $\boldsymbol{a}_{2,t}$ and $a_{3,t}$ of the auxiliary function $g_t(\boldsymbol{\theta}; \boldsymbol{\delta})$ can be determined by running a regression of $h_t(\boldsymbol{\theta}_t + \boldsymbol{f}_{4,t}(\boldsymbol{\delta}))$ on a constant, $\boldsymbol{\theta}_t + \boldsymbol{f}_{4,t}(\boldsymbol{\delta})$ and $(\boldsymbol{\theta}_t + \boldsymbol{f}_{4,t}(\boldsymbol{\delta}))'\boldsymbol{A}_t^*(\boldsymbol{\theta}_t + \boldsymbol{f}_{4,t}(\boldsymbol{\delta}))$ based on draws from the posterior $p(\boldsymbol{\theta}, \boldsymbol{\delta}|\boldsymbol{y})$. Similar to the simple case in Section 2.1, we require $a_{3,t} \leq 0$ for all t.

Such a structure of the model likelihood function allows easy construction of the importance sampler and, as will be illustrated in the examples, covers interesting models in practice.

2.3 The Gelfand-Dey Method

In essence, the approach described in Section 2.1 and 2.2 calibrates a distribution for model parameters that approximates their joint posterior distribution. This approach should

be equally useful in other settings where an approximation to a complicated posterior distribution is in need. One such example is the Gelfand-Dey (GD) method of Gelfand and Dey (1994) for computing the marginal likelihood.

The GD method introduces an auxiliary distribution, which is often called a tuning function in the literature, to facilitate the computation of the marginal likelihood. Assuming that the model parameters are $\{\boldsymbol{\theta}, \boldsymbol{\delta}\}$ as in Section 2.2, the GD method uses the following equation:

$$\frac{1}{p(\boldsymbol{y})} = \int \frac{q(\boldsymbol{\theta}, \boldsymbol{\delta})}{p(\boldsymbol{y}|\boldsymbol{\theta}, \boldsymbol{\delta})p(\boldsymbol{\theta}, \boldsymbol{\delta})} p(\boldsymbol{\theta}, \boldsymbol{\delta}|\boldsymbol{y}) d\boldsymbol{\theta} d\boldsymbol{\delta} \tag{18}$$

where $q(\boldsymbol{\theta}, \boldsymbol{\delta})$ is the tuning function satisfying $\int q(\boldsymbol{\theta}, \boldsymbol{\delta}) d\boldsymbol{\theta} d\boldsymbol{\delta} = 1$. The Monte Carlo estimate of marginal likelihood is:

$$\hat{p}_{\mathrm{GD}}(\boldsymbol{y}) = \left(\frac{1}{m} \sum_{i=1}^{m} \frac{q(\boldsymbol{\theta}^{(i)}, \boldsymbol{\delta}^{(i)})}{p(\boldsymbol{y}|\boldsymbol{\theta}^{(i)}, \boldsymbol{\delta}^{(i)})p(\boldsymbol{\theta}^{(i)}, \boldsymbol{\delta}^{(i)})} \right)^{-1} \tag{19}$$

where $\{\boldsymbol{\theta}^{(i)}, \boldsymbol{\delta}^{(i)}\}_{i=1}^{m}$ are draws from the posterior. High efficiency is obtained when the tuning function $q(\boldsymbol{\theta}, \boldsymbol{\delta})$ is close to the posterior $p(\boldsymbol{\theta}, \boldsymbol{\delta}|\boldsymbol{y}) \propto p(\boldsymbol{\theta}, \boldsymbol{\delta})p(\boldsymbol{y}|\boldsymbol{\theta}, \boldsymbol{\delta})$, which is exactly what the proposed approach aims to achieve. In the empirical examples, the GD estimator of marginal likelihood is also applied and is compared to the IS estimator where the tuning function of the GD method and the importance sampler of the IS method are calibrated by the proposed approach in the same way.

3 Approaches for Comparison

This section describes two representatives of existing approaches for constructing importance samplers, which will be used as the benchmarks to evaluate the performance of the proposed approach.

For the simple case described in Section 2.1, a straightforward importance sampler for the model parameters $\boldsymbol{\theta}$ would be a Gaussian distribution with the mean $\frac{1}{m}\sum_{i=1}^{m} \boldsymbol{\theta}^{(i)}$

and the covariance matrix $\frac{1}{m}\sum_{i=1}^{m}\boldsymbol{\theta}^{(i)}(\boldsymbol{\theta}^{(i)})' - \frac{1}{m}\sum_{i=1}^{m}\boldsymbol{\theta}^{(i)}\left(\frac{1}{m}\sum_{i=1}^{m}\boldsymbol{\theta}^{(i)}\right)'$, where $\{\boldsymbol{\theta}^{(i)}\}_{i=1}^{m}$ are posterior draws of $\boldsymbol{\theta}$. This simple approach is often effective in practice when the dimension of $\boldsymbol{\theta}$ is modest. For easier reference, I will call this approach as the *direct calibration* approach.

For the more general case of models with latent variables $\boldsymbol{\theta} = \{\boldsymbol{\theta}_t\}_{t=1}^{T}$ and fixed parameters $\boldsymbol{\delta}$ (see Section 2.2), I consider a recent contribution in the literature by Chan (2023) that proposes a cross entropy approach to construct importance samplers. Basically, minimizing the Kullback-Leibler divergence (i.e. the cross entropy plus a constant) between the posterior $p(\boldsymbol{\theta}, \boldsymbol{\delta}|\boldsymbol{y})$ and the importance sampler $q(\boldsymbol{\theta}, \boldsymbol{\delta})$ leads to a maximum likelihood estimator of $q(\boldsymbol{\theta}, \boldsymbol{\delta})$ based on posterior draws from $p(\boldsymbol{\theta}, \boldsymbol{\delta}|\boldsymbol{y})$. Chan (2023) constructs an importance sampler for $\boldsymbol{\theta}$ by specifying an order-1 autoregressive (AR) process for $\boldsymbol{\theta}_t$ with time-dependent parameters to maximize flexibility and shows that the resulting importance sampler is effective in models with stochastic volatility.

To apply the cross entropy approach of Chan (2023) in the setup of this paper, I specify a vector AR(1) model $\boldsymbol{\theta}_t = \boldsymbol{a}_t + \boldsymbol{B}_t\boldsymbol{\theta}_{t-1} + \boldsymbol{C}_t\boldsymbol{\delta} + N(\boldsymbol{0}, \boldsymbol{D}_t)$ to construct the component $q(\boldsymbol{\theta}|\boldsymbol{\delta})$ of the importance sampler, where the parameter $\boldsymbol{a}_t$ is a K-by-1 vector, $\boldsymbol{B}_t$ is a K-by-K matrix, $\boldsymbol{C}_t$ is a K-by-L matrix and $\boldsymbol{D}_t$ is a K-by-K covariance matrix.[4] The parameters of the vector AR(1) model are estimated at each t based on posterior draws $\{\boldsymbol{\theta}_t^{(i)}, \boldsymbol{\theta}_{t-1}^{(i)}, \boldsymbol{\delta}^{(i)}\}_{i=1}^{m}$ by first running ordinary least squares (OLS) regressions for each element of $\boldsymbol{\theta}_t$ and then using the residuals from the OLS regressions to estimate the covariance matrix $\boldsymbol{D}_t$. The other component $p(\boldsymbol{\delta})$ of the importance sampler is a Gaussian distribution whose mean and covariance matrix equal the posterior mean and covariance matrix of $\boldsymbol{\delta}$. It is useful to note here that the dependence between $\boldsymbol{\theta}$ and $\boldsymbol{\delta}$ is linear in the importance sampler constructed by the cross entropy approach thus described, while the importance sampler constructed by the proposed approach described in Section 2.2 allows non-linear dependence between

[4]The initial value is $\boldsymbol{\theta}_1 = \boldsymbol{a}_1 + \boldsymbol{C}_1\boldsymbol{\delta} + N(\boldsymbol{0}, \boldsymbol{D}_1)$.

$\boldsymbol{\theta}$ and $\boldsymbol{\delta}$.

4 Empirical Examples

The proposed approach is illustrated in 3 models that have been routinely applied in empirical studies of economic data. The first is a binary probit regression model applied to predict economic recessions, which would correspond to the simple case discussed in Section 2.1. The second and third examples concern the linear regression model with stochastic volatility and correspond to the general case of Section 2.2. The second example specifies Gaussian model residual for the regressand while the third example allows fat fails by specifying a student-t distribution for the model residual. Appendix E analyzes the stability of the marginal likelihood estimates to alleviate the concern of the infinite variance problem (Geweke (1989)).

4.1 Binary Probit Regression

Let $y_t \in \{0, 1\}$ be the binary response and $\boldsymbol{x}_t$ be a J-by-1 vector of regressors including the constant. The probit model specifies the model likelihood function $p(\boldsymbol{y}|\boldsymbol{\theta}) = \prod_{t=1}^{T} p(y_t|\boldsymbol{\theta})$, where $p(y_t|\boldsymbol{\theta}) = \Phi(\boldsymbol{x}_t'\boldsymbol{\theta})^{y_t} \left(1 - \Phi(\boldsymbol{x}_t'\boldsymbol{\theta})\right)^{1-y_t}$, $\boldsymbol{\theta}$ is the regression coefficients and $\Phi(\cdot)$ is the cumulative density function of a standard Gaussian distribution. A conventional prior for $\boldsymbol{\theta}$ would be $N(\mathbf{0}, \sigma^2 \mathrm{I}_J)$, where σ^2 is the prior variance common to all regression coefficients and I_J is a J-by-J identity matrix.

The probit model is studied in the context of predicting economic recessions. See Nissila (2020) for a recent survey on probit-based models for recession prediction. Data on the binary response is the NBER-based U.S. recession indicator. Table C1 in Appendix C lists the 11 employed predictors including macroeconomic variables such as the inflation rate as well as financial variables such as the stock market return and the credit spread. The data

sample is quarterly from Q3 1953 to Q2 2021 with a total of 272 observations. Including a constant, there are a total of 12 regressors in the model. The prediction horizon is 4 quarters. Setting the prior variance $\sigma^2 = 100$, posterior draws of $\boldsymbol{\theta}$ are produced by the method in Liu and Wu (1999). A total of 10,000 posterior draws are kept after 2,000 burn-ins.

The probit model under study corresponds to the simple case discussed in Section 2.1. The auxiliary function is $g_t(\boldsymbol{\theta}) = a_{1,t} + \boldsymbol{a}'_{2,t}\boldsymbol{\theta} + \boldsymbol{\theta}'\boldsymbol{A}_t\boldsymbol{\theta}$ where $\boldsymbol{A}_t$ should be negative semi-definite. To simplify the estimation of $\boldsymbol{A}_t$, one can take inspiration from a second-order Taylor expansion of the log likelihood function where the quadratic term is $\boldsymbol{\theta}'\boldsymbol{A}_P\boldsymbol{\theta}$ with $\boldsymbol{A}_P \propto \boldsymbol{x}_t\boldsymbol{x}'_t$. By specifying $\boldsymbol{A}^*_t = \boldsymbol{x}_t\boldsymbol{x}'_t$ and $\boldsymbol{A}_t = a_{3,t}\boldsymbol{A}^*_t$, I run the following regression:

$$\log\left(p(y_t|\boldsymbol{\theta}^{(i)}\right) = a_{1,t} + \boldsymbol{a}'_{2,t}\boldsymbol{\theta}^{(i)} + a_{3,t}\left(\boldsymbol{\theta}^{(i)}\right)'\boldsymbol{A}^*_t\boldsymbol{\theta}^{(i)} + \text{residual} \tag{20}$$

for each $t = 1, ..., T$ where $\{\boldsymbol{\theta}^{(i)}\}_{i=1}^m$ are posterior draws of $\boldsymbol{\theta}$. To ensure $a_{3,t} \leq 0$, I drop the quadratic term and re-run the regression whenever the original regression of Equation (20) returns a positive $a_{3,t}$. The R-squared of the regressions across $t = 1, ..., T$ ranges from 50% to over 99% with the average of 94%. Let $g(\boldsymbol{\theta}) = \sum_{t=1}^T g_t(\boldsymbol{\theta})$ as defined in Equation (6), the importance sampler $q(\boldsymbol{\theta})$ can be easily constructed by using Lemma 2.1.

Table 1 compares the log marginal likelihood (LML) estimates by the direct calibration approach and the new one under both the IS and GD methods. The new approach produces fairly close LML estimates under the IS and GD methods (-128.9365 versus -128.9343), while those by the direct calibration approach shows larger difference (-128.9413 versus -128.9714). Thus there is less finite sample bias in the LML estimates by the new approach than those by the direct calibration one. Moreover, the numerical standard error of the LML estimate by the new approach is 38% smaller than that by the direct calibration one under the IS method and is 29% smaller under the GD method.

4.2 Regression with Stochastic Volatility and Gaussian Residual

For time series studies, the linear regression model is commonly augmented with stochastic volatility (SV) to capture the phenomenon of volatility clustering often observed in economic time series data. A typical form of the model is:

$$y_t = \boldsymbol{x}_t'\boldsymbol{\beta} + \exp\left(\frac{\theta_t}{2}\right)\epsilon_t \tag{21}$$

$$\theta_t = (1-\rho)\mu + \rho\theta_{t-1} + s\eta_t$$

for $t = 1, ..., T$, where y_t is the scalar regressand, $\boldsymbol{x}_t$ is a J-by-1 vector of regressors, ϵ_t and η_t are i.i.d. standard Gaussian residuals, and θ_t is the log volatility with the initial value $\theta_1 = \mu + \sqrt{\frac{s^2}{1-\rho^2}}\eta_1$. For Bayesian analysis, the priors for the fixed model parameters are $\boldsymbol{\beta} \sim N(\mathbf{0}, \sigma^2 \mathrm{I}_J)$, $\mu \sim N(0, \sigma_\mu^2)$, $\frac{1+\rho}{2} \sim Beta(a_{\rho,0}, b_{\rho,0})$, and $s \sim N(0, \sigma_s^2)$. The autoregressive parameter ρ is transformed by $\rho^* = \log\left(\frac{1+\rho}{1-\rho}\right)$ such that it takes values on the whole real line. For the parameter s, the re-parametrization is $s^* = \log(s^2)$.[5] The priors of the transformed parameters are adjusted by using the change-of-variables formula.

The regression model is applied to predict the equity premium with the predictors examined in Welch and Goyal (2008). The dependent variable is the value-weighted quarterly return of the S&P500 index minus the corresponding risk free rate. A total of 12 economic predictors are selected based on Welch and Goyal (2008) that include stock characteristics, interest rates and other macroeconomic indicators and are listed in Table D1 of Appendix D. In estimation, I also include an intercept and an AR 1 lag of the equity premium. The sample runs from Q1 1947 to Q4 2020 with a total of 296 quarterly observations. The prediction horizon is 1 quarter. The hyper parameters of the priors are $\sigma^2 = 100$, $\sigma_\mu^2 = 10$, $a_{\rho,0} = 8$, $b_{\rho,0} = 2$ and $\sigma_s^2 = 1$. An Metropolis-within-Gibbs sampler is applied to estimate the model where estimation of the SV process is based on the algorithm of Kastner

[5]Though the prior of s is Gaussian, its posterior is often bi-model. The re-parametrization $s^* = \log(s^2)$ helps restore a bell-shaped posterior distribution to facilitate constructing the importance sampler.

and Fruhwirth-Schnatter (2014). Analysis is based on 10,000 posterior draws after 2,000 burn-ins.

Let $\boldsymbol{\delta} = \{\boldsymbol{\beta}, \mu, \rho^*, s^*\}$ collect the fixed model parameters and $\boldsymbol{\theta} = \{\theta_t\}_{t=1}^T$ the latent variables. The marginal likelihood can be computed by the IS and GD methods following the discussions in Section 2.2 and 2.3. The importance sampler is $q(\boldsymbol{\theta}, \boldsymbol{\delta}) = q(\boldsymbol{\theta}|\boldsymbol{\delta})q(\boldsymbol{\delta})$ where the component $q(\boldsymbol{\delta})$ is a Gaussian density with its mean and covariance matrix equal to the posterior mean and covariance matrix of $\boldsymbol{\delta}$. To calibrate the component $q(\boldsymbol{\theta}|\boldsymbol{\delta})$, the log likelihood function $\log(p(y_t|\boldsymbol{\theta}, \boldsymbol{\delta})) = -\frac{\log(2\pi)}{2} - \frac{\theta_t}{2} - \frac{(y_t - \boldsymbol{x}_t'\boldsymbol{\beta})^2}{2}\exp(-\theta_t)$ is matched with the structure of Equation (15), which gives $f_{1,t}(\boldsymbol{\delta}) = -\frac{\log(2\pi)}{2}$, $f_{2,t}(\boldsymbol{\delta}) = -\frac{1}{2}$, $f_{3,t}(\boldsymbol{\delta}) = \frac{(y_t - \boldsymbol{x}_t'\boldsymbol{\beta})^2}{2}$, $f_{4,t}(\boldsymbol{\delta}) = 0$, and $h_t(\theta_t + f_{4,t}(\boldsymbol{\delta})) = -\exp(-\theta_t)$. Inserting $f_{4,t}(\boldsymbol{\delta}) = 0$ into Equation (16) leads to the auxiliary function $g_t^*(\boldsymbol{\theta}; \boldsymbol{\delta}) = a_{1,t} + a_{2,t}\theta_t + a_{3,t}\theta_t^2$, where the parameter $\boldsymbol{A}_t^*$ in Equation (16) equals 1. Thus one can determine the free parameters of $g_t^*(\boldsymbol{\theta}; \boldsymbol{\delta})$ by running the following regression:

$$h_t\left(\theta_t^{(i)} + f_{4,t}(\boldsymbol{\delta}^{(i)})\right) = a_{1,t} + a_{2,t}\theta_t^{(i)} + a_{3,t}\left(\theta_t^{(i)}\right)^2 + \text{residual}$$

subject to $a_{3,t} \leq 0$ based on posterior draws $\{\theta_t^{(i)}\}_{i=1}^m$ for each $t = 1, ..., T$.[6] With $g_t^*(\boldsymbol{\theta}; \boldsymbol{\delta})$ at hand, one can use Equation (17) to compute:

$$g_t(\boldsymbol{\theta}; \boldsymbol{\delta}) = \frac{a_{1,t}(y_t - \boldsymbol{x}_t'\boldsymbol{\beta})^2 - \log(2\pi)}{2} + \frac{a_{2,t}(y_t - \boldsymbol{x}_t'\boldsymbol{\beta})^2 - 1}{2}\theta_t + \frac{a_{3,t}(y_t - \boldsymbol{x}_t'\boldsymbol{\beta})^2}{2}\theta_t^2$$

and then compute $g(\boldsymbol{\theta}; \boldsymbol{\delta}) = \sum_{t=1}^T g_t(\boldsymbol{\theta}; \boldsymbol{\delta})$. On the other hand, the AR(1) process for log volatility θ_t in Equation (21) would lead to the prior $p(\boldsymbol{\theta}|\boldsymbol{\delta}) = p(\theta_1|\boldsymbol{\delta})\prod_{t=2}^T p(\theta_t|\theta_{t-1}, \boldsymbol{\delta})$ with $p(\theta_1|\boldsymbol{\delta}) = N\left(\mu, \frac{s^2}{1-\rho^2}\right)$ and $p(\theta_t|\theta_{t-1}, \boldsymbol{\delta}) = N\left((1-\rho)\mu + \rho\theta_{t-1}, s^2\right)$ for $t > 1$. Combining $p(\boldsymbol{\theta}|\boldsymbol{\delta})$ and $g(\boldsymbol{\theta}; \boldsymbol{\delta})$, one can readily apply Lemma 2.2 to calibrate $q(\boldsymbol{\theta}|\boldsymbol{\delta})$.

Table 2 compares the log marginal likelihood (LML) estimates by the cross entropy approach and the new one under both the IS and GD methods. The new approach is able to produce LML estimate with small finite sample bias (LML = 298.4089 and 298.3604

[6]The R-squared of the regressions ranges from 5% to 55% with the average of 36%.

under IS and GD respectively) while the cross entropy approach under the IS and GD methods would produce LML estimates with noticeably larger difference (LML = 297.8849 and 298.1574 under IS and GD respectively). In terms of numerical stability, the numerical standard error of the LML estimate by the new approach is 65% smaller than that by the cross entropy one under the IS method and is 86% smaller under the GD method.

To see the efficiency of the new approach more intuitively, Figure 1 compares the log ratio $\{w^{(i)}\}_{i=1}^m$ that are computed by the cross entropy approach and the new one under both the IS and GD methods, where $w^{(i)} = \log\left(\frac{p(\boldsymbol{y}|\boldsymbol{\theta}^{(i)},\boldsymbol{\delta}^{(i)})p(\boldsymbol{\theta}^{(i)}|\boldsymbol{\delta}^{(i)})p(\boldsymbol{\delta}^{(i)})}{q(\boldsymbol{\theta}^{(i)},\boldsymbol{\delta}^{(i)})}\right)$ and $\{\boldsymbol{\theta}^{(i)},\boldsymbol{\delta}^{(i)}\}_{i=1}^m$ are draws from the importance sampler under the IS method and are posterior draws under the GD method. It is clear from Equation (11) and (19) that, by substituting the log ratio $\{w^{(i)}\}_{i=1}^m$, the IS estimator of marginal likelihood can be re-written as:

$$\hat{p}_{IS}(\boldsymbol{y}) = \frac{1}{m}\sum_{i=1}^{m}\exp\left(w^{(i)}\right)$$

where $\{w^{(i)}\}_{i=1}^m$ are based on $\{\boldsymbol{\theta}^{(i)},\boldsymbol{\delta}^{(i)}\}_{i=1}^m$ from the importance sampler, and the GD estimator as:

$$\hat{p}_{GD}(\boldsymbol{y}) = \left(\frac{1}{m}\sum_{i=1}^{m}\exp\left(-w^{(i)}\right)\right)^{-1}$$

where $\{w^{(i)}\}_{i=1}^m$ are based on posterior draws $\{\boldsymbol{\theta}^{(i)},\boldsymbol{\delta}^{(i)}\}_{i=1}^m$. A more efficient importance sampler or tuning function would return draws of the log ratio $\{w^{(i)}\}_{i=1}^m$ that are more concentrated and less volatile. Compared to the summand $\{\exp\left(w^{(i)}\right)\}_{i=1}^m$ or $\{\exp\left(-w^{(i)}\right)\}_{i=1}^m$, the distribution of the log ratio $\{w^{(i)}\}_{i=1}^m$ is usually much closer to a bell shape and is more convenient to analyze. In Figure 1, it can be seen that draws of the log ratio $\{w^{(i)}\}_{i=1}^m$ by the new approach are clearly much more stable than those by the cross entropy approach under both the IS and GD methods. The standard deviation of the log ratio $\{w^{(i)}\}_{i=1}^m$ by the new approach, under either the IS or GD method, is an order of magnitude smaller than that by the cross entropy approach.

4.3 Regression with Stochastic Volatility and Student-t Residual

Fatter tails than the Gaussian distribution are often found in economic time series data. To allow fat tails, the regression model of Equation (21) can be modified to specify a student-t distribution for the model residual $\epsilon_t \sim t(v)$, where $v > 0$ is the degrees of freedom parameter. The prior for v is $IG(a_{v,0}, b_{v,0})$ where IG denotes the inverse gamma distribution. We re-parameterize $v^* = \log(v)$ such that the support of the new parameter is the whole real line. In estimation, the hyper parameters for v are $a_{v,0} = 7$ and $b_{v,0} = 60$. The setup of other model parameters are the same as in the Gaussian residual case of Section 4.2.

The model is applied to the same dataset of equity premium in Section 4.2 and is estimated by a Metropolis-within-Gibbs sampler. In the model estimation, the student-t distribution $\epsilon_t \sim t(v)$ is re-framed as a hierarchical Gaussian distribution $\epsilon_t \sim N(0, d_t)$ and $d_t \sim IG\left(\frac{v}{2}, \frac{v}{2}\right)$ for $t = 1, ..., T$. To improve sampling efficiency, the parameters μ and s in the SV process are sampled by integrating out the linear coefficients $\boldsymbol{\beta}$ and are drawn by a Metropolis-Hastings step with a random walk proposal which is tuned by the approach of Garthwaite et al. (2016). Similar to the Gaussian residual case of Section 4.2, the interweaving strategy of Kastner and Fruhwirth-Schnatter (2014), which originates from Yu and Meng (2011), is applied to further boost the sampling efficiency of μ and s. The degrees of freedom parameter v is sampled by a Metropolis-Hastings step with the proposal tuned by the approach of Garthwaite et al. (2016). A total of 10,000 posterior draws are collected for analysis after 2,000 burn-ins.

Denote $\boldsymbol{\delta} = \{\boldsymbol{\beta}, \mu, \rho^*, s^*, v^*\}$. The calibration approach for the component $q(\boldsymbol{\delta})$ of the importance sampler is the same as in the Gaussian residual case of Section 4.2. The steps of calibrating the other component $q(\boldsymbol{\theta}|\boldsymbol{\delta})$ are also the same as in Section 4.2 except that now the log likelihood function becomes $\log(p(y_t|\boldsymbol{\theta}, \boldsymbol{\delta})) = \log\left(\frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi v}\Gamma\left(\frac{v}{2}\right)}\right) - \frac{\theta_t}{2} -$

$\frac{1+v}{2}\log\left(1+\frac{(y_t-\boldsymbol{x}_t'\boldsymbol{\beta})^2\exp(-\theta_t)}{v}\right)$. Matching the log likelihood function with the structure of Equation (15) produces $f_{1,t}(\boldsymbol{\delta})=\log\left(\frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi v}\Gamma\left(\frac{v}{2}\right)}\left(\frac{(y_t-\boldsymbol{x}_t'\boldsymbol{\beta})^2}{v}\right)^{-\frac{1+v}{2}}\right)$, $f_{2,t}(\boldsymbol{\delta})=\frac{v}{2}$, $f_{3,t}(\boldsymbol{\delta})=\frac{1+v}{2}$, $f_{4,t}(\boldsymbol{\delta})=-\log\left(\frac{(y_t-\boldsymbol{x}_t'\boldsymbol{\beta})^2}{v}\right)$ and $h_t(\theta_t+f_{4,t}(\boldsymbol{\delta}))=-\log\left(1+\exp(\theta_t+f_{4,t}(\boldsymbol{\delta}))\right)$. The auxiliary function $g_t^*(\boldsymbol{\theta};\boldsymbol{\delta})$ is $a_{1,t}+a_{2,t}(\theta_t+f_{4,t}(\boldsymbol{\delta}))+a_{3,t}(\theta_t+f_{4,t}(\boldsymbol{\delta}))^2$. The free parameters $a_{1,t}$, $a_{2,t}$ and $a_{3,t}$ of $g_t^*(\boldsymbol{\theta};\boldsymbol{\delta})$ are determined by running the following regression:

$$h_t\left(\theta_t^{(i)}+f_{4,t}(\boldsymbol{\delta}^{(i)})\right)=a_{1,t}+a_{2,t}\left(\theta_t^{(i)}+f_{4,t}(\boldsymbol{\delta}^{(i)})\right)+a_{3,t}\left(\theta_t^{(i)}+f_{4,t}(\boldsymbol{\delta}^{(i)})\right)^2+\text{residual}$$

subject to $a_{3,t}\leq 0$ based on posterior draws $\{\theta_t^{(i)},\boldsymbol{\delta}^{(i)}\}_{i=1}^m$ for each $t=1,...,T$.[7]

The LML estimates by the cross entropy approach and the new one are provided in the right columns of Table 2. Similar to the Gaussian residual case, the difference in the LML estimates between the IS and GD methods is smaller when the new approach is used. The new approach is also able to push the numerical standard error of the LML estimate 88% smaller under the IS method and 71% smaller under the GD method than the cross entropy approach. The log ratio $\{w^{(i)}\}_{i=1}^m$ shown in Figure 2 shows a similar pattern as in the Gaussian residual case and is much less volatile when the new approach is used.

It is also interesting to note that, while the LML estimates by the cross entropy approach show a mixed message regarding the relative merit of the student-t specification, the LML estimates by the new approach unanimously support, albeit slightly, that the student-t specification actually under-performs the Gaussian one in this example.

5 Conclusion

In summary, this paper has developed and demonstrated a new approach to construct importance samplers for computing the marginal likelihood for models including those with latent variables. The new approach is guided by the principle of minimizing the Monte Carlo variance of the IS estimator and is simple to implement. The computational

[7]The R-squared of the regressions ranges from 5% to 59% with the average of 41%.

cost is low as only a series of low dimensional linear regressions based on posterior draws of model parameters are required. Encouraging results are found in the examples examined and should encourage further efforts to apply the ideas of this paper for model specification analysis.

Table 1: Log Marginal Likelihood Estimates for the Probit Regression Model

	Direct Calibration	New Approach
Importance Sampling	-128.9413	-128.9365
	(0.0039)	(0.0024)
Gelfand-Dey	-128.9714	-128.9343
	(0.0048)	(0.0034)

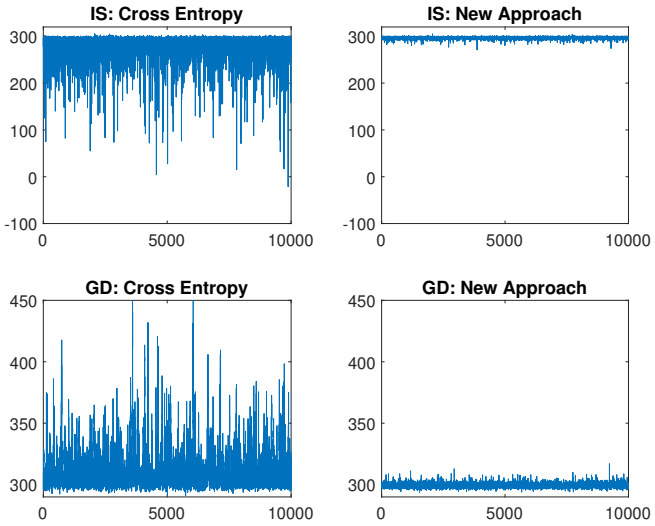
Note: This table compares the log marginal likelihood estimates for the probit regression model of Section 4.1 that are computed by the direct calibration approach and the new one under both the importance sampling and Gelfand-Dey methods. The number of draws is 10,000 in all cases. Numbers in the brackets are the numerical standard errors of the log marginal likelihood estimates computed by the delta method. For the Gelfand-Dey method, the standard error is calculated by the Newey-West method (Newey and West (1987)) with the number of lags $floor\left(4(m/100)^{2/9}\right)$ where m is the number of draws.

Table 2: Log Marginal Likelihood Estimates for the Linear Regression Model with SV

	Gaussian Residual		Student-t Residual	
	Cross Entropy	New Approach	Cross Entropy	New Approach
Importance Sampling	297.8849	298.4089	298.1101	297.3956
	(0.2631)	(0.0927)	(0.5407)	(0.0626)
Gelfand-Dey	298.1574	298.3604	297.5448	297.7603
	(0.2195)	(0.0308)	(0.2705)	(0.0796)

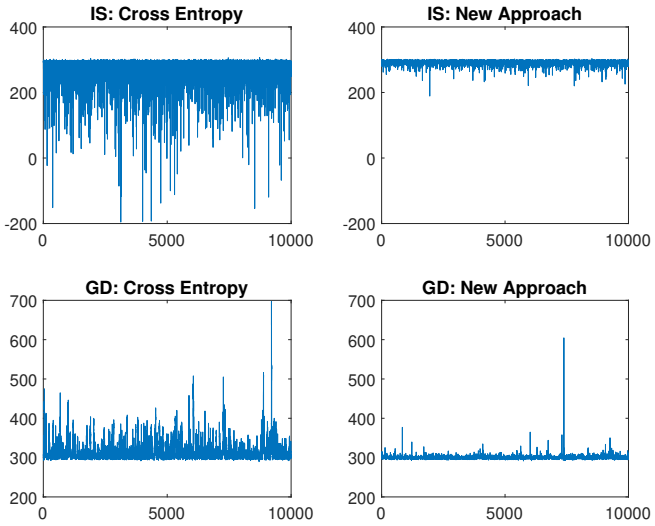
Note: This table compares the log marginal likelihood estimates for the linear regression model with stochastic volatility and Gaussian/student-t residuals (Section 4.2 and 4.3 respectively) that are computed by the cross entropy approach and the new one under both the importance sampling and Gelfand-Dey methods. The number of draws is 10,000 in all cases. Numbers in the brackets are the numerical standard errors of the log marginal likelihood estimates computed by the delta method. For the Gelfand-Dey method, the standard error is calculated by the Newey-West method (Newey and West (1987)) with the number of lags $floor\left(4(m/100)^{2/9}\right)$ where m is the number of draws.

Figure 1: Log Ratio for the SV-Gaussian Linear Regression Model



Note: This figure compares the log ratio $\{w^{(i)}\}_{i=1}^m$ for the linear regression model with stochastic volatility and Gaussian residual (Section 4.2) that are computed by the cross entropy approach and the new one under both the importance sampling (IS) and Gelfand-Dey (GD) methods, where $w^{(i)} = \log\left(\frac{p(\boldsymbol{y}|\boldsymbol{\theta}^{(i)},\boldsymbol{\delta}^{(i)})p(\boldsymbol{\theta}^{(i)}|\boldsymbol{\delta}^{(i)})p(\boldsymbol{\delta}^{(i)})}{q(\boldsymbol{\theta}^{(i)},\boldsymbol{\delta}^{(i)})}\right)$ and $\{\boldsymbol{\theta}^{(i)},\boldsymbol{\delta}^{(i)}\}_{i=1}^m$ are draws from the importance sampler under the IS method and are posterior draws under the GD method.

Figure 2: Log Ratio for the SV-Student Linear Regression Model



Note: This figure compares the log ratio $\{w^{(i)}\}_{i=1}^m$ for the linear regression model with stochastic volatility and student-t residual (Section 4.3) that are computed by the cross entropy approach and the new one under both the importance sampling (IS) and Gelfand-Dey (GD) methods, where $w^{(i)} = \log\left(\frac{p(\boldsymbol{y}|\boldsymbol{\theta}^{(i)},\boldsymbol{\delta}^{(i)})p(\boldsymbol{\theta}^{(i)}|\boldsymbol{\delta}^{(i)})p(\boldsymbol{\delta}^{(i)})}{q(\boldsymbol{\theta}^{(i)},\boldsymbol{\delta}^{(i)})}\right)$ and $\{\boldsymbol{\theta}^{(i)},\boldsymbol{\delta}^{(i)}\}_{i=1}^m$ are draws from the importance sampler under the IS method and are posterior draws under the GD method.

A Proof of Lemma 2.1

Proof. Since $p(\boldsymbol{\theta})$ is the Gaussian density for $N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, it follows:

$$p(\boldsymbol{\theta}) \propto \exp\left(-\frac{1}{2}\boldsymbol{\theta}'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\theta} + \boldsymbol{\theta}'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0\right)$$

Multiplying $p(\boldsymbol{\theta})$ by $\exp(g(\boldsymbol{\theta}))$ and re-arranging items give:

$$p(\boldsymbol{\theta})\exp(g(\boldsymbol{\theta})) \propto \exp\left(-\frac{1}{2}\boldsymbol{\theta}'(\boldsymbol{\Sigma}_0^{-1} - 2\boldsymbol{A})\boldsymbol{\theta} + \boldsymbol{\theta}'(\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0 + \boldsymbol{a}_2)\right)$$

which is the kernel of the Gaussian density $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}_0^{-1} - 2\boldsymbol{A}$ and $\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0 + \boldsymbol{a}_2$. □

B Proof of Lemma 2.2

Proof. Given the assumption $p(\boldsymbol{\theta}) = p(\boldsymbol{\theta}_1)\prod_{t=2}^{T} p(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1})$ where $p(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1}) = N(\boldsymbol{\mu}_0 + \boldsymbol{\Phi}_0\boldsymbol{\theta}_{t-1}, \boldsymbol{\Sigma}_0)$ for $t > 1$ and $p(\boldsymbol{\theta}_1) = N(\tilde{\boldsymbol{\mu}}_0, \tilde{\boldsymbol{\Sigma}}_0)$, we have:

$$p(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1}) \propto \exp\left(-\frac{1}{2}\boldsymbol{\theta}_t'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\theta}_t + \boldsymbol{\mu}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\theta}_t + \boldsymbol{\theta}_{t-1}'\boldsymbol{\Phi}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\theta}_t - \frac{1}{2}\boldsymbol{\theta}_{t-1}'\boldsymbol{\Phi}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Phi}_0\boldsymbol{\theta}_{t-1} - \boldsymbol{\mu}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Phi}_0\boldsymbol{\theta}_{t-1}\right)$$

for $t > 1$ and

$$p(\boldsymbol{\theta}_t) \propto \exp\left(-\frac{1}{2}\boldsymbol{\theta}_t'\tilde{\boldsymbol{\Sigma}}_0^{-1}\boldsymbol{\theta}_t + \tilde{\boldsymbol{\mu}}_0'\tilde{\boldsymbol{\Sigma}}_0^{-1}\boldsymbol{\theta}_t\right)$$

for $t = 1$.

Similarly, given that $q(\boldsymbol{\theta}) = q(\boldsymbol{\theta}_1)\prod_{t=2}^{T} q(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1})$ where $q(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1}) = N(\boldsymbol{\mu}_t + \boldsymbol{\Phi}_t\boldsymbol{\theta}_{t-1}, \boldsymbol{\Sigma}_t)$ for $t > 1$ and $q(\boldsymbol{\theta}_1) = N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$, the functional forms of $q(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1})$ and $q(\boldsymbol{\theta}_1)$ would be the same as those for $p(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1})$ and $p(\boldsymbol{\theta}_1)$ except for the notational change of replacing $\boldsymbol{\mu}_0$, $\boldsymbol{\Phi}_0$, $\boldsymbol{\Sigma}_0$, $\tilde{\boldsymbol{\mu}}_0$ and $\tilde{\boldsymbol{\Sigma}}_0$ by appropriate $\boldsymbol{\mu}_t$, $\boldsymbol{\Phi}_t$ and $\boldsymbol{\Sigma}_t$.

Let $g_t(\boldsymbol{\theta}) = a_{1,t} + \boldsymbol{a}_{2,t}'\boldsymbol{\theta}_t + \boldsymbol{\theta}_t'\boldsymbol{A}_t\boldsymbol{\theta}_t$. Start with $t = T$. Retain only the items involving $\boldsymbol{\theta}_T$

in $p(\boldsymbol{\theta}_T|\boldsymbol{\theta}_{T-1})\exp(g_T(\boldsymbol{\theta}))$ and match them with the items in $q(\boldsymbol{\theta}_T|\boldsymbol{\theta}_{T-1})$. This would leave:

$$\boldsymbol{\Sigma}_t^{-1} = \boldsymbol{\Sigma}_0^{-1} - 2\boldsymbol{A}_T$$

$$\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_T = \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0 + \boldsymbol{a}_{2,T}$$

$$\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\Phi}_t = \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Phi}_0$$

Next for $t = T-1$, collect the items involving $\boldsymbol{\theta}_{T-1}$ in $p(\boldsymbol{\theta}_{T-1}|\boldsymbol{\theta}_{T-2})\exp(g_{T-1}(\boldsymbol{\theta}))$ as well as the items involving $\boldsymbol{\theta}_{T-1}$ from $p(\boldsymbol{\theta}_T|\boldsymbol{\theta}_{T-1})\exp(g_T(\boldsymbol{\theta}))$. Do the same for $q(\boldsymbol{\theta}_{T-1}|\boldsymbol{\theta}_{T-2})$. By matching the items, one obtains:

$$\boldsymbol{\Sigma}_{T-1}^{-1} + \boldsymbol{\Phi}_T'\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\Phi}_T = \boldsymbol{\Sigma}_0^{-1} - 2\boldsymbol{A}_{T-1} + \boldsymbol{\Phi}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Phi}_0$$

$$\boldsymbol{\Sigma}_{T-1}^{-1}\boldsymbol{\mu}_{T-1} - \boldsymbol{\Phi}_T'\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\mu}_T = \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0 + \boldsymbol{a}_{2,T-1} - \boldsymbol{\Phi}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0$$

$$\boldsymbol{\Sigma}_{T-1}^{-1}\boldsymbol{\Phi}_{T-1} = \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Phi}_0$$

Repeating these steps backward until $t = 1$ gives the formulas in Lemma 2.2.

□

C Regressors of the Probit Regression Model

The regressors of the probit regression model in Section 4.1 are listed in Table C1.

D Regressors of the Linear Regression Models

The regressors of the linear regression models in Section 4.2 and 4.3 are listed in Table D1.

E Stability of Marginal Likelihood Estimates

A concern of the importance sampling estimator of marginal likelihood is that, in extreme cases, the denominator may be too thin-tailed for the Monte Carlo estimate to have a finite

Table C1: List of Predictors for Economic Recessions

Name	Description
Term spread	Difference between 10-year Treasury constant maturity rate and 3-month Treasury bill rate
Long spread	Difference between 20- and 10-year Treasury constant maturity rates
Short rate	Change of 3-month Treasury bill rate
Long rate	Change of 20-year Treasury constant maturity rate
Inflation	Change of inflation rate measured as log change of GDP deflator
Expenditure	Log change of real government consumption expenditures and gross investment
Debt	Log change of consumer credit to households and non-profit organizations
Mortgage	Log change of one-to-four-family residential mortgages
S&P500	Log change of average daily closing price
AAA	Log change of average monthly Moody's Aaa corporate bond yield relative to 10-year Treasury constant maturity rate
BAA	Log change of average monthly Moody's Baa corporate bond yield relative to 10-year Treasury constant maturity rate

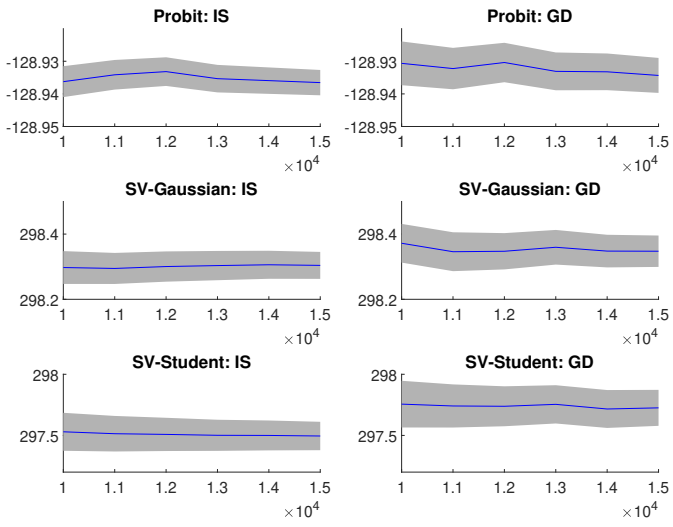
Note: Data on the S&P500 index is from Robert Shiller's website http://www.econ.yale.edu/ shiller/data.htm. Data on all other variables are from the FRED database of the U.S. Federal Reserve Bank of St. Louis.

Table D1: List of Predictors for Equity Premium

Name	Description
Dividend price ratio	Log dividends minus log price
Dividend payout ratio	Log dividends minus log earnings
Stock variance	Log of sum of squared daily returns on the S&P500 index
Book-market ratio	Ratio of book to market value for the Dow Jones Industrial Average index
Equity expansion	Ratio of 12-month moving sums of net issues by NYSE listed stocks divided by the total end-of-year market capitalization of NYSE stocks
Treasury bill rate	Quarterly change of 3-month secondary market Treasury bill rate
Long term yield	Quarterly change of long-term government bond yield from Ibbotson's *Stocks, Bonds, Bills and Inflation Yearbook*
Term spread	Long term yield minus treasury bill rate
Default yield spread	Difference between BAA and AAA-rated corporate bond yields
Default return spread	Difference between long-term corporate and government bond returns
Inflation rate	Consumer price index (all urban consumers)
Investment-capital ratio	Ratio of aggregate (private non-residential fixed) investment to aggregate capital

Note: The data is publicly available from Amit Goyal's website https://sites.google.com/view/agoyal145/?redirpath=/. Detailed descriptions of the variables can be found in Welch and Goyal (2008).

Figure E1: Sequence of LML Estimates



Note: This figure shows the sequences of log marginal likelihood (LML) estimates and their 95% confidence intervals based on the sample sizes m = 10,000, 11,000, ..., 15,000 by the proposed approach under the importance sampling (IS) and Gelfand-Dey (GD) methods in the 3 empirical examples: probit regression (Probit), linear regression with stochastic volatility and Gaussian residual (SV-Gaussian), and linear regression with SV and student-t residual (SV-Student). In each panel, the solid line is the sequence of LML estimates while the shaded area is the sequence of confidence intervals.

variance (Geweke (1989)). The same concern applies to the Gelfand-Dey estimator. To alleviate such concerns, we use 6 different sample sizes m = 10,000, 11,000, ..., 15,000 to compute a sequence of LML estimates as well as their standard deviations. A sequence of confidence intervals are computed as the LML estimates plus and minus 1.96 times the standard deviations. For well-behaved estimates, the sequences of LML estimates and confidence intervals should be flat across different m. Figure E1 shows these sequences by the proposed approach in the 3 empirical examples. We find that these sequences are largely stable and do not seem to show signs of the infinite variance problem.

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