Time Dependent Shrinkage of Time-Varying Parameter Regression Models

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Abstract

This paper studies the time-varying parameter (TVP) regression model in which the regression coefficients are random walk latent states with time dependent conditional variances. This TVP model is flexible to accommodate a wide variety of time variation patterns but requires effective shrinkage on the state variances to avoid over-fitting. A Bayesian shrinkage prior is proposed based on reparameterization that translates the variance shrinkage problem into a variable shrinkage one in a conditionally linear regression with fixed coefficients. The proposed prior allows strong shrinkage for the state variances while maintaining the flexibility to accommodate local signals. A Bayesian estimation method is developed that employs the ancilarity-sufficiency interweaving strategy to boost sampling efficiency. Simulation study and an empirical application to forecast inflation rate illustrate the benefits of the proposed approach.

Keywords: TVP, Bayesian shrinkage, MCMC, Horseshoe, ASIS
JEL Codes: C01, C11, C22, E37

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1 Introduction

The time-varying parameter (TVP) regression model allows the coefficients of a linear regression model to vary over time to capture possible model instability and has been widely applied in econometric studies of time series data (Cogley and Sargent (2005), Primiceri (2005), Dangl and Halling (2012), Belmonte et al. (2014)). The commonly used configuration of the TVP model specifies the time varying coefficients as latent states that follow independent random walk processes with constant variances. Bitto and Fruhwirth-Schnatter (2019) is a recent contribution that shrinks the variances of the time varying coefficients and is able to automatically discriminate constant coefficients from time varying ones[1]. However the assumption of homoskedastic latent states is inherently restrictive and permits only two types of time variation patterns for each coefficient: either staying constant all the time or changing continuously. Allowing for infrequent parameter shifts and episodical combinations of constant and time-varying coefficients remains a challenge for the TVP model with homoskedastic latent states.

Recent developments in the literature have begun to allow time varying variances in the latent states of TVP models. By allowing the latent state variance to take different values from approximately zero to very large ones at each time t, both constant parameters, continuous and discrete parameter shifts as well as their episodical combinations could be accommodated in the TVP framework. Hence the practitioners can be relieved of the difficult task to pre-determine if potential parameter shifts are continuous as in the conventional TVP models or are infrequent as in the change point literature (e.g. Chib (1998), Dufays and Rombouts (2020)). To allow for heteroskedastic latent states, one strand of the literature directly applies Bayesian global-local shrinkage priors to the differenced latent states[2]. The variance of each differenced latent state is specified as the product of a "global"' shrinkage parameter that remains constant and a "local"' shrinkage parameter

[1]Other recent examples of shrinkage TVP models with homoskedastic latent states include Cadonna et al. (2020), Chan et al. (2020) etc.

[2]Another strand of the literature allows heteroskedastic latent states by applying time-dependent spike-and-slab mixture priors for state variances (e.g. Giordani and Kohn (2008), Chan et al. (2012), Hauzenberger (2021), Rockova and McAlinn (2021)) but faces the computational hurdle due to the combinatorial complexity of sampling the mixture indicators of the spike-and-slab priors.

that can vary over time. The global shrinkage parameter pushes the variance of the latent state towards zero and hence favors a constant coefficient, while the local shrinkage parameter adapts to local signals at each time t and could result in a large overall variance at time t to accommodate possible local parameter shifts. For example, Kowal et al. (2019) and Huber and Pfarrhofer (2021) contain TVP models where the horseshoe prior (Carvalho et al. (2010)) is applied to each differenced latent state. Kowal et al. (2019) further proposes a dynamic horseshoe prior that allows an autoregressive structure in the local shrinkage parameter of each latent state variance and shows superior performance of the resulting TVP models over alternative dynamic shrinkage priors such as the dynamic normal-gamma one of Kalli and Griffin (2014)[3]. In this paper, I will show that both approaches can be improved by borrowing from the "non-centered" parametrization strategy of Fruhwirth-Schnatter and Wagner (2010) and in a computationally efficient fashion.

Shrinking the state variances towards zero amounts to the "variance selection" problem in the literature (Harvey (1989)). Pioneering work on the variance selection problem in the Bayesian framework includes Fruhwirth-Schnatter (2004), Fruhwirth-Schnatter and Wagner (2010), Nakajima and West (2013) and Kalli and Griffin (2014) etc. An attractive idea from Fruhwirth-Schnatter and Wagner (2010) is that by appropriate reparameterization of the original TVP model, the variance selection problem can be recast as a variable selection problem in a conditionally linear regression with the signed square root of the latent state variance being the constant coefficients. Hence the vast literature on variable selection by shrinkage priors can be borrowed on for efficient model estimation. I build on this insight from Fruhwirth-Schnatter and Wagner (2010) and proposes a new hierarchical prior to model the heteroskedasticity in the latent states of TVP models.

Specifically, rather than directly applying shrinkage priors to the time dependent state variances, I consider reparametrizing the random walk TVP model and apply horseshoe shrinkage priors to the signed square root of the state variances. In a nutshell, the resulting prior on each latent state variance is a gamma distribution in which the scale parameter is the product of a time-invariant global shrinkage parameter and a time-varying local shrinkage parameter. While the local shrinkage parameters follow independent inverted

[3]See Hauzenberger et al. (2020) for similar strategies for versions of TVP models where latent states follow independent Gaussian distributions rather than random walks.

beta distributions as in the horseshoe prior, the global shrinkage parameter is a scale-mixture gamma distribution. I label this new shrinkage prior as a *gamma horseshoe prior*. I show that the gamma horseshoe prior results in a prior distribution of the state variances that allows significantly more probability mass near zero than directly placing horseshoe priors on the state variances and hence offers stronger shrinkage while retaining a heavy tail for flexibility. Dynamic structure of the local shrinkage parameter as in Kowal et al. (2019) can be easily added to the gamma horseshoe prior to produce a dynamic version.

Estimating the TVP model with the gamma horseshoe prior can be challenging due to the high dependence between the the latent states and their time-varying variances. A standard Gibbs sampler applied to this TVP model produces posterior draws of model parameters that are slow to converge and mix poorly. To boost sampling efficiency, the ancilarity-sufficiency interweaving strategy (ASIS) of Yu and Meng (2011) is adopted in this paper. A straightforward implementation of the ASIS, however, involves repeated large-scale matrix operations within each MCMC iteration and would be impractical in realistic applications. I develop an alternative MCMC scheme that avoids repeated large-scale matrix operations and instead breaks the parameters of the state variances into blocks and apply the ASIS to each block in a component-wise fashion. The resulting sampler is efficient and computationally tractable. With the use of the horseshoe prior, an advantage of the proposed gamma horseshoe prior is that few user-specified hyper-parameter or manual intervention is needed and hence allows for convenient automated model estimation.

For evaluating out-of-sample forecast performance, the Kalman filter is used conditional on simulations of relevant parameters to approximate the one-step-ahead predictive likelihood. A similar strategy was used in Bitto and Fruhwirth-Schnatter (2019) for the random walk TVP model with homoskedastic latent states and found that the resulting estimate of the one-step-ahead predictive likelihood tends to be more accurate than a pure simulation-based approximation.

The proposed approach is investigated in two applications. In a simulation study where the regression coefficients of the data generating process include both constants, continuous and discrete parameter shifts and their episodical combinations, the in-sample estimation accuracy of the coefficients under the gamma horseshoe priors is found to significantly outperform the horseshoe prior and be comparable to that under the dynamic horseshoe prior of Kowal et al. (2019). In contrast to the horseshoe prior, the benefit of adding an autoregressive structure to the local parameters is diminished in the gamma horseshoe prior. It is also found that posterior draws of the regression coefficients from the gamma horseshoe prior mix better than those from the dynamic horseshoe prior.

In an empirical illustration, the TVP model is applied to forecast quarterly U.S. inflation rates. The regressors include autoregressive lags and lagged interest rate and unemployment rate. Model estimates based on the gamma horseshoe prior, the horseshoe prior and their dynamic versions are compared. Similar to the findings in the simulation study, posterior draws of the regression coefficients based on the gamma horseshoe prior exhibit better mixing behavior than those from the dynamic horseshoe prior. In out-of-sample forecasts, the gamma horseshoe prior outperforms its dynamic version, the horseshoe prior and the dynamic horseshoe prior based on log predictive likelihoods.

The remainder of the paper is organized as follows. Section 2 provides the details of the TVP model and the shrinkage priors. Estimation details are provided in Section 3. Section 4 and 5 present the simulation study and the empirical application respectively. Section 6 concludes. Additional details of the shrinkage priors and estimation algorithm are provided in appendices.

2 The Model

The TVP model under study is a linear regression model where the regression coefficients follow independent random walk processes:

$$
\begin{aligned}
y_t &= x_t'\beta_t + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_t^2), \\
\beta_t &= \beta_{t-1} + \eta_t, \quad \eta_t \sim N(0, \mathrm{diag}(w_t)), \\
\beta_0 &\sim N(0, \mathrm{diag}(w_0))
\end{aligned} \tag{1}
$$

where y_t is a scalar dependent variable, x_t is a K-dimensional vector of regressors, β_t is the corresponding time-varying coefficients and $\mathrm{diag}(w_t)$ is a diagonal matrix with the K-by-1 vector w_t in its diagonal for $t = 1, 2, ..., n$. The initial value β_0 plays the role of fixed regression coefficients[4] and follows a zero-mean normal distribution with the covariance matrix diag(w_0).

Viewed as a state space system, ϵ_t and η_t are the Gaussian disturbances in the measurement and state equations respectively. For economic time series, the measurement equation disturbance ϵ_t is often allowed to have a time dependent variance. In empirical studies of this paper, I use the stochastic volatility (SV) specification for the measurement equation variance σ_t^2:

$$\log(\sigma_t^2) = (1-\rho)\mu + \rho \log(\sigma_{t-1}^2) + \epsilon_{y,t}, \quad \epsilon_{y,t} \sim N(0, \sigma_y^2), \tag{2}$$

where $\log(\sigma_1^2) \sim N(\mu, \sigma_y^2/(1-\rho^2))$. The homoskedastic case can be accommodated by imposing $\sigma_t^2 = \sigma^2$.

The state variance w_t is allowed to be time dependent. Different types of time variations in the regression coefficients β_t can be captured by varying the value of w_t. For example, a constant non-zero value $w_t = w$ leads to the commonly used homoskedastic specification of TVP models (Bitto and Fruhwirth-Schnatter (2019)). The change-point specification that assumes discrete shifts in the regression coefficients can be accommodated by setting $w_t > 0$ at the break points while $w_t = 0$ at other times. Constant regression coefficients can be accommodated by $w_t = 0$ for $t = 1, 2, ..., n$. As argued in Hauzenberger et al. (2020), time variations in TVP models often occur only episodically and only for a subset of the regression coefficients in economic time series data. Such empirical patterns amount to episodical combinations of constant and time-varying parameters, which can be easily accommodated in the TVP model of Equation (1).

2.1 Prior for State Variance

The key parameter in the TVP model of Equation (1) is the state variance w_t. In a Bayesian framework, the task is to develop a suitable prior for w_t that should, on one hand, allow significant likelihoods for w_t close to zero to reduce the risk of overfitting and, on the other hand, contain a heavy tail for w_t to retrieve possible local signals of each time t. Recent developments in the field of Bayesian shrinkage priors provide an

[4]To see this, let $\beta_t^* = \beta_t$ - β_0. The TVP model can be rewritten as $y_t = x_t'\beta_0 + x_t'\beta_t^* + \epsilon_t$, $\beta_t^* = \beta_{t-1}^*$ + η_t and $\beta_0^* = 0$.

ideal toolkit for this task. In this paper, I focus on the horseshoe prior (Carvalho et al. (2010), Polson and Scott (2012)) that is computationally convenient and has shown good performance in many applications (Bhadra et al. (2019))[5].

A straightforward approach would be to directly impose a horseshoe prior on the differenced latent states $\Delta\beta_{j,t} = \beta_{j,t} - \beta_{j,t-1}$ for $j = 1, 2, ..., K$. That amounts to specify $w_{j,t} = r_j s_{j,t}$ with the "global"component r_j and the "local" component $s_{j,t} \sim IB(0.5, 0.5)$, where IB denotes the inverted beta distribution[6]. This is the "static horseshoe" specification investigated in recent studies such as Kowal et al. (2019) and Huber and Pfarrhofer (2021).

To improve the shrinkage property of the prior for w_t, this paper considers an alternative that translates such a variance shrinkage problem into a variable shrinkage one in a (conditionally) linear regression model by reparameterization. To motivate the prior specification for the state variance w_t, let $\eta_t^* = \text{diag}(\frac{1}{\tilde{w}_t})\eta_t$ denote the normalized state disturbance where $\tilde{w}_t = \pm\sqrt{w_t}$ is the signed square root of the state variance w_t. Substituting the normalized state disturbance η_t^* into the TVP model of Equation (1) gives:

$$
\begin{aligned}
& y_t = x_t'\beta_0 + \tilde{x}_t'\tilde{w} + \epsilon_t, \\
& \eta_t^* \sim N(0, I_K)
\end{aligned} \tag{3}
$$

where $\tilde{x}_t$ is a nK-by-1 vector with the first tK elements being $x_t \odot \eta_1^*$, $x_t \odot \eta_2^*$, ..., $x_t \odot \eta_t^*$ and the remaining elements being zero. The notation $\odot$ denotes the element-wise product of matrices of the same size. $\tilde{w} = [\tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_n]$ is a nK-by-1 vector stacking the signed square roots of the state variances w_1, ..., w_n. Equation (3) is the "non-centered" parametrization of the TVP model of Equation (1) in the spirit of Fruhwirth-Schnatter and Wagner (2010). Conditional on the normalized state disturbance η_t^*, the measurement equation in the reparametrized TVP model of Equation (3) becomes a linear regression model with the

[5]Alternative shrinkage priors for linear regressions include the spike-and-slab one (George and McCulloch (1993), Ishwaran and Rao (2005)) and the normal-gamma one (Griffin and Brown (2010)) etc. A comprehensive comparison of the various shrinkage priors in the current TVP context is left for future research.

[6]The density of an inverted beta distribution $IB(a, b)$ is $p(x) = \frac{x^{a-1}(1+x)^{-a-b}}{B(a,b)} I\{x > 0\}$ where $B(\cdot, \cdot)$ is the beta function and a and b are positive real numbers. If $x \sim IB(0.5, 0.5)$, then $\sqrt{x} \sim C^+(0, 1)$ and vice versa, where $C^+(0, 1)$ is a standard half-Cauchy distribution with the density $p(z) = \frac{2}{\pi(1+z^2)} I\{z > 0\}$.

signed square roots $\tilde{w}$ being the fixed coefficients. Hence the task of specifying a shrinkage prior for the variance parameter w_t is transformed into one of placing a shrinkage prior on fixed coefficients of a linear regression, which connects with the rich literature on variable selection in linear regression models. Indeed the horseshoe prior was originally proposed for shrinking fixed coefficients in a linear regression form.

With these considerations, I specify a hierarchical prior for the signed square roots $\tilde{w}$ as $\tilde{w}_{j,t}|v_j, d_{j,t} \sim N(0, v_j d_{j,t})$ with the *global parameter* v_j and the *local parameter* $d_{j,t} \sim IB(0.5, 0.5)$ for $j = 1, ..., K$ and $t = 1, ..., n$. The resulting prior for the state variance is a gamma distribution $w_{j,t}|v_j, d_{j,t} \sim G(0.5, 2v_j d_{j,t})$[7].

The remaining piece is to specify the prior for the global parameter v_j. Again I use the fact that the signed square root $\tilde{v}_j = \pm\sqrt{v_j}$ is the fixed coefficient in a conditionally linear regression model by reparametrizing the TVP model of Equation (1):

$$
\begin{aligned}
y_t &= x_t'\beta_0 + (x_t \odot \beta_t^*)'\tilde{v} + \epsilon_t, \\
\beta_t^* &\sim N(\beta_{t-1}^*, \text{diag}(\phi_t)), \\
\beta_0^* &= 0 \qquad (4)
\end{aligned}
$$

where the transformed latent state is $\beta_{j,t}^* = (\beta_{j,t} - \beta_{j,0})/\tilde{v}_j$, the *scaled state variance* is $\phi_{j,t} = \frac{w_{j,t}}{v_j}$, $\beta_t^* = [\beta_{1,t}^*, \beta_{2,t}^*, ..., \beta_{K,t}^*]$, $\phi_t = [\phi_{1,t}, \phi_{2,t}, ..., \phi_{K,t}]$ and $\tilde{v} = [\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_K]$. A horseshoe prior for the fixed coefficients $\tilde{v}$ results in $\tilde{v}_j|\tau_0, \tau_j \sim N(0, \tau_0\tau_j)$ with the common component $\tau_0 \sim IB(0.5, 0.5)$ and the individual component $\tau_j \sim IB(0.5, 0.5)$. The equivalent prior for the global parameter $v_j = \tilde{v}_j^2$ is $v_j|\tau_0, \tau_j \sim G(0.5, 2\tau_0\tau_j)$ for $j = 1, ..., K$.

In summary, the proposed prior for the state variance w_t is $w_{j,t}|v_j, d_{j,t} \sim G(0.5, 2v_j d_{j,t})$ with $d_{j,t} \sim IB(0.5, 0.5)$, $v_j|\tau_0, \tau_j \sim G(0.5, 2\tau_0\tau_j)$, $\tau_0 \sim IB(0.5, 0.5)$ and $\tau_j \sim IB(0.5, 0.5)$ for $j = 1, ..., K$ and $t = 1, ..., n$. I label this prior as a *gamma horseshoe* (GHS) prior since the local parameter $d_{j,t}$ follows an inverted beta distribution as in a horseshoe prior while the global parameter v_j follows a gamma distribution conditionally. Appendix A contains a detailed comparison of the properties of the proposed gamma horseshoe prior with the horseshoe one. It is found that the gamma horseshoe prior allocates significantly more

[7]If $\pm\sqrt{x} \sim N(0, a)$, then $x \sim G(0.5, 2a)$ and vice versa, where the gamma distribution $G(\alpha, \beta)$ has the density $\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}\exp(-\frac{x}{\beta})$.

probability mass around the point of zero than the horseshoe prior and thus offers stronger shrinkage ability while containing a heavy tail to avoid the risk of over-shrinking.

A dynamic version of the gamma horseshoe prior in the vein of Kowal et al. (2019) can be easily incorporated by specifying $d_{j,t} = d_{j,t-1}^{\rho_j} e_{j,t}$ where $d_{j,0} = 1$, $e_{j,t} \sim IB(0.5, 0.5)$ captures the serially uncorrelated component of $d_{j,t}$ and ρ_j is the autoregressive coefficient. I will refer to the dynamic version as the dynamic gamma horseshoe (DGHS) prior to differentiate from the static one with $\rho_j = 0$.

Using the framework of Equation (1), Table 1 compares the absolutely continuous shrinkage priors for the state variances that appear in the literature for random-walk TVP models with heteroskedastic latent states. There are different ways to present the priors. I choose to write them in a common format in Table 1 to facilitate comparison. Relative to the horseshoe (HS) and dynamic horseshoe (DHS) priors, the proposed GHS and DGHS priors add an extra layer of gamma distributions for both the global and local parameters and hence offers greater flexibility.

2.2 Prior for Other Model Parameters

Besides the state variance w_t, other parameters in the TVP model of Equation (1) include the measurement equation variance σ_t^2 and the initial state β_0.

For the SV specification of the measurement equation variance σ_t^2 (Equation (2)), the priors are $\mu \sim N(0, 10)$, $\rho \sim N(0.95, 0.04)I_{\{-1<\rho<1\}}$. Note that an informative prior is used for the autoregressive coefficient ρ to reflect the empirical regularity that volatilities of economic time series tend to be highly persistent. For the variance parameter σ_y^2 in the SV model, I follow Kastner and Fruhwirth-Schnatter (2014) and specify a Gaussian prior for its signed square root $\sigma_y = \pm\sqrt{\sigma_y^2}$ as $\sigma_y | s_y \sim N(0, s_y)$ to facilitate estimation. It is possible to specify a fixed number (e.g. 10) for the variance s_y in the prior of σ_y. In this paper, I use a prior $s_y \sim IB(0.5, 0.5)$ to determine s_y in a data-driven way.

In the case where the measurement equation disturbance is homoskedastic $\sigma_t^2 = \sigma^2$, the Jeffery's prior $\sigma^2 \propto \frac{1}{\sigma^2}$ is used.

For the initial state β_0, a horseshoe prior is imposed to push insignificant elements towards zero. Specifically $\beta_{j,0} | \tau_{0,0}, \tau_{j,0} \sim N(0, \tau_{0,0}\tau_{j,0})$ with $\tau_{0,0} \sim IB(0.5, 0.5)$ and $\tau_{j,0} \sim$

$IB(0.5, 0.5)$ for $j = 1, ..., K$.

3 Estimation

Given the priors described in Section 2, a standard Gibbs sampler can be developed for estimating the TVP model of Equation (1). However the resulting posterior draws suffer from slow convergence and poor mixing, particularly when the state variances are close to zero. In this paper, I adopt the ancillarity-sufficiency interweaving strategy (ASIS) of Yu and Meng (2011) to design a more efficient Gibbs sampler. The ASIS boosting strategy has shown good performance in many applications of state space models (e.g. Simpson et al. (2017) and Bitto and Fruhwirth-Schnatter (2019)). A discussion of applying the ASIS is provided in Section 3.1. The details of the proposed Gibbs sampler are presented in Section 3.2.

3.1 ASIS Boosting

The ASIS considers two parametrizations of a hierarchical latent variable model: a sufficient augmentation (SA) in which the latent variable is a sufficient statistic for the relevant parameters, and an ancillary augmentation (AA) in which the same parameters are all in the distribution of the observation and hence the latent variable is an ancillary statistic for them. By moving between a SA-AA pair, Yu and Meng (2011) found that the resulting Markov chain Monte Carlo (MCMC) algorithm is often more efficient and, under certain conditions, provides the fastest converging algorithm within a general class of data augmentation schemes.

The ASIS can be implemented in a component-wise fashion. That is, a *conditional* SA-AA pair for a subset of model parameters can be formed by treating other parameters as fixed, while the conditional SA-AA pair can be different for different subsets of model parameters. Hence the ASIS is quite flexible for use in practical applications.

For the TVP model of Equation (1), the key to implementing the ASIS is to find a suitable (conditional) SA-AA pair or SA-AA pairs of model parametrizations for the parameters that are difficult to estimate in a vanilla Gibbs sampler, namely, the state variances and associated hyper-parameters.

Return to the TVP model of Equation (1). It is clear that the latent state β_t is a sufficient statistic for the state variance w_t and for that matter the signed square root $\tilde{w}_t$. Since the reparametrization of Equation (3) moves the signed square roots $\tilde{w}$ to the measurement equation and leaves the new latent state η_t^* free of $\tilde{w}_t$, one may be tempted to directly apply the ASIS to the signed square roots $\tilde{w}$ by treating Equation (1) and Equation (3) as a SA-AA pair. The drawback of this strategy however is that drawing the signed square roots $\tilde{w}$ in Equation (3) will be working with an enormous linear regression with nK regressors. Even with the state-of-art algorithm in Bhattacharya et al. (2016), the computational complexity is in the order of $\mathcal{O}(n^3K)$ in each MCMC iteration and could be overwhelming in typical economic studies with hundreds of data points[8].

This paper proposes an alternative strategy to avoid repeated large-scale matrix operations while boosting the sampling quality. Instead of directly boosting the state variance, two separate component-wise ASISs are applied to the global parameter v_j and the local parameter $d_{j,t}$ in the prior of the state variance.

Recall from Equation (4) the auxiliary variable of the scaled state variance $\phi_{j,t} = \frac{w_{j,t}}{v_j}$. It follows $w_{j,t} = v_j \phi_{j,t}$ with $\phi_{j,t}|v_j, d_{j,t} \sim G(0.5, 2d_{j,t})$. Conditional on the scaled state variance $\phi_{j,t}$, the latent state $\beta_{j,t}$ is a sufficient statistic for the global parameter v_j and hence the TVP model of Equation (1) can be viewed as a SA representation for v_j. On the other hand, the reparametrization of Equation (4) has the signed squared root $\tilde{v}_j$ of the global parameter v_j in its measurement equation, while its state equation $\beta_{j,t}^* \sim N(\beta_{j,t-1}^*, \phi_{j,t})$ is conditionally free of the global parameter v_j. Therefore the reparametrization of Equation (4) constitutes a conditional AA representation for v_j. An ASIS boosting for the global parameter v_j can be conducted based on this conditional SA-AA pair.

For the local parameter $d_{j,t}$, consider the subset of Equation (4) along with the conditional distribution of the scaled state variance $\phi_{j,t}$:

$$\begin{aligned} \Delta\beta_{j,t}^* &= \beta_{j,t}^* - \beta_{j,t-1}^* \sim N(0, \phi_{j,t}) \\ \phi_{j,t}|v_j, d_{j,t} &\sim G(0.5, 2d_{j,t}) \end{aligned} \tag{5}$$

[8] See Johndrow et al. (2020) and Hauzenberger et al. (2020) for methods that use approximations to the exact algorithm of Bhattacharya et al. (2016).

Equation (5) can be viewed as a nested latent variable model with $\Delta\beta^*_{j,t}$ being the observation, $\phi_{j,t}$ being the latent variable and $d_{j,t}$ being the parameter of interest. In this nested model, the latent variable $\phi_{j,t}$ is a sufficient statistic for $d_{j,t}$ while the observation $\Delta\beta^*_{j,t}$ is independent of $d_{j,t}$ conditional on $\phi_{j,t}$. Hence Equation (5) constitutes a conditional SA representation for $d_{j,t}$. Now let $\phi^*_{j,t} = \frac{\phi_{j,t}}{d_{j,t}}$. It follows that the conditional distribution $\phi^*_{j,t}|v_j, d_{j,t} \sim G(0.5, 2)$ is free of $d_{j,t}$. Substituting $\phi_{j,t} = \phi^*_{j,t} d_{j,t}$ into Equation (5) gives a conditional AA representation for $d_{j,t}$:

$$
\begin{aligned}
&\Delta\beta^*_{j,t} \sim N(0, \phi^*_{j,t} d_{j,t}) \\
&\phi^*_{j,t}|v_j, d_{j,t} \sim G(0.5, 2)
\end{aligned}
\tag{6}
$$

A second ASIS boosting can be applied to this conditional SA-AA pair for the local parameter $d_{j,t}$.

3.2 MCMC Estimation

The proposed Gibbs sampler divides the model parameters into the following blocks:

1. Measurement equation variance:
    (a) SV specification: σ_t^2, μ, ρ, σ_y^2 and associated hyper-parameters including s_y,
    (b) Homoskedastic specification: σ^2,
2. Hyper-parameters $\{\tau_0, \tau_j\}$ and $\{\tau_{0,0}, \tau_{j,0}\}$ for the global parameter v_j and the initial state $\beta_{j,0}$ respectively,
3. Hyper-parameters for the local parameter $d_{j,t}$,
4. Transformed latent states $\beta^*_{j,t}$, global parameter v_j and initial state β_0,
5. Scaled state variance $\phi_{j,t}$ and local parameter $d_{j,t}$,

where $j = 1, 2, ..., K$ and $t = 1, 2, ..., n$.

In block 1, sampling of the measurement equation variance in the SV specification is similar to Kastner and Fruhwirth-Schnatter (2014) by adopting the log linearization strategy of Kim et al. (1998) and Omori et al. (2007). The details are given in Appendix

B. In the homoskedastic case $\sigma_t^2 = \sigma^2$ with the Jeffery's prior $\sigma^2 \propto \frac{1}{\sigma^2}$, the posterior is $\sigma^2 | y, x, \beta \sim \text{IG}\left(\frac{n}{2}, \frac{1}{2}\sum_{t=1}^{n} \epsilon_t^2\right)$ where $\epsilon_t = y_t - x_t'\beta_t$, $y = [y_1, ..., y_n]$, $x = [x_1, ..., x_n]$, $\beta = [\beta_1, ..., \beta_n]$ and IG denotes the inverse gamma distribution.

Block 2 is the hyper-parameters for two horseshoe priors. By adopting the hierarchical inverse gamma representation of Makalic and Schmidt (2016), the posteriors of these hyper-parameters are inverse gamma distributions. Appendix C provides the details.

Block 3 contains the hyper-parameters for the local parameter $d_{j,t}$. In the static case, an auxiliary variable $e_{j,t}$ is introduced to represent $d_{j,t} \sim IB(0.5, 0.5)$ as a hierarchical inverse gamma distribution $d_{j,t} \sim IG\left(0.5, \frac{1}{e_{j,t}}\right)$ and $e_{j,t} \sim IG(0.5, 1)$(Makalic and Schmidt (2016)). The posterior for the auxiliary variable is $e_{j,t} | d_{j,t} \sim IG\left(1, 1 + \frac{1}{d_{j,t}}\right)$. In the dynamic case, use the log linearization $\log(d_{j,t}) = \rho_j \log(d_{j,t-1}) + \log(\xi_{j,t})$ where $\xi_{j,t} \sim IB(0.5, 0.5)$. Following Polson et al. (2013) and Kowal et al. (2019), $\log(\xi_{j,t})$ can be represented as a precision mixture of normals $\log(\xi_{j,t}) \sim N\left(0, \frac{1}{e_{j,t}}\right)$, where $e_{j,t} \sim PG(1, 0)$ is an auxiliary variable and PG denotes the polya-gamma distribution (Barndorff-Nielsen et al. (1982)). The posteriors of the parameters ρ_j and $e_{j,t}$ are derived in Appendix E.

Block 4 and 5 implement two component-wise ASISs. For block 4, the steps are as follows:

i Conditional on ϕ_t, β_0, $\tilde{v}$ and σ_t^2, draw the transformed latent states β_t^* from the AA represention (Equation (4)) by the simulation smoother of Durbin and Koopman (2002)[9].

ii Let α be a $2K$-by-1 vector stacking β_0 and $\tilde{v}$. Conditional on β_t^*, σ_t^2 and the hyper-parameters in block 2, draw α from a linear regression with the posterior $N(b, B)$, where $B^{-1} = B_0^{-1} + \sum_{t=1}^{n} \frac{1}{\sigma_t^2} \tilde{x}_t \tilde{x}_t'$, $B^{-1}b = \sum_{t=1}^{n} \frac{1}{\sigma_t^2} \tilde{x}_t y_t$, B_0 is a diagonal matrix with the diagonal elements $\tau_{0,0}\tau_{1,0}$, ..., $\tau_{0,0}\tau_{K,0}$, $\tau_0\tau_1$, ..., $\tau_0\tau_K$, and $\tilde{x}_t = [x_t, x_t \odot \beta_t^*]'$.

iii Keep the sign of $\tilde{v}$. Calculate the latent state $\beta_{j,t} = \beta_{j,0} + \tilde{v}_j \beta_{j,t}^*$ of the SA representation (Equation (1)).

iv Update v and β_0 in the SA representation of Equation (1). It is possible to use a

[9] Alternative approaches to simulate the latent states from a linear Gaussian state space system include Rue (2001) and McCausland et al. (2011) etc.

nested Gibbs sampler to alternate updating v and β_0. In this paper, I marginalize over β_0 when updating v to improve sampling efficiency. The details are provided in Appendix D.

v Update $\tilde{v}_j$ as the positive square root of v_j times the sign from Step iii. Update the transformed latent state $\beta^*_{j,t} = \frac{\beta_{j,t}-\beta_{j,0}}{\tilde{v}_j}$.

Sampling the parameters of block 5 has the following steps:

i Draw the transformed latent variable $\phi^*_{j,t} = \frac{\phi_{j,t}}{d_{j,t}}$ in the AA representation of Equation (6):

$$\phi^*_{j,t}|d_{j,t}, \beta^*_{j,t} \sim GIG\left(0, 1, \frac{\left(\Delta\beta^*_{j,t}\right)^2}{d_{j,t}}\right)$$

ii Based on the AA representation of Equation (6), draw

$$d_{j,t}|e_{j,t}, \phi^*_{j,t}, \beta^*_{j,t} \sim IG\left(1, \frac{1}{e_{j,t}} + \frac{1}{2\phi^*_{j,t}}\left(\Delta\beta^*_{j,t}\right)^2\right)$$

in the static case. In the dynamic case, draw $d_{j,t}$ based on the algorithm in Kowal (2019). The key is to treat $d_{j,t}$ as the stochastic volatility of a SV model and use the log linearization strategy of Kim et al. (1998) and Omori et al. (2007) to sample $\log(d_{j,t})$. The details are given in Appendix E.

iii Compute the latent variable $\phi_{j,t} = \phi^*_{j,t} d_{j,t}$ for the SA representation of Equation (5).

iv Based on the SA representation of Equation (5), update $d_{j,t}$ from the following posterior in the static case:

$$d_{j,t}|e_{j,t}, \phi_{j,t} \sim IG\left(1, \frac{1}{e_{j,t}} + \frac{\phi_{j,t}}{2}\right)$$

In the dynamic case, update $d_{j,t}$ based on the algorithm in Kowal et al. (2019) with the details in Appendix E.

v Update the transformed latent variable $\phi^*_{j,t} = \frac{\phi_{j,t}}{d_{j,t}}$.

This completes the description of the MCMC algorithm for estimating the TVP model with the proposed gamma horseshoe priors. It should be noted that the proposed algorithm mostly samples from closed-form distributions and scales linearly with the number of observations n and the number of regressors K. No manual intervention or difficult user-specified hyper-parameters are required. Thus model estimation can be performed efficiently in an automated fashion and facilitates practical applications.

3.3 Predictive Likelihoods

In this paper, the predictive likelihood is used to compare different prior specifications of TVP models. See Geweke and Amisano (2010) for a reviews of Bayesian predictive analysis. Specifically, given a model specification $\mathcal{M}$ and a data sample of the dependent variable $y^n = \{y_t\}_{t=1}^n$ and regressors $x^{n+1} = \{x_t\}_{t=1}^{n+1}$, the one-step-ahead predictive likelihood is $p(y_{n+1}|y^n, x^{n+1}, \mathcal{M})$ that integrates out all the parameters in model $\mathcal{M}$. For expositional convenience, the regressors in x_t are treated as exogenous. It should be understood that in the case of predictive regressions, x_t are variables observed at time $t-1$.

To compute the one-step-ahead predictive likelihood $p(y_{n+1}|y^n, x^{n+1}, \mathcal{M})$, I use the strategy that is labeled as the *conditionally optimal Kalman mixture approximation* in Bitto and Fruhwirth-Schnatter (2019). Let θ^n collect the parameters $\{w_t\}_{t=1}^n$, $\{\sigma_t^2\}_{t=1}^n$ and other model hyper-parameters. It is straightforward to derive the conditional distribution:

$$y_{n+1}|x_{n+1}, w_{n+1}, \sigma_{n+1}^2, b_n, B_n \sim N\left(x_{n+1}' b_n,\ \sigma_{n+1}^2 + x_{n+1}' \left(B_n + \mathrm{diag}(w_{n+1})\right) x_{n+1}\right) \tag{7}$$

where b_n and B_n are the mean and covariance matrix of the filtering distribution $\beta_n|y^n, x^n, \theta^n \sim N(b_n, B_n)$. One can approximate the one-step-ahead predictive likelihood as:

$$p\left(y_{n+1}|y^n, x^{n+1}, \mathcal{M}\right) \approx \frac{1}{M}\sum_{i=1}^{M} p\left(y_{n+1}|x_{n+1}, w_{n+1}^{(i)}, \left(\sigma_{n+1}^2\right)^{(i)}, b_n^{(i)}, B_n^{(i)}\right) \tag{8}$$

where $w_{n+1}^{(i)}$ and $\left(\sigma_{n+1}^2\right)^{(i)}$ are simulated based on posterior draws of the gamma horseshoe prior and the SV model for the measurement equation disturbance. The filtered mean $b_n^{(i)}$ and filtered covariance matrix $B_n^{(i)}$ are computed based on a Kalman filter with their derivations provided in Appendix F.

Given the one-step-ahead predictive likelihood $p(y_t|y^{t-1}, x^t, \mathcal{M})$, the cumulative log predictive likelihood $\sum_{i=1}^m \log p(y_{n+i}|y^{n+i-1}, x^{n+i}, \mathcal{M})$ over a prediction sample $t = n + 1$, ..., $n + m$ is the criterion for comparing the performance of model $\mathcal{M}$ with alternatives.

4 Simulation Study

A sample of 300 data points are simulated from a linear regression model with 6 coefficients exhibiting different types of time variation over $t = 1, ..., 300$:

1 Random walk: $\beta_{1,t} = \sum_{j=1}^{t} u_j$ with $u_j \sim N(0, 0.01)$.

2 Change point: $\beta_{2,t} = I_{\{100<t\leq 200\}} - I_{\{t>200\}}$.

3 Mixture of constant parameter, random walk and change point:

$$\beta_{3,t} = \left(\sum_{j=1}^{t} u_j\right) I_{\{100<t\leq 200\}} + I_{\{t>200\}}$$

with $u_j \sim N(0, 0.01)$.

4 Ones: $\beta_{4,t} = 1$.

5 Small constant: $\beta_{5,t} = 0.1$.

6 Zeros: $\beta_{6,t} = 0$.

The regressors are from standard normal distributions. The dependent variables is generated by adding a noise from $N(0, \sigma^2)$ where σ^2 is calibrated such that the ratio of σ^2 to the variance of the dependent variable is 0.2.[10] The 4 TVP priors in Table 1 are applied to the simulated data. In estimation, the measurement equation disturbance is taken to be homoskedastic. A sample of 10,000 posterior draws are kept for analysis after a burn-in of length 5,000.[11]

Figure 2 shows the point-wise posterior medians and 90% credible sets of the regression coefficients estimated by the various TVP priors. The posterior medians match the true coefficients reasonably well. Among the 4 TVP priors, the HS specification leads to markedly more volatile posterior medians and wider credible sets of the coefficient estimates than the other 3 prior specifications, particularly for the constant coefficients. The

[10] In experiments, another two data generating processes are studied where the ratio of σ^2 to the variance of the dependent variable is 0.5 and 0.8 respectively. The estimation results are qualitatively similar.

[11] Generating 1,000 posterior draws from the GHS and DGHS prior specifications takes about 20 and 26 seconds respectively on a standard desktop computer with a 3.0 GHz Intel Core i5 CPU, running in MATLAB R2020b.

coefficient estimates of the GHS, DGHS and DHS priors are close with the exception of the change-point coefficient where the DGHS and DHS priors produce sharper estimates of the change points than the GHS prior.

To quantify the accuracy of coefficient estimates from the TVP priors, I compute the point-wise root mean square error (RMSE) for each coefficient j at point t:

$$\text{RMSE}_{j,t} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(\hat{\beta}_{j,t}^{(i)} - \beta_{j,t}\right)^2} \tag{9}$$

where $\hat{\beta}_{j,t}^{(i)}$ denotes a posterior draw of the coefficient from a given model and $\beta_{j,t}$ the true coefficient value. Box plots of the distributions of $\text{RMSE}_{j,t}$ across $t = 1, ..., 300$ for each coefficient $j = 1, ..., 6$ are shown in Figure 3. A concentrated distribution of $\text{RMSE}_{j,t}$ implies a stable level of estimation precision across points t and hence is more desirable.

In Figure 3, the RMSEs from the HS prior clearly lag behind those from the other 3 priors. The DGHS and DHS priors show very similar levels of estimation precision, while the RMSEs of the GHS prior are comparable to those of the DGHS and DHS priors except in the case of the change-point coefficient where the two dynamic priors tend to perform slightly better.

It may appear counter intuitive that the GHS prior performs as well as the DGHS prior as the autoregressive structure in the DGHS prior is more flexible. However this observation ignores that the gamma distributions placed on the global and local components of the GHS and DGHS priors already allows an extra layer of flexibility and hence the autoregressive structure becomes less important than in the case of the HS and DHS priors. To see this, note that, in both the GHS and DGHS priors, the state variance $w_{j,t} = v_j \phi_{j,t}$ can be re-written as $w_{j,t} = v_j d_{j,t} \phi^*_{j,t}$ where $\phi^*_{j,t} = \frac{\phi_{j,t}}{d_{j,t}}$ and $\phi^*_{j,t} | v_j, d_{j,t} \sim G(0.5, 2)$. Hence local behaviors in the GHS and DGHS priors are controlled by two parameters $d_{j,t}$ and $\phi^*_{j,t}$. In contrast, the HS and DHS priors have a single parameter $d_{j,t}$ to adjust local behavior and hence the extra flexibility rendered by an autoregressive structure on $d_{j,t}$ in the DHS prior makes larger difference relative to the HS prior.

In terms of the mixing efficiency of posterior draws, I first show the benefit of the ASIS boosting in Figure 4, which plots the point-wise effective sample size (ESS) of the regression coefficient estimates by the GHS prior with and without the ASIS boosting[12]. ASIS boosting improves the ESS over the standard Gibbs sampler for all coefficients and particularly for the constant ones. Without boosting, the posterior draws for the constant coefficients from a standard Gibbs sampler have extremely low ESSs and are hardly usable for model inference. The computational cost of adding the ASIS steps is minimal, increasing the running time of generating 1,000 posterior draws by about 1 second in this simulation example.

In Figure 5, the ESSs of the regression coefficient estimates by the GHS, DGHS and DHS priors are compared. For clearer exposition, the figures shows the logarithm of the ratio of the point-wise ESSs of the GHS and DGHS priors to those of the DHS prior. The DGHS prior leads to comparable or slightly better mixed posterior draws of the time-varying coefficients than the DHS prior but shows greater sampling efficiency for the constant coefficients. The GHS prior performs the best among the three priors and shows uniformly and significantly higher sampling efficiency for all coefficients. It is noted that, for both the DGHS and DHS priors, posterior draws of the autoregressive coefficients in the dynamic shrinkage process are found to mix unsatisfactorily, which hinders the sampling efficiency of these two dynamic priors.

5 Empirical Illustration

To illustrate the proposed methodology, I apply the TVP model of Equation (1) to forecast the quarterly U.S. inflation rate. The dependent variable is the quarter-to-quarter change of annualized log quarterly inflation rate. The regressors include a constant, lags 1 to 6 of the dependent variable, lagged quarter-to-quarter change of quarterly average 3-month U.S. treasury bill rate, and lagged quarter-to-quarter change of quarterly average U.S. unemployment rate[13]. The data sample runs from Q2 1957 to Q4 2020 with a total of 255

[12]The ESS is computed by the initial monotone sequence method of Geyer (1992) and is normalized by dividing by the number of posterior draws.

[13]The data source is the FRED database of the U.S. federal reserve bank of St. Louis. The series names are CPILFESL, TB3MS and UNRATE for consumer price index, 3-month treasury bill rate and unemployment rate respectively. Quarterly average is computed as the average monthly values within each quarter.

observations. Figure 6 shows the data used in the estimation. The high volatilities in late 1970s and 2020 are evident in the inflation rate data.

MCMC estimation is performed with 30,000 posterior draws after a burn-in of length 5,000. The SV specification of Equation (2) for the measurement equation variance is used for the TVP model. The time variation patterns of the estimated coefficients are similar across the TVP prior specifications. As an illustration, Figure 7 shows the point-wise posterior medians and 90% credible sets of the coefficients β_t from the GHS prior, along with the OLS estimate for comparison. The intercept shows a small yet prolonged dip in the 1990s. For the autoregressive lags, coefficients on lag 1 and 3 trend upwards while coefficients on lag 2, 4, 5 and 6 decline gradually over time, though the magnitude of changes in the autoregressive coefficients is relatively small. The coefficient on the lagged interest rate variable is markedly smaller than the OLS estimate and contains zero within its 90% credible sets for most of the time periods. For the lagged unemployment rate, its coefficient shows a V shape over the data sample. The V bottom is reached around mid 1970s when the posterior of the coefficient on the lagged unemployment rate mostly falls in the negative region, suggesting a pattern consistent with the Phillips curve. Since then, the posterior median of the coefficient on the lagged unemployment rate steadily goes up towards zero.

As will be shown in Section 5.1, forecasting performance of the HS prior significantly lags behind the other 3 TVP priors. So I focus on comparing the sampling efficiency of the coefficients β_t by the GHS, DGHS and DHS priors. Similar to the simulation study, Figure 8 shows the logarithm of the ratio of the point-wise ESSs of estimated β_t by the GHS and DGHS priors to those of the DHS prior. Between the pair of DGHS and DHS priors, the ESSs of estimated β_t by the DGHS prior are generally higher than those of the DHS prior, though occasionally they under-perform. In contrast, the GHS prior leads to uniformly higher sampling efficiency for all coefficients than both the DGHS and DHS priors.

5.1 Comparing TVP Priors by Out-of-Sample Forecasts

The TVP priors are evaluated via out-of-sample forecasts. The log predictive likelihoods discussed in Section 3.3 are used to compare the performance of the 4 TVP priors. An iterative prediction exercise is conducted. First the sample from Q2 1957 to Q4 2003 is used to estimate the TVP models under the 4 priors and predictive likelihoods for Q1 2004 are computed. Next the estimation sample is expanded by one observation to include Q1 2004 and generates the forecast for Q2 2004. This procedure is repeated until Q4 2020 with a total of 68 out-of-sample forecasts.

Figure 9 shows the evolution of the difference in the cumulative log predictive likelihoods of the GHS and DGHS priors relative to those of the HS and DHS priors. The dominance of the GHS and DGHS priors in predictive likelihoods over the HS prior is evident to the extent that the difference between the predictive likelihoods of the GHS and DGHS priors is hardly visible in the upper panel of Figure 9. Relative to the DHS prior, the GHS and DGHS priors steadily accumulate gains in predictive likelihoods throughout the forecast sample. The volatile movements of inflation rates in 2020 appear to result in large predictive gains for the GHS and DGHS priors relative to the DHS prior. Among the two gamma horseshoe priors, the GHS prior performs better than the DGHS prior and shows steady improvement in predictive likelihoods over the forecast sample. These forecast results show the benefits of the GHS prior that has a simpler structure than the dynamic priors yet produces effective shrinkage for the regression coefficients.

As a commonly used gauge of predictive improvement in the forecasting literature, Diebold-Mariano tests are conducted and find that the averages of the log predictive likelihood differences in the GHS-DHS and DGHS-DHS pairs are highly significant with p-values of 0.001 and 0.003 respectively. Hence the Diebold-Mariano tests support significant predictive improvement of the gamma horseshoe priors, static or dynamic, over the dynamic horseshoe prior of Kowal et al. (2019). A similar Diebold-Mariano test for the GHS-DGHS pair returns a p-value of 0.09 and provides modest support of the more parsimonious GHS prior specification.

6 Conclusion

The TVP model with time-dependent state variance offers a flexible framework to accommodate a wide variety of time variation patterns of parameter changes but is prone to over-parameterization. Effective shrinkage on the state variance is crucial to reduce the risk of model over-fitting. This paper introduces a hierarchical horseshoe prior named *gamma horseshoe prior* on the state variance that allows for stronger shrinkage than the conventional horseshoe prior while maintaining flexibility to accommodate local signals. The new prior is motivated by reparametrizing the TVP model that translates a variance shrinkage problem into a variable shrinkage one borrowing on the insight from Fruhwirth-Schnatter and Wagner (2010). Dynamic version of the gamma horseshoe prior in the vein of Kowal et al. (2019) is also provided.

An MCMC algorithm is developed to estimate the TVP model under the proposed gamma horseshoe priors that employs the ASIS of Yu and Meng (2011) as the critical ingredient to boost sampling efficiency. The novelty of the algorithm is that it avoids repeated large-scale matrix operations and instead applies the ASIS in a tractable component-wise fashion. In both simulation and an inflation rate forecast exercise, the gamma horseshoe priors are shown to provide effective shrinkage for the regression coefficients of the TVP model. It is also noted that the need for dynamic structure is reduced in the gamma horseshoe priors relative to the case of the horseshoe priors. In out-of-sample forecasts, the gamma horseshoe prior outperforms the horseshoe prior and their dynamic versions.

This paper focuses on the univariate TVP model. Extension to the multivariate case is possible. For example, a computationally convenient approach could be the time-varying Cholesky decomposition introduced in Lopes et al. (2018) and Carriero et al. (2019) that factorizes the covariance matrix of the measurement equation disturbance through Cholesky decomposition and reduces the multivariate TVP model into a system of independent univariate TVP models. Given its computational efficiency, studying the proposed gamma horseshoe prior in the multivariate context could be an interesting venue of future research.

Appendix

A Comparing Gamma Horseshoe Prior with Horseshoe Prior

Consider a simplified version of the horseshoe prior $\beta_H \sim N(0, w_H)$ with $w_H \sim IB(0.5, 0.5)$ and the gamma horseshoe prior $\beta_G \sim N(0, w_G)$ with the scale mixture $w_G \sim G(0.5, 2v)$ and $v \sim IB(0.5, 0.5)$. Following Carvalho et al. (2010), the shrinkage property of the priors is analyzed via the shrinkage parameter $\kappa_j = \frac{1}{1+w_j}$ where $j \in \{H, G\}$. $\kappa \to 1$ implies shrinking the parameter β towards zero while $\kappa \to 0$ implies minimal shrinkage for β.

The density of κ_G has no closed form. However, based on the fact (Makalic and Schmidt (2016)):

$$v \sim IB(0.5, 0.5) \Longleftrightarrow v|\lambda \sim IG\left(0.5, \frac{1}{\lambda}\right), \quad \lambda \sim IG(0.5, 1)$$

one can derive the density of κ_G conditional on the auxiliary variable λ as:

$$p_G(\kappa_G|\lambda) = \frac{1}{\pi(2\kappa_G + \lambda(1-\kappa_G))}\sqrt{\frac{2\lambda}{\kappa_G(1-\kappa_G)}}$$

Similarly, the conditional density of κ_H for the horseshoe prior can be written as:

$$p_H(\kappa_H|\lambda) = \frac{1}{\sqrt{\pi\lambda\kappa_H(1-\kappa_H)^3}} \exp\left(-\frac{\kappa_H}{\lambda(1-\kappa_H)}\right)$$

The first 3 panels of Figure 1 compares the logarithm of the conditional densities of κ_G and κ_H conditional on $\lambda = 1.2$, 4.4 and 31.2 respectively, which are about the 20^{th}, 50^{th} and 80^{th} percentiles of the distribution $IG(0.5, 1)$. The conditional density of κ_G remains a desirable "U" shape under various values of λ that contains large probability mass around the two ends of $\kappa_G = 0$ and 1, while the conditional density of κ_H becomes very small towards the end $\kappa_H = 1$ and hence offers limited shrinkage.

Let $f(\kappa, \lambda) = \frac{p_G(\kappa|\lambda)}{p_H(\kappa|\lambda)}$. It is useful to note that $\lim_{\kappa \to 0} f(\kappa, \lambda) = \sqrt{\frac{2}{\pi}} \approx 0.8$ while $\lim_{\kappa \to 1} f(\kappa, \lambda) = +\infty$ for a given finite λ. Therefore the gamma horseshoe prior allocates more probability mass towards the end of complete shrinkage $\kappa = 1$ than the horseshoe prior without much reduction in its probability mass at the other end of minimal shrinkage $\kappa = 0$, avoiding the risk of "over-shrinking".

Turning to the marginal distribution of κ_G and κ_H, it can be derived that $\kappa_H \sim Beta(0.5, 0.5)$ and hence its density has a symmetric "U" shape. For the gamma horseshoe prior, the density of κ_G is approximated based on a million draws of w_G from its scale-mixture Gamma distribution. The 4$^{\text{th}}$ panel of Figure 1 compares the densities of κ_G and κ_H. The density of κ_G has a "U" shape that is tilted towards the end $\kappa_G = 1$. The empirical density mass of κ_G located in the three disjoint regions $(0, 0.1]$, $(0.1, 0.9)$ and $[0.9, 1)$ is 16%, 47% and 37% respectively while the corresponding density mass is 21%, 58% and 21% for κ_H under the horseshoe prior. Hence the gamma horseshoe prior offers an appreciably stronger shrinkage for the parameter β.

The density of β under the simplified horseshoe and gamma horseshoe priors is compared based on a million simulations. Checking the two tail regions of β: $(-\infty, -25] \cup [25, \infty)$ and $(-\infty, -100] \cup [100, \infty)$, the density mass of β in these regions under the gamma horseshoe prior is 1.6% and 0.4% respectively, which is lower than under the horseshoe prior (2.0% and 0.5% respectively) but remains appreciable to accommodate extreme values of β.

B SV Model Estimation

Estimating the SV model of Equation (2) follows Kastner and Fruhwirth-Schnatter (2014). There are two key ingredients in the method of Kastner and Fruhwirth-Schnatter (2014). The first is the log linearization strategy of Omori et al. (2007) that approximates the logarithm of a $\chi^2(1)$-distributed variable by a mixture of normal distributions and hence transforms a non-linear state space model into a linear one. The second is applying the ASIS strategy of Yu and Meng (2011) that directly boosts the sampling efficiency of the long-run mean and the variance parameter of the volatility process. The details of the method can be found in Kastner and Fruhwirth-Schnatter (2014) and are not repeated here to save space.

The main difference in this paper from Kastner and Fruhwirth-Schnatter (2014) is the prior of the variance parameter in the volatility process. Use the notation of Equation (2). Kastner and Fruhwirth-Schnatter (2014) sets a fixed value for the scale parameter s_y in the gamma prior of the variance parameter $\sigma_y^2 \sim G(0.5, 2s_y)$. This paper instead specifies a prior $s_y \sim IB(0.5, 0.5)$ to determine s_y in a data driven way. The conditional posterior of s_y can be easily derived by applying the hierarchical inverse gamma representation in Makalic and Schmidt (2016): $s_y|a_y, \sigma_y^2 \sim IG\left(1, \frac{1}{a_y} + \frac{\sigma_y^2}{2}\right)$ where a_y is an auxiliary variable with the prior $a_y \sim IG(0.5, 1)$ and the posterior $a_y|s_y \sim IG\left(1, 1 + \frac{1}{s_y}\right)$.

C Hyper-Parameters of Horseshoe Prior

The horseshoe prior for the square root of the global parameter is $\tilde{\upsilon}_j = \pm\sqrt{\upsilon_j} \sim N(0, \tau_0\tau_j)$ with $\tau_0 \sim IB(0.5, 0.5)$ and $\tau_j \sim IB(0.5, 0.5)$ for $j = 1, ..., K$. Following Makalic and Schmidt (2016), the inverted beta distributions are represented as hierarchical inverse gamma ones by introducing auxiliary variables:

$$\tau_0 \sim IB(0.5, 0.5) \Longleftrightarrow \tau_0 \sim IG\left(0.5, \frac{1}{a_0}\right), \quad a_0 \sim IG(0.5, 1)$$
$$\tau_j \sim IB(0.5, 0.5) \Longleftrightarrow \tau_j \sim IG\left(0.5, \frac{1}{a_j}\right), \quad a_j \sim IG(0.5, 1)$$

The posteriors can be derived as follows:

$$\begin{aligned}
&\tau_0|a_0, \{\tilde{\upsilon}_j\}_{j=1}^K, \{\tau_j\}_{j=1}^K \sim IG\left(\frac{1+K}{2}, \frac{1}{a_0} + \frac{1}{2}\sum_{j=1}^K \frac{1}{\tau_j}\tilde{\upsilon}_j^2\right), \\
&a_0|\tau_0 \sim IG\left(1, 1 + \frac{1}{\tau_0}\right), \\
&\tau_j|\tilde{\upsilon}_j, \tau_0, a_j \sim IG\left(1, \frac{1}{a_j} + \frac{1}{2\tau_0}\tilde{\upsilon}_j^2\right), \\
&a_j|\tau_j \sim IG\left(1, 1 + \frac{1}{\tau_j}\right).
\end{aligned}$$

Similarly for the horseshoe prior of the initial state $\beta_{j,0} \sim N(0, \tau_{0,0}\tau_{j,0})$ with $\tau_{0,0} \sim IB(0.5, 0.5)$ and $\tau_{j,0} \sim IB(0.5, 0.5)$ for $j = 1, ..., K$, the hierarchical inverse gamma representation is:

$$\tau_{0,0} \sim IB(0.5, 0.5) \Longleftrightarrow \tau_{0,0} \sim IG\left(0.5, \frac{1}{b_0}\right), \quad b_0 \sim IG(0.5, 1)$$
$$\tau_{j,0} \sim IB(0.5, 0.5) \Longleftrightarrow \tau_{j,0} \sim IG\left(0.5, \frac{1}{b_j}\right), \quad b_j \sim IG(0.5, 1)$$

with the following posteriors:

$$\tau_{0,0}|b_0, \{\beta_{j,0}\}_{j=1}^K, \{\tau_{j,0}\}_{j=1}^K \sim IG\left(\frac{1+K}{2}, \frac{1}{b_0} + \frac{1}{2}\sum_{j=1}^K \frac{1}{\tau_{j,0}}\beta_{j,0}^2\right),$$
$$b_0|\tau_{0,0} \sim IG\left(1, 1 + \frac{1}{\tau_{0,0}}\right),$$
$$\tau_{j,0}|\beta_{j,0}, \tau_{0,0}, b_j \sim IG\left(1, \frac{1}{b_j} + \frac{1}{2\tau_{0,0}}\beta_{j,0}^2\right),$$
$$b_j|\tau_{j,0} \sim IG\left(1, 1 + \frac{1}{\tau_{j,0}}\right).$$

D Sampling v and β_0 from the SA Representation

Let $\phi = \{\phi_t\}_{t=1}^n$, $\theta_{0,v} = \{\tau_0, \tau_1, ..., \tau_K\}$, $\theta_{0,\beta} = \{\tau_{0,0}, \tau_{0,1}, ..., \tau_{0,K}\}$ and $\mathcal{D} = \{\beta, \phi, \theta_{0,v}, \theta_{0,\beta}\}$. The target is to sample from $p(v, \beta_0|\mathcal{D}) = p(v|\mathcal{D})p(\beta_0|\mathcal{D}, v)$ based on the conditional SA representation of Equation (1).

Given the independent priors $p(v_j|\theta_{0,v}) = G(0.5, 2\tau_0\tau_j)$ for $j = 1, ..., K$, the marginal distribution $p(v|\mathcal{D}) \propto p(v|\theta_{0,v})\ p(\beta|v, \phi, \theta_{0,\beta})$ has no closed form. However one can decompose the likelihood as $p(\beta|v, \phi, \theta_{0,\beta}) = p(\beta_1|v, \phi, \theta_{0,\beta})p(\beta_2, ..., \beta_n|v, \phi, \beta_1)$. The partial posterior $p(v|\theta_{0,v})p(\beta_2, ..., \beta_n|v, \phi, \beta_1)$ would constitute independent generalized inverse Gaussian (GIG) distributions for elements in v. Specifically one can derive:

$$p(v_j|\theta_{0,v})\prod_{t=2}^n p(\beta_{j,t}|v_j, \phi_{j,t}, \beta_{j,t-1}) \propto (v_j)^{-n/2}\exp\left(-\frac{v_j}{2\tau_0\tau_j} - \frac{1}{2v_j}\sum_{t=2}^n \frac{(\Delta\beta_{j,t})^2}{\phi_{j,t}}\right) \tag{D1}$$

where $\Delta\beta_{j,t} = \beta_{j,t} - \beta_{j,t-1}$ for $j = 1, ..., K$ and $t = 2, ..., n$. Recognizing that the right-hand side of Equation (D1) is the kernel of the distribution $GIG\left(1 - \frac{n}{2}, \frac{1}{\tau_0\tau_j}, \sum_{t=2}^n \frac{(\Delta\beta_{j,t})^2}{\phi_{j,t}}\right)$ for v_j, an independent Metropolis-Hastings step can be readily applied to sample from $p(v_j|\mathcal{D})$ by using the proposal $GIG\left(1 - \frac{n}{2}, \frac{1}{\tau_0\tau_j}, \sum_{t=2}^n \frac{(\Delta\beta_{j,t})^2}{\phi_{j,t}}\right)$ for $j = 1, ..., K$[14]. The initial item of the likelihood $p(\beta_1|v, \phi, \theta_{0,\beta}) = \int p(\beta_1|v, \phi, \beta_0)p(\beta_0|\theta_{0,\beta})d\beta_0 = N(0, \mathrm{diag}(v \odot \phi_1 + v_{\beta,0}))$ can be computed by recognizing $p(\beta_1|v, \phi, \beta_0) = N(\beta_0, \mathrm{diag}(v \odot \phi_1))$ and $p(\beta_0|\theta_{0,\beta}) = N(0, \mathrm{diag}(v_{\beta,0}))$, where the j^{th} element of $v_{\beta,0}$ is $\tau_{0,0}\tau_{0,j}$ for $j = 1, ..., K$.

[14]Sampling from the GIG distribution is by adapting the Matlab function **gigrnd** written by Enes Makalic and Daniel Schimidt that implements an algorithm from Devroye (2014).

The conditional distribution $p(\beta_0|\mathcal{D}, v) \propto p(\beta_0|\theta_{0,\beta})p(\beta_1|v, \phi, \beta_0)$ is normal thanks to the normal prior $p(\beta_0|\theta_{0,\beta})$ and the normal likelihood $p(\beta_1|v, \phi, \beta_0)$. It is straightforward to derive the posterior $p(\beta_{0,j}|\mathcal{D}, v) = N(b_j, B_j)$ where $B_j = \frac{\tau_{0,0}\tau_{j,0}v_j\phi_{j,1}}{\tau_{0,0}\tau_{j,0}+v_j\phi_{j,1}}$ and $b_j = \frac{\tau_{0,0}\tau_{j,0}\beta_{j,1}}{\tau_{0,0}\tau_{j,0}+v_j\phi_{j,1}}$ for $j = 1, ..., K$.

E Dynamic Gamma Horseshoe Prior

For convenience, reproduce the SA parametrization of Equation (5) and the equation for $\log(d_{j,t})$:

$$\begin{aligned}
&\Delta\beta^*_{j,t} \sim N(0, \phi_{j,t}) \\
&\phi_{j,t} \sim G(0.5, 2d_{j,t}) \\
&\log(d_{j,t}) = \rho_j \log(d_{j,t-1}) + \log(\xi_{j,t})
\end{aligned}$$

where $\log(d_{j,0}) = 0$, $\log(\xi_{j,t}) \sim N\left(0, \frac{1}{e_{j,t}}\right)$ and $e_{j,t} \sim PG(1, 0)$ for $j = 1, ..., K$ and $t = 1, ..., n$. The posterior of the scaled state variance can be derived as:

$$\phi_{j,t}|\beta^*_{j,t}, d_{j,t} \sim GIG\left(0, \frac{1}{d_{j,t}}, \left(\Delta\beta^*_{j,t}\right)^2\right)$$

If one starts the ASIS block from the AA representation (Equation (6)):

$$\begin{aligned}
&\Delta\beta^*_{j,t} \sim N(0, \phi^*_{j,t}d_{j,t}) \\
&\phi^*_{j,t} \sim G(0.5, 2) \\
&\log(d_{j,t}) = \rho_j \log(d_{j,t-1}) + \log(\xi_{j,t})
\end{aligned}$$

The posterior of the transformed latent variable is:

$$\phi^*_{j,t}|\beta^*_{j,t}, d_{j,t} \sim GIG\left(0, 1, \frac{\left(\Delta\beta^*_{j,t}\right)^2}{d_{j,t}}\right)$$

To sample the local parameter $d_{j,t}$ from the SA representation of Equation (5), note that it can be viewed as a stochastic volatility following a SV process with the long-run mean of zero. The strategy of Kastner and Fruhwirth-Schnatter (2014) can be applied here to estimate the SV model. Specifically, applying the log linearization strategy of Omori et al. (2007) approximates a $\log\left(\chi^2(1)\right)$ distribution by a 10-component mixture of normals

$\sum_{i=1}^{10} \pi_i N(\mu_i^*, h_i^*)$ and transforms the distribution $\phi_{j,t}|d_{j,t} \sim G(0.5, 2d_{j,t})$[15] into a normal one: $\log(\phi_{j,t})|d_{j,t} \sim N\left(\mu^*_{s_{j,t}} + \log(d_{j,t}), h^*_{s_{j,t}}\right)$ where $s_{j,t} \in \{1, 2, ..., 10\}$ is an indicator for the 10-component-mixture-normal distribution. The posterior for the indicator is:

$$p(s_{j,t} = i|\phi_{j,t}, d_{j,t}) \propto (h_i^*)^{-\frac{1}{2}} \exp\left(-\frac{1}{2h_i^*}\left(\log(\phi_{j,t}) - \mu_i^* - \log(d_{j,t})\right)^2\right)\pi_i$$

To sample $\log(d_{j,t})$, the precision-based algorithm of Rue (2001) and McCausland et al. (2011) is used. Let $\tilde{d}_j$ be a n-by-1 vector collecting $\log(d_{j,1})$, ..., $\log(d_{j,n})$, $\tilde{y}_j$ be a n-by-1 vector collecting $\log(\phi_{j,1}) - \mu^*_{s_{j,1}}$, ..., $\log(\phi_{j,n}) - \mu^*_{s_{j,n}}$, H be a n-by-n lower bi-diagonal matrix with ones in the diagonal and $-\rho_j$ in each $(i, i-1)$ element for $i = 2$, ..., n, P be a n-by-n diagonal matrix with diagonal elements $h^*_{s_{j,1}}$, ..., $h^*_{s_{j,n}}$ and Q be a n-by-n diagonal matrix with diagonal elements $\frac{1}{e_{j,1}}$, ..., $\frac{1}{e_{j,n}}$. It follows that the posterior for $\tilde{d}_j$ is $N(b_d, B_d)$ where $B_d^{-1} = H'Q^{-1}H + P^{-1}$ and $B_d^{-1}b_d = P^{-1}\tilde{y}_j$. The key for efficient computation is to note that the matrix B_d^{-1} is a tri-diagonal band matrix. Hence a Cholesky decomposition of B_d^{-1} can be efficiently computed by back-substitution and be directly used to draw $\tilde{d}_j$ from the posterior $N(b_d, B_d)$.

Similarly one can sample the local parameter $\log(d_{j,t})$ from the AA representation of Equation (6). Conditional on $\phi^*_{j,t}$, one can write $\frac{\Delta\beta^*_{j,t}}{\sqrt{\phi^*_{j,t}}} \sim N(0, d_{j,t})$. Hence the local parameter $d_{j,t}$ can be recast as the stochastic volatility in a SV process with $\frac{\left(\Delta\beta^*_{j,t}\right)^2}{\phi^*_{j,t}}$ being the observation variable. Sampling of $\log(d_{j,t})$ follows the same steps as in the SA representation by replacing the observation variable $\phi_{j,t}$ with the new one $\frac{\left(\Delta\beta^*_{j,t}\right)^2}{\phi^*_{j,t}}$.

Conditional on $d_{j,t}$, its hyper-parameters ρ_j and $e_{j,t}$ can be sampled as follows. Given the prior $\rho_j \sim N(\mu_\rho, h_\rho)I_{\{-1<\rho_j<1\}}$, a Metropolis-Hasting step is used to sample ρ_j by a proposal $N(a_\rho, b_\rho)$ where $b_\rho^{-1} = h_\rho^{-1} + \sum_{t=1}^n e_{j,t}\left(\log(d_{j,t-1})\right)^2$ and $b_\rho^{-1}a_\rho = h_\rho^{-1}\mu_\rho + \sum_{t=1}^n e_{j,t}\log(d_{j,t-1})\log(d_{j,t})$. This paper sets $\mu_\rho = 0.95$ to reflect the prior belief that $d_{j,t}$, being the stochastic volatility of a SV process, is highly persistent and at the same time sets $h_\rho = 1$ to allow a large degree of uncertainty for this prior belief.

Following Kowal et al. (2019), the posterior for $e_{j,t}$ is $e_{j,t}|d_{j,t}, \rho_j \sim PG(1, \log(d_{j,t}) - \rho_j\log(d_{j,t-1}))$[16]. This completes the sampling steps for the dynamic gamma horseshoe

[15] A Gamma distribution $G(0.5, 2)$ is equivalent to a $\chi^2(1)$ distribution.

[16] Sampling from the polya-gamma distribution is by the Matlab function **pgdraw** written by Enes Makalic and Daniel Schimidt that implements an algorithm from Windle (2013).

prior.

F Kalman Filter

The iterations to derive the conditional filtering distribution $p(\beta_n|y^n, x^n, \theta^n)$ are adapted from Bitto and Fruhwirth-Schnatter (2019) and are provided below using the notation of Equation (1):

- Start with the initial state $\beta_0 \sim N(b_0, B_0)$ where $b_0 = 0$ and $B_0 = \text{diag}(\tau_0\tau_1, ..., \tau_0\tau_K)$.
- Iterate forward over a prediction step and a filtering step for $t = 1, ..., n$:
  - $\beta_t|\mathcal{I}_{t-1} \sim N(b_{t-1}, R_t)$ with $R_t = B_{t-1} + \text{diag}(w_t)$;
  - $\beta_t|\mathcal{I}_t \sim N(b_t, B_t)$ with $b_t = b_{t-1} + K_t(y_t - \hat{y}_t)$ and $B_t = (I_K - K_t x_t')R_t$, where $\hat{y}_t = x_t' b_{t-1}$, $K_t = R_t x_t S_t^{-1}$ and $S_t = x_t' R_t x_t + \sigma_t^2$.

where $\mathcal{I}_t$ denotes the information set $\{y^t, x^t, \theta^n\}$ at time t.

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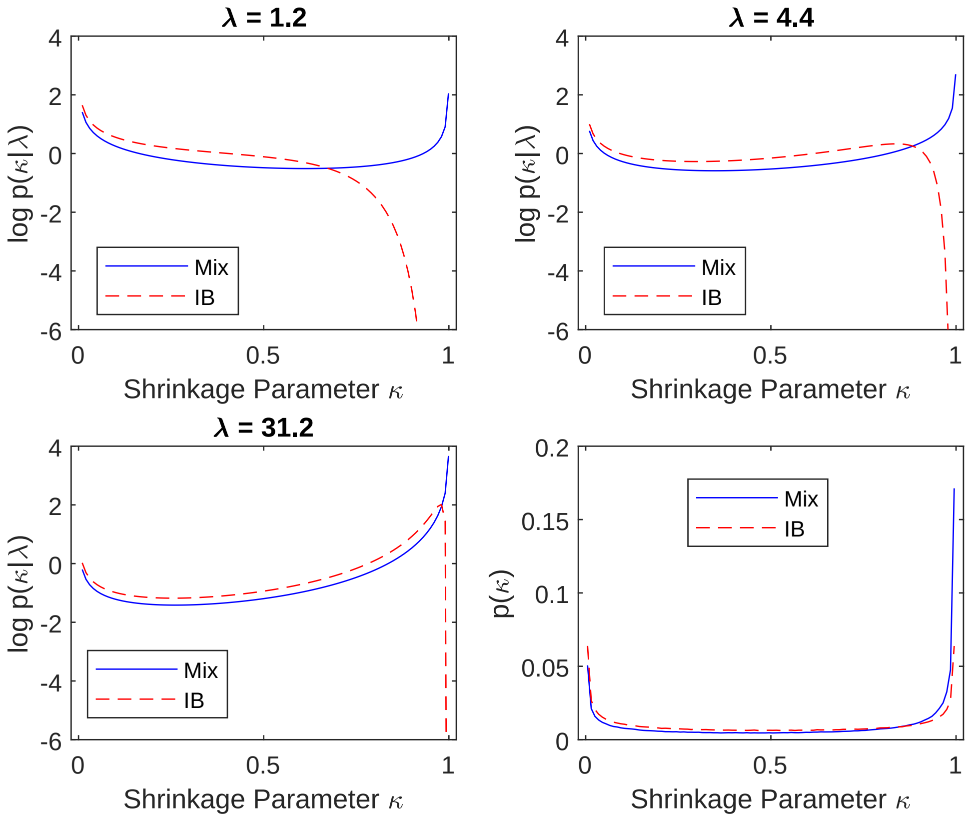
Table 1: Comparing Priors for State Variances

	State Variance	Hierarchical Prior
Gamma horseshoe (GHS)	$w_{j,t} = v_j \phi_{j,t}$	$v_j \sim G(0.5, 2\tau_0\tau_j)$, $\phi_{j,t} \sim G(0.5, 2d_{j,t})$ $\tau_0 \sim IB(0.5, 0.5)$, $\tau_j \sim IB(0.5, 0.5)$ $d_{j,t} \sim IB(0.5, 0.5)$
Dynamic gamma horseshoe (DGHS)	$w_{j,t} = v_j \phi_{j,t}$	$v_j \sim G(0.5, 2\tau_0\tau_j)$, $\phi_{j,t} \sim G(0.5, 2d_{j,t})$ $\tau_0 \sim IB(0.5, 0.5)$, $\tau_j \sim IB(0.5, 0.5)$ $d_{j,t} = d_{j,t}^{\rho_j} e_{j,t}$, $e_{j,t} \sim IB(0.5, 0.5)$
Horseshoe (HS)	$w_{j,t} = \tau_0 \tau_j d_{j,t}$	$\tau_0 \sim IB(0.5, 0.5)$, $\tau_j \sim IB(0.5, 0.5)$ $d_{j,t} \sim IB(0.5, 0.5)$
Dynamic horseshoe (DHS)	$w_{j,t} = \tau_0 \tau_j d_{j,t}$	$\tau_0 \sim IB(0.5, 0.5)$, $\tau_j \sim IB(0.5, 0.5)$ $d_{j,t} = d_{j,t}^{\rho_j} e_{j,t}$, $e_{j,t} \sim IB(0.5, 0.5)$

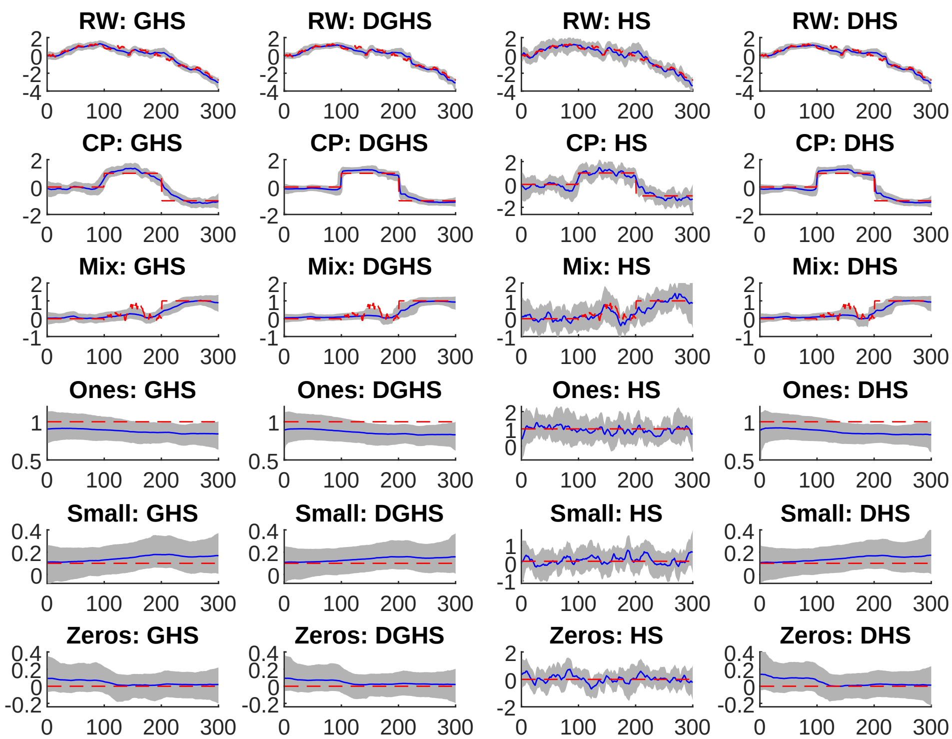
Note: This table compares the priors for states variances in TVP models with heteroskedastic latent states as in Equation (1). When writing the hierarchical priors, the conditioning variables in conditional distributions are omitted to save space.

Figure 1: Comparing Shrinkage Parameter



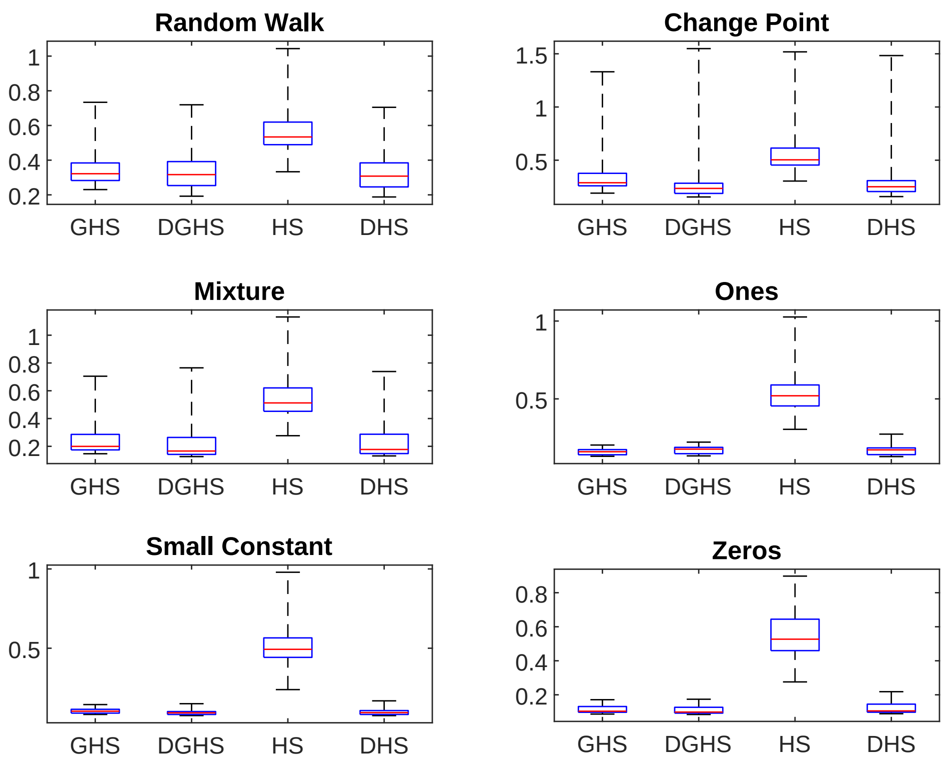
Note: The shrinkage parameter is $\kappa = \frac{1}{1+w}$ where w is the variance of the prior. The parameter λ arises from the hierarchical inverse gamma representation of an inverted beta distribution. The details are in Appendix A. The label "IB" refers to the inverted beta distribution under the horseshoe prior (dash line) while the label "Mix" refers to the scale-mixture gamma distribution under the gamma horseshoe prior (solid line). The y-axis of the log conditional densities are truncated from below for better exposition.

Figure 2: Regression Coefficient Estimates: Simulation



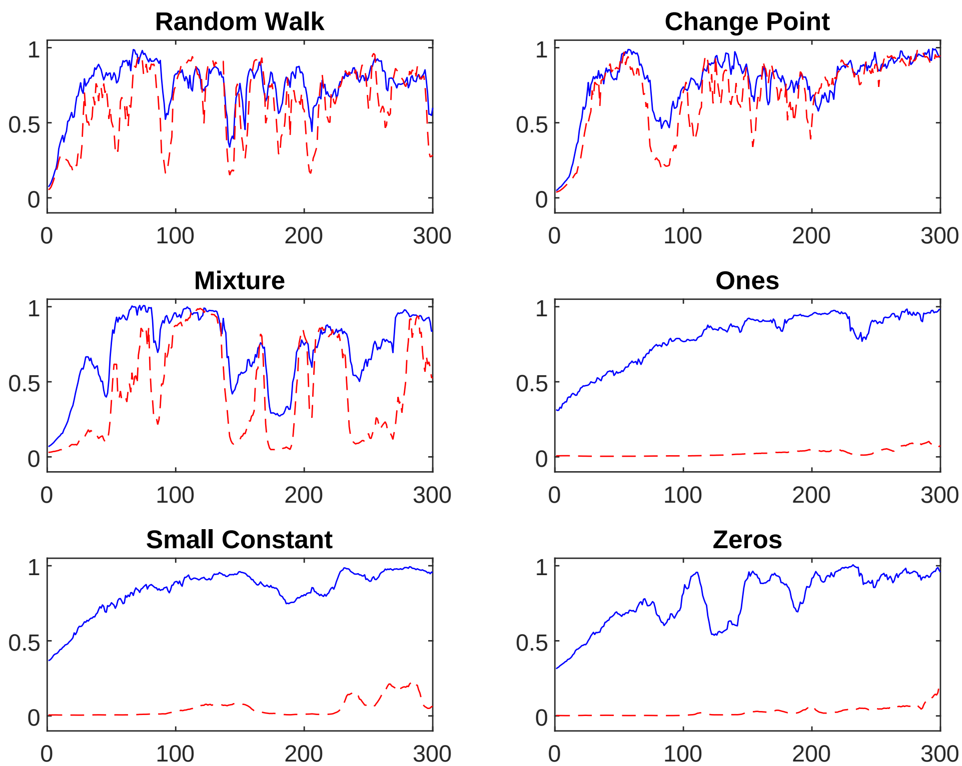
Note: The figure shows the point-wise posterior medians and 90% credible sets of estimated coefficient $\beta_{j,t}$ over time points t = 1, ..., 300 for each coefficient j = 1, ..., 6 under the 4 TVP priors: gamma horseshoe (GHS), dynamic gamma horseshoe (DGHS), horseshoe (HS) and dynamic horseshoe (DHS). In each subplot, the solid line is the posterior median with the dash line being the true coefficient value and the grey area being the 90% credible set.

Figure 3: Root Mean Squared Error of Coefficient Estimates: Simulation



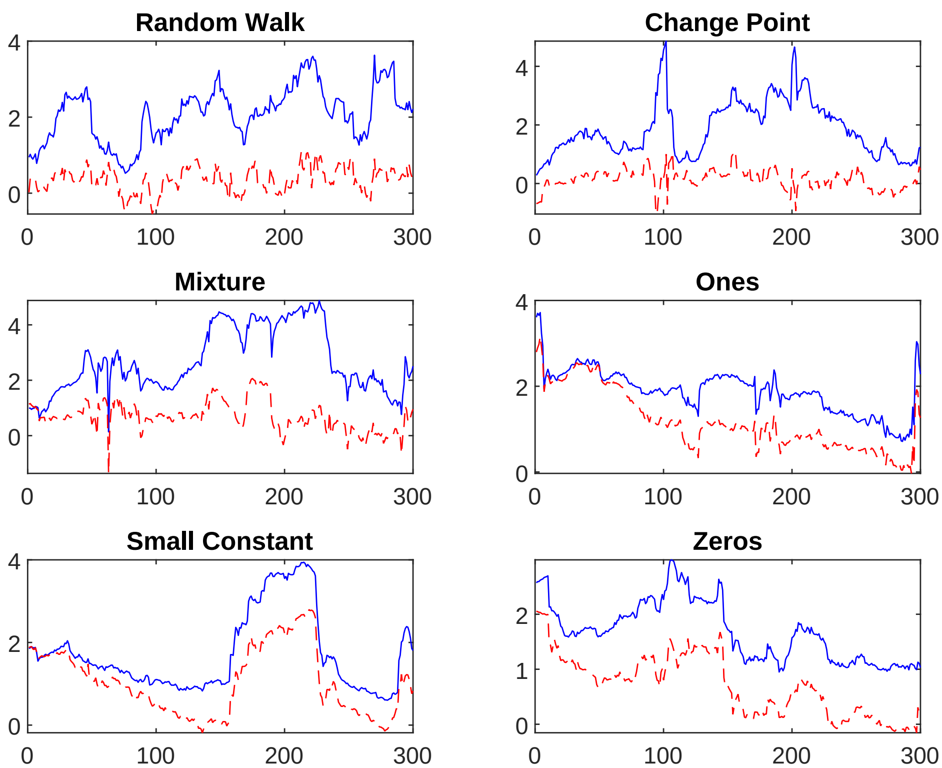
Note: The figure shows the box plots of the root mean squared errors (Equation (9)) of estimated coefficient $\beta_{j,t}$ over time points $t = 1, ..., 300$ for each coefficient $j = 1, ..., 6$ under the 4 TVP priors: gamma horseshoe (GHS), dynamic gamma horseshoe (DGHS), horseshoe (HS) and dynamic horseshoe (DHS). On each box, the central mark indicates the median, and the bottom and top edges of the box indicate the 25^{th} and 75^{th} percentiles respectively. The whiskers extend to the most extreme data points.

Figure 4: Sampling Efficiency Gain by ASIS Boosting: Simulation



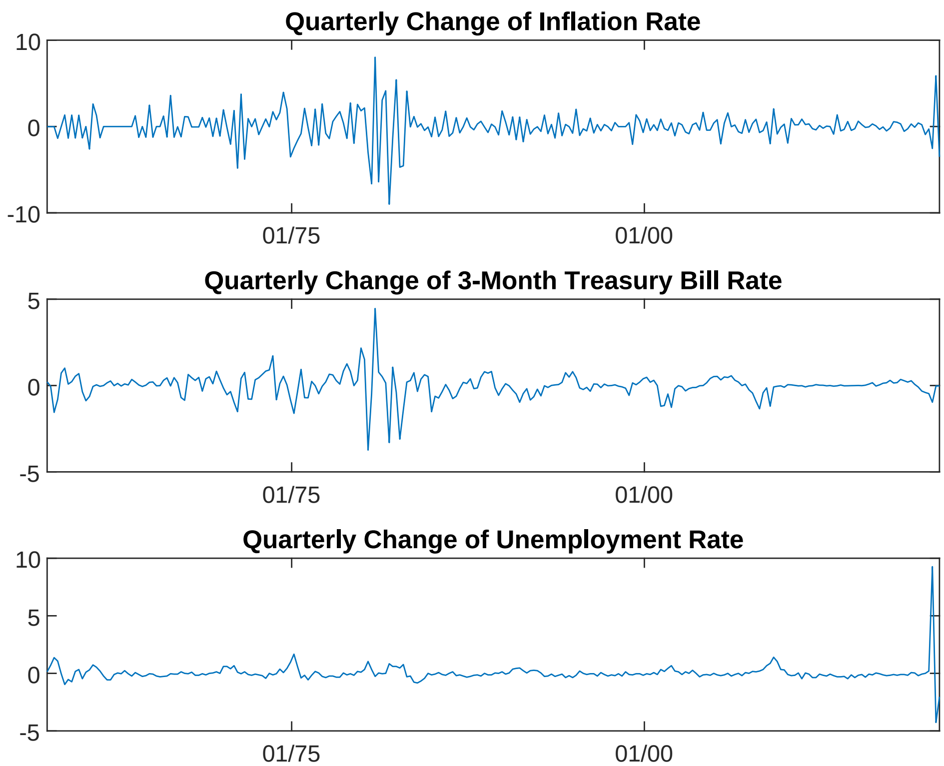
Note: The figure compares the point-wise effective sample size (ESS) of estimated coefficient $\beta_{j,t}$ by a standard Gibbs sampler (dash line) and the ASIS-boosted algorithm (solid line) over time points t = 1, ..., 300 for each coefficient j = 1, ..., 6 under the gamma horseshoe (GHS) prior. The ESS is normalized by dividing by the number of posterior draws. A higher ESS indicates greater sampling efficiency.

Figure 5: Comparing Sampling Efficiency of Regression Coefficients: Simulation



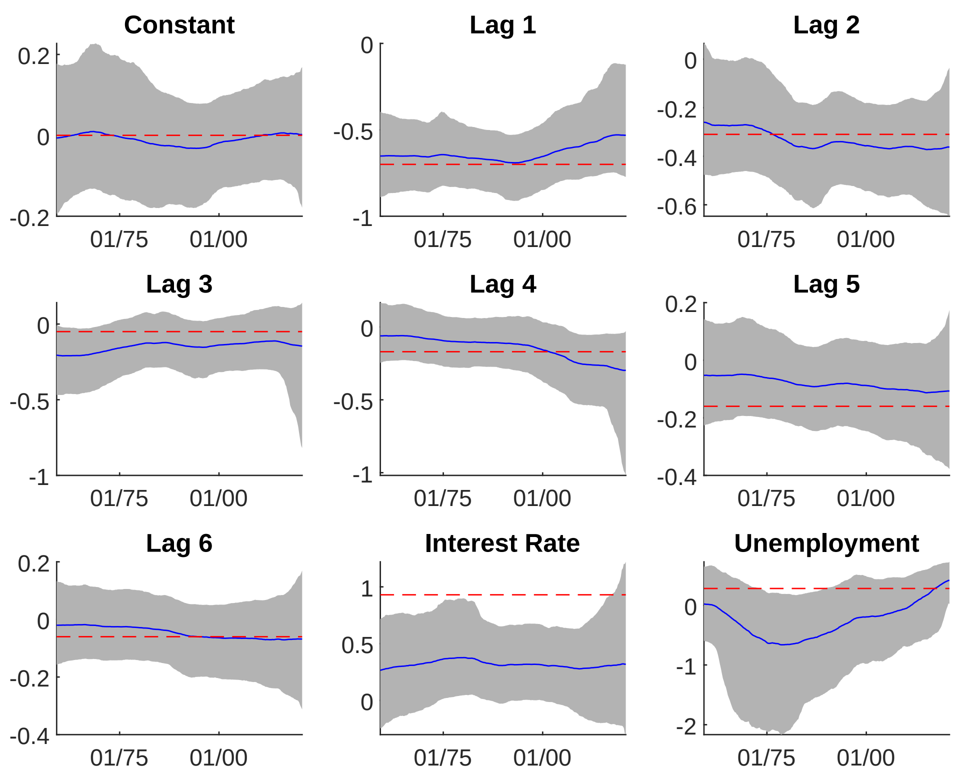
Note: The figure shows the logarithm of the ratio of the point-wise effective sample size (ESS) of estimated coefficient $\beta_{j,t}$ under the gamma horseshoe (GHS, solid line) and dynamic gamma horseshoe (DGHS, dash line) priors to that under the dynamic horseshoe (DHS) prior over time points $t = 1, ..., 300$ for each coefficient $j = 1, ..., 6$. A positive value of the log ESS ratio indicates greater sampling efficiency over the DHS prior.

Figure 6: Data for Empirical Application: Inflation Rate



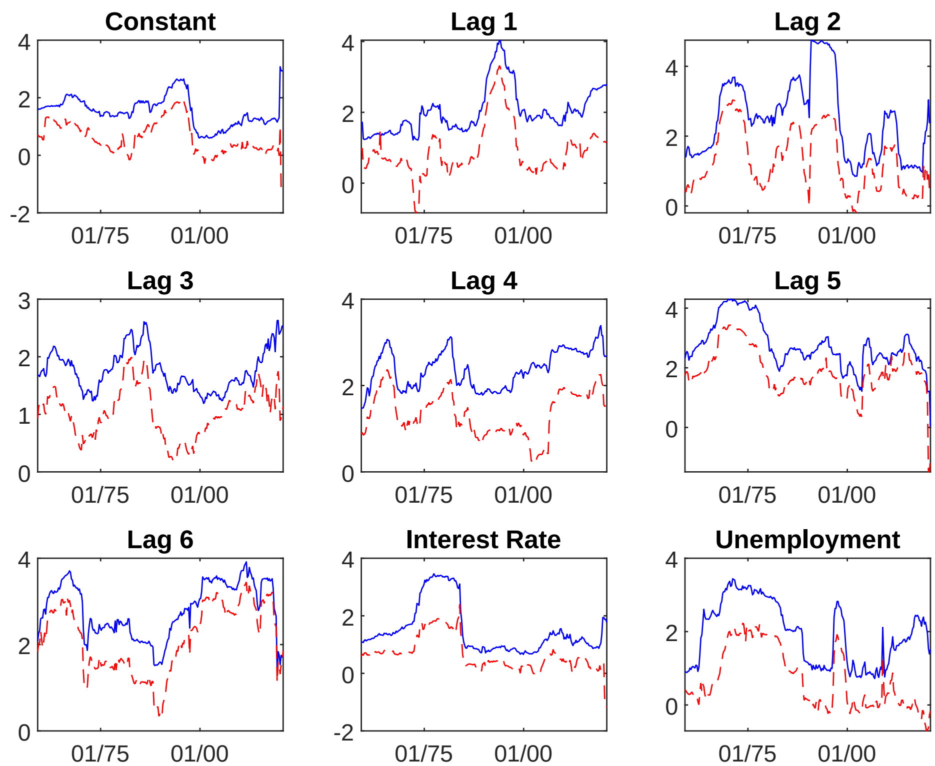
Note: The data source is the FRED database of the U.S. federal reserve bank of St. Louis. The data sample is from Q2 1957 to Q4 2020. The series "inflation rate" is the annualized log quarterly inflation rate based on the U.S. core consumer price index. For the interest rate and unemployment rate, their quarterly values are the average monthly values within each quarter.

Figure 7: Coefficient Estimates by the Gamma Horseshoe Prior: Inflation Rate



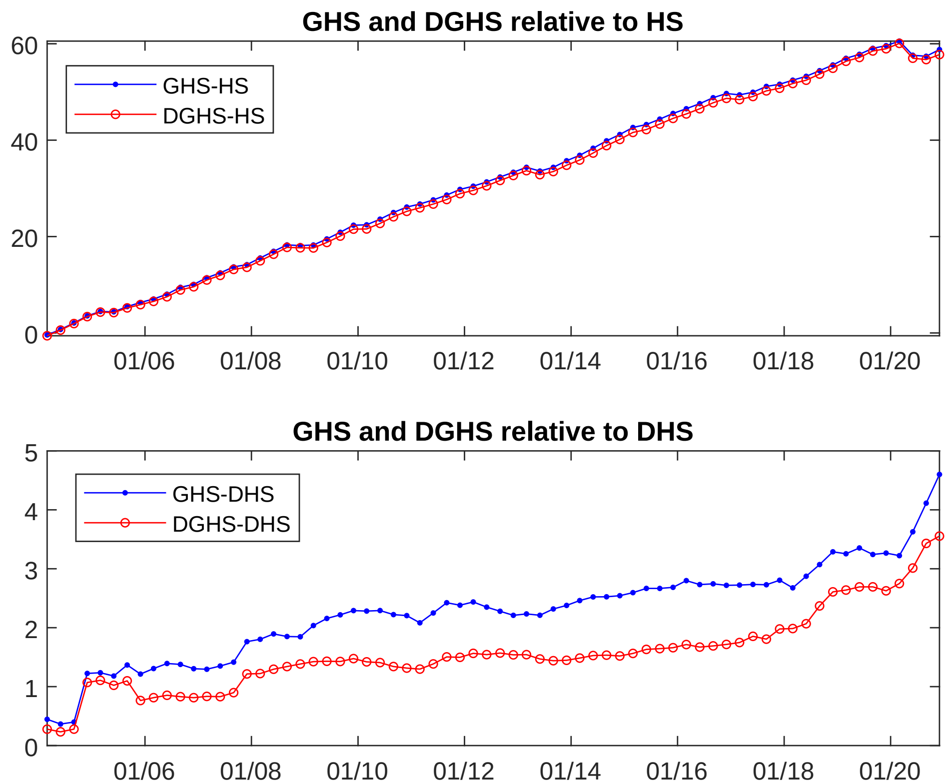
Note: The solid line is the point-wise posterior median of the coefficients with the grey shade being the point-wise 90% credible sets. The dash line is the OLS estimate.

Figure 8: Comparing Sampling Efficiency of Regression Coefficients: Inflation



Note: The figure shows the logarithm of the ratio of the point-wise effective sample size (ESS) of estimated coefficient $\beta_{j,t}$ under the gamma horseshoe (GHS, solid line) and dynamic gamma horseshoe (DGHS, dash line) priors to that under the dynamic horseshoe (DHS) prior for each coefficient $j = 1, ..., 6$. A positive value of the log ESS ratio indicates greater sampling efficiency over the DHS prior.

Figure 9: Difference in Cumulative Log Predictive Likelihoods: Inflation Rate



Note: The line with the point marker (.) is the difference in the cumulative log predictive likelihood of the gamma horseshoe (GHS) prior relative to that of the horseshoe (HS) prior (the upper panel) and the dynamic horseshoe (DHS) prior (lower panel). The line with the circle marker (∘) is the difference in the cumulative log predictive likelihood of the dynamic gamma horseshoe (DGHS) prior relative to that of the HS prior (the upper panel) and the DHS prior (lower panel).