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Invited Paper

Physical representation of universal quantum gates for a trapped ion

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ABSTRACT

We discuss possibilities of quantum manipulations of trapped ions from the perspective of quantum information processing. We show how from a restricted set of feasible microscopic interactions a physical implementation of an arbitrary unitary operator can be realized. An example is given for the discrete Fourier transform. The synthesis of operators enables to realize universal quantum gates to process unknown input qubits which can be encoded into vibrational levels of trapped ions. We discuss also problem of an arbitrary measurement which can be performed using quantum gates.

Keywords: trapped ion, vibrational motion, quantum gates, quantum information processing

1 INTRODUCTION

The deterministic control of dynamics of quantum system is one of the desired goals of the experimental physics in quantum domain. Recent advantages in laser cooling and trapping of ions and atoms which have been originally stimulated by demands for high precision spectroscopy and ultimate frequency standards have led to a realization of such well-controlled quantum systems. Nowadays, a significant reduction of decoherence effects have made these systems also very attractive candidates for implementation of ideas of quantum computation and information processing.[1–4] In 1995 Cirac and Zoller[5] proposed a realization of a quantum computer based on a linear ion trap as a system with a highly coherent control of quantum degrees of freedom. In particular, the motional quantum dynamics of a trapped ion can be controlled by sequences of laser pulses which are tuned to appropriate vibrational sidebands.[4] Therefore we will focus our attention on quantized vibrational motion of an ion confined in harmonic trapping potential.

An important step in quantum information processing represents a proper initialization of quantum systems. For this purpose various quantum state preparation (synthesis) procedures have been proposed also for trapped ions.[6,7] In general, prepared (multiparticle) entangled states can serve to demonstrate the non-local character of quantum theory and to test its fundamentals.[1] However, the key ingredients for quantum information processing with trapped ions are one-qubit and two-qubit quantum logic gates. Such quantum logic elements can be used to build complex quantum networks for specific tasks. Some of the quantum logic elements has been already demonstrated experimentally.

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In this paper we propose a realization of universal quantum gates for a quantized vibrational motion of a trapped ion confined in 1D harmonic trap. Our aim is to represent an arbitrary unitary operator $\hat{V}$ which transforms any unknown vibrational state $|\psi\rangle$ of a trapped ion to $|\psi'\rangle = \hat{V}|\psi\rangle$ as an appropriate sequence of feasible microscopic interactions. It is shown that this approach which is based on a specific decomposition of the desired transformation $\hat{V}$ requires a restriction of the dynamics to a finite dimensional Fock subspace. Otherwise the chosen approach suffers from a "leakage" of probabilities to undesired higher Fock states. We discussed also problem of an arbitrary measurement which can be performed using quantum gates.

2 ENGINEERING OF UNITARY OPERATORS

One of the most important task in quantum information processing is to design quantum gates which take an arbitrary input and process it according to a given prescription. In a rather trivial case it is possible to realize the desired transformation $\hat{V}$ as the time evolution operator associated with a specific Hamiltonian $\hat{H}$, i.e. $\hat{V} = \exp(-i\hat{H}t/\hbar)$. However, only a restricted set of microscopic interactions is available in practice. This fact leads to a necessity to represent a desired transformation via an appropriate sequence of available interactions. In other words, the desired transformation $\hat{V}$ has to be decomposed into a sequence of feasible transformations. An algorithmic proof that any discrete finite-dimensional unitary matrix can be factorized into a sequence of two-dimensional beam splitter transformations was given by Reck et al..[8] In the terminology of quantum computation, any transformation can be represented by a quantum network built of two elementary primitives: one-qubit and two-qubit logic gates.[9] The problem of controlled dynamics of quantum systems has been addressed recently also by Harel and Akulin[11] and by Lloyd and Braunstein.[10] Harel and Akulin[11] have demonstrated a method to attain any desired unitary evolution of the 1-D translational motion of a cold atom by switching on and off alternatively two distinct constant perturbations. To represent arbitrary 1D quantum gates for an ion confined in 2D trap a constructive algorithm has been proposed recently by Hladký et al.[7] which involves a direct transparent encoding of transformation matrices into the "lattice" of two–mode vibrational number states. However, a conditional selection of right outputs via observation of fluorescence signal is required.

To realize quantum gates for trapped ion we utilize the deterministic approach by Harel and Akulin[11] based on the decomposition of the desired transformation into an appropriate sequence of two feasible interactions. An advantage of this treatment is that it is not necessary to find a specific quantum network (i.e. decomposition) for each particular transformation $\hat{V}$. The task is reduced to an appropriate setting of parameters of the fixed sequence of elementary transformation. In particular, we assume that the desired unitary operator $\hat{V}$ can be written as an alternating sequence of switching between two evolution operators $\hat{U}^{(p)} = \exp[-i\hat{H}^{(p)}\tau]$

$$\hat{V} = \hat{U}^{(b)}_{\xi_{M^2}} \hat{U}^{(a)}_{\xi_{M^2-1}} \dots \hat{U}^{(b)}_{\xi_2} \hat{U}^{(a)}_{\xi_1} \tag{1}$$

associated with two feasible interactions described by corresponding Hamiltonians $\hat{H}^{(p)}$ $(p = a, b)$. To synthesize $M \times M$ unitary matrix which represents $\hat{V}$ we need to adjust M^2 independent real interaction parameters $\{\xi_j\}_{j=1}^{M^2}$. Obviously, other decompositions are possible too. To ensure the validity of the decomposition (1) the set of commutators

$$[\hat{H}^{(a)}, [\hat{H}^{(a)}, \dots [\hat{H}^{(a)}, \hat{H}^{(b)}], \quad [\hat{H}^{(b)}, [\hat{H}^{(b)}, \dots [\hat{H}^{(b)}, \hat{H}^{(a)}] \tag{2}$$

of all orders up to M^2 has to form a complete basis for the $M \times M$ unitary matrices as suggested by Harel and Akulin.[10,11] The sequence (1) represents the system of nonlinear equations for M^2 real interaction parameters ξ_j.

Let us consider an implementation of quantum gates for an ion confined in 1D harmonic trapping potential. The motional degrees of freedom of the ion can be well controlled via irradiating the ion by laser beams. The frequency difference of two properly aligned lasers is tuned to the electronic transition or appropriate vibrational sidebands so stimulating a Raman transition. In particular, our choice of two interactions which are experimentally

feasible in laboratory[4,12] reads:

$$\begin{aligned}\hat{H}^{(a)} &= g_a\mathcal{F}(\hat{a}_x^\dagger\hat{a}_x)|g\rangle\langle e| + g_a^\star\mathcal{F}(\hat{a}_x^\dagger\hat{a}_x)|e\rangle\langle g|,\\ \hat{H}^{(b)} &= g_b\hat{a}_x^\dagger\mathcal{F}(\hat{a}_x^\dagger\hat{a}_x)|g\rangle\langle e| + g_b^\star\mathcal{F}(\hat{a}_x^\dagger\hat{a}_x)\hat{a}_x|e\rangle\langle g|.\end{aligned} \tag{3}$$

We assume two internal electronic levels $|g\rangle$ and $|e\rangle$; $\hat{a}_x$ ($\hat{a}_x^\dagger$) denotes the annihilation (creation) operator for the vibrational mode x. The nonvanishing commutators (2) require to operate outside of the Lamb–Dicke regime with the important nonlinear term

$$\mathcal{F}(\hat{a}_x^\dagger\hat{a}_x) = \mathrm{e}^{-\eta_x^2/2}\sum_{k=0}\frac{(-1)^k\eta_x^{2k}}{(k+1)!k!}\hat{a}_x^{\dagger k}\hat{a}_x^k. \tag{4}$$

The Lamb–Dicke parameter for the ion with the mass m_a is defined as $\eta_x = \delta k_x\sqrt{\hbar/2m_a\nu_x}$ where ν_x is the vibrational frequency and δk_x is the difference of wavevectors of applied Raman lasers (with the lasers aligned so that $\delta\vec{k} = (\delta k_x, 0, 0)$; $\delta x_0 = \sqrt{\hbar/2m_a\nu_x}$ means the rms size of the ground vibrational state).

Let us assume that the desired transformation $\hat{V}$ acts only on the Fock subspace $\mathcal{H}_N$ spanned by N number states $|0\rangle, |1\rangle, \ldots, |N-1\rangle$. With two internal electronic levels forming the subspace $\mathcal{H}_i$ we should take the decomposition (1) with $M = 2N$. Note that the decomposition (1) is valid for finite-dimensional systems. However, the subspace $\mathcal{H}_N \otimes \mathcal{H}_i$ is not dynamically invariant with respect to the action of considered interaction Hamiltonians (3). For particular choice of the interactions (3) the decomposition (1) can be treated as an approximation. The task is to minimize the deviation of the decomposition (3) from the desired transformation $\hat{V}$ via an appropriate choice of the independent complex interaction parameters which are given as $\xi_j = |g_p|\mathrm{e}^{-i\varphi_j}\tau_j$ with $j = 1, \ldots, 2N^2$. It means that $2N^2$ real parameters are represented by interaction "lengths" $|g_p|\tau_j$ and the additional set of $2N^2$ real parameters are phases $\{\varphi_j\}$ of the interaction constants g_p of the interaction Hamiltonians $\hat{H}^{(p)}$ (3) which can be properly adjusted. Note that with adjustable complex interaction parameters the decomposition (1) is twice shorter with terms $1, \ldots, \frac{1}{2}M^2$. In the case of laser stimulated Raman transition the interaction constants are given as $g_p \sim \mathcal{E}_a^*\mathcal{E}_b/\Delta$ where $\mathcal{E}_p$ are complex amplitudes of applied laser fields and Δ is detuning of lasers from off-resonant level of the Raman transition. Therefore the phases φ_j can be adjusted as phase differences of the applied laser beams.

As an important example for synthesis of quantum gates for 1D vibrational motion of the trapped ion we consider the quantum Fourier transform (QFT) which enters the Shor's algorithm to factorize numbers on quantum computers. The action of QFT is restricted to $\mathcal{H}_N \otimes \mathcal{H}_i$ being defined as

$$\hat{V}_{QFT} = \frac{1}{\sqrt{M-1}}\sum_{m,n=0}^{M}\exp\left(2\pi i\frac{mn}{M}\right)\,\|m\rangle\rangle\,\langle\langle n\| \tag{5}$$

where $\|2m\rangle\rangle = |m\rangle \otimes |g\rangle$ and $\|2m+1\rangle\rangle = |m\rangle \otimes |e\rangle$ $(m = 0, \ldots, N-1)$. The numerically optimalized realization of the QFT is visualized in Figure 1 which shows the absolute values of the real and imaginary part of the matrix elements of the desired (Figs. 1a,b) and the numerically fitted transformations (Figs. 1c,d). Values $N = 4$ and $\eta_x = 0.4$ have been assumed. It is seen that the realization of $\hat{V}_{QFT}$ is only approximate due to the transfer of probabilities outside of the subspace $\mathcal{H}_N$ to higher number states. Moreover, the transfer of amplitudes which leads even to temporal occupations of higher Fock states invokes further questions about the influence of decoherence effects on higher number states as well as the validity of the Hamiltonians (3) which assume the rotating wave approximation at the vibrational frequency.

From the above given example it is clearly seen that our approach based on the decomposition (1) of the desired transformation $\hat{V}$ requires a restriction of the dynamics to a finite–dimensional Fock subspace to avoid a "leakage" of probabilities to undesired higher Fock states.

To make the system under consideration finite-dimensional it is important to consider the nonlinear dependence of the generalized Rabi frequencies on the Lamb-Dicke parameter outside of the Lamb–Dicke regime. In particular,

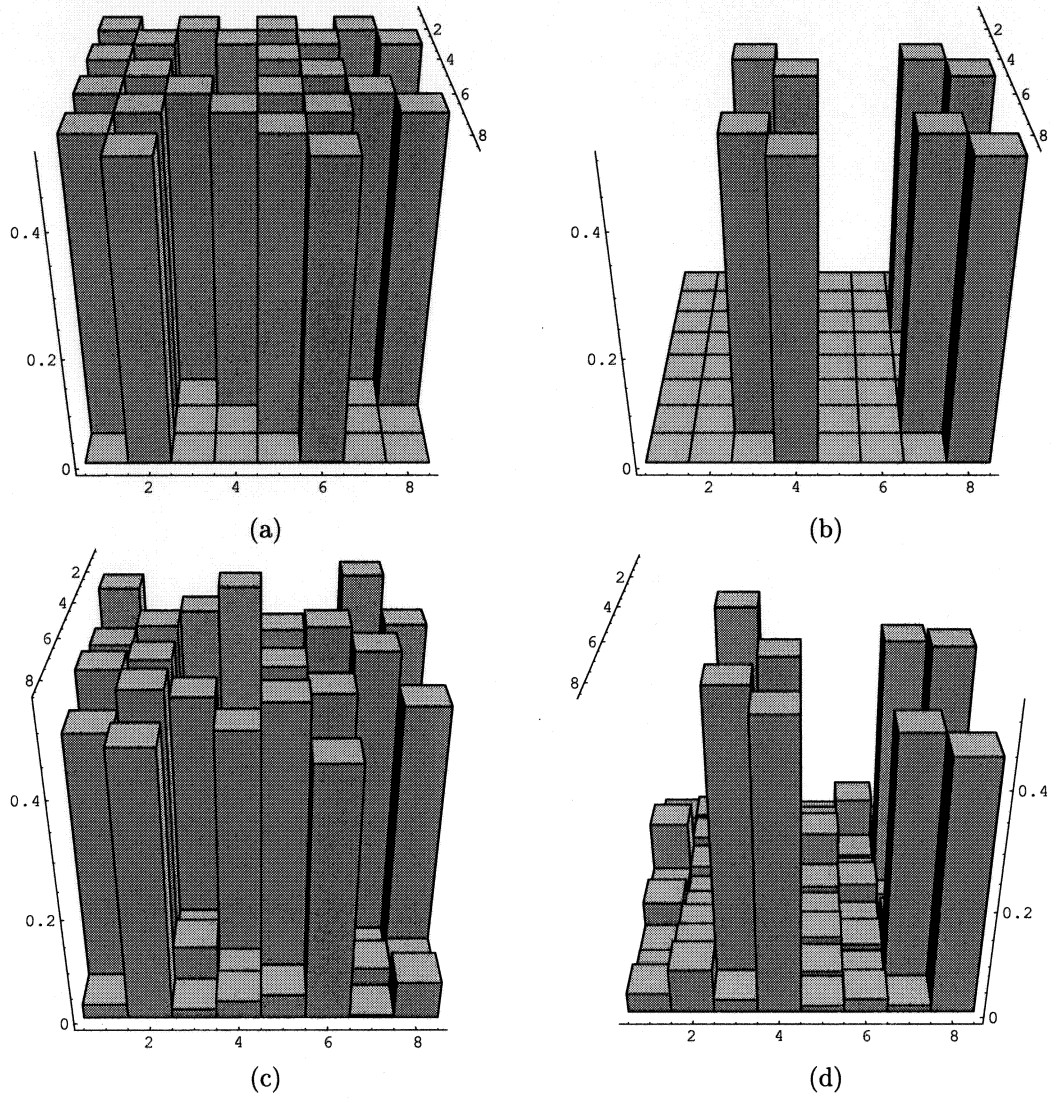

Figure 1: The absolute values of the real (a) and imaginary parts (b) of the matrix elements of the ideal $\hat{V}_{QFT}$. The corresponding values of the numerically fitted decomposition (1) are shown in (c) and (d), respectively. Matrix elements are labeled in the basis $\|j\rangle\rangle$ where $\|2m\rangle\rangle = |m\rangle \otimes |g\rangle$ and $\|2m+1\rangle\rangle = |m\rangle \otimes |e\rangle$ $(m = 0, \ldots, N-1)$.

the transition amplitude between states $|m\rangle \otimes |e\rangle \leftrightarrow |m+1\rangle|g\rangle$ is given as

$$\Omega_{m,m+1} \equiv \langle m+1; g|\hat{H}^{(b)}|m; e\rangle = g_b \mathrm{e}^{-\eta_x^2/2} L_m^1(\eta_x^2)/\sqrt{m+1} \tag{6}$$

where $L_m^1(x)$ denotes the m-th generalized Laguerre polynomial. The Lamb-Dicke parameters can be controlled by varying the geometry of the laser beams.[13] Therefore it is possible to choose η_x such that $L_{N-1}^1(\eta_x) = 0$, i.e., the transition amplitude $\Omega_{N-1,N} = 0$. However, for $N = 4$ this natural "truncation" of the Fock space requires $\eta_x > 1$ which is not desirable because it leads to appearance of additional effects (such as micromotion, recoil heating, and even trap instabilities). On the other hand, the "truncation" at higher N decreases the required η_x.

To operate at practical values of the Lamb–Dicke parameter ($\eta_x < 0.2$) an arrangement with two pairs of

Raman lasers seems to be more appropriate. In particular, two pairs of Raman lasers can stimulate processes described by the effective interaction Hamiltonians $\hat{H}^{(b1)}$, $\hat{H}^{(b2)}$ [see Eq.(2)] with different interaction constants g_{b1}, g_{b2} and Lamb–Dicke parameters η_{x1}, η_{x2}. The resulting Hamiltonian $\hat{H}^{(b)} = \hat{H}^{(b1)} + \hat{H}^{(b2)}$ can be designed to form a finite-dimensional subspace if mutual phases of interaction constants are shifted by π. When the condition

$$|g_{b1}|\mathrm{e}^{-\eta_{x1}^2/2} L_{N-1}^1(\eta_{x1}) - |g_{b2}|\mathrm{e}^{-\eta_{x2}^2/2} L_{N-1}^1(\eta_{x2}) = 0$$

is fulfilled then the Rabi frequency for transition $|N-1\rangle \otimes |e\rangle \leftrightarrow |N\rangle|g\rangle$ equals to zero. Here the Lamb-Dicke parameters are adjusted by appropriate geometry of the laser beams and absolute values of the interaction constants are controlled via intensities of the applied laser beams.

Alternatively, to avoid even temporal excitations of the number states outside of the desired finite-dimensional subspace we can operate on an extended system. Besides of the vibrational mode x we can utilize also vibrational mode y. In particular, we consider subspace with a fixed total number of vibrational quanta $N-1$. As one of the basic interactions we can again use $\hat{H}^{(a)}$ but instead of the one–mode Hamiltonian $\hat{H}^{(b)}$ in (3) we use the effective two–mode interaction Hamiltonian[14] which has been used in an algorithm for synthesis of 2D vibrational states[7]:

$$\hat{H}'^{(b)} = g_b' \hat{a}_x^\dagger \mathcal{F}(\hat{a}_x^\dagger \hat{a}_x)\mathcal{F}(\hat{a}_y^\dagger \hat{a}_y)\hat{a}_y |g\rangle\langle e| + g_b'^{\star} \hat{a}_y^\dagger \mathcal{F}(\hat{a}_y^\dagger \hat{a}_y)\mathcal{F}(\hat{a}_x^\dagger \hat{a}_x)\hat{a}_x |e\rangle\langle g|. \tag{7}$$

Both Hamiltonians $\hat{H}^{(a)}$ and $\hat{H}'^{(b)}$ commute with the operator of the total number of vibrational quanta $\hat{N} = \hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y$. It means that by the sequence of switchings between these two interaction channels we can represent arbitrary unitary quantum gates (operator) which act on the finite-dimensional subspace spanned by N two–mode number states $\{|0, N-1\rangle, \ldots |N-1, 0\rangle\}$. This approach completely avoids temporal occupations of the states outside of the original Hilbert subspace.

2.1 Engineering of arbitrary quantum measurement

In this part we briefly discuss a possibility to design an arbitrary measurement using quantum gates. Let assume that we want to perform a quantum-mechanical measurement associated with an arbitrary Hermitian operator $\hat{B}$. As output of the measurement performed on an ensemble of identically prepared system which represent an unknown quantum state $|\psi\rangle$ eigenvalues b_μ of the eigenstates $|b_\mu\rangle$ of $\hat{B}$ with probabilities $p_\mu = |\langle b_\mu|\psi\rangle|^2$ are obtained. However, it can be impossible to design an apparatus associated with measurement of arbitrary observable $\hat{B}$. On the other hand, we can include quantum gates in a measurement scheme based on an available (feasible) apparatus associated with the observable $\hat{A}$. As shown in Figure 2, quantum gates can perform an appropriate mapping of the eigenstates of $\hat{B}$ on the eigenstates of $\hat{A}$. The required transformation takes the form $\hat{V} = \sum_\mu |a_\mu\rangle\langle b_\mu|$. The output eigenvalues a_μ obtained with probabilities p_μ thus correspond uniquely to b_μ. The required mapping $\hat{V}$ can be physically represented via a sequence of feasible interactions [see Eq.(1)]. In our case of a trapped ion it is represented via a sequence of laser pulses.

3 CONCLUSIONS

We have presented synthesis of unitary operators for a trapped ion based on a sequential switching of two different interactions. The approach is applicable outside of the Lamb–Dicke regime with the number of elementary operations which scales as N^2. To avoid temporal occupations of the states outside of the "target" Fock subspace an appropriate setup with pairs of Raman lasers have to be used to adjust Lamb–Dicke parameters. One of important applications of the operator synthesis is a realization of universal quantum gates for qubits which are encoded in vibrational levels. As an example we have considered realization of the discrete Fourier transform.

A simplified task comparing to synthesis of quantum gates is preparation of quantum state. Especially entangled (correlated) states are important for quantum information processing and quantum communication.

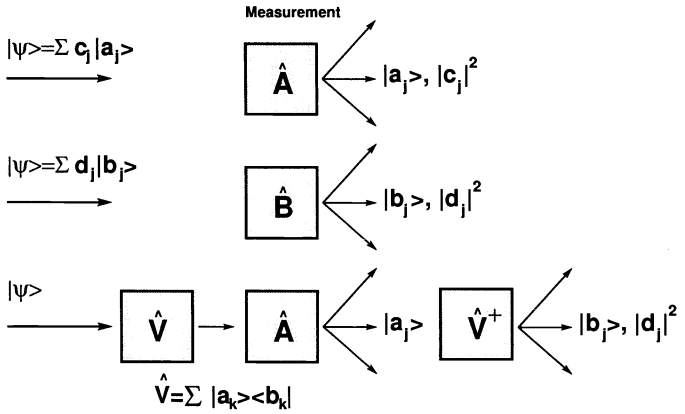

Figure 2: A measurement associated with a desired observable $\hat{B}$ can be transformed on a feasible one associated with $\hat{A}$ using quantum gates which perform an appropriate mapping of the eigenstates of the involved observables.

The decomposition of type (1) can be used also to create a desired entangled quantum state $|\psi_{out}\rangle$ from a given initial state $|\psi_{in}\rangle$, i.e. $|\psi_{out}\rangle = \hat{V}|\psi_{in}\rangle$. Moreover, the length of the sequence (1) is dramatically reduces as just M terms with M independent complex parameters are required to create the target state $|\psi_{out}\rangle$.

The presented decomposition procedure is not unique and further optimalization is possible. Another problem which deserves attention is the stability of the algorithm with respect to noise.

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