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Multi-qubit non-adiabatic holonomic controlled quantum gates in decoherence-free subspaces

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Abstract Non-adiabatic holonomic quantum gate in decoherence-free subspaces is of greatly practical importance due to its built-in fault tolerance, coherence stabilization virtues, and short run-time. Here, we propose some compact schemes to implement two- and three-qubit controlled unitary quantum gates and Fredkin gate. For the controlled unitary quantum gates, the unitary operator acting on the target qubit is an arbitrary single-qubit gate operation. The controlled quantum gates can be directly implemented by utilizing non-adiabatic holonomy in decoherence-free subspaces and the required resource for the decoherence-free subspace encoding is minimal by using only two neighboring physical qubits undergoing collective dephasing to encode a logical qubit.

Keywords Multi-qubit controlled gate · Quantum holonomy · Decoherence-free subspace

1 Introduction

Based on the quantum parallelism, quantum computation is believed to speed up the solution of a number of mathematical tasks and has attracted more and more

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interests. The key step to implement effective quantum computation is the construction of robust quantum gates. Geometric phase [1], which only depends on global geometric properties of the evolution paths, is robust against certain types of errors in the control process and has been used to realize fault-tolerant quantum gates [2,3]. Holonomic quantum computation (HQC) first proposed by Zanardi and Rasetti [4] based on non-Abelian extension of geometric phase, which is also called quantum holonomy [5], is also a promising way to implement universal sets of robust gates [6–9]. Unfortunately, however, the long run-time requirement for the desired parametric control associated with adiabatic evolution makes the quantum gates become vulnerable to open system effects and parameter fluctuations that may lead to loss of coherence. In order to remove the problem of long run-time requirement associated with the adiabatic evolution, Wang and Keiji [10] designed a scheme for the non-adiabatic conditional geometric phase shift gate with nuclear magnetic resonance (NMR), and Zhu and Wang [11] achieved universal quantum gates based on non-adiabatic geometric phase. For the non-Abelian case, Sjöqvist et al. [12] developed a non-adiabatic generalization of HQC recently, in which high-speed universal quantum gates can be implemented with non-adiabatic non-Abelian geometric phase [13]. Non-adiabatic HQC has been also experimentally demonstrated in different physical systems, such as the three-level transmon qubit [14], nuclear magnetic resonance (NMR) quantum information processor [15], and diamond nitrogen-vacancy centers [16,17].

Besides errors from the control of the quantum system, decoherence, arisen from the inevitable interaction between the quantum system and environment, is an another main challenge in implementing robust quantum gates. Decoherence will destruct the desired coherence of the system, so it is harmful for effective quantum computation. One of the promising strategies to avoid decoherence is decoherence-free subspaces (DFSs) which utilize the symmetry structure of the system-environment interaction [18]. The basic idea of DFS is that information encoded in it still undergoes unitary evolution even taking the decoherence caused by environment into account. In addition, DFSs have been experimentally demonstrated in a host of physical systems [19–23].

Many efforts have been devoted to combining the fault tolerance of HQC and the quantum coherence stabilization virtues of DFSs [24–27]. In 2005, Wu et al. [26] implemented HQC in DFSs which was robust against some stochastic errors and collective dephasing. However, the long run-time requirement associated with the adiabatical control of the parameters and by using of four neighboring physical qubits undergoing collective dephasing to encode a logical qubit are great challenges in experiment. After that, Xu et al. [28] developed a non-adiabatic generalization of HQC in DFSs which can overcome the long run-time requirement of its adiabatic counterpart. Later, some other schemes for non-adiabatic HQC in DFSs in different physical systems have been also proposed [29–31]. However, all the schemes above focused only on one- and two-qubit gates. As we all know, it is too complex to implement most algorithms with the increase of the number of qubits if only one- and two-qubit gates are available. The direct implementation of multi-qubit gates, which is generally believed to provide a simpler design, a faster operation, and a lower decoherence, is thus of greatly practical importance.

In this paper, inspired by above works, we propose some compact schemes to implement non-adiabatic holonomic two- and three-qubit controlled unitary quantum gates and Fredkin gate in DFSs. Here, the unitary operator acting on the target qubit in controlled unitary quantum gates is an arbitrary single-qubit gate operation by varying the parameters independently. These controlled quantum gates can be directly implemented, which avoids the extra work of combining two gates into one. Furthermore, they are robust against certain types of errors in the control process and the decoherence caused by environment and can be implemented at high speed. This is the first scheme for implementing three-qubit controlled quantum gates by utilizing non-adiabatic holonomy in DFSs. Moreover, an attractive feature of our schemes is that the resources cost for the DFSs encoding is minimal by using only two neighboring physical qubits to encode a logical qubit.

2 Quantum holonomy and physical model

We now briefly show how quantum holonomy can arise in non-adiabatic unitary evolution before introducing our physical model. Consider a quantum system described by an N-dimensional state space and governed by Hamiltonian $H(t)$. Assume that there is a time-dependent M-dimensional subspace $S(t)$ spanned by the orthonormal basis vectors $\{|\psi_m(t)\rangle\}_{m=1}^{M}$. The evolution operator $\mathcal{U}(\tau, 0)$ is a holonomic matrix acting on $S(0)$ spanned by $\{|\psi_m(0)\rangle\}_{m=1}^{M}$ if $|\psi_m(t)\rangle$ satisfy the following conditions [28]:

$$\text{(i)} \sum_{m=1}^{M} |\psi_m(\tau)\rangle\langle\psi_m(\tau)| = \sum_{m=1}^{M} |\psi_m(0)\rangle\langle\psi_m(0)|, \tag{1}$$

$$\text{(ii)} \ \langle\psi_m(t)|H(t)|\psi_l(t)\rangle = 0, \quad m, l = 1, 2, ..., M, \tag{2}$$

where τ is the evolution period, $|\psi_m(t)\rangle = \mathcal{U}(t, 0)|\psi_m(0)\rangle = \mathbf{T}\exp(-i\int_0^t H(t')dt')$ $|\psi_m(0)\rangle$, $\mathbf{T}$ is time ordering. Here, condition (i) ensures that the evolution of subspace $S(0)$ is cyclic, while condition (ii) means that the evolution is purely geometric.

In order to combine the fault tolerance of HQC and the quantum coherence stabilization virtues of DFSs, we consider the following physical model. The quantum system consists of N physical qubits interacting collectively with a dephasing environment. The interaction between the quantum system and its environment is described by the interaction Hamiltonian

$$H_I = \left(\sum_{k=1}^{N} Z_k\right) \otimes B, \tag{3}$$

where Z_k is the Pauli Z operator for the kth physical qubit and B is an arbitrary environment operator. Due to the symmetry of the interaction, we can find a DFS to protect quantum information against decoherence. For the simplest case, i.e., the number of physical qubits is two, there exists a DFS:

$$S^D = \text{Span}\{|01\rangle, |10\rangle\}. \tag{4}$$

We can use this subspace to encode a logical qubit, i.e., $|0\rangle_L = |01\rangle$, $|1\rangle_L = |10\rangle$, and the subscript L denotes the logical states. Obviously, the resources cost for the DFS encoding is minimal by using only two neighboring physical qubits, which undergo collective dephasing to encode a logical qubit. In the following, we will use this encoding to implement controlled quantum gates.

3 Two-qubit controlled unitary gate

In this section, we demonstrate how to implement a non-adiabatic holonomic two-qubit controlled unitary gate, denoted as C_1-U gate, in DFS. Here, U is an arbitrary single-qubit unitary gate operation acting on the target qubit, whose matrix form is given by

$$U = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}. \tag{5}$$

To this end, we consider four physical qubits interacting collectively with the dephasing environment and there exists a six-dimensional DFS:

$$S^{D_1} = \text{Span}\Big\{|0101\rangle, |0110\rangle, |1001\rangle, |1010\rangle, |0011\rangle\ |1100\rangle\Big\}. \tag{6}$$

We encode logical qubits in the subspace

$$S^{L_1} = \text{Span}\Big\{|0101\rangle, |0110\rangle, |1001\rangle, |1010\rangle\Big\}, \tag{7}$$

where the logical qubit states are denoted as $|0\rangle_L|0\rangle_L = |0101\rangle$, $|0\rangle_L|1\rangle_L = |0110\rangle$, $|1\rangle_L|0\rangle_L = |1001\rangle$, and $|1\rangle_L|1\rangle_L = |1010\rangle$. S^{L_1} is a subspace of S^{D_1}, and the remaining vectors $|0011\rangle$ and $|1100\rangle$ are used as ancillary states, denoted as $|a_1\rangle = |0011\rangle$ and $|a_2\rangle = |1100\rangle$ for convenience. Under the basis $\{|0\rangle_L|0\rangle_L, |0\rangle_L|1\rangle_L, |1\rangle_L|0\rangle_L, |1\rangle_L|1\rangle_L\}$, the C_1-U gate is written as [32]

$$C_1 - U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{pmatrix}. \tag{8}$$

In order to implement C_1-U gate, we consider the following Hamiltonian

$$\begin{aligned} H_1 = \frac{1}{2}\Big\{&(I_2 + Z_2)\Big[\Delta_1(I_1 + Z_1) + (\Omega_1 R^x_{13} + \Omega_2 R^x_{14} + \text{H.c.})\Big] \\ &+(I_1 - Z_1)\Big[\Delta_2(I_2 - Z_2) + (\Omega_3 R^x_{23} + \Omega_4 R^x_{24} + \text{H.c.})\Big]\Big\}, \end{aligned} \tag{9}$$

where $R^x_{lm} = \frac{1}{4}(X_l - iY_l)(X_m + iY_m)$, I is the one-qubit identity matrix, X, Y, and Z are Pauli matrices acting on corresponding physical qubit, H.c. means Hermitian conjugate. The basic element of Hamiltonian H_1, R^x_{lm} + H.c., represents the XY interaction term. It can be found in a variety of quantum systems, e.g., quantum dots [33], atoms in cavities [34], and trapped ions [35]. Δ_i and Ω_i are controllable coupling parameters, with

$$\begin{aligned}
\Delta_1 &= -\Omega\sin\xi, & \Delta_2 &= -\Omega\sin\gamma,\\
\Omega_1 &= \Omega\cos\xi\cos\frac{\alpha}{2}, & \Omega_3 &= -\Omega\cos\gamma\cos\frac{\alpha}{2},\\
\Omega_2 &= \Omega e^{i\beta}\cos\xi\sin\frac{\alpha}{2}, & \Omega_4 &= \Omega e^{i\beta}\cos\gamma\sin\frac{\alpha}{2}.
\end{aligned}\tag{10}$$

The Hamiltonian H_1 can be rewritten as

$$\begin{aligned}
H_1' = &-2\Omega\big(\sin\xi|a_1\rangle\langle a_1| + \sin\gamma|a_2\rangle\langle a_2|\big)\\
&+\Omega\big(\cos\xi|1\rangle_L|+\rangle_L\langle a_1| + \cos\gamma|1\rangle_L|-\rangle_L\langle a_2| + \text{H.c.}\big),
\end{aligned}\tag{11}$$

where we have used two orthogonal states $|+\rangle_L = \cos\frac{\alpha}{2}|0\rangle_L + e^{i\beta}\sin\frac{\alpha}{2}|1\rangle_L$ and $|-\rangle_L = e^{-i\beta}\sin\frac{\alpha}{2}|0\rangle_L - \cos\frac{\alpha}{2}|1\rangle_L$. The subspace spanned by $\{|+\rangle_L, |-\rangle_L\}$ is the same as that by $\{|0\rangle_L, |1\rangle_L\}$. The evolution operator associated with H_1 is $\mathcal{U}_1(t) = e^{-iH_1t}$. With the choice of $\Omega\tau_1 = \pi$, the resulting evolution operator is given by

$$\mathcal{U}_1(\tau_1) = \begin{pmatrix}
e^{i(\delta-\frac{\theta}{2})} & 0 & 0 & 0 & 0 & 0\\
0 & e^{i(\delta+\frac{\theta}{2})} & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & e^{i(\delta-\frac{\theta}{2})} & 0\\
0 & 0 & 0 & 0 & 0 & e^{i(\delta+\frac{\theta}{2})}
\end{pmatrix},\tag{12}$$

in the basis $\{|a_1\rangle, |a_2\rangle, |0\rangle_L|+\rangle_L, |0\rangle_L|-\rangle_L, |1\rangle_L|+\rangle_L, |1\rangle_L|-\rangle_L\}$, where $\delta - \theta/2 = \pi + \pi\sin\xi$ and $\delta + \theta/2 = \pi + \pi\sin\gamma$. Since the parameters ξ and γ are mutually independent, we can vary the parameters δ and θ independently.

Therefore, for the states in the logical subspace S^{L_1}, the action of the evolution operator $\mathcal{U}_1(\tau_1)$ is equivalent to C_1-U gate and the single-qubit unitary gate operation U is written as

$$U = e^{i(\delta-\frac{\theta}{2})|+\rangle_L\langle+| + i(\delta+\frac{\theta}{2})|-\rangle_L\langle-|}.\tag{13}$$

Under the basis $\{|0\rangle_L, |1\rangle_L\}$, defining the Pauli operators as $\sigma_x = |0\rangle_L\langle 1| + |1\rangle_L\langle 0|$, $\sigma_y = -i|0\rangle_L\langle 1| + i|1\rangle_L\langle 0|$, and $\sigma_z = |0\rangle_L\langle 0| - |1\rangle_L\langle 1|$, then U can be rewritten as

$$U = \exp(i\delta)R_{\hat{n}}(\theta), \quad R_{\hat{n}}(\theta) = \exp\left(-i\frac{\theta}{2}\hat{n}\cdot\boldsymbol{\sigma}\right),\tag{14}$$

with $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and the unit vector $\hat{n} = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$. In the above, $R_{\hat{n}}(\theta)$ represents a single-qubit rotation around the direction $\hat{n}$ with angle θ. Thus U corresponds to an arbitrary single-qubit gate operation by varying the parameters δ, θ, α, and β independently [36]. In particular, when setting $\delta = \theta/2 = \alpha = \pi/2$ ($\xi = \pi$, $\gamma = 0$, $\alpha = \pi/2$) and $\beta = 0$, we can implement a two-qubit controlled-NOT (CNOT) gate.

Since S^{D_1} is an invariant subspace of the evolution operator, $\mathcal{U}_1(\tau_1)$ has decoherence-free property. Next, we use conditions (i) and (ii) to check that $\mathcal{U}_1(\tau_1)$ is a holonomic matrix acting on S^{L_1}. For condition (i), the subspace spanned by $\{\mathcal{U}_1(\tau_1)|0\rangle_L|0\rangle_L, \mathcal{U}_1(\tau_1)|0\rangle_L|1\rangle_L, \mathcal{U}_1(\tau_1)|1\rangle_L|0\rangle_L, \mathcal{U}_1(\tau_1)|1\rangle_L|1\rangle_L\}$ coincides with S^{L_1}, it is satisfied. While for condition (ii), considering that $\mathcal{U}_1(t)$ commutes with H_1, condition (ii) reduces to $\langle k|H_1|k^{'}\rangle = 0$, where $|k\rangle, |k^{'}\rangle \in \{|0\rangle_L|0\rangle_L, |0\rangle_L|1\rangle_L, |1\rangle_L|0\rangle_L, |1\rangle_L|1\rangle_L\}$. From Eq. (11), it is easy to find that condition (ii) is also satisfied. Therefore, $\mathcal{U}_1(\tau_1)$ is a holonomic matrix acting on S^{L_1} with decoherence-free property.

Through the above illustration, a non-adiabatic holonomic C_1-U gate in which U is an arbitrary single-qubit gate operation in DFS with two- and three-body interactions have been directly and successfully implemented. It is worth pointing out that one needs four-body interaction [28] or the combination of a single-qubit gate and a two-qubit nontrivial gate [30] to implement a non-adiabatic holonomic CNOT gate in DFS.

4 Three-qubit controlled gate

It is well known that by using two CNOT gates, two C_1-V gates ($V^2 = U$), and a C_1-$V^\dagger$ gate, one can get a three-qubit controlled unitary gate with two control qubits and a unitary operator U acting on a target qubit, which is denoted as C_2-U gate [32]. Obviously, this combination is very complex and it is more desirable to implement C_2-U gate directly. In this section we will show how to implement the C_2-U gate directly in DFS. To this end, we need six physical qubits interacting collectively with the dephasing environment to construct a ten-dimensional DFS:

$$S^{D_2} = \mathrm{Span}\Big\{|010101\rangle, |010110\rangle, |011001\rangle, |011010\rangle, |100101\rangle,$$
$$|100110\rangle, |101001\rangle, |101010\rangle, |100011\rangle, |101100\rangle\Big\}. \qquad (15)$$

Similar to the case of C_1-U gate, we encode logical qubits in the subspace

$$S^{L_2} = \mathrm{Span}\Big\{|010101\rangle, |010110\rangle, |011001\rangle, |011010\rangle,$$
$$|100101\rangle, |100110\rangle, |101001\rangle, |101010\rangle\Big\}, \qquad (16)$$

and the logical qubit states are denoted as

$$\begin{aligned} |0\rangle_L|0\rangle_L|0\rangle_L &= |010101\rangle, & |0\rangle_L|0\rangle_L|1\rangle_L &= |010110\rangle, \\ |0\rangle_L|1\rangle_L|0\rangle_L &= |011001\rangle, & |0\rangle_L|1\rangle_L|1\rangle_L &= |011010\rangle, \\ |1\rangle_L|0\rangle_L|0\rangle_L &= |100101\rangle, & |1\rangle_L|0\rangle_L|1\rangle_L &= |100110\rangle, \\ |1\rangle_L|1\rangle_L|0\rangle_L &= |101001\rangle, & |1\rangle_L|1\rangle_L|1\rangle_L &= |101010\rangle. \end{aligned} \tag{17}$$

In the case of three-qubit C_2-U gate, we also use only two neighboring physical qubits to encode a logical qubit and $|a_3\rangle = |100011\rangle$ and $|a_4\rangle = |101100\rangle$ are as ancillary states. The Hamiltonian H_2 for implementing the C_2-U gate is

$$\begin{aligned} H_2 = &\frac{1}{4}\Big\{(I_1 - Z_1)(I_4 + Z_4)\Big[\Delta_1(I_3 + Z_3) + (\Omega_1 R^x_{35} + \Omega_2 R^x_{36} + \text{H.c.})\Big] \\ &+(I_1 - Z_1)(I_3 - Z_3)\Big[\Delta_2(I_4 - Z_4) + (\Omega_3 R^x_{45} + \Omega_4 R^x_{46} + \text{H.c.})\Big]\Big\} \\ = &\Big[2\Delta_1|a_3\rangle\langle a_3| + (\Omega_1|1\rangle_L|1\rangle_L|0\rangle_L\langle a_3| + \Omega_2|1\rangle_L|1\rangle_L|1\rangle_L\langle a_3| + \text{H.c.}) \\ &+2\Delta_2|a_4\rangle\langle a_4| + (\Omega_3|a_4\rangle_L\langle 1|_L\langle 1|_L\langle 1| + \Omega_4|a_4\rangle_L\langle 1|_L\langle 1|_L\langle 0| + \text{H.c.})\Big], \end{aligned} \tag{18}$$

where the controllable coupling parameters are chosen the same as in the case of C_1-U gate [see Eq. (10)]. In this way the Hamiltonian in Eq. (18) can be rewritten as

$$\begin{aligned} H_2' = &-2\Omega(\sin\xi|a_3\rangle\langle a_3| + \sin\gamma|a_4\rangle\langle a_4|) \\ &+\Omega(\cos\xi|1\rangle_L|1\rangle_L|+\rangle_L\langle a_3| + \cos\gamma|1\rangle_L|1\rangle_L|-\rangle_L\langle a_4| + \text{H.c.}). \end{aligned} \tag{19}$$

The Hamiltonian H_2' has the same structure as H_1' and the states $|+\rangle_L$ and $|-\rangle_L$ are the same as that in Eq. (11). Similar to the case of C_1-U gate, it is easy to get the evolution operator associated with H_2 under the basis $\{|0\rangle_L|0\rangle_L|0\rangle_L, |0\rangle_L|0\rangle_L|1\rangle_L, |0\rangle_L|1\rangle_L|0\rangle_L, |0\rangle_L|1\rangle_L|1\rangle_L, |1\rangle_L|0\rangle_L|0\rangle_L, |1\rangle_L|0\rangle_L|1\rangle_L, |1\rangle_L|1\rangle_L|0\rangle_L, |1\rangle_L|1\rangle_L|1\rangle_L\}$

$$\mathcal{U}_2(\tau_2) = \text{Diag}\,[1, 1, 1, 1, 1, 1, U]\,, \tag{20}$$

with evolution time satisfying $\Omega\tau_2 = \pi$. From Eq. (20), one can easily find that $\mathcal{U}_2(\tau_2)$ acts as a C_2-U gate on the states of S^{L_2} and U is given by Eq. (5). A Toffoli gate, which can perform a NOT operation on the target qubit or not, depending on the states of two control qubits [37], is an important C_2-U gate. It is widely used in phase estimation [36], complex quantum algorithms [38], quantum error correction [39], and fault-tolerant quantum circuits [40]. One can get a Toffoli gate by using at least six CNOT gates in principle [41]. Here, the Toffoli gate can be directly implemented by utilizing the same parameters in the case of implementing CNOT gate. The decoherence-free and holonomy properties of the gate can now easily be verified.

Since the verification exactly parallels the one for the case of C_1-U gate discussed in the last section and we do not present here.

Now we turn to the implementation of a Fredkin gate, which is another important three-qubit controlled gate that can perform a swap operation on two target qubits or not, depending on the state of the control qubit. In order to achieve the Fredkin gate, we consider the following Hamiltonian

$$\begin{aligned} H_3 &= \frac{1}{2\sqrt{2}}\eta(I_1 - Z_1)(R^x_{35} - R^x_{46} + \text{H.c.}) \\ &= \eta\frac{1}{\sqrt{2}}(|1\rangle_L|1\rangle_L|0\rangle_L\langle a_3| + |1\rangle_L|0\rangle_L|1\rangle_L\langle a_4| \\ &\quad -|1\rangle_L|0\rangle_L|1\rangle_L\langle a_3| - |1\rangle_L|1\rangle_L|0\rangle_L\langle a_4| + \text{H.c.}) \\ &= \eta(|1\rangle_L|1\rangle_L|0\rangle_L - |1\rangle_L|0\rangle_L|1\rangle_L)\langle a_-| + \text{H.c.}, \end{aligned} \tag{21}$$

where η is a controllable coupling parameter and $|a_-\rangle = \frac{1}{\sqrt{2}}(|a_3\rangle - |a_4\rangle)$. Here the encoding is as the same as the situation in C_2-U gate [see Eq. (18)]. The Hamiltonian H_3 is in the Λ-type with ancillary state $|a_-\rangle$ at the top, while the logical qubit states $|1\rangle_L|1\rangle_L|0\rangle_L$ and $|1\rangle_L|0\rangle_L|1\rangle_L$ at the bottom. The state orthogonal to $|a_-\rangle$ is denoted as $|a_+\rangle = \frac{1}{\sqrt{2}}(|a_3\rangle + |a_4\rangle)$, and it decouples from the evolution of the system. The subspace spanned by $\{|a_+\rangle, |a_-\rangle\}$ is the same to that by $\{|a_3\rangle, |a_4\rangle\}$. When the evolution time τ_3 meets $\eta\tau_3 = \pi/\sqrt{2}$, the resulting evolution operator in the basis $\{|0\rangle_L|0\rangle_L|0\rangle_L$, $|0\rangle_L|0\rangle_L|1\rangle_L$, $|0\rangle_L|1\rangle_L|0\rangle_L$, $|0\rangle_L|1\rangle_L|1\rangle_L$, $|1\rangle_L|0\rangle_L|0\rangle_L$, $|1\rangle_L|0\rangle_L|1\rangle_L$, $|1\rangle_L|1\rangle_L|0\rangle_L$, $|1\rangle_L|1\rangle_L|1\rangle_L\}$ is given by

$$\mathcal{U}_3(\tau_3) = \text{Diag}\left[1, 1, 1, 1, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]. \tag{22}$$

One can find from Eq. (22) that $\mathcal{U}_3(\tau_3)$ acts as a Fredkin gate on the states in the logic subspace S^{L_2} and its decoherence-free and holonomy properties can be demonstrated easily. In this way we implement a non-adiabatic holonomic three-qubit Fredkin gate in DFS with three-body interaction.

5 Discussion and conclusion

So far, we have succeeded in constructing C_1-U, C_2-U, and Fredkin gates. We now introduce a few concepts from differential geometry to understand the nature of the above holonomic gates. The set of K-dimensional subspaces of an N-dimensional Hilbert space is a Grassmann manifold $G(N; K)$. The closed path $\mathcal{C}$ of K-dimensional subspaces is a loop in $G(N; K)$. We now consider the holonomic gates described above. The C_1-U, C_2-U, and Fredkin gates are associated with loops in $G(4;2)$ [42], where the Hilbert spaces relevant for the holonomy are spanned by $\{|a_1\rangle, |a_2\rangle, |1\rangle_L|0\rangle_L, |1\rangle_L|1\rangle_L\}$, $\{|a_3\rangle, |a_4\rangle, |1\rangle_L|1\rangle_L|0\rangle_L, |1\rangle_L|1\rangle_L|1\rangle_L\}$, and $\{|a_3\rangle, |a_4\rangle, |1\rangle_L|0\rangle_L|1\rangle_L, |1\rangle_L|1\rangle_L|0\rangle_L\}$, respectively. However, the previous schemes were almost associated with loops in $G(3;2)$ [12]. It is worth noting that the schemes proposed here can be generalized. For the C_1-U gate between the mth and the nth logic qubits, the Hamiltonian has the same structure as H_1 but with the exchanging $R^x_{13} \rightarrow R^x_{2m-1,2n-1}$, $R^x_{14} \rightarrow R^x_{2m-1,2n}$, $R^x_{23} \rightarrow R^x_{2m,2n-1}$, $R^x_{24} \rightarrow R^x_{2m,2n}$, $(I_1 + Z_1) \rightarrow (I_{2m-1} + Z_{2m-1})$, and $(I_2 + Z_2) \rightarrow (I_{2m} + Z_{2m})$. For the C_2-U gate between the mth, nth, and lth logic qubits, the Hamiltonian has the same structure as H_2 but with the exchanging $R^x_{35} \rightarrow R^x_{2n-1,2l-1}$, $R^x_{36} \rightarrow R^x_{2n-1,2l}$, $R^x_{45} \rightarrow R^x_{2n,2l-1}$, $R^x_{46} \rightarrow R^x_{2n,2l}$, $(I_1 - Z_1) \rightarrow (I_{2m-1} - Z_{2m-1})$, $(I_3 + Z_3) \rightarrow (I_{2n-1} + Z_{2n-1})$, and $(I_4 + Z_4) \rightarrow (I_{2n} + Z_{2n})$. At last, for the Fredkin gate between the mth, nth, and lth logic qubits, the Hamiltonian has the same structure as H_3 but with the exchanging $R^x_{35} \rightarrow R^x_{2n-1,2l-1}$, $R^x_{46} \rightarrow R^x_{2n,2l}$ and $(I_1 - Z_1) \rightarrow (I_{2m-1} - Z_{2m-1})$.

In conclusion, we have proposed schemes for implementing C^1-U, C^2-U, and Fredkin gates directly by using non-adiabatic holonomy in DFSs. Our schemes combine the coherence stabilization virtues of DFSs and the built-in fault tolerance of holonomic control. These gate operations can be implemented in a high speed which avoids the extra errors and decoherence involved in adiabatic case due to long time evolution. Moreover, the resource cost for the DFSs encoding is minimal by using only two neighboring physical qubits undergoing collective dephasing to encode a logical qubit.

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