SUPPLEMENTAL MATERIAL

Atomic structure

As described in the main text, the gate is performed between two trapped $^{171}\mathrm{Yb}^+$ ions which sit in different magnetic fields due to a magnetic-field gradient. The Hamiltonian consists of three terms $H = H_{\mathrm{int}} + H_{\mathrm{ext}} + H_{\mathrm{couple}}$, where H_{int} describes the four internal states ($|0\rangle$, $|0'\rangle$, $|-1\rangle$, and $|+1\rangle$) of each atom, H_{ext} describes the axial stretch mode of the ion pair, and H_{couple} describes the coupling between the internal and external degrees of freedom due to the field gradient.

$$\begin{aligned} H_{\mathrm{int}} = \sum_{i=1,2} & -\omega_i^0 |0\rangle_i\langle 0|_i - \omega_i^- |-1\rangle_i\langle -1|_i \\ & + \omega_i^+ |+1\rangle_i\langle +1|_i \end{aligned} \tag{1}$$

$$H_{\mathrm{ext}} = \nu \hat{a}^\dagger \hat{a} \tag{2}$$

$$H_{\mathrm{couple}} = \sum_{i=1,2} \nu \eta_i (\hat{a}^\dagger + \hat{a}) \hat{\sigma}_{zi} \tag{3}$$

where all the Hamiltonians presented here are normalized by $\hbar$, ω_i^0 and $\omega_i^\pm$ are the energies of the states $|0\rangle$ and $|\pm 1\rangle$ with respect to $|0'\rangle$, $\nu = \sqrt{3}\nu_z$ is the axial stretch mode frequency, $\hat{a}^\dagger$ and $\hat{a}$ the creation and annihilation operators for that mode, $\hat{\sigma}_{zi} = |+1\rangle_i\langle +1|_i - |-1\rangle_i\langle -1|_i$, and η_1 and η_2 are the effective Lamb-Dicke parameters for the two ions, describing the strength of the coupling between the atoms' internal states and the mode due to the field gradient. $\eta_1 = -\eta_2 = z_0 \mu_B \partial_z B/\sqrt{2}\hbar\nu$ where $\partial_z B$ is the axial magnetic-field gradient (the same for both ions) and $z_0 = \sqrt{\hbar/2m\nu}$. For our system $\nu = 2\pi \times 459.34(1)$ kHz and $\partial_z B = 23.6\,\mathrm{Tm}^{-1}$ giving $\eta_1 = 0.0041$. The couplings between $|0\rangle$ and $|0'\rangle$ and the motion, a consequence of the second-order Zeeman effect, are small and therefore are not considered.

Preparation and detection errors

The magnetic-field gradient separates the frequencies ω_i^0 of the clock transitions $|0\rangle \leftrightarrow |0'\rangle$ in the two ions by 11.9 kHz due to the second-order Zeeman shift. This allows the states $|00\rangle$, $|00'\rangle$, $|0'0\rangle$ and $|0'0'\rangle$ to be individually prepared using both optical pumping to prepare $|00\rangle$, and microwave π pulses, resonant with the desired clock transition, to prepare $|0'\rangle$ [1]. We estimate that each of the states is prepared with infidelity less than 10^{-3}. We measure the ion fluorescence using a photomultiplier tube and discriminate between the cases of 0, 1 and 2 ions fluorescing by setting two thresholds. We record the histograms after preparing each of the four spin states, allowing us to extract a linear map between the probabilities P_0, P_1 and P_2 obtained by thresholding, and the spin state probabilities P_{00}, $P_{00'} + P_{0'0}$ and $P_{0'0'}$. This mapping is then used to normalize data in subsequent experiments.

Motional coupling due to magnetic-field gradient

The effect of the magnetic-field gradient is to allow transitions between internal states to affect the motional state of the ions. A microwave field oscillating at frequency $\omega_{\mu\mathrm{w}} = \omega_1^0 + \omega_1^+ \pm \nu + \delta$ close to one of the motional sidebands of the transition $|0\rangle_1 \leftrightarrow |+1\rangle_1$, adds the following term to the Hamiltonian: $H_{\mu\mathrm{w}} = \Omega_{\mu\mathrm{w}}(|+1\rangle_1\langle 0|_1 + |0\rangle_1\langle +1|_1)\cos[\omega_{\mu\mathrm{w}} t]$. By making a polaron-like (Schrieffer-Wolff) transformation [2]

$$U_p = \exp\left(\sum_{i=1,2} \eta_i \left[\hat{a}^\dagger - \hat{a}\right] \hat{\sigma}_{zi}\right), \tag{4}$$

the internal and external states are now coupled [3, 4], such that the driven term $H_{\mu\mathrm{w}}$ transforms to an (anti-) Jaynes-Cummings Hamiltonian

$$H_{\mu\mathrm{w}} = \begin{cases} \frac{\eta_1 \Omega_{\mu\mathrm{w}}}{2}(|+1\rangle_1\langle 0|_1 \hat{a}^\dagger e^{-i\delta t} + |0\rangle_1\langle +1|_1 \hat{a} e^{i\delta t}) & \text{for } \omega_{\mu\mathrm{w}} = \omega_1^0 + \omega_1^+ + \nu + \delta \\ -\frac{\eta_1 \Omega_{\mu\mathrm{w}}}{2}(|+1\rangle_1\langle 0|_1 \hat{a} e^{-i\delta t} + |0\rangle_1\langle +1|_1 \hat{a}^\dagger e^{i\delta t}) & \text{for } \omega_{\mu\mathrm{w}} = \omega_1^0 + \omega_1^+ - \nu + \delta \end{cases} \tag{5}$$

depending on the field being tuned close to either the blue or red sideband, and where we have gone into the interaction picture and dropped all fast rotating terms.

Sideband cooling

The experimental two-qubit gate sequence is preceded by a sideband cooling sequence similar to that described in Ref. [5], however, here we use a microwave field instead of an RF field, to drive the red sideband. The ions are initially Doppler laser cooled for 4 ms using near-resonant light at 369 nm and prepared in $|00\rangle$ after 30 μs of optical pumping. The sideband cooling sequence then consists of applying a microwave field pulse of frequency $\omega_{\mu\mathrm{w}} = \omega_1^0 + \omega_1^+ - \nu$, driving the red sideband transition with a carrier Rabi frequency $\Omega/2\pi = 74\,\mathrm{kHz}$. Optical re-pumping then reinitialises the ions in $|00\rangle$. We apply a total of 500 repetitions of this sideband cooling sequence, each repetition applying an increasing microwave sideband pulse time which corresponds to the sideband Rabi frequencies of different populated n levels. Using this sequence we achieve a final temperature of $\bar{n} = 0.14(3)$.

Tunable quantum-engineered clock qubit

Qubits formed of the states $|{+1}\rangle_i$, $|0\rangle_i$ would rapidly decohere due to magnetic field fluctuations. Countering the magnetic field noise is done by applying four microwave fields of frequencies $\omega_1^0 + \omega_1^+$, $\omega_1^0 - \omega_1^-$, $\omega_2^0 + \omega_2^+$, and $\omega_2^0 - \omega_1^-$, resonant with the $|{+1}\rangle_1 \leftrightarrow |0\rangle_1$, $|{-1}\rangle_1 \leftrightarrow |0\rangle_1$, $|{+1}\rangle_2 \leftrightarrow |0\rangle_2$, and $|{-1}\rangle_2 \leftrightarrow |0\rangle_2$ transitions respectively. If all four microwave fields are driven with equal Rabi frequencies $\Omega_{\mu\mathrm{w}}$, the internal state Hamiltonian in the interaction picture becomes

$$\begin{aligned} H_{\mathrm{int}} =& \frac{\Omega_{\mu\mathrm{w}}}{2} \sum_{i=1,2} (|0\rangle_i \langle{+1}|_i + |0\rangle_i \langle{-1}|_i \\ &+ |{+1}\rangle_i \langle 0|_i + |{-1}\rangle_i \langle 0|_i), \end{aligned} \tag{6}$$

and then transforms to the dressed state basis

$$H_{\mathrm{int}} = \frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}} \sum_{i=1,2} (|u\rangle_i \langle u|_i - |d\rangle_i \langle d|_i), \tag{7}$$

where $|u\rangle_i = \frac{1}{2}|{+1}\rangle_i + \frac{1}{2}|{-1}\rangle_i + \frac{1}{\sqrt{2}}|0\rangle_i$, $|d\rangle_i = \frac{1}{2}|{+1}\rangle_i + \frac{1}{2}|{-1}\rangle_i - \frac{1}{\sqrt{2}}|0\rangle_i$. These resonantly driven transitions operate as a form of continuous dynamical decoupling for the Λ-system, and result in dark states $|D\rangle_i = \frac{1}{\sqrt{2}}(|{+1}\rangle_i - |{-1}\rangle_i)$, which are protected both against magnetic field noise and microwave amplitude fluctuations, such that together with the $|0'\rangle_i$ states constitute robust effective clock qubits [6, 7].

Each qubit can be manipulated by an RF field

$$H_{\mathrm{rf}} = \Omega_{\mathrm{rf}} \left(|{+1}\rangle_i \langle 0'|_i + |0'\rangle_i \langle{+1}|_i\right) \cos \omega_i^{\mathrm{rf}} t. \tag{8}$$

By setting $\omega_i^{\mathrm{rf}} = \omega_i^+$, and if $\Omega_{\mathrm{rf}} \ll |\omega_i^+ - \omega_i^-|$, then

$$\begin{aligned} H_{\mathrm{rf}} &= \frac{\Omega_{\mathrm{rf}}}{2\sqrt{2}} \left(|D\rangle_i \langle 0'|_i + |0'\rangle_i \langle D|_i\right) \\ &= \frac{\Omega_0}{2} \left(|D\rangle_i \langle 0'|_i + |0'\rangle_i \langle D|_i\right), \end{aligned} \tag{9}$$

in the interaction picture and after dropping fast rotating terms [8]. We have defined a Rabi frequency $\Omega_0 = \Omega_{\mathrm{rf}}/\sqrt{2}$ which is the Rabi frequency for driving the engineered qubit.

The concept for a trapped-ion quantum computer presented in this Letter requires well-protected qubits with a tunable transition frequency in the MHz range to allow for high-fidelity individual addressing within a set of global radiation fields. Traditionally, qubits consisting of a first-order magnetic field sensitive transition would have to be used, however such qubits are highly sensitive to ambient magnetic field fluctuations, limiting the achievable coherence time. As shown above the effective clock qubit which is well-protected from ambient magnetic field fluctuations can be manipulated by setting $\omega_j^{\mathrm{rf}} \approx \omega_i^+$ for ion i and field j. Therefore, in order to change the transition frequency one simply changes the magnetic field environment at the ion to shift the first-order magnetic field sensitive state $|{+1}\rangle_i$ such that ω_i^+ is resonant with the desired global radiation field.

Multi-qubit gate

If instead of setting the RF to the resonant frequency $\omega_i^{\mathrm{rf}} = \omega_i^+$, four RF fields are set to be equally detuned from the red and blue sidebands, $\omega_i^{\mathrm{rf}} = \omega_i^+ \pm (\nu + \delta)$, Eq. 8 in the interaction picture becomes

$$H_{\mathrm{rf}} = \Omega_{\mathrm{rf}} \sum_{i=1,2} \left(|{+1}\rangle_i \langle 0'|_i + |0'\rangle_i \langle{+1}|_i\right) \cos\left[(\nu + \delta)\, t\right]. \tag{10}$$

This gives a Mølmer-Sørensen interaction, if coupling to the motional degrees of freedom is present. To show such coupling exists, a polaron-like transformation (Eq. 4) is made to the Hamiltonian containing the stretch mode (Eq. 2), the magnetic-field gradient (Eq. 3), the microwave (Eq. 6) and the RF driving fields (Eq. 10),

$$\begin{aligned} H =& \nu \hat{a}^\dagger \hat{a} \\ &+ \sum_{i=1,2} \nu \eta_i \left(\hat{a}^\dagger + \hat{a}\right) \hat{\sigma}_{zi} \\ &+ \frac{\Omega_{\mu\mathrm{w}}}{2} \sum_{i=1,2} \left(|0\rangle_i \langle{+1}|_i + |0\rangle_i \langle{-1}|_i + |{+1}\rangle_i \langle 0|_i + |{-1}\rangle_i \langle 0|_i\right) \\ &+ \Omega_{\mathrm{rf}} \sum_{i=1,2} \left(|{+1}\rangle_i \langle 0'|_i + |0'\rangle_i \langle{+1}|_i\right) \cos\left[(\nu + \delta)\, t\right], \end{aligned} \tag{11}$$

which is therefore transformed to

$$
\begin{aligned}
H_p = U_p H U_p^\dagger &= \nu \hat{a}^\dagger \hat{a} \\
&+ \nu \sum_{i,j=1,2} \eta_i \eta_j \hat{\sigma}_{zi} \hat{\sigma}_{zj} \\
&+ \frac{\Omega_{\mu\mathrm{w}}}{2} \sum_{i=1,2} \left[\left(|{+1}\rangle_i \langle 0|_i + |0\rangle_i \langle -1|_i \right) e^{\eta_i \left(\hat{a}^\dagger - \hat{a} \right)} + \mathrm{h.c.} \right] \\
&+ \Omega_{\mathrm{rf}} \sum_{i=1,2} \left(|{+1}\rangle_i \langle 0'|_i e^{\eta_i \left(\hat{a}^\dagger - \hat{a} \right)} + \mathrm{h.c.} \right) \cos\left[(\nu + \delta) t \right]
\end{aligned}
\tag{12}
$$

where the RF fields in the last term couple the internal degrees of freedom to the external ones. In the Lamb-Dicke regime, where $\eta_i \sqrt{\bar{n}+1} \ll 1$, the displacement operator $D(\eta_i) = e^{\eta_i \left(\hat{a}^\dagger - \hat{a} \right)}$ is expanded in orders of the Lamb-Dicke parameter η_i. After transforming to the dressed state basis, and moving to the interaction picture with respect to the dressed state energy gap (Eq. 7) and the stretch mode (Eq. 2), the first-order expansion of the RF transition term in Eq. 12 yields

$$
\begin{aligned}
\Omega_{\mathrm{rf}} \sum_{i=1,2} \eta_i \Bigg(& \left[\frac{|u\rangle_i e^{i \frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}} t} + |d\rangle_i e^{-i \frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}} t}}{2} - \frac{|D\rangle_i}{\sqrt{2}} \right] \langle 0'|_i \\
& \times \left(\hat{a}^\dagger e^{i\nu t} - \hat{a} e^{-i\nu t} \right) + \mathrm{h.c.} \Bigg) \cos\left[(\nu + \delta) t \right].
\end{aligned}
\tag{13}
$$

Under the assumption $\delta \ll \frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}} \ll \nu$, the main contribution of Eq. 13 is

$$
\begin{aligned}
H_{\mathrm{gate}} = &- \frac{\eta_1 \Omega_0}{2} \left(|D\rangle_1 \langle 0'|_1 - |0'\rangle_1 \langle D|_1 \right. \\
& \left. - |D\rangle_2 \langle 0'|_2 + |0'\rangle_2 \langle D|_2 \right) \left(\hat{a} e^{i\delta t} - \hat{a}^\dagger e^{-i\delta t} \right),
\end{aligned}
\tag{14}
$$

which drives a Mølmer-Sørensen gate [9, 10], whereas all the other terms are dropped in the rotating wave approximation. If such a Hamiltonian is applied for a time $\tau = 2\pi/\delta$ then the qubit states are subject to the unitary transformation

$$
U = \exp\left[i \frac{\pi \eta_1^2 \Omega_0^2}{\delta^2} \hat{\sigma}_{y1} \hat{\sigma}_{y2} \right],
\tag{15}
$$

with $\hat{\sigma}_{yi} = -i \left(|D\rangle_i \langle 0'|_i - |0'\rangle_i \langle D|_i \right)$. By setting the detuning $\delta = 2\eta_1 \Omega_0$, the initial state $|0'0'\rangle$ is ideally transformed into the required maximally entangled state $\frac{1}{\sqrt{2}} (|0'0'\rangle - i|DD\rangle)$.

Corrections to the gate

The above derivation of H_{gate} considers only the slowest rotating terms. Terms dropped from the derivation lead to both sources of infidelity and lightshifts of the qubit levels. We discuss these in this section, as well as methods to counteract such terms.

Off-resonant carrier excitation due to the rf driving fields

In the standard Mølmer-Sørensen gate performed using trapped ions, off-resonant excitation of the carrier transitions can lead to a reduction in the average gate fidelity. There are a total of 6 RF-driven carrier transitions per ion, $|0'\rangle_i \leftrightarrow |D\rangle_i$, $|0'\rangle_i \leftrightarrow |u\rangle_i$ and $|0'\rangle_i \leftrightarrow |d\rangle_i$, each via both $|{+1}\rangle$ and $|{-1}\rangle$. The RF driven transitions originating from $|0'\rangle_i \leftrightarrow |{-1}\rangle$, not considered in the derivation above due to the additional detuning of the second-order Zeeman splitting $\Delta_i = \omega_i^+ - \omega_i^- \sim O(10\,\mathrm{kHz})$, result in a term

$$
\begin{aligned}
H_{\mathrm{off}} = & \frac{\Omega_0}{\sqrt{2}} \sum_{i=1,2} |{-1}\rangle_i \langle 0'|_i e^{\eta_i \left(a^\dagger e^{i\nu t} - a e^{-i\nu t} \right)} \\
& \times \left[e^{i(\nu + \delta + \Delta_i) t} + e^{-i(\nu + \delta - \Delta_i) t} \right] + \mathrm{h.c.}
\end{aligned}
\tag{16}
$$

As is the case with many other gate implementations, the effect of off-resonant carrier excitation is to produce a rapid oscillation in the gate fidelity. Each carrier transition introduces an average infidelity of approximately Ω^2/ν^2, where Ω is the carrier Rabi frequency, which varies depending on if the transition is to $|D\rangle$ or to $|u\rangle/|d\rangle$ [8]. Summing over the six carrier transitions, the total infidelity due to carrier excitation is $4\Omega_0^2/\nu^2$. In principle, it should be possible to time the gate to minimise the infidelity, however, this is difficult in practise. The standard approach to counteracting this infidelity of using pulse shaping of the gate fields [11] will work far more effectively for this gate, and is therefore implemented to effectively remove this effect. Instead of switching the gate pulses on and off near-instantaneously we shape the pulses in a way that at the beginning and end of the pulse the amplitude rises and falls adiabatically within a window length of $10\,\mu$s which was found to be sufficient using numerical simulations of the full gate dynamics.

Lightshifts and leakage outside the qubit subspace

The RF-driven off-resonant carrier transitions in Eq. 16 will produce a net lightshift term arising from non-vanishing A.C Stark shifts in the rotating frame of the dressed state energy gap in Eq. 7:

$$
\begin{aligned}
H_{\mathrm{rf}_{-1}}^{\mathrm{shift}} =& \frac{\Omega_0^2}{4}\sum_{i=1,2}\left(\frac{1}{\nu+\Delta_i}-\frac{1}{\nu-\Delta_i}\right)\left(|D\rangle_i\langle D|_i-|0'\rangle_i\langle 0'|_i\right)\\
&-\frac{\Omega_0^2}{8}\sum_{i=1,2}\left(\frac{1}{\nu+\Omega_{\mu\mathrm{w}}/\sqrt{2}+\Delta_i}-\frac{1}{\nu-\Omega_{\mu\mathrm{w}}/\sqrt{2}-\Delta_i}+\frac{1}{\nu-\Omega_{\mu\mathrm{w}}/\sqrt{2}+\Delta_i}-\frac{1}{\nu+\Omega_{\mu\mathrm{w}}/\sqrt{2}-\Delta_i}\right)|0'\rangle_i\langle 0'|_i,
\end{aligned}
\tag{17}
$$

where we have omitted δ, since $\delta \ll \Delta_i, \Omega_{\mu\mathrm{w}}/\sqrt{2}, \nu$. This can be approximated as

$$
H_{\mathrm{rf}_{-1}}^{\mathrm{shift}} \approx -\frac{3\Omega_0^2}{4\nu^2}\sum_{i=1,2}\Delta_i\hat{\sigma}_{zi}, \tag{18}
$$

under the assumption $\Delta_i, \Omega_{\mu\mathrm{w}}/\sqrt{2} \ll \nu$. The leading terms of the first-order expansion of the RF transitions in Eq. 13 yield in the second order of perturbation the following terms:

$$
H_{\mathrm{rf}}^{\mathrm{shift}} = \underbrace{\frac{(\eta_1\Omega_0)^2}{2}\left(\frac{1}{\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}-\delta}+\frac{1}{\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}+\delta}\right)}_{g_{\mathrm{rf\ shift}}}\sum_{i=1,2}|0'\rangle_i\langle 0'|_i, \tag{19}
$$

which is a net lightshift term, as well as

$$
\begin{aligned}
H_{\mathrm{rf}}^{\mathrm{leak}} =& \underbrace{\frac{(\eta_1\Omega_0)^2}{2}\left(\frac{1}{\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}-\delta}-\frac{1}{\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}+\delta}\right)}_{g_{\mathrm{rf\ leak}}}\\
&\times\left(|u\rangle_1\langle 0'|_1-|0'\rangle_1\langle d|_1\right)\left(|0'\rangle_2\langle u|_2-|d\rangle_2\langle 0'|_2\right)+\mathrm{h.c}
\end{aligned}
\tag{20}
$$

which operates as a leakage outside of the qubit space.

The magnetic-field gradient (second term in Eq. 12) together with the microwave driving fields (third term in Eq. 12) also produce similar contributions. After transforming to the dressed state basis and moving to the interaction picture with respect to the dressed state energy gap (Eq. 7) and the stretch mode (Eq. 2), we obtain in second-order expansion of η

$$
\begin{aligned}
H_I =& -\sum_{i,j=1}^{2}\frac{\nu\eta_i\eta_j}{4}\left(\hat{S}_{+i}e^{i\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}t}+\mathrm{h.c.}\right)\left(\hat{S}_{+j}e^{i\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}t}+\mathrm{h.c.}\right)\\
&-\frac{\Omega_{\mu\mathrm{w}}}{2\sqrt{2}}\sum_{i=1,2}\left[\eta_i\left(\hat{S}_{+i}e^{i\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}t}-\mathrm{h.c.}\right)\left(\hat{a}^\dagger e^{i\nu t}-\hat{a}e^{-i\nu t}\right)\right]
\end{aligned}
\tag{21}
$$

where $\hat{S}_{+i} = \sqrt{2}(|u\rangle_i\langle D|_i + |D\rangle_i\langle d|_i)$ and $\hat{S}_{-i} = \hat{S}_{+i}^\dagger$. The leading unwanted terms originating from Eq. 21 give rise to

$$
H_{\mu\mathrm{w}}^{\mathrm{leak}} = \underbrace{-\frac{\eta_1^2\nu^3}{2\left(\frac{\Omega_{\mu\mathrm{w}}{}^2}{2}-\nu^2\right)}}_{g_{\mu\mathrm{w}}}\left(\hat{S}_{+1}\hat{S}_{-2}+\mathrm{h.c}\right), \tag{22}
$$

which leads to leakage outside of the qubit space, in addition to

$$
H_{\mu\mathrm{w}}^{\mathrm{shift}} = \underbrace{\frac{\eta_1^2\nu^3}{\frac{\Omega_{\mu\mathrm{w}}{}^2}{2}-\nu^2}}_{-2g_{\mu\mathrm{w}}}\left(|D\rangle_1\langle D|_1+|D\rangle_2\langle D|_2\right), \tag{23}
$$

which is a lightshift term.

For the gate described in this Letter, the coupling parameters for the leakage terms would be $g_{\mathrm{rf\ leak}} = 2\pi \times 0.05\,\mathrm{Hz}$ and $g_{\mu\mathrm{w}} = 2\pi \times 3.8\,\mathrm{Hz}$ and we find that $g_{\mathrm{rf\ shift}} = 2\pi \times 2.2\,\mathrm{Hz}$. We suppress their effect by introducing an additional energy gap between the coupled states, suppressing the couplings by making them off-resonant. We accomplish this by making the Rabi frequencies of the dressing fields for the two ions different by a small amount $\delta_0 = \Omega_{\mu\mathrm{w}1} - \Omega_{\mu\mathrm{w}2}$ [4]. If $\delta_0 \gg \eta^2\nu$, this coupling is energetically suppressed. The dressing field Rabi frequencies used for the gate presented here are $\Omega_{\mu\mathrm{w}1} = 2\pi \times 20.5\,\mathrm{kHz}$ and $\Omega_{\mu\mathrm{w}2} = 2\pi \times 21.6\,\mathrm{kHz}$. The lightshifts are compensated by shifting the frequencies of the RF gate fields.

Expansion to higher orders in η of Eq. 16 yields the following terms

$$
\begin{aligned}
H_{\mathrm{rf}}^{\mathrm{higher}} &= \frac{\Omega_0}{2\sqrt{2}} \sum_{j=1,2} \left(\eta_j |u\rangle_j \langle 0'|_j \left(\hat{a}^\dagger e^{i\left(\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}+\Delta_j-\delta\right)t} - \hat{a} e^{i\left(\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}+\Delta_j+\delta\right)t} \right) + \mathrm{h.c.} \right) \\
&+ \frac{\Omega_0}{2\sqrt{2}} \sum_{j=1,2} \left(\eta_j |d\rangle_j \langle 0'|_j \left(\hat{a}^\dagger e^{i\left(-\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}+\Delta_j-\delta\right)t} - \hat{a} e^{i\left(-\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}+\Delta_j+\delta\right)t} \right) + \mathrm{h.c.} \right).
\end{aligned}
\tag{24}
$$

These terms give rise to another A.C. Stark shift, also leading also to an additional lightshift term that is phonon-number dependent, and so cannot be compensated by a simple change in gate field frequencies, given by

$$
\begin{aligned}
H_{\mathrm{phonon}}^{\mathrm{ls}} = \sum_{i=1,2} &\underbrace{\frac{(\eta_1\Omega_0)^2}{8} \left(\frac{1}{\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}} - \Delta_i} - \frac{1}{\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}} + \Delta_i} \right)}_{g_{\mathrm{ph}}} \\
&\times \left(2\hat{a}^\dagger\hat{a}+1\right) |0'\rangle_i \langle 0'|_i .
\end{aligned}
\tag{25}
$$

The lightshift fluctuates from shot-to-shot due to the thermal spread in the phonon number, which scales as the square root of the mean phonon number. The strength of this term for our gate parameters is $g_{\mathrm{ph}} = 2\pi \times 0.7\,\mathrm{Hz}$ and $2\pi \times 0.3\,\mathrm{Hz}$ for ions 1 and 2 respectively. The effect of this shift is already small, and could be further reduced by an increase in microwave Rabi frequency and reduction of the mode temperature.

Different Zeeman splitting due to the magnetic-field gradient

The two ions are aligned along the magnetic-field gradient which determines the z-axis. Therefore they feel a different magnetic field, resulting in a difference in Zeeman splitting $\Delta_B = g\mu_B \partial_z B_z \Delta Z$, where $\Delta Z = \left(e^2/2\pi\epsilon_0 M\nu^2\right)^{1/3}$ is the distance between the two ions. This difference yields additional terms in the Hamiltonian in the rotating frame corresponding to the bare energy structure. Due to the four microwave driving fields the following additional terms are obtained:

$$
\begin{aligned}
H_{\mathrm{Zeeman}}^{\mu\mathrm{w}} =& \frac{\Omega_{\mu\mathrm{w}}}{2} (|{+1}\rangle_1 \langle 0|_1 + |0\rangle_1 \langle {-1}|_1 \\
&+ |{-1}\rangle_2 \langle 0|_2 + |0\rangle_2 \langle {+1}|_2) e^{i\Delta_B t} + \mathrm{h.c.}
\end{aligned}
\tag{26}
$$

which in second order results in

$$
H_{\mathrm{Zeeman}}^{\mu\mathrm{w\ eff}} = \frac{\Omega_{\mu\mathrm{w}}^2}{4\Delta_B} \left(F_{z1} - F_{z2}\right), \tag{27}
$$

where $F_{zi} = |{+1}\rangle_i \langle {+1}|_i - |{-1}\rangle_i \langle {-1}|_i$. This should be taken into account when determining the microwave and RF frequencies.

Due to the four RF driving fields, the following terms should be added to the Hamiltonian:

$$
\begin{aligned}
H_{\mathrm{Zeeman}}^{\mathrm{rf}} = \frac{\Omega_0}{\sqrt{2}} \Big(& (|{+1}\rangle_1 \langle 0'|_1 + |0'\rangle_2 \langle {+1}|_2) \\
& \times \left(e^{i(\Delta_B+\nu+\delta)t} + e^{i(\Delta_B-\nu-\delta)t} \right) \\
& + |{-1}\rangle_1 \langle 0'|_1 \left(e^{-i(\Delta_B-\Delta+\nu+\delta)t} + e^{-i(\Delta_B-\Delta-\nu-\delta)t} \right) \\
& + |{-1}\rangle_2 \langle 0'|_2 \left(e^{i(\Delta_B+\Delta+\nu+\delta)t} + e^{i(\Delta_B+\Delta-\nu-\delta)t} \right) \\
& + \mathrm{h.c.} \Big).
\end{aligned}
\tag{28}
$$

As a result, the four RF driving fields give rise to additional lightshifts given by

$$
H_{\mathrm{Zeeman}}^{\mathrm{rf\ eff}} = \frac{\Delta\Omega_0^2}{\Delta_B^2 - \nu^2} \left(|0'\rangle_1 \langle 0'|_1 + |0'\rangle_2 \langle 0'|_2\right). \tag{29}
$$

Together with the lightshift terms that were derived above, the net lightshift is compensated by shifting the frequencies of the gate fields.

Higher-order contribution to the gate transition

An additional term arises when considering the following two terms in higher orders of perturbation:
(1.) the first-order expansion in η of the microwave transition in Eq. 12

$$
-\frac{\Omega_{\mu\mathrm{w}}}{2\sqrt{2}} \sum_{i=1,2} \eta_i \left(S_{+i} e^{i\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}t} - \mathrm{h.c.} \right) \left(\hat{a}^\dagger e^{i\nu t} - \mathrm{h.c.} \right) \tag{30}
$$

(2.) the RF carrier transition (last term in Eq. 12)

$$
\begin{aligned}
\frac{\Omega_0}{\sqrt{2}} \sum_{i=1,2} &\left(\left[|u\rangle_i \langle 0'|_i e^{i\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}t} + |d\rangle_i \langle 0'|_i e^{-i\frac{\Omega_{\mu\mathrm{w}}}{\sqrt{2}}t} \right] + \mathrm{h.c.} \right) \\
&\times \cos\left[(\nu+\delta)\,t\right].
\end{aligned}
\tag{31}
$$

These two terms oscillate almost with the same frequency $\approx \nu \pm \Omega_{\mu\mathrm{w}}/\sqrt{2}$, where the difference is exactly δ, such that Raman transitions are obtained

$$
\begin{aligned}
H_{\mathrm{h.o.}} &= \frac{\Omega_{\mu\mathrm{w}}^2}{\Omega_{\mu\mathrm{w}}^2 - 2\nu^2} \sum_{j=1,2} \frac{\eta_j \Omega_0}{2} \left(|D\rangle_j \langle 0'|_j - |0'\rangle_j \langle D|_j \right) \\
&\qquad \times \left(\hat{a} e^{i\delta t} - \hat{a}^\dagger e^{-i\delta t} \right) \\
&= \frac{\Omega_{\mu\mathrm{w}}^2}{2\nu^2 - \Omega_{\mu\mathrm{w}}^2} H_{\mathrm{gate}}
\end{aligned}
\tag{32}
$$

This produces a Mølmer-Sørensen coupling equivalent to Eq. 14, that changes the required gate duration as the Rabi frequency of the microwave driving fields $\Omega_{\mu\mathrm{w}}$ is increased. In the limit $\Omega_{\mu\mathrm{w}} \gg \nu$, this term yields an opposite sign to the desired coupling, cancelling the gate completely due to destructive interference with these Raman transitions. In our experiment $\Omega_{\mu\mathrm{w}} \ll \nu$, and the change in gate time is negligible.

Imperfections in the dressing fields

Introducing a small imbalance in the amplitudes of the two microwave driving fields $\Delta\Omega_{\mu\mathrm{w_i}} = \Omega^{+}_{\mu\mathrm{w_i}} - \Omega^{-}_{\mu\mathrm{w_i}}$, with $\Delta\Omega_{\mu\mathrm{w_i}} \ll \Omega_{\mu\mathrm{w_i}}$, yields an additional term:

$$\sum_{i=1,2} \frac{\Delta\Omega_{\mu\mathrm{w_i}}}{2} \left(|{+1}\rangle_i \langle 0|_i - |{-1}\rangle_i \langle 0|_i + \mathrm{h.c.} \right). \tag{33}$$

In the interaction picture with respect to the dressed state energy, this yields an A.C. Stark shift which does not operate in our qubit subspace:

$$\sum_{i=1,2} \frac{\Delta\Omega^2_{\mu\mathrm{w_i}}}{2\sqrt{2}\Omega_{\mu\mathrm{w_i}}} \left(|u\rangle_i \langle u|_i - |d\rangle_i \langle d|_i \right). \tag{34}$$

However, this amplitude imbalance together with the ambient magnetic field fluctuations $\delta B(t)$ gives rise to another noise-inducing term that survives the RWA:

$$\begin{aligned} \sum_{i=1,2} & \frac{\sqrt{2}\left(\frac{\Delta\Omega_{\mu\mathrm{w_i}}}{2} - \frac{\delta B}{\sqrt{2}}\right)^2}{\Omega_{\mu\mathrm{w_i}}} |u\rangle_i \langle u|_i \\ & - \frac{\sqrt{2}\left(\frac{\Delta\Omega_{\mu\mathrm{w_i}}}{2} + \frac{\delta B}{\sqrt{2}}\right)^2}{\Omega_{\mu\mathrm{w_i}}} |d\rangle_i \langle d|_i \\ & + \frac{2\delta B \Delta\Omega_{\mu\mathrm{w_i}}}{\Omega_{\mu\mathrm{w_i}}} |D\rangle_i \langle D|_i. \end{aligned} \tag{35}$$

The first two terms do not operate in our qubit subspace, while the latter is another σ_{zi} lightshift term coupled to the ambient magnetic noise, and therefore causes dephasing. This term equals exactly the original dephasing term multiplied by $\frac{\Delta\Omega_{\mu\mathrm{w_i}}}{\Omega_{\mu\mathrm{w_i}}}$, which means that the original dephasing term is being prolonged by a factor of $\left(\frac{\Delta\Omega_{\mu\mathrm{w_i}}}{\Omega_{\mu\mathrm{w_i}}}\right)^2$. This factor is typically smaller than 10^{-4}.

Further increasing the gate fidelity

The infidelity of our demonstrated gate is dominated by heating of the motional mode, and depolarization of the qubit. Increasing the speed of the gate reduces both sources of infidelity. In addition, the heating rate can be further reduced by increasing the trap frequency. By modifying our gate parameters (including the size of the magnetic field gradient) the infidelities can be dramatically reduced and operations with an error below the relevant fault-tolerant threshold are achievable.

We have modeled the following parameters: $\Omega_{\mu\mathrm{w}}/2\pi = 10\,\mathrm{kHz}$, $\Omega_0/2\pi = 198\,\mathrm{kHz}$, $\nu/2\pi = 1.1\,\mathrm{MHz}$, $\eta = 0.0071$ (which corresponds to a magnetic field gradient $\partial_z B_z = 150\,\mathrm{T/m}$), with pulse shaping using a $\sin^2(t)$ profile with rise and fall duration $t_{shaping} = 10\pi/\nu$. Instead of applying an imbalance between $\Omega_{\mu\mathrm{w}1}$ and $\Omega_{\mu\mathrm{w}2}$, we have detuned the microwave transitions equally by $2\pi \times 0.5\,\mathrm{kHz}$ with respect to the $|0\rangle_i$ levels [4]. To check these changes do not substantially increase the gate's intrinsic infidelities discussed above, we have simulated the gate performance, namely the gate state fidelity for each one of the four possible states in the code space $\{|D,D\rangle, |D,0'\rangle, |0',D\rangle, |0',0'\rangle\}$, from which we calculate the gate process fidelity as their average. In the simulation, we consider a vibrational mode with a cutoff $n_{cut} = 15$ and no further approximations have been made. Taking into account a depolarisation time of 2 s (as previously measured using our dressed-state system [12]) as well as a stretch-mode heating rate of $1.3\,\mathrm{s}^{-1}$ (heating rate as measured in the relevant apparatus and scaled to 1.1 MHz; if using an ion chip with 150 μm ion-electrode distance, the expected tenfold increase in heating rate could be compensated by light cooling of the trap electrodes to liquid nitrogen temperature [13]), we calculate a total fidelity of 0.999 and a gate time of $361\mu s$. Further improvement is of course possible with higher magnetic field gradients as well as with longer coherence times and lower heating rates of the type seen in other experiments.

Extension to a large-scale architecture

Our method can be applied to construct a large scale quantum computer. We have developed a detailed engineering blueprint for this purpose [14]. Here we discuss important considerations relevant to our method. Individual addressing of ions in the same entanglement zone is achieved using the local magnetic field gradients, while individual addressing of ions in different zones is achieved by applying local voltages to position ions in different zones in a different magnetic offset field. The crosstalk between ions i and j for a square pulse resonant with ion i can be characterised by the time-averaged excitation probability of ion j, given by $C_{ij} \approx \Omega_j^2/2\Delta_{ij}^2$, where Ω_j is the Rabi frequency of the desired transition in ion j and Δ_{ij} is the frequency separation between the transitions in the two ions [15]. In a single entanglement zone, the frequency separation between the Zeeman sublevels of the ion pair for the parameters in the example case in the previous section is 9.8 MHz. For the microwave dressing fields, Rabi frequencies $\Omega_{\mu\mathrm{w}}/2\pi = 10$ kHz are used, and therefore the crosstalk values for the dressing fields are $C_{12} = C_{21} = 5.2 \times 10^{-7}$. For the rf fields used to drive the two-qubit gate, the crosstalk values would be $C_{12} = C_{21} = 2.0 \times 10^{-4}$ for square pulse shapes, however, the field amplitudes would be shaped with a $\sin^2$ profile as demonstrated in this work. Shaping the pulse amplitudes further reduces the crosstalk by several orders of magnitude. To see this, a numerical simulation of a two-level system driven by a field with Rabi frequency $\Omega(t)$ and detuning δ was performed. The Rabi frequency was varied in time starting with a $\sin^2$ shape ramp from $\Omega = 0$ to $\Omega = \Omega_{\mathrm{max}}$ for a time t_{w}, followed by a hold at Ω_{max} for a time t_{h}, and finally a second $\sin^2$ shape ramp down to $\Omega = 0$ in time t_{w}. It was found that for $\delta > 10\Omega_{\mathrm{max}}$ and $t_{\mathrm{w}} > \pi/\Omega_{\mathrm{max}}$, the error is reduced to $< 10^{-7}$. This detuning requirement is fulfilled for both the rf and microwave dressing fields in this example. Therefore the crosstalk between ions in a single zone is $< 10^{-6}$, and is therefore negligible compared to other error sources.

As mentioned, individual addressing of ions in different zones is achieved by positioning the ions in different zones in a different local static magnetic offset field achieved making use of the position dependent magnetic field originating from the local static magnetic field gradient within each zone. Ions that are not being addressed sit at magnetic field B_1 corresponding to position z_1, while ions that require to be addressed are moved to position z_2 resulting in a magnetic field B_2. As an example, if $B_2 - B_1 = 2$ G, the Zeeman states of the ions that are not being addressed are 2.8 MHz off-resonant. The crosstalk for such a frequency separation with the parameters in the example case is 6.4×10^{-6} for the microwave dressing fields, and $< 10^{-7}$ for the shaped rf gate field pulse. Other types of gates can then be introduced by positioning the ions in different locations resulting in additional magnetic offset fields B_3, B_4 etc.. The minimal set of gates required for a universal quantum computer following the surface code error correction scheme described in Ref. [16] consists of two single qubit gates (Hadamard + $\pi/8$ σ_z-rotation) and a two-qubit entangling gate such as the one presented in this work. Therefore there are four offset magnetic fields required: No interaction, single qubit Hadamard, single qubit $\pi/8$ σ_z-rotation and two-qubit gate. The total required range of magnetic field offsets is therefore approximately 6 G for an arbitrarily large processor. Additional operations could be added by increasing the range of magnetic field offsets if required.

The currents creating the static magnetic field gradients local to each gate zone are applied permanently and are not switched on or off. An alternative method to select arbitrary gate zones for gate execution is to add an additional current carrying wire to each gate zone. The low current passing through this wire in each gate zone creating the required different levels of magnetic field need to be switched in order to individually address ions in different zones. Here, currents of ≈ 100 mA need to be applied to the wires to ramp the local offset B-field from B_1 to B_2 (a difference of ≈ 2 G) which would then shift the qubit frequency into resonance with a particular set of global gate fields. Different levels of current are applied to relevant coils to shift the qubit frequencies into resonance with different global gate fields. On-chip digital-to-analogue converters can be used to control the currents with 2 MS/s and 16 bit precision. A realistic timing sequence for a two-qubit gate operation for example would then be a 5 μs ramp from B_1 to B_2, followed by the gate operation which is then followed by a second 5 μs ramp from B_2 back to B_1 where an integrated filter produces a smooth waveform.

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