CS5010 Artificial Intelligence Principles

Lecture 11 Uncertainty 2

Probabilistic Inference

Lei Fang
University of St Andrews

Last time

- Random variable and its distribution

$$P(X) > 0 \ \text{ and } \ \sum_x P(X = x) = 1$$

- We can have multiple random variables and their joint distribution (still a distribution)

$$P(X_1, X_2, \ldots, X_n) > 0 \ \text{ and}$$
$$\sum_{x_1} \sum_{x_2} \ldots \sum_{x_n} P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = 1$$

Two rules

Sum rule:

$$P(X=x)=\sum_y P(X=x, Y=y);$$

Product rule:

$$P(X,Y)=P(X)P(Y|X)=P(Y)P(X|Y)$$

Conditional probability

Based on the product rule, we have

$$P(X|Y) = \frac{P(X,Y)}{P(Y)}$$

This time

- Probabilistic inference using joint distribution and the two rules
- Baye's rule
- Examples of probabilistic inference
    - Concept learning

Probabilistic inference by joint distribution

Given two coins with different probabilities of head turning up:

- Coin 1 is a fair coin $p = 0.5$
- Coin 2 is bent with $p = 0.2$

Your friend randomly pick one coin (which you do not know) and toss it 2 times and record the results,

$$Y_1 \in \{head, tail\}, \quad Y_2 \in \{head, tail\}.$$

Was the fair coin used or not given the following observations?

- $Y_1 = h, Y_2 = h$
- $Y_1 = t, Y_2 = t$
- $Y_1 = t$

Step 1, figure out the random variables

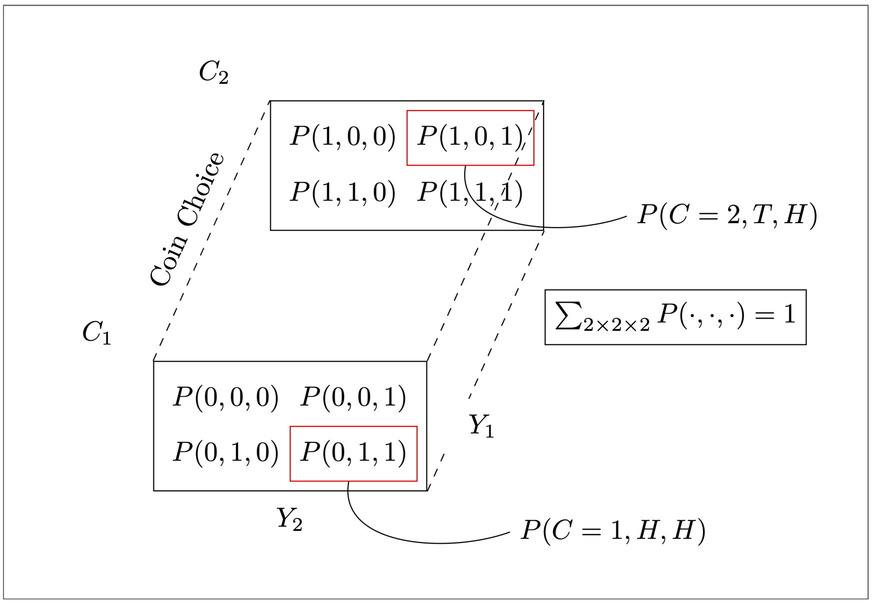
Firstly, identify the random variables

$$C,\ Y_1,\ Y_2$$

- $C = 1, 2$ denote which coin has been used
- Y_1, Y_2 are the tossing outcomes

Step 2, figure out the joint distribution

The joint distrution $P(C, Y_1, Y_2)$ is over $2 \times 2 \times 2$ possible combinations e.g $P(C = 1, Y_1 = head, Y_2 = head)$



- We need to specify the $2 \times 2 \times 2$ entries
- Due to product rule, we have

$$P(C = c, Y_1 = y_1, Y_2 = y_2) = P(C = c)P(Y_1 = y_1, Y_2 = y_2|C = c)$$

- As the two tosses are independent, we have

$$P(Y_1, Y_2|C = c) = P(Y_1|C = c)P(Y_2|C = c)$$

  - it is called **conditional independence**: condition on knowing the coin choise, the two tosses are independent

- then we can populate the joint distribution table e.g.

$$P(C=1,Y_1=t,Y_2=t)=P(C=1)\times P(Y_1=t|C=1)\times P(Y_2=t|C=1)=0.5*0.$$

$$\vdots$$

$$P(C=2,Y_1=h,Y_2=h)=P(C=2)\times P(Y_1=h|C=2)\times P(Y_2=h|C=2)=0.5*$$

We can equivalently represent the 3-dimensional-shaped distribution as a big table (similar to truth table)

C	Y1	Y2	P
1	tail	tail	0.125
1	head	tail	0.125
1	tail	head	0.125
1	head	head	0.125
2	tail	tail	0.32
2	head	tail	0.08
2	tail	head	0.08
2	head	head	0.02

Note that the sum is one as expected

$$\sum_{c,y_1,y_2} P(\cdot,\cdot,\cdot) = 1$$

Step three, probabilistic inference by using the joint distribution

Essentially, we want to figure out

$$P(C = 1|Y_1 = y_1, Y_2 = y_2)$$

- the distribution of some **unknown** C given evidence Y_1, Y_2
- we can simply use probability rules

$$P(C = c|Y_1 = y_1, Y_2 = y_2) = \frac{P(C = c, Y_1 = y_1, Y_2 = y_2)}{P(Y_1 = y_1, Y_2 = y_2)} = \frac{P(C = c, Y_1 = y_1, Y_2 =}{\sum_c P(C = c, Y_1 = y_1, Y_2 =}$$

|C | Y1 | Y2 | P | | --- | --- | --- | --- | | 1 | tail | tail | 0.125 | | 1 | head | tail | 0.125 | | 1 | tail | head | 0.125 | | 1 | head | head | 0.125 | | 2 | tail | tail | 0.32 | | 2 | head | tail | 0.08 | | 2 | tail | head | 0.08 | | 2 | head | head | 0.02 |

- e.g.

$$P(C = 1|Y_1 = h, Y_2 = h) = \frac{P(C = 1, Y_1 = h, Y_2 = h)}{\sum_{c=1,2} P(C = c, Y_1 = h, Y_2 = h)} = \frac{0.125}{0.125 + 0.02} = 0.862$$

C	Y1	Y2	P
1	tail	tail	0.125
1	head	tail	0.125
1	tail	head	0.125
1	head	head	0.125
2	tail	tail	0.32
2	head	tail	0.08
2	tail	head	0.08
2	head	head	0.02

Similarly, if $Y_1 = t, Y_2 = h$

$$P(C=1|Y_1=t,Y_2=h) = \frac{P(C=1,Y_1=t,Y_2=h)}{\sum_{c=1,2} P(C=c,Y_1=t,Y_2=h)} = \frac{0.125}{0.125+0.08} = 0.61$$

if $Y_1 = t, Y_2 = t$

$$P(C=1|Y_1=t,Y_2=t) = \frac{P(C=1,Y_1=t,Y_2=t)}{\sum_{c=1,2} P(C=c,Y_1=t,Y_2=t)} = \frac{0.125}{0.125+0.32} = 0.281$$

- Very reasonable, if more Tails are observed, your friend is more likely to have used the bent coin (second coin) or cheating

Joint distribution contains all the information one needs

You can calculate **everything** based on the joint distribution
For example, we can predict the second toss based on the first toss

$$P(Y_2|Y_1) = \frac{P(Y_1, Y_2)}{P(Y_1)} = \frac{\sum_c P(C=c, Y_1, Y_2)}{\sum_c \sum_{y_2} P(C=c, Y_1, Y_2=y_2)}$$

e.g.

$$P(Y_2=h|Y_1=t) = \frac{P(Y_1=t, Y_2=h)}{P(Y_1=t)} = \frac{\sum_c P(C=c, Y_1=t, Y_2=h)}{\sum_c \sum_{y_2} P(C=c, Y_1=h, Y_2=y_2)} = \frac{0.}{0.125+0.}$$

Note that

$$P(Y_2=t|Y_1=t) = \frac{\sum_c P(C=c, Y_1=t, Y_2=t)}{\sum_c \sum_{y_2} P(C=c, Y_1=h, Y_2=y_2)} = \frac{0.125+0.32}{0.125+0.125+0.32+0.08} = 1$$

- $P(Y_2|Y_1=t)$: conditional distribution is a distribution $\sum P(Y_2|Y_1=t) = 1$

We can also calculate marginal probability on the second toss only

$$P(Y_2) = \sum_c \sum_{y_1} P(C = c, Y_1 = y_1, Y_2)$$

e.g.

$$P(Y_2 = h) = 0.125 + 0.125 + 0.08 + 0.02 = 0.35$$

Again it is easy to verify

$$P(Y_2 = t) = 1 - 0.35 = 0.65$$

Conditional Independence

For the above example, we made the conditional independence assumption: i.e. knowing the coin in use, the tosses become independent:

$$P(Y_1, Y_2|C) = P(Y_1|C)(Y_2|C)$$

Note that marginally, or without the condition, the two tosses are not independent !

$$P(Y_1, Y_2) \neq P(Y_1)P(Y_2)$$

- knowing the result of the first toss Y_1 changes our belief on the coin used
    - if $Y_1 = t$, then we are more likely to have used the bent coin rather than the fair one
    - therefore, the chance to observe another Tail would be increased!
    - $P(Y_2 = t|Y_1 = t) > P(Y_2 = t)$

We have verified it already in previous slides

$$P(Y_2 = t|Y_1 = t) = 1 - P(Y_2 = h|Y_1 = t) = 0.685, \quad P(Y_2 = t) = 0.65$$

Equivalently, conditional independence can be writen as

$$P(Y_1|C, Y_2) = P(Y_1|C) \text{ or } P(Y_2|C, Y_1) = P(Y_2|C)$$

- knowing Y_2 does not provide any more information if C is known
- can you also prove it from the normal CI definition $P(Y_1, Y_2|C) = P(Y_1|C)(Y_2|C)$?

Problem with joint distribution

- Joint distribution contains all the information but not efficient
- Its size increases **exponentially** with respect to the size of random variables
- Suppose your friend toss a random chosen coin 5 times
    - The random variables are $C, Y_1, Y_2, \ldots, Y_5$
    - $P(C, Y_1, Y_2, \ldots, Y_5)$ has $2 \times 2 \times 2 \times 2 \times 2 \times 2$ rows !
    - What if toss 20 times ? 2 million entries!
- Baye's rule comes to rescue

Baye’s rule

Baye's rule provides an alternative way to do probabilistic inference

$$P(Y|X) = \frac{P(Y)P(X|Y)}{\sum_y P(Y=y)P(X|Y=y)}$$

Nothing new, just manipulation of the two rules and conditional probability

- remember conditional probability

$$P(Y|X) = \frac{P(X,Y)}{P(X)}$$

- the **numerator** has used chain rule: $P(Y)P(Y|X) = P(X,Y)$

- the **denumerator** has used both summation and chain rule: $P(X) = \sum_y P(X, Y=y) = \sum_y P(Y=y)P(X|Y=y)$

One often writes

$$\underbrace{P(Y|X)}_{Posterior} \propto \underbrace{P(Y)}_{\text{Prior}}\underbrace{P(X|Y)}_{\text{Likelihood}} \text{ or } \underbrace{P(Y|X)}_{Posterior} = \alpha\underbrace{P(Y)}_{\text{Prior}}\underbrace{P(X|Y)}_{\text{Likelihood}}$$

- $\propto$ means proportional to
- because $P(X) = \sum_y P(Y=y)P(X|Y=y)$ is a normalising constant
- i.e. from $Y = y$'s perspective, it is a constant (written as α); indeed it does not depend on y

Probabilistic inference: by Baye's rule

Again two coins, one fair the other bent with $p_1 = 0.5$ and $p_2 = 0.2$,
Your friend randomly pick one then toss 5 times and made the following observations

$$Y_1 = t, Y_2 = h, Y_3 = t, Y_4 = t, Y_5 = t$$

Which coin he/she has used ?

According to Baye's rule,

$$
\begin{aligned}
P(C|Y_1, Y_2, Y_3, Y_4, Y_5) &\propto P(C)P(Y_1, Y_2, Y_3, Y_4, Y_5|C) && (1) \\
&= P(C) \underbrace{\prod_{i=1}^{5} P(Y_i|C)}_{\text{Conditional Independence!}} && (2)
\end{aligned}
$$

So

$$P(C=1|t,h,t,t,t) \propto p(C=1) \times (1-p_1)p_1(1-p_1)(1-p_1)(1-p_1) = 0.5 \cdot 0.5^5 = 0.0156$$

$$P(C=2|t,h,t,t,t) \propto p(C=2) \times (1-p_2)p_2(1-p_2)(1-p_2)(1-p_2) = 0.5 \cdot 0.2^1 \cdot 0.8^4 = 0.$$

Normalise (conditional distribution is a distribution, therefore sum to one), i.e. find α

$$P(C=1|t,h,t,t,t) = \frac{0.5 \cdot 0.5^5}{0.5^6 + 0.5 \cdot 0.2 \cdot 0.8^4} = 0.276$$

and

$$P(C=2|t,h,t,t,t) = \frac{0.5 \cdot 0.2 \cdot 0.8^4}{0.5^6 + 0.5 \cdot 0.2 \cdot 0.8^4} = 0.724$$

The computation scales easily to larger case, say 20 observations $Y_1, \ldots, Y_{20}$

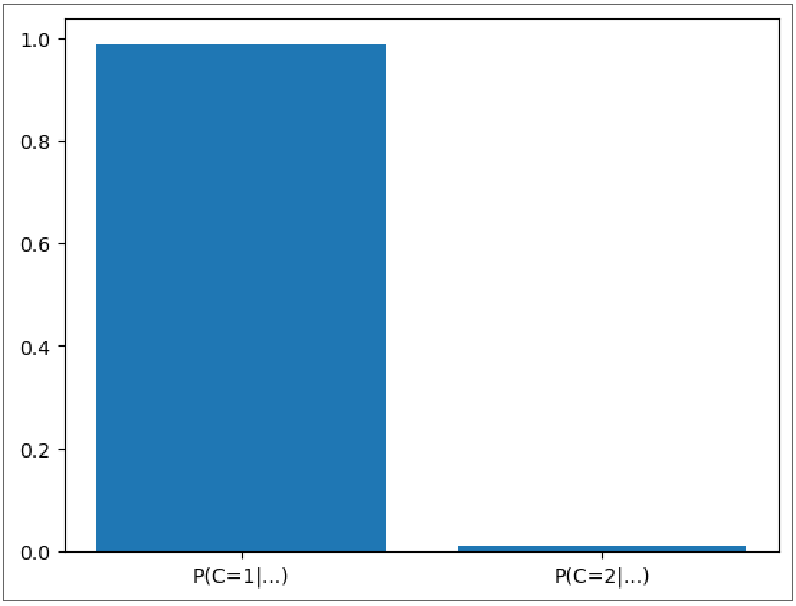
- where there are 10 heads and 10 tails

In [10]:

```
# a method to calculate the posterior probability of C given the model parameters and observation
def calculatePosterior(pc1, p1, p2, nheads, ntails):
    posteriors1 = pc1 * p1**nheads * (1-p1)**ntails
    posteriors2 = (1-pc1) * p2**nheads * (1-p2)**ntails
    return posteriors1/(posteriors1+posteriors2), posteriors2/(posteriors1+posteriors2)
```

In [7]:

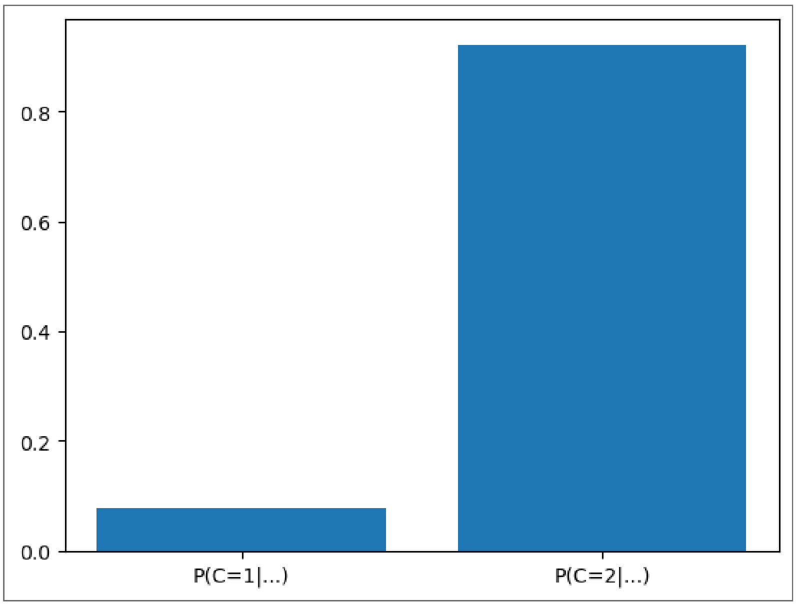
```
nheads = 10
ntails = 10
plt.bar(["P(C=1|...)", "P(C=2|...)"] , calculatePosterior(0.5, 0.5, 0.2, nheads, ntails))
plt.show();
```



- what if there are 5 heads and 15 tails ?

In [9]:

```
nheads = 5; ntails = 15;
plt.bar(["P(C=1|...)", "P(C=2|...)"] , calculatePosterior(0.5, 0.5, 0.2, nheads, ntails)); plt.show()
```



Probabilistic inference $P(Query|Evidence)$

Probabilistic inference in a nutshell: $P(Query|Evidence)$
For example, $P(C|Y_1 = h, Y_2 = h)$

- Query: $\{C\}$: coin choice
- Evidence: $\{Y_1, Y_2\}$, i.e. both tosses are heads

Inference with **nuiance** or **hidden** r.v.: $P(Y_2|Y_1 = h)$

- Query:s $\{Y_2\}$
- Evidence: $\{Y_1\}$

- **Nuisance**, or **hidden** r.v. $\{C\}$, all the other r.v. except $Query$ and $Evidence$: $\{C, Y_1, Y_2\}/\{Y_1, Y_2\}$
- we need to use sum rule to sum them out (so called nuiance r.v.)

$$P(Y_2|Y_1) = \sum_c P(Y_2, C = c|Y_1) = \sum_c P(Y_2|C = c, Y_1)P(C = c|Y_1) \tag{3}$$
$$= \sum_c \underbrace{P(Y_2|C = c)}_{\text{CI assumption!}} P(C = c|Y_1) \tag{4}$$

- it makes perfect sense: to make prediction, we need to consider both possible but hidden coin choices and weight them accordingly (the weights are $P(C|Y_1)$)

How to specify the probability model

The probability model $P(X_1, X_2, \ldots, X_n)$ can be specified by either

1. fully specify the joint distribution: **not practical** in real world applications
    - need $O(2^n)$ number of rows or entries (assuming all r.v. are binary)

1. or simplied by making conditional independence assumption
    - CI usually encodes some causal relations
    - e.g. given knowing the coin of choice, the tosses are independent

$$P(C, Y_1, Y_2, \ldots) = P(C) \prod P(Y_i|C)$$

- we only need 3 parameters for this example rather than ($2^n - 1$); namely $P(C), P(Y_i|C=1), P(Y_i|C=2)$
- note that we only need one parameter for each distribution as the outcome is binary and the sum needs to be one e.g. $P(C=2) = 1 - P(C=1)$

More examples of inference and applying Baye's rule

A classic example of applying Baye's rule:
You are tested positive for COVID by rapid antigen test, with an sensitity of 79% (some call it accuracy) and specificity 98%. Assume you live in the UK, how likely you are really infected with COVID?

- *sensitivity*: the probability of detecting the disease when there is one
- *specificity*: "how well does it detect the absence of disease"

Firstly, identify the random variables,

- $T = \{0, 1\}$: testing result is negative (0) or positive (1)
- $C = \{0, 1\}$: you are truely infected by COVID

Secondly, specify the model $P(C,T)$, essentially to decide which way to apply chain rule

$$P(C,T) = P(C)P(T|C) \text{ or } P(C,T) = P(T)P(C|T)$$

1. Consider the "generating process"; we need the the causal relationship, obviously

$$C \Rightarrow T$$

   i.e. a test result depends on whether you got the disease or not, and we know

$$P(T=1|C=1) = 0.79$$

   - *sensitivity*: the probability of detecting the disease when there is one

$$P(T=0|C=0) = 0.98$$

   - *specificity*: "how well does it detect the absence of disease"
2. then $P(C,T) = P(C)P(T|C)$, we still need to specify the prior $P(C)$
   - $P(C=1) = 0.02$: the prior probability that a person is infected with COVID in UK
     - I used the currect active case number (1.33 million)/the total UK population (67.22 millon) (**https://www.worldometers.info/coronavirus/count**
     - or random pick one from the UK now, the chance he has COVID

Lastly, do the inference, we are interested in the posterior

$$P(C|T=1)$$

- Apply Baye's rule

$$P(C=0|T=1) \propto P(C=0)P(T=1|C=0) = (1-0.02)*(1-0.98) = 0.0196,$$
$$P(C=1|T=1) \propto P(C=1)P(T=1|C=1) = 0.02*0.79 = 0.0158$$

- $P(C|T=1) = \begin{cases} \frac{0.0196}{0.0196+0.0158} = 55.4\%, & C=0 \\ \frac{0.0158}{0.0196+0.0158} = 44.6\%, & C=1 \end{cases}$

So you only have 44.6% chance to really infected with COVID given a positive rapid test result.
That's why you usually will be given another confirming PCR (lab) test (gold standard) to confirm.

Exercise: how likely you are infected after a confirming PCR positive result ? What if the confirming result is negative ?
Assume a PCR test has sensitivity of 90\% and specificity of 99.8\%.

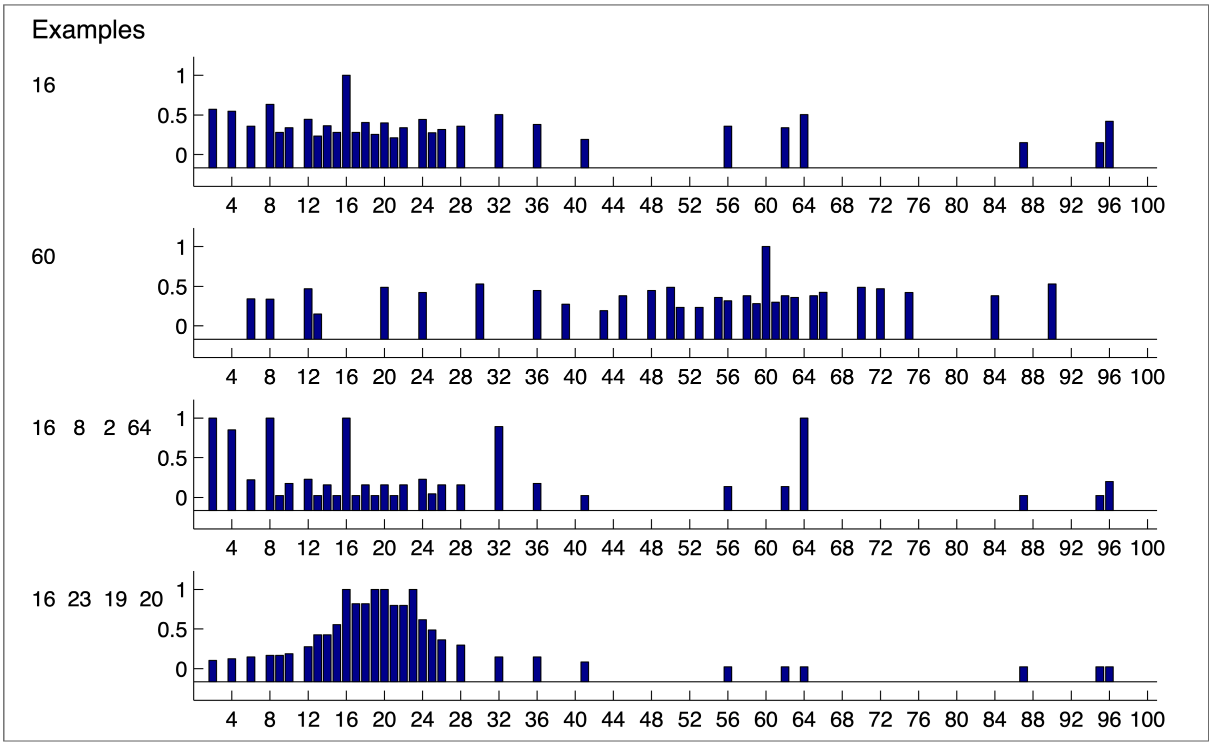
- what r.v.s are there ?
- any conditional independence assumption you can make ?
- what are the parameters ?

Bayesian concept learning

An interesting example of applying **Bayesian inference** to replicate human's **concept learning** process.
Human are good at summarising or learning concepts from examples.
Suppose, you are given a bunch of numbers between 1 and 100:

- if the data are $\{2, 4, 6, 8, 10\}$, you may deduce the *concept* is **even numbers**,
  - then you may predict the next observation is $12\ldots$
- if the data are $\{15, 20, 25\}$, you may deduce the *concept* is **multiples of 5**,
  - then you may predict the next is e.g. 50
- if the data is $\{16\}$, you may deduce the *concept* is **power of 4** or **multiples of 4** or **even** number
  - so you may predict the next is again 64 or maybe $20, 24$ or $18, \ldots$.
  - not likely you would predict e.g. 3

Prediction made by human (averaged over 8 humans)



The question: can we **train** a machine to **learn the concepts** and **make predictions** like humans ?

- i.e. feed observations to a machine, ask the machine learn the **concept** correctly ?
- it turns out machine can learn concept well by using a simple model

Probabilistic model for concept learning

We adopt a simple model on the following r.v. H and $\mathcal{D}$

- H: hypothesis or concepts
    - e.g. $h_{\text{two}} \triangleq \text{"power of two"}$, $h_{\text{even}} \triangleq \text{"even numbers"}$
    - so $h \in \{h_{\text{even}}, h_{\text{odd}}, h_{\text{square}}, h_{\text{two}}, h_{\text{three}}, h_{\text{all}}, \ldots\}$
- $\mathcal{D}$ is the observations, say $\mathcal{D} = \{16, 2, 8\}$

We need to specify the joint distribution: $P(H, \mathcal{D})$

1. specify P by chain rule: $P(H, \mathcal{D}) = P(H)P(\mathcal{D}|H)$
    - causal relationship: $H \Rightarrow \mathcal{D}$
    - easier than the other way around
2. assume conditional independent assumption over $\mathcal{D}$ given h, i.e.

$$P(\mathcal{D}|h) = P(d_1, d_2, \ldots, d_N|h) = \prod_{i=1}^{N} P(d_i|h)$$

    - reasonable assumption, given knowing the concept, previous observations no longer influence the later

The parameters for $P(h)$ and $P(D|h)$

For prior $P(H=h), h \in \{h_{\text{two}}, h_{\text{even}}, h_{\text{odd}}, h_{\text{three}}, h_{\text{all}}, \ldots\}$,

- due to igorance, we can simply use largely uniform prior over concepts of: square, multiples, powers
- but assign higher probability to some common concepts, say $h_{\text{even}}, h_{\text{odd}}$
- and penalise some very rare concepts, say $h_{\texttt{powers of 2+\{37\}}}$, $h_{\texttt{powers of 2-\{32\}}}$

We use a uniform distributed likelihood for $P(d_i|h)$:

$$P(d_i|h) = \begin{cases} \left[\frac{1}{\text{size}(h)}\right], & \text{if } d_i \in h \\ 0 & \text{otherwise} \end{cases}$$

- $\text{size}(h)$ is the size of qualifying set of the concept
  - e.g. $h_{\text{even}} = \{2, 4, 6, \ldots, 100\}$, $\text{size}(h_{\text{even}}) = 50$
  - $h_{\text{end in 9}} = \{9, 19, 29, \ldots, 99\}$, $\text{size}(h_{\text{even}}) = 10$
- therefore,

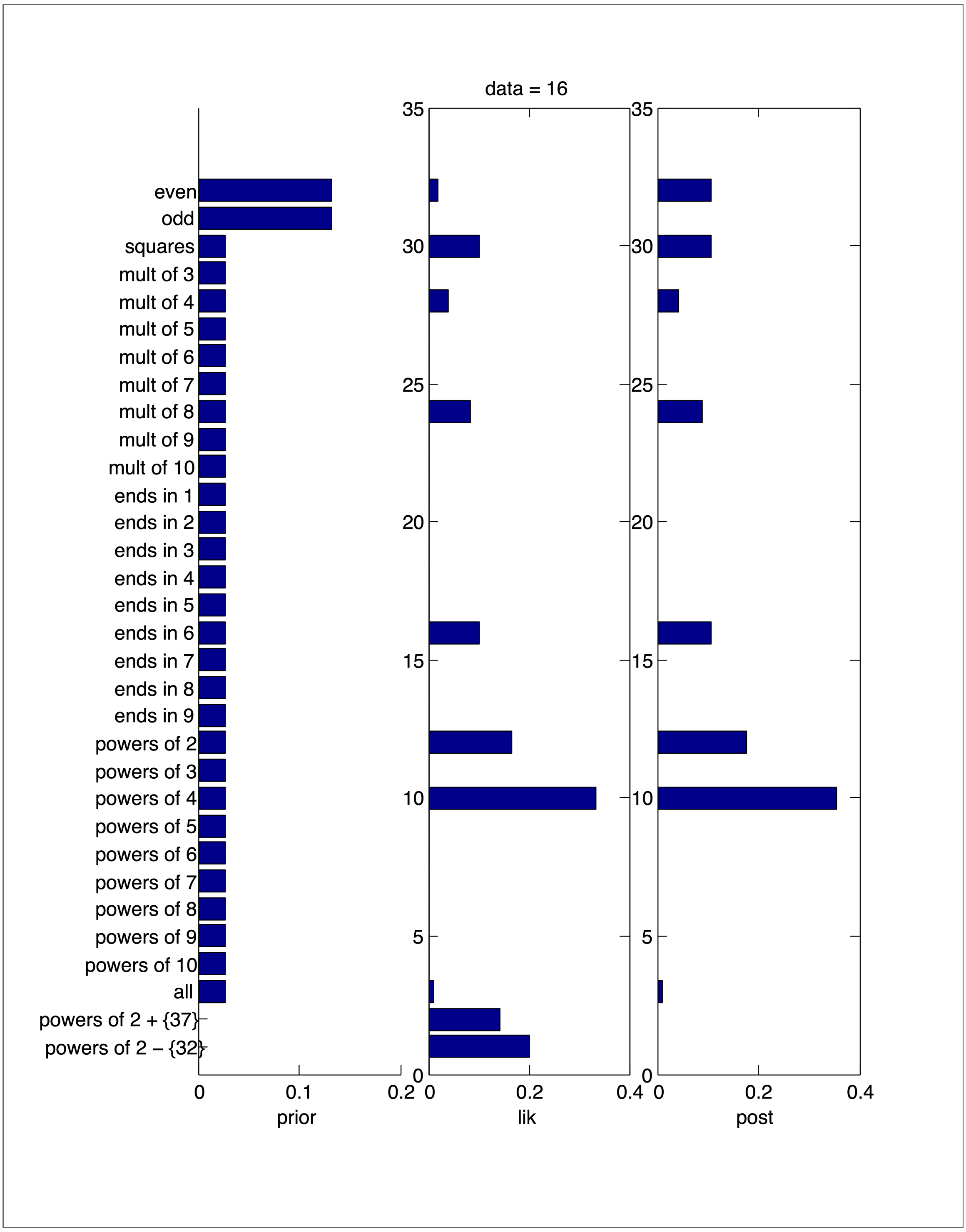
$$P(\{16\}|h_{\text{even}}) = 1/50, P(\{16\}|h_{\text{power of 4}}) = 1/3, P(\{16\}|h_{\text{end in 9}}) = 0,$$

$$P(\{16, 8, 2, 64\}|h_{\text{even}}) = (1/50)^4, P(\{16, 8, 2, 64\}|h_{\text{power of 4}}) = 0$$

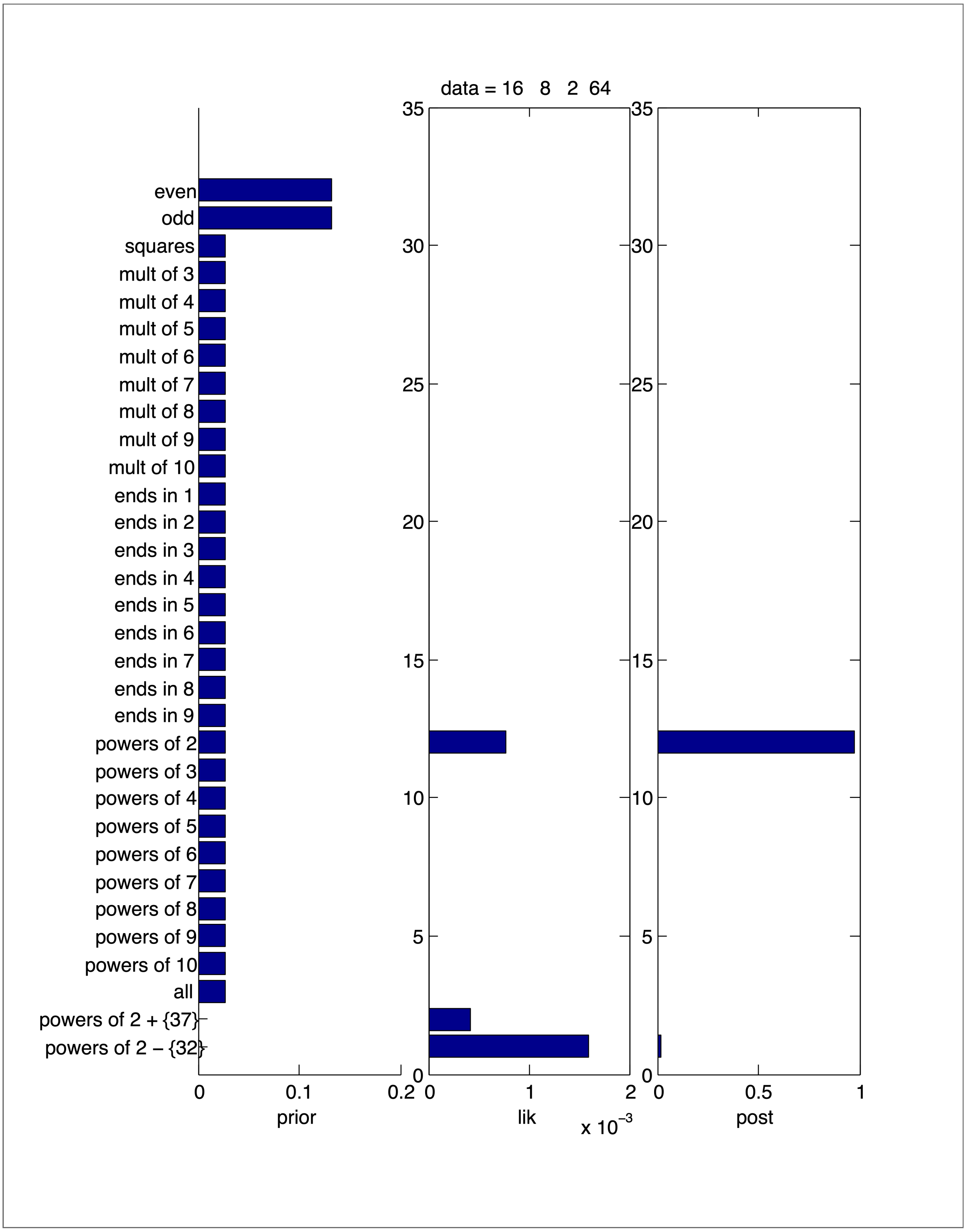
Concept inference by Baye’s rule with $\mathcal{D} = \{16\}$

Baye's rule:

$$P(H|\mathcal{D}) \propto \underbrace{P(H)}_{\text{prior}}\underbrace{P(\mathcal{D}|h)}_{\text{Likelihood}}$$



Concept inference by Baye's rule with $\mathcal{D} = \{16, 8, 2, 4\}$



Bayesian predictive distribution (not examinable)

We also want to predict future observations:

$$P(d_{N+1}|\mathcal{D})$$

- conditional on the observation so far, the posterior distribution over a future observation d_{N+1}

Use the two rules: sum and chain rule

$$P(d_{N+1}|\mathcal{D}) = \sum_h P(d_{N+1}, h|\mathcal{D}) = \sum_h P(d_{N+1}|h, \mathcal{D})P(h|\mathcal{D})$$

We further apply conditional independent assumption

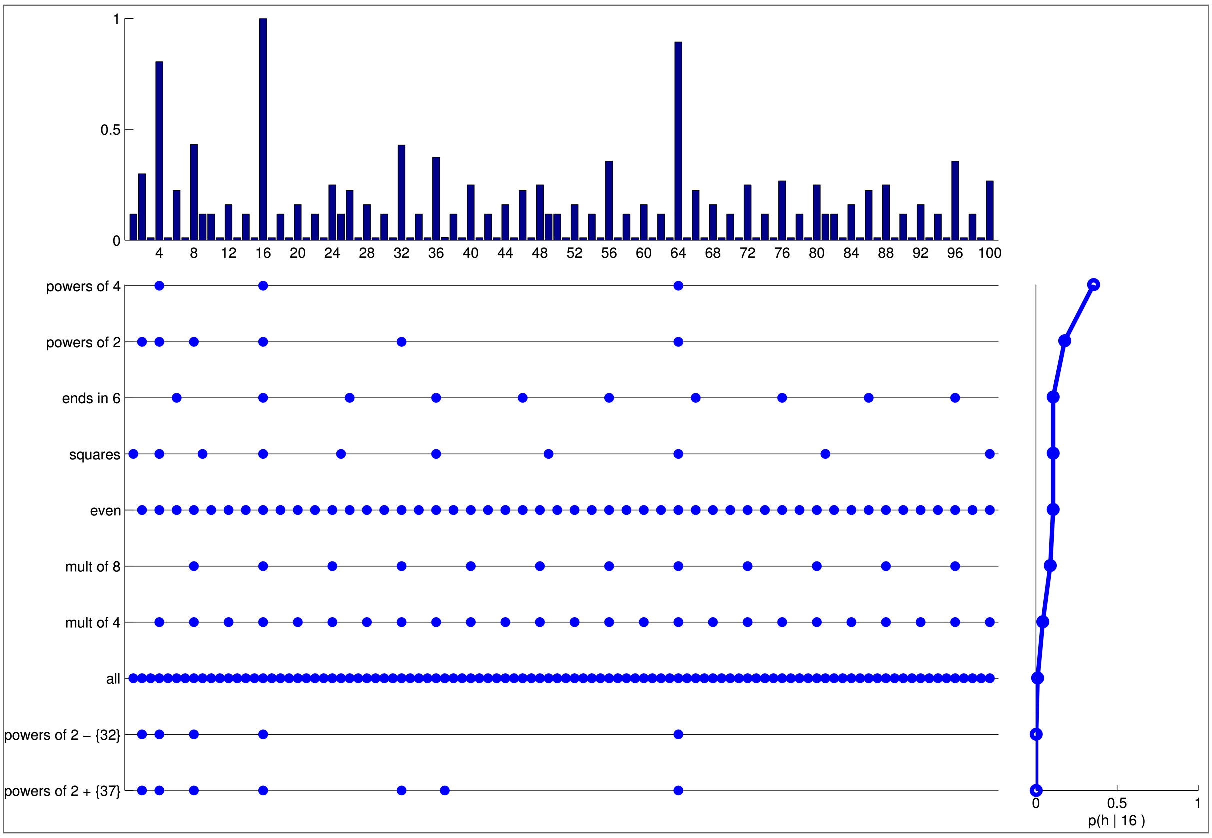
$$P(d_{N+1}|h, \mathcal{D}) = P(d_{N+1}|h)$$

Combined the above we have

$$P(d_{N+1}|\mathcal{D}) = \sum_h P(d_{N+1}|h)P(h|\mathcal{D})$$

- a weighted average over $P(d_{N+1}|h)$
- weights are $P(h|\mathcal{D})$

Bayesian predictive distribution on $\mathcal{D} = \{16\}$



Machine’s prediction on concept learning based on Bayesian inference

Compare with human's prediction (left), machine's prediction (right) is pretty much indistinguishable !

Human's prediction	Machine's prediction
.	.

Summary

- Using joint probability distribution
  - contains all the information we need
  - but not practical: too many parameters
- Conditional independence
  - simplifies the joint distribution $P(\mathcal{D}|H=h) = \prod P(d|H=h)$
- Inference is inverse engineering
  - specify the distribution (generating process): by using chain rule and CI assumption

$$P(\mathcal{D}, H) = P(H)P(\mathcal{D}|H)$$

  - inference is doing the reverse engineering: given the observation what are the unknowns (by Baye's rule)

$$P(H=h|\mathcal{D}) \propto P(H=h)P(\mathcal{D}|H=h)$$