Software Implementation for Estimation of Optimal Cost Density Function with Cost Cumulants

Salvatore J. Giorgi, Firdous Saleheen, and Chang-Hee Won

Abstract— **Statistical control considers the problem of finding a controller which manages the value of linear combinations of multiple cost cumulants associated with a class of stochastic dynamical systems. With this in mind, we consider deriving a full statistical characterization of a random cost function, namely, its probability density function, from the cost cumulants. This paper presents the development of software to compute and plot the cost density function. By plotting this density function one can monitor each cumulants impact on the overall cost function, which will, in turn, help to increase the depth of understanding of statistical control theory.**

I. Introduction

Starting in 1965, Sain first examined minimal cost variance for the open-loop situation [1]. In 1971, Sain and Liberty considered minimizing the performance variance with a pre-constrained mean for an open loop system [2]. Liberty continued to investigate characteristic function of integral quadratic forms. He with Hartwig, developed methods of generating cumulants in the time domain [3]. In 1992, Won solved the state-feedback minimum-cost-variance (MCV) problem for the closed-loop case. The solution gave rise to a MCV control law that is composed of two matrix functions that satisfy a system of two coupled Riccati equations. [4]. In 2004, Pham generalized the MCV control with his kCC control which minimizes a linear combination of first k cumulants [5]. He explored the cumulants of finite-horizon integral quadratic cost associated with a linear stochastic system. The cumulants were obtained by the generalized Karhunen-Loeve expansion presented in [3]. Pham showed that if these cost cumulants resulted from the series expansion, are further expressed in terms of the iterated-integral kernels then the numerical values of these cost cumulants of any order can be calculated exactly by solving the corresponding set of backward-in-time differential equations.

Continuing in this progression towards total probabilistic control, we consider the complete specification of the probability density function when formulating an optimal control law. Liberty and Hartwig developed a computer-aided software package in [3] to provide a complete statistical description of random cost function associated with a Gauss-Markov linear dynamical process in the form of a plot of the probability density function (pdf). However, a good approximation for the first characteristic function for the random cost function is often computationally demanding. Akin and Counts [6] showed a rational inversion procedure for rational approximation of inverse Laplace transform. The procedure utilized series expansions and Viskovatov's method as discussed by Khovanskii [7] for the generation of a continued fraction. Burrows [8] demonstrated a technique for rational approximation of probability density functions using Viskovatov method formulated in [7].

The purpose of this study is to develop a full software package which generates a graphical model of an approximated pdf of an integral quadratic cost function in a more efficient and accurate way. A finite horizon stochastic linear time invariant system is considered. The software uses Pham's results [5] to calculate a user specified number of cumulants. From the cumulant values the first and second characteristic functions are approximated as demonstrated in [8]. Finally, the pdf is obtained from the first characteristic function.

A graphical software package which fully illustrates the complete probabilistic description of a random cost will greatly increase the utility of statistical control methods. One will be able to determine exactly how each statistic affects the overall performance of the controller as well as shape the cost density through fine tuning of control parameters.

The paper is organized as follows: Section II states the optimal control problem based on calculating cost cumulants as in addition to stating the problem of reconstructing the pdf. Section III formulates both of these problems. A continued fractions approximation technique is discussed in Section IV. Section V describes the software framework for the solution of pdf calculation. Section VI presents several numerical examples, while Section VII discusses the capabilities of the developed software. Finally, Section VIII concludes our work and suggests future work on the subject.

II. Preliminaries and Problem Statement

Consider the stochastic linear dynamical system modeled on $[t_0,t_f]$ by,

$$d\left(x\left(t\right)\right)=\left(Ax\left(t\right)+Bu\left(t\right)\right)dt+G(t)dw\left(t\right),\quad x(t_0)=x_0 \quad (1)$$

where continuous coefficients $A\in\mathscr{C}([t_0,t_f];\mathbb{R}^{n\times n})$; $B\in\mathscr{C}([t_0,t_f];\mathbb{R}^{n\times m})$; $G\in\mathscr{C}([t_0,t_f];\mathbb{R}^{n\times p})$. The process $w(t)$ is a stationary Weiner process on complete probability space $(\Omega,\mathscr{F},\mathscr{P})$ over $[t_0,t_f]$ with the correlation of increments $E\{[w(\tau)-w(\zeta)][w(\tau)-w(\zeta)]^T\}=W|\tau-\zeta|$ and $W>0$. Also assume that the initial condition, $x(t_0)$ is known.

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For the given (t_0, x_0), the system state is $x(t)$ is uniquely determined from $u(t)$. Thus a finite horizon integral quadratic form random cost $J : \mathscr{C}([t_0,t_f];\mathbb{S}^n) \mapsto \mathbb{R}^+$ such that

$$J = \int_{t_0}^{t_f} (x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau))d\tau + x^T\left(t_f\right)Q_f x\left(t_f\right) \tag{2}$$

where $Q \in \mathscr{C}([t_0,t_f];\mathbb{R}^{n\times n}$ is positive semidefinite; $R \in \mathscr{C}([t_0,t_f];\mathbb{R}^{m\times m}$ is positive definite. Also $Q_f \in \mathbb{R}^{n\times n}$ is symmetric and positive semidefinite.

Suppose further that both $k \in \mathbb{Z}^+$ and the sequence $\gamma = \{\gamma_i \geq 0\}_{i=1}^k$ with $\gamma_1 > 0$. Then the finite horizon optimal state-feedback kCC control is achieved [5] by the gain

$$K^*(\alpha) = -R^{-1}B^T \sum_{r=1}^{k} \hat{\gamma}_r H^*(\alpha, r), \quad \alpha \in [t_0, t_f], \tag{3}$$

where the real constants $\hat{\gamma}_r = \frac{\gamma_i}{\gamma_1}$ represent parametric control design freedom and $\{H^*(t_0,r) \geq 0\}_{r=1}^k$ are symmetric solutions of the backward differential Riccati-type equations [5]

$$\begin{aligned} -\frac{d}{d\alpha}H^*(\alpha,1) &= [A+BK^*(\alpha)]^T H^*(\alpha,1) + \\ & H^*(\alpha,1)[A+BK^*(\alpha)] + K^{*T}(\alpha)RK^*(\alpha) + Q \end{aligned} \tag{4}$$

$$\begin{aligned} -\frac{d}{d\alpha}H^*(\alpha,r) &= [A+BK^*(\alpha)]^T H^*(\alpha,r) + H^*(\alpha,r) \\ & [A+BK^*(\alpha)] + \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!} H^*(\alpha,s)GWG^T H^*(\alpha,r-s), \end{aligned} \tag{5}$$

with terminal conditions $H^*(t_f,1) = Q_f$, and $H^*(t_f,r) = 0$ when $2 \leq r \leq k$, whenever these solutions exist.

Now the problem is to estimate probability density of the cost function of (2) using (3), (4), and (5).

III. Problem Formulation

The kth cost cumulant, $\kappa_k \in \mathbb{R}^+ \cup \{0\}$, of J is determined by [5]

$$\kappa_k = k!2^{k-1}{x_0}^T H(t_0,k)x_0 + (k-1)!2^{k-1}D(t_0,k), \tag{6}$$

The cost cumulant building entity $D(t_0,k)$ is given by [5]

$$D(t_0,k) = \int_{t_0}^{t_f} Tr\{H(t_0,k)\}GWG^T dt \tag{7}$$

To find $D(t_0,k)$, a symmetric solution of (4) and (5) is used. Using (4), (5) and (7) κ_k is obtained.

The second characteristic function or cumulant generating function $\psi_J(\omega)$ has a MacLaurin series expansion in a neighborhood about the origin and is given by [9]

$$\psi_J(\omega) = \sum_{k=1}^{\infty} \kappa_k \frac{(j\omega)^k}{k!} \tag{8}$$

where the coefficients κ_k are k-th cost cumulants.

The first characteristic function also has a series expansion

$$C_J(\omega) = M_J(\omega) = \sum_{i=0}^{\infty} \frac{m_i \omega^i}{i!}, \tag{9}$$

where M_J is the moment generating function, $m_0 = 1$, and m_i, for $i = 1,2,...$ are the ith moments. The first characteristic function is related to the second characteristic function by [9]

$$C_J(\omega) = e^{\psi_J(\omega)}. \tag{10}$$

The moments can be expressed in terms of cumulants using Bell polynomials [9]. The nth moments can be expressed in terms of n cumulants, $\kappa_1, \kappa_2, ..., \kappa_n$, using the nth complete Bell polynomial

$$m_n = B_n(\kappa_1, ..., \kappa_n) = \sum_{k=1}^{n} B_{n,k}(\kappa_1, ..., \kappa_{n-k+1}), \tag{11}$$

where

$$\begin{aligned} & B_{n,k}(\kappa_1, ..., \kappa_{n-k+1}) = \\ & \sum \frac{n!}{j_1!j_1!...j_{n-k+1}!}\left(\frac{\kappa_1}{1!}\right)^{j_1}\left(\frac{\kappa_2}{2!}\right)^{j_1}...\left(\frac{\kappa_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}. \end{aligned} \tag{12}$$

Here, the sum is taken over all sequences $j_1, j_2, ..., j_{n-k+1}$ of non negative integers such that $j_1 + j_2 + ... = k$ and $j_1 + 2j_2 + 3j_3 + ... = n$.

The probability density function is related to the first characteristic function by [9]

$$C_J(\omega) = E\{e^{j\omega\zeta}\} = \int_{-\infty}^{\infty} e^{j\omega\zeta} f_J(\zeta)d\zeta, \tag{13}$$

where f_J is the probability density function or cost density function of (2). Thus, the cost density function f_J can be found by taking inverse Fourier transform of $C_J(\omega)$

$$f_J(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\zeta} C_J(\omega)d\omega. \tag{14}$$

IV. Approximation Technique

Since (8) is an infinite sum power series, the use of such a power series to compute the cost density can be quite cumbersome. Finding inverse Fourier transform (14) for each term in the power series can yield slowly convergent or even divergent series which may not be suited for practical calculations [6]. Hence, an inversion procedure is needed that is as simple to generate as a series expansion and which can be easily inverted to yield an accurate rational approximation to the inverse. A method of continued fractions is viewed in part as a way of obtaining rational functions. The rational functions such that their own power series expansion matches that of the original series associated with the second characteristic function up to some prescribed number of terms, is used to circumvent these computational difficulties.

Now we shall use the Viskovatov continued fraction technique as formulated in [7]. The Viskovatov technique develops a continued fractions approximation from a ratio

of two power series. For the general case we consider a function

$$F(s) = \frac{\alpha_{10} + \alpha_{11}s + \alpha_{12}s^2 + ...}{\alpha_{00} + \alpha_{01}s + \alpha_{02}s^2 + ...}. \quad (15)$$

The continued fractions expansion is as follows:

$$F(s) = \frac{1}{\frac{\alpha_{00}+\alpha_{01}s+\alpha_{02}s^2+...}{\alpha_{10}+\alpha_{11}s+\alpha_{12}s^2+...}}$$

$$= \frac{1}{\frac{\alpha_{00}}{\alpha_{10}} + \frac{\alpha_{00}+\alpha_{01}s+\alpha_{02}s^2+...}{\alpha_{10}+\alpha_{11}s+\alpha_{12}s^2+...} - \frac{\alpha_{00}}{\alpha_{10}}}$$

$$= \frac{\alpha_{10}}{\alpha_{00} + \frac{\alpha_{10}(\alpha_{00}+\alpha_{01}s+\alpha_{02}s^2+...)}{\alpha_{10}+\alpha_{11}s+\alpha_{12}s^2+...} - \frac{\alpha_{00}(\alpha_{10}+\alpha_{11}s+\alpha_{12}s^2+...)}{\alpha_{10}+\alpha_{11}s+\alpha_{12}s^2+...}}$$

$$= \frac{\alpha_{10}}{\alpha_{00} + \frac{(\alpha_{10}\alpha_{01}-\alpha_{00}\alpha_{11})+(\alpha_{10}\alpha_{02}-\alpha_{00}\alpha_{12})s+...}{\alpha_{10}+\alpha_{11}s+\alpha_{12}s^s+...}s}$$

$$= \frac{\alpha_{10}}{\alpha_{00} + \frac{\alpha_{20}+\alpha_{21}s+\alpha_{22}s^2...}{\alpha_{10}+\alpha_{11}s+\alpha_{12}s^s+...}s}$$

where $\alpha_{2j} = \alpha_{10}\alpha_{0,j+1} - \alpha_{00}\alpha_{1,j+1}$ for $j = 0,1,2,....$

Continuing in this fashion we have

$$F(s) = \frac{\alpha_{10}}{\alpha_{00} + \frac{\alpha_{20}}{\alpha_{10}+\frac{\alpha_{30}+\alpha_{31}s+\alpha_{32}s^2+...}{\alpha_{20}+\alpha_{21}s+\alpha_{22}s^s+...}s}}$$

where $\alpha_{3j} = \alpha_{20}\alpha_{1,j+1} - \alpha_{10}\alpha_{2,j+1}$ for $j = 0,1,2,....$ Finally, we generalize the above by

$$F(s) = \frac{\alpha_{10}}{\alpha_{00} + \frac{\alpha_{20}s}{\alpha_{10}+\frac{\alpha_{30}s}{\alpha_{20}+\frac{\alpha_{40}s}{\alpha_{30}+...}}}} \quad (16)$$

where $\alpha_{ij} = \alpha_{i-1,0}\alpha_{i-2,j+1} - \alpha_{i-2,0}\alpha_{i-1,j+1}$ for $i \geq 2$ and $j \geq 0$. We will use the following notation for (16)

$$F(s) = \frac{\alpha_{10}}{\alpha_{00}} + \frac{\alpha_{20}s}{\alpha_{10}} + \frac{\alpha_{30}s}{\alpha_{20}} + ... \quad (17)$$

Next, we consider the rational polynomials that are generated when the continued fraction is terminated after a finite number of terms, which we define as

$$\frac{N_L}{D_L} \equiv \frac{\alpha_{10}}{\alpha_{00}} + \frac{\alpha_{20}s}{\alpha_{10}} + \frac{\alpha_{30}s}{\alpha_{20}} + ... + \frac{\alpha_{L,0}s}{\alpha_{L-1,0}}. \quad (18)$$

From the definition above we have the following recurrence relationship, as seen in [7],

$$\frac{N_1}{D_1} = \frac{\alpha_{10}}{\alpha_{00}},$$

and

$$\frac{N_2}{D_2} = \frac{\alpha_{10}}{\alpha_{00} + \frac{\alpha_{20}s}{\alpha_{10}}} = \frac{\alpha_{10}\alpha_{10}}{\alpha_{00}\alpha_{10} + \alpha_{20}s} = \frac{\alpha_{10}N_1}{\alpha_{10}D_1 + \alpha_{20}s},$$

which is generalized by

$$\frac{N_L}{D_L} = \frac{\alpha_{L-1,0}N_{L-1} + \alpha_{L,0}N_{L-2}s}{\alpha_{L-1,0}D_{L-1} + \alpha_{L,0}D_{L-2}s}, \quad (19)$$

with $N_0 = 0$ and $D_0 = 1$. From this we get an approximation to the original rational polynomial

$$\frac{N_L}{D_L} \approx F(s). \quad (20)$$

In our case, we will use two approximations, to see which gives the most statistical insight. For the first case we approximate the second characteristic function. Here, the numerator of $F(s)$ equals (8), the second characteristic function, and the denominator is one. Thus, $\alpha_{00} = 1$ and $\alpha_{0i} = 0$, for $i = 1,2,....$ We let $Z_L(s)$ be a truncated form of (8) with L terms,

$$Z_L(s) = s\sum_{r=0}^{L-1} \frac{\kappa_{r+1}}{(r+1)!}s^r = s\sum_{r=0}^{L-1} B(r)s^r \quad (21)$$

where

$$B(0) = \frac{\kappa_1}{1!}, B(1) = \frac{\kappa_2}{2!}, \cdots, B(L-1) = \frac{\kappa_L}{L!}, \quad (22)$$

and construct the terms in (16) as follows:

$\alpha_{00} = 1,$
$\alpha_{0q} = 0, \quad 1 \leq q \leq L,$
$\alpha_{1q} = B(q), \quad 0 \leq q \leq L-1,$
$\alpha_{pq} = \alpha_{p-1,0}\alpha_{p-2,q+1} - \alpha_{p-2,0}\alpha_{p-1,q+1},$

for $2 \leq p \leq L-1$ and $0 \leq q \leq L-p$. Finally, we construct (19) using the α_{pq} values above to get our approximation to the second characteristic function (8). Then combining (10) and (14) we get the following

$$f_J(\zeta) = \mathscr{F}^{-1}\{C_J(\omega)\} \approx \mathscr{F}^{-1}\{e^{s\frac{N_L}{D_L}}\}. \quad (23)$$

For the second case we approximate the first characteristic function, where the moments are generated with the n cumulants from the complete Bell polynomials (11), and then use the continued fractions approximation. The numerator of $F(s)$ equals (9), the first characteristic function, and the denominator is one. Thus, $\alpha_{00} = 1$, $\alpha_{0i} = 0$, for $i = 1,2,...,$ and

$\alpha_{00} = 1,$
$\alpha_{0q} = 0, \quad 1 \leq q \leq L,$
$\alpha_{10} = 1,$
$\alpha_{1q} = \frac{m_q}{q!}, \quad 1 \leq q \leq L-1,$
$\alpha_{pq} = \alpha_{p-1,0}\alpha_{p-2,q+1} - \alpha_{p-2,0}\alpha_{p-1,q+1},$

for $2 \leq p \leq L-1$ and $0 \leq q \leq L-p$. Finally, we construct (19) using the α_{pq} values above to get our approximation to the first characteristic function. Then combining (10) and (14) we get the following

$$f_J(\zeta) = \mathscr{F}^{-1}\{C_J(\omega)\} \approx \mathscr{F}^{-1}\{\frac{N_L}{D_L}\}. \quad (24)$$

V. Software Framework

The software framework is as follows. First, the user inputs the values of their system (including system matrices, covariance matrices, initial states, etc.), cost function constraints, number of cumulants k to calculate, and values of gamma γ. Next, the program calculates the solutions to (4), (5), also determines (7). Then the program calculates the cost cumulants. With these cumulant values, the software then determines the second characteristic function. Next, it calculates the first characteristic function and, finally, calculates and plots the pdf. The details of each step are discussed below.

Step 0: Initialize system matrices A and B, cost function matrices Q and R, W, G, number of cumulants to solve for k, γ_r for $r = 1,2,\cdots k$, terminal conditions $H(t_f,k) = 0$, for $i = 1,2,\cdots k$, and initial state x_0.

Step 1: Solve k coupled differential Riccati-type equations backward in time;

$$\frac{dX}{dt} = \begin{bmatrix} \frac{dH(t_0,1)}{dt} & \frac{dH(t_0,2)}{dt} & \cdots & \frac{dH(t_0,k)}{dt} \end{bmatrix}^T$$

$$X = \begin{bmatrix} H(t_0,1) & H(t_0,2) & \cdots & H(t_0,k) \end{bmatrix}^T$$

Approximate solution of *dX/dt* using Runge-Kutta methods, with *initial conditions* = **0** and time = [0 10];

Flip solution X;

Step 2: Calculate D matrices from (7). Use trapezoidal approximation to integral using time values of each $H(t_0,k)$;

Step 3: Calculate k cumulants using (6);

Step 4: Calculate second characteristic (8) ith series approximation;

Initialize $firstchar = 1$;

Loop (for i = 1:k)

$\left\{ firstchar = firstchar * exp(s^{(i)} * z(i)) \right\}$

Step 5: Calculate second characteristic equation (8) with continued fractions approximation;

The algorithm for this is discussed in Section IV;

Step 6: Calculate first characteristic equation (10) by taking the exponential of the results from Step 4 power series approximation;

Step 7: Calculate first characteristic equation (10) by taking the exponential of the results from Step 5 (continued fractions approximation);

Step 8: Take inverse Fourier transform of the results of both Step 6 and 7;

Step 9: Plot the inverse Fourier transforms.

As mentioned in the introduction, the set of coupled Riccati-type equations are solved backward in time. This can be done using the Runge-Kutta method and considering our known terminal conditions as the initial conditions needed for the algorithm. The Runge-Kutta method integrates forward in time, thus, the the solution to this technique needs to be flipped to represent our backward-in-time integration. The solution X is an $l \times kn^2$ matrix, where the entries in row p correspond to approximated values of each of the entries of the $H(t_0,i)$, for $i = 1,...,k$, matrices at the pth moment in time. Since (6) depends on time values of $H(t_0,i)$ on the interval $[t_0\ t_f]$, by taking the ith block of n^2 entries from each row we can reconstruct $H(t_0,i)$ at each moment in time. With the time values of the reconstructed matrices $H(t_0,i)$ we can use a trapezoidal integration approximation to calculate $D(t_0,i)$. Once $D(t_0,i)$ and $H(t_0,i)$ have been calculated, the cumulants immediately follow.

VI. Numerical Examples

Here we present three numerical examples, including a scalar case and matrix case. Both graphical examples consider the first two cumulants.

TABLE I: Cumulant Values for Scalar System

γ_2	First Cumulant	Second Cumulant
0.1	8.03	4.16
0.5	8.79	3.98
1	9.80	3.94
2.5	13.58	4.33
5	30.84	7.83

A. Scalar Case

For the scalar example, we set $A = B = Q = R = W = 1$, $G = 0.3$, $x_0 = 0$, $\gamma_1 = 1$, and $k = 2$.

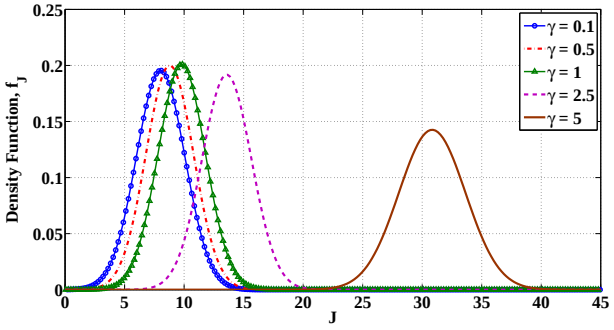

Fig. 1: Cost Density Function for Scalar Example with $k = 2$ and Varying γ_2

Fig. 1 shows the pdf of the cost function for γ_2 = 0.1, 0.5, 1, 2.5, and 5. The numerical values of the cumulants are shown in Table I.

B. Matrix Case

Following parameters were used for matrix example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \end{bmatrix}^T, R = 1, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, x_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T,$$

$\gamma_1 = 1$

Fig. 2 shows the pdf of the cost function for $\gamma_2 = 1, 5, 10, 15, 25$.

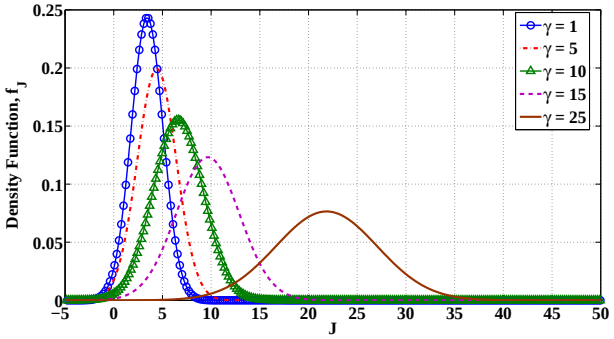

Fig. 2: Cost Density Function for Matrix Example with $k = 2$ and Varying γ

TABLE II: Cumulant Values for Matrix System

γ_2	First Cumulant	Second Cumulant
1	3.43	2.68
5	4.22	4.00
10	6.61	6.60
15	9.65	10.49
25	21.85	27.19

TABLE III: Higher Order Cumulant Values for Matrix System

Cumulant	Value	Cumulant	Value
1	4.62	6	7778.79
2	0.59	7	-11866.45
3	1.04	8	-83571.24
4	3.68	9	791686.51
5	19.91	10	-9635313.45

C. Higher Order Cumulants

Though we are currently unable to plot the pdf for higher order cumulants, we are able to compute the values of the cumulants numerically. Table III shows the values for the first ten cumulants for the matrix system described above. For this example, γ_r, for $r = 1, 2, \cdots, 10$ are all equal to 1.

VII. Discussion

All of the pdfs above are calculated from the first two cumulants, i.e. mean and variance, using a truncated power series expansion of the second characteristic function. It is important to note that ((8)) is an infinite series which we are approximating by only two terms. This is equivalent to considering a distribution whose cumulants greater than two are equal to zero. For example, consider the third cumulant. This cumulant is directly related to the distribution's skewness and is a measure of lopsidedness. Any distribution with a third moment equal to zero will be symmetric. This is exactly what is seen in the above figures. Each pdf is symmetric about the first cumulant. Similar arguments can be made for the fourth and higher order cumulants. Therefore, we know that significant statistical information is being lost by only using the first two cumulants to reconstruct the pdf.

The software does not come without benefits, though. A great deal of insight can be gained from density functions calculated from only mean and variance. By varying gamma we still retain the ability to move the distribution's center and change its variance. Additionally, while the software cannot graph pdfs constructed from higher order cumulants, we still have the ability to calculate numerical value of the cumulants. Simulations have shown that for any size system A and B there is no limit to the number of cumulants we are able to calculate. Combining these values with available graphs we can get a better idea of what the true pdf will look like. Again, using the third cumulant as an example, we are able to infer to which side of the mean the distribution is skewed.

Currently, our software has full functionality for any size system A and B, with $k = 1, 2$ with a direct power series computation for the second characteristic function. By full functionality, we mean that the software computes the k cumulants, generates the pdf, and produces a plot of the pdf. Additionally, we can compute any number of cumulants for any size system. Problems arise when trying to compute a closed form of the pdf from the inverse Fourier Transform for k greater than 2. Because the continued fractions approximation introduces higher order s terms, we are not able to compute the pdf for any k using this method.

To resolve the issue with Fourier transform, one alternative way can be using Fast Fourier transform [10]. The approximated first characteristic function is a continuous function in the frequency domain and must therefore be converted to a series of finite, discrete samples via a means of bilinear transformation before the cost density can be obtained. Since the data input to the fast Fourier transform is a finite sequence, the output is also a finite sequence. The values of sampling rate and resolution in frequency domain must be chosen carefully to minimize the possible errors due to aliasing.

VIII. Conclusion and Future Work

Without the ability to plot pdfs for $k > 2$ our software has limited capability. While we do have the ability to calculate each cumulant, as a visual tool, our current plots show a limited approximation to the cost distribution. Therefore, future work includes developing a method to approximate an inverse Fourier transform for a characteristic function which includes multiple high order terms. Once this is complete, one can compare the continued fractions method to the direct power series computation. At that point, effects of higher order cumulants can be studied in detail.

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