Lecture 6, Sept. 19

Tautology

Let F and G be formula and let S be a set of formulas

Notation. We say that F is a tautology, and we write $\models F$, when $\alpha(F) = 1$ for every assignment α

We say that F is a contradiction when $\alpha(F) = 0$ for every assignment α, or equivalently when $\models \neg F$

We say that F is equivalent to G, and we write $F \equiv G$ when $\alpha(F) = \alpha(G)$ for every assignment α

We say that argument "S therefore G" is valid, or that "S induces G" or that "G is a consequence of S", when for every assignment α for which $\alpha(F) = 1$ for every $F \in S$ we have $\alpha(G) = 1$.

When $S = \{F_1, F_2, \dots, F_n\}$ we have $S \models G$ is equivalent to $\{((F_1 \wedge F_2) \wedge \cdots \wedge F_n)\} \models G$ which is equivalent to $\models (((F_1 \wedge F_2) \wedge \cdots \wedge F_n) \to G)$

When we consider an argument "S therefore G", the formula in S are called the premises for the hypothesis or the assumption and the formula G is called the conclusion of the argument.

Here are some examples of tautology.

1. $\models F \vee \neg F$
2. $\models P \to P$
3. $\models P \leftrightarrow P$
4. $\models \neg(P \wedge \neg P)$
5. $\models \neg P \to (P \to Q)$
6. $\models Q \to (P \to Q)$

Here are some truth equivalences

1. $P \equiv P$
2. $P \equiv \neg\neg P$
3. $P \vee Q \equiv Q \vee P$
4. $P \wedge Q \equiv Q \wedge P$
5. $P \leftrightarrow Q \equiv Q \leftrightarrow P$
6. $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
7. $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
8. $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
9. $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Here are some valid argument

1. $\{P\} \models P$
2. $\{P \wedge Q\} \models P$
3. $\{P \wedge Q\} \models Q$
4. $P \models \{P \wedge Q\}$
5. $Q \models \{P \wedge Q\}$
6. $\{\neg P\} \models P \to Q$
7. $\{Q\} \models P \to Q$
8. $\{P, Q\} \models P \leftrightarrow Q$
9. $\{\neg P, \neg Q\} \models P \leftrightarrow Q$
10. $\{P, P \to Q\} \models Q$

6.1 Example. Determine whether

$$\models (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$$

Solution. We make a truth table for

$$F = (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$$

P	Q	R	$(P \to (Q \to R))$	$((P \to Q) \to (P \to R))$
1	1	1	1	1
1	1	0	0	0
1	0	1	1	1
1	0	0	1	1
0	1	1	1	1
0	1	0	1	1
0	0	1	1	1
0	0	0	1	1

Here are some relationships between tautologies, equivalences and validity.

$$\begin{aligned} F \equiv G &\Leftrightarrow \models (F \leftrightarrow G) \\ &\Leftrightarrow \models ((F \to G) \wedge (G \to F)) \\ &\Leftrightarrow \{F\} \models G \text{ and } \{G\} \models F \end{aligned}$$

When $S = \{F_1, F_2, \ldots, F_n\}$,

$$\begin{aligned} S \models G &\Leftrightarrow \{F_1, F_2, \ldots, F_n\} \models G \\ &\Leftrightarrow (((F_1 \wedge F_2) \wedge \cdots \wedge F_n) \models G \\ &\Leftrightarrow \models (((F_1 \wedge F_2) \wedge \cdots \wedge F_n) \to G \end{aligned}$$

Also

$$\models F \Leftrightarrow \emptyset \models F$$

$\models F$ means for all assignment α, $\alpha(F) = 1$

$\emptyset \models F$ means for all assignment α, if (for every $G \in \emptyset, \alpha(G) = 1$) then $\alpha(F) = 1$

Notation. For a set A and a statement or formula F

$$\forall x \in A \;\; F \text{ means } \forall x (x \in A \to F)$$

and

$$\exists x \in A \;\; F \text{ means } \forall x (x \in A \wedge F)$$

So (for every $G \in \emptyset, \alpha(G) = 1$) is always true. This proves that $\models F \Leftrightarrow \emptyset \models F$.

6.2 Example. Determine whether

$$(P \vee Q) \to R \equiv (P \to R) \wedge (Q \to R)$$

Solution.

P	Q	R	$(P \vee Q) \to R$	$(P \to R) \wedge (Q \to R)$
1	1	1	1	1
1	1	0	0	0
1	0	1	1	1
1	0	0	0	0
0	1	1	1	1
0	1	0	0	0
0	0	1	1	1
0	0	0	1	1

6.3 Example. Determine whether

$$\{P \to (Q \vee \neg R), Q \to \neg P\} \models R \to \neg P$$

Solution.

P	Q	R	$\neg R$	$Q \vee \neg R$	$\neg P$	$P \to (Q \vee \neg R)$	$Q \to \neg P$	$R \to \neg P$
1	1	1	0	1	0	1	0	0
1	1	0	1	1	0	1	0	1
1	0	1	0	0	0	0	1	0
1	0	0	1	1	0	1	1	1
0	1	1	0	1	1	1	1	1
0	1	0	1	1	1	1	1	1
0	0	1	0	0	1	1	1	1
0	0	0	1	1	1	1	1	1