Lecture 21, Oct. 28

EA 3 due Fri Nov. 4

WA 3 due Wed Nov. 9

21.1 Definition (Continuity). We say that $f(x)$ is **continuous** at $x = a$ if

1. $\lim_{x\to a} f(x)$ exists
2. $\lim_{x\to a} f(x) = f(a)$

Equivalently, we say that $f(x)$ is continuous at $x = a$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$.

If $f(x)$ is not continuous at $x = a$ we say that f is **discontinuous** at $x = a$. We write

$$D(f) = \{a \in \mathbb{R} \mid f \text{ is discontinuous at } x = a\}$$

21.2 Theorem (Sequential Characterization of Limit). *Assume that $f(x)$ is defined on an open interval I containing $x = a$, Then the following are equivalent:*

1. *$f(x)$ is continuous at $x = a$*
2. *If $\{x_n\}$ with $x_n \to a$, we have $f(x_n) \to f(a)$*

Proof. Assume that $f(x)$ is continuous at $x = a$. Let $\{x_n\}$ be such that $x_n \to a$. Let $\epsilon > 0$. Since $f(x)$ is continuous at $x = a$, there exists a $\delta > 0$ such that for all $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$. Since $\{x_n\}$ converges to a, there exists a $N_0 > 0$ such that for all $n > N_0$ we have $|x_n - a| < \delta$. Then if $n \geq N_0$, we have $|f(x_n) - f(a)| < \epsilon$.

Conversely, for a contraposition, that $f(x)$ is not continuous at $x = a$. Then there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ there exists $x_\delta \in (a - \delta, a + \delta)$ with $|f(x_\delta) - f(a)| \geq \epsilon_0$. In particular, there exists a $x_n \in (a - \frac{1}{n}, a + \frac{1}{n})$ with $|f(x_n) - f(a)| > \epsilon_0$. Hence $f(x_n)$ does not converge to $f(a)$. ☐

21.3 Theorem (Arithmetic Rules). *Assume $f(x)$ and $g(x)$ are continuous at $x = a$, then*

1. *$(cf)(x)$ is continuous at $x = a$ for $c \in \mathbb{R}$*
2. *$(f + g)(x)$ is continuous at $x = a$*
3. *$(fg)(x)$ is continuous at $x = a$*
4. *$(f/g)(x)$ is continuous at $x = a$ provided that $g(a) \neq 0$.*

21.4 Question. Let $f\colon \mathbb{R} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$. Let $h(x) = g \circ f(x) = g(f(x))$. Assume that $\lim_{x\to a} f(x) = L$ and $\lim_{y\to L} g(y) = M$.

Is $\lim_{x\to a} g \circ f(x) = \lim_{x\to a} h(x) = M$?

21.5 Theorem. *If $f(x)$ is continuous at $x = a$, and $g(y)$ is continuous at $y = f(a)$, then $h(x) = g \circ f(x)$ is continuous at $x = a$.*

Proof. Let $x_n \to a$, then $f(x_n) \to f(a)$, hence $g(f(x_n)) \to g(f(a))$ $\square$

21.6 Example. Show that $\sin x$ is continuous.

Observation:

1. $\sin x$ is continuous at $x = 0$ since $\lim_{x\to 0} \sin x = 0$.
2. If we can show that $\lim_{h\to 0} \sin(x_0 + h) = \sin x_0$ then $\sin x$ is continuous at x_0.

$$\begin{aligned}
\lim_{h\to 0} \sin(x_0 + h) &= \lim_{h\to 0} [\sin x_0 \cos h + \sin h \cos x_0] \\
&= \sin x_0
\end{aligned}$$

Nature of Discontinuity

21.7 Example.

$$f(x) = \frac{x^2 - 1}{x - 1}$$

$f(x)$ is not continuous at $x = 1$.

Let

$$g(x) = \begin{cases} f(x) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

21.8 Definition. If $\lim_{x\to a} f(x) = L$ exists but $L \neq f(a)$, then we say that $f(x)$ has a **removable discontinuity** at $x = a$. Let

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ L & \text{if } x = a \end{cases}$$

21.9 Definition. If $\lim_{x\to a} f(x)$ does not exists, then $x = a$ is called an **essential discontinuity** for $f(x)$.

3 Types of Essential Discontinuities

1. Finite jump discontinuity: $\lim_{x\to a^+} f(x) = L$, $\lim_{x\to a^-} f(x) = M$ and $L \neq M$
2. Vertical Asymptote: $\lim_{x\to a^\pm} f(x) = \pm\infty$
3. Oscillatory Discontinuity: $\lim_{x\to 0} \sin(1/x)$