Lecture 15, Oct. 17

Women in Pure Math/Math Finance

Lunch/Workshop

Tuesday, Oct.25

12:30-1:20

MC 5417

Limits of Functions

15.1 Example.

$$f(x) = \frac{x^2 - 1}{x - 1}$$

$domain(f) = \{x \in \mathbb{R} \mid x \neq 1\}$.

Note.

$$f(x) = \frac{(x+1)(x-1)}{x-1} = (x+1) \text{ if } x \neq 1.$$

What can we say about the values of $f(x)$ as x approaches 1? As x gets closer and closer to 1, $f(x)$ gets closer and closer to 2. We want to say that 2 is the limit of $f(x)$ as x approaches 1.

15.2 Definition. Heuristic Definition of Limit I If $f(x)$ is defined on an open interval around $x = a$, except possibly at $x = a$, then we say that L is the limit of $f(x)$ as x approaches a if as x gets closer and closer to a, f(x) gets closer and closer to L

15.3 Definition. Heuristic Definition of Limit II We say that L is the limit of $f(x)$ as x approaches a, if for every positive tolerance $\epsilon > 0$, $f(x)$ approximates L with an error less than ϵ provided that x is close enough to a, and not equal to a.

15.4 Definition. Formal Definition for a Limit of a Function We say that L is the limit of $f(x)$ as x approaches a, if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then

$$|f(x) - L| < \epsilon.$$

In this case we write

$$\lim_{x \to a} f(x) = L.$$

15.5 Example. Show that

$$\lim_{x \to 2} 3x + 1 = 7.$$

Solution. Let $\epsilon > 0$

$$|3x + 1 - 7| = |3x - 6| = 3\,|x - 2|\,.$$

We want $|3x + 1 - 7| < \epsilon$. We can make this happen if $|x - 2| < \epsilon/3$

Hence if $\delta = \epsilon/3$, then

$$0 < |x - 2| < \delta = \epsilon/3 \Rightarrow |3x + 1 - 7| = 3\,|x - 2| < 3\epsilon/3 = \epsilon$$

15.6 Example. $f(x) = mx + b, m \neq 0$

$$\lim_{x \to a} f(x) = ma + b$$

Solution. Given $\epsilon > 0$, chooose $\delta = \epsilon / |m|$

15.7 Example. Show that

$$\lim_{x \to 3} x^2 = 9$$

Solution.

$$\left|x^2 - 9\right| = |x+3|\,|x-3|$$

Let $\epsilon > 0$. We can assume $\delta < 1$.

If $0 < |x - 3| < 1 \Rightarrow x \in (2, 4)$.

Hence $|x + 3| < 7$.

Hence for any $\delta < 1$,

$$0 < |x-3| < 1 \Rightarrow \left|x^2 - 9\right| < 7\,|x-3|\,.$$

Let $\delta = min\{1, \epsilon/7\}$

If $0 < |x - 3| < \delta \Rightarrow \left|x^2 - 9\right| \leq 7\,|x-3| = \epsilon$

15.8 Example. Show that

$$\lim_{x \to 1} x^7 + 4x^5 - 3x + 2 = 1$$

Solution. Don't want to do this by $\epsilon - \delta$.

15.9 Example.

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

What is

$$\lim_{x \to 0} f(x)$$

Solution. $\lim_{x \to 0} f(x)$ does not existst.

Assume $\lim_{x \to 0} f(x) = L$. Let $\epsilon = 1/2$. Suppose that $\delta > 0$ is such that $0 < |x - 0| < \delta \Rightarrow |f(x) - L| < \epsilon = 1/2$

Let $x = \delta/2$. Then $L \in (1/2, 3/2)$. Let $x = -\delta/2$. Then $L \in (-3/2, -1/2)$.

$$L \in (1/2, 3/2) \cap (-3/2, -1/2) = \emptyset$$

15.10 Theorem. *If* $\lim_{x \to a} f(x) = L$ *and* $\lim_{x \to a} f(x) = M$*, then* $L = M$*.*

15.11 Theorem. $\lim_{x \to a} f(x) = L$ *if and only if whenever* $\{x_n\}$ *is a sequence with* $x_n \to a$; $x_n \neq a$ *we have that* $f(x_n) \to L$