Lecture 23, Nov. 2

23.1 Theorem (Intermediate Value Thm (IVT))**.** *If $f(x)$ is continuous on $[a, b]$, $f(a) < 0$ and $f(b) > 0$, then there exists $c \in (a, b)$ with $f(c) = 0$.*

Proof. Let $E = \{x \in [a, b] \mid f(x) \leq 0\}$. Then $E \neq \emptyset$ since $a \in E$. Since E is bounded, it has a lub which we call c (Note: $c \in [a, b]$). We claim $f(c) = 0$ We can find $x_n \in E$ with $x_n \to c$. By the Sequential Characterization of Continuity $f(x_n) \to f(c)$. Since $f(x_n) \leq 0$ for all n, $f(c) \leq 0$. Observe that $c < B$ for each $n \in \mathbb{N}$. We choose $y_n \in [a, b]$ so that $c < y_n \leq b$ and $|c - y_n| < \frac{1}{n}$. Since $y_n \to c$, we have $f(y_n) \to f(c)$. But $f(y_n) > 0$ for all n, so $f(c) \geq 0$. Thus $f(c) = 0$. □

Note. A similar statement holds if $f(a) > 0$ and $f(b) < 0$.

23.2 Corollary (Intermediate Value Theorem II)**.** *If $f(x)$ is continuous on $[a, b]$, and if $f(a) < \alpha < f(b)$ or $f(b) < \alpha < f(a)$, then there exists $c \in (a, b)$ with $f(c) = \alpha$.*

Proof. Let $g(x) = f(x) - \alpha$ and apply the theorem 23.1. □

23.3 Question. Assume $f : [a, b] \to \mathbb{R}$ is 1-1. What can we say about f if $f(x)$ is also continuous? Is f strictly monotonic?

23.4 Definition. We say that $f(x) : [a, b]$ is non-decreasing on $[a, b]$ if whenever $x, y \in [a, b]$ with $x < y$, we have $f(x) \leq f(y)$. We say that f is strictly increasing if whenever $x, y \in [a, b]$ with $x < y$ we have $f(x) < f(y)$. Similarly we could define non-increasing and strictly decreasing.

f is monotonic on $[a, b]$ if it is either non-decreasing or non-increasing. f is strictly monotonic if it is strictly increasing or strictly decreasing.

23.5 Corollary.

TO BE FINISHED