Lecture 17, Oct. 7

Binomial Theorem

17.1 Definition. For $n, k \in \mathbb{N}$ with $0 \le k \le n$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$$

The number of ways to choose k of n objects,

1. If we choose the k objects with replacement (or with repetition), and if order matters, is n^k
2. If we choose the k objects without replacement, and if order matters, is $\dfrac{n!}{(n-k)!}$. (In particular the number of ways to arrange n objects is $n!$)
3. If we choose the k objects without replacement, and if order does not matters (so we form a k-element set), is $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Note. For $n, k \in \mathbb{N}$ with $0 \le k \le n$, $\binom{n}{0} = 1$, $\binom{n}{n} = 1$, $\binom{n}{k} = \binom{n}{n-k}$, $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

Proof.

$$\begin{aligned}\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!}\\ &= \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)!}{(k+1)!(n-k)!}\\ &= \frac{n!(k+1+n-k)}{(k+1)!(n-k)!}\\ &= \frac{(n+1)!}{(k+1)!(n+1-k-1)!}\\ &= \binom{n+1}{k+1}\end{aligned}$$

$\square$

Pascal Triangle

17.2 Example.

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

17.3 Theorem. Binomial Theorem *For $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$ we have the following formula*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof. When $n = 0$,

$$(a+b)^0 = 1$$

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \binom{0}{0} a^0 b^0 = 1$$

When $n = 1$,

$$(a+b)^1 = a + b$$

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1 = a + b$$

Let $n \geq 1$ be arbitrary.

Suppose, inductively, that

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} b^n$$

Then

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b)(\binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} b^n) \\
&= \binom{n}{0} a^{n+1} + \binom{n}{1} a^n b + \cdots + \binom{n}{n-1} a^2 b^{n-1} + \binom{n}{n} a b^n \\
&\qquad + \binom{n}{0} a^n b + \binom{n}{1} a^{n-1} b^2 + \cdots + \binom{n}{n-1} a^1 b^n + \binom{n}{n} b^{n+1} \\
&= \binom{n+1}{0} a^{n+1} + \binom{n+1}{1} a^n b + \cdots + \binom{n+1}{n} a b^n + \binom{n+1}{n+1} b^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n}{k} a^k b^{n-k}
\end{aligned}$$

By induction, it follows that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ for all $n \geq 0$ □

17.4 Example.

$$(x+2)^6 = x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$$

17.5 Example. Find the coefficient of x^8 in the expansion of

$$(5x^3 - \frac{2}{x^2})^{11}$$

Solution.

$$\begin{aligned}
(5x^3 - \frac{2}{x^2})^{11} &= \sum_{k=0}^{11} \binom{11}{k} (5x^3)^{11-k} (-\frac{2}{x^2})^{11} \\
&= \sum_{k=0}^{11} (-1)^k \binom{11}{k} 5^{11-k} 2^k x^{3(11-k)-2k}
\end{aligned}$$

To get $3(11-k)-2k=8$ that is $k=5$

So that the coefficient of x^8 is $(-1)^5\binom{11}{5}5^{11-5}2^5=-231000000$

17.6 Example. Find

$$\sum_{k=0}^{n}\binom{2n}{2k}\frac{1}{2^k}$$

Solution.

$$(1+\frac{1}{2})^{2n}=\binom{2n}{0}+\binom{2n}{1}\frac{1}{2}+\dots$$

$$(1-\frac{1}{2})^{2n}=\binom{2n}{0}-\binom{2n}{1}\frac{1}{2}+\dots$$

And replace 2 by $\sqrt{2}$