Lecture 33, Nov. 21

Exponential and Logarithmic Functions

33.1 Definition (a^x)**.** Let $a > 0$ We have

1. $a^0 = 1$
2. $a^n = a \cdot a \cdot a \cdots a$ if $n \in \mathbb{N}$
3. $a^{n/m} = \sqrt[m]{a^n}$
4. if $\alpha \in \mathbb{R}$, $\alpha > 0$, let $a^\alpha = \lim_{r_n \to \alpha} a^{r_n}$ where $r_n \subset \mathbb{Q}$, $r_n \geq 0$
5. If $\alpha < 0$, let $a^\alpha = \frac{1}{a^{-\alpha}}$

33.2 Theorem (Properties of a^x)**.**

1. *$a^{x+y} = a^x a^y$*
2. *$a^{x^y} = a^{xy}$*
3. *$f(x) = a^x$ is differentiable and $f'(x) = f'(0)f(x) = f'(0)a^x$*
4. *There exist a unique base "e" for which if $f(x) = e^x$ then $f'(0) = 1$.*

Note. The derivative of $f(x) = a^x$ at $x = 0$ varies continuously with a. It also increase with a.

33.3 Theorem (The function e^x)**.** *Properties*

1. *Domain $e^x = \mathbb{R}$*
2. *Range $e^x = \mathbb{R}^+ = y \in \mathbb{R} \mid y \geq 0$*
3. *e^x is strictly increasing and hence invertible.*
4. *$f'(x) = f'(0)f(x) = 1 \cdot e^x = e^x$ (Inverse for $f(x) = e^x$)*

33.4 Definition (Natural log)**.** We define $g(y) = \ln y : \mathbb{R}^+ \to \mathbb{R}$ by $g(y) = x$ if and only if $e^x = y$

From the Inverse Function Theorem,

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{e_0^x} = \frac{1}{y_0}$$

Thus if $g(y) = \ln y$ then $g'(y) = \frac{1}{y}$.

Note. If $a > 0$, then $a = e^{\ln a}$, then $a^x = e^{\ln a^x} = e^{x \ln a}$. If $h(x) = a^x$,then the Chain Rule shows that

$$h'(x) = \frac{d}{dx} e^{x \ln a} = \ln a e^{x \ln a} = \ln a \cdot a^x$$

In particular, $h'(0) = \ln a = \lim\limits_{h \to 0} \dfrac{a^n - 1}{h}$

Note. If $a \neq 1$, $a > 0$, then $f(x) = a^x$ is 1-1 from $\mathbb{R}$ onto $\mathbb{R}^+$

33.5 Definition. $g(y) = \log_a y : \mathbb{R}^+ \to \mathbb{R}$ by $g(y) = x$ iff $a^x = y$.

$\log_a y = x \Leftrightarrow a^x = y \Rightarrow e^{x \ln a} = y \Rightarrow \ln(e^{x \ln a}) = \ln y$ and $x \ln a = \ln y$, then $x = \frac{\ln y}{\ln a}$.

Hence, $\log_a(y) = \dfrac{\ln y}{\ln a} \Rightarrow \dfrac{d}{dx}(\log_a(x) = \dfrac{d}{dx}(\dfrac{\ln x}{\ln a}) = \dfrac{1}{\ln ax}$

33.6 Example (On the final exam)**.** Let $f(x) = x^x = (e^{\ln x})^x = e^{x \ln x}$

Domain $f = \mathbb{R}^+$

Note. If $g(x) = x \ln x$

$g'(x) = \frac{x}{x} + \ln x = 1 + \ln x = 0 \Rightarrow x = \frac{1}{e}$

$$f'(x) = e^{x \ln x} \frac{d}{dx} x \ln x = (1 + \ln x) e^{x \ln x} = (1 + \ln x) x^x$$

33.7 Example.

$$g(x) = x^{\sin x} = e^{\ln x \sin x}$$

33.8 Example (Mean Value Theorem)**.** Question: Suppose that a car travels a distance of 110km in exactly 1hr. If the posted speed limit on the road is 100km/h. Can you prove that the car was speeding at some point.