Lecture 13, Sept. 30

13.1 Example.

$$\{F \to (G \wedge H), (F \wedge G) \vee H\} \vDash H$$

Proof. Proof by contradiction. Suppose H is false

$$\begin{array}{l}
(F \wedge G) \vee H, \neg H \quad \therefore F \wedge G \\
(F \wedge G) \quad \therefore F \\
F \to (G \wedge H), F \quad \therefore G \wedge H \\
G \wedge H \quad \therefore H \\
\neg H, H \quad \textit{gives the contradiction} \\
\therefore H
\end{array}$$

□

Here is a derivation for the valid argument

$$\begin{array}{ll}
S = \{F \to (G \wedge H), (F \wedge G) \vee H, \neg H\} \vDash F \to (G \wedge H) & \textit{by V1} \\
S \vDash (F \wedge G) \vee H & \textit{by V1} \\
S \vDash \neg H & \textit{V1} \\
S \vDash F \wedge G & \textit{V18 on line 2,3} \\
S \vDash F & \textit{V11 on line 4} \\
S \vDash G \wedge H & \textit{V23 on line 1,5} \\
S \vDash H & \textit{by V12 on 6} \\
\{F \to (G \wedge H), (F \wedge G) \vee H\} \vDash H & \textit{V5 on line 3,7}
\end{array}$$

Here is another proof

Proof. Let α be an arbitrary assignment

Suppose that $F \to (G \wedge H)$ is true (under α)

Suppose that $(F \wedge G) \vee H$ is true

Note that wither $F \wedge G$ is true or H is true

Case 1. Suppose $F \wedge G$ is true. [V14]

Since $F \wedge G \therefore F$ [V11]

Since $F \to (G \wedge H)$ and $F \therefore F$ [V23]

Since $G \wedge H \therefore H$ [V12]

Case 2. Support that H is true. [V14]

Then H is true. [V1]

In either case, we have proven H [V14]

□

Here is a corresponding derivation of valid argument

1. $S = \{F \to (G \wedge H), (F \wedge G)\} \vDash F \to (G \wedge H)$
2. $S \vDash F \wedge G$
3. $S \vDash F$
4. $S \vDash G \wedge H$
5. $S \vDash H$
6. $\{F \to (G \wedge H), H\} \vDash H$
7. $\{F \to (G \wedge H), (F \wedge G) \vee H\} \vDash H$

13.2 Example. Show that
$$\{(F \vee \neg G) \to H, F \leftrightarrow (G \wedge \neg H)\} \vDash \neg(H \to F)$$

Solution: We need to show that

for every assignment α

if $(F \vee \neg G) \to H$ is true under α

and $F \leftrightarrow (G \wedge \neg H)$ is true

then $H \to F$ is false

Proof. Let α be arbitrary assignment

Suppose $(F \vee \neg G) \to H$ is true

Suppose $F \leftrightarrow (G \wedge \neg H)$ is true.

[We need to show that $H \to F$ is false. Notice that $\neg(H \to F) \equiv H \wedge \neg F$. So we need to show that H is true and F is false.]

Suppose, for a contradiction, that H is false.

Since $(F \vee \neg G) \to H$ and $\neg H$ $\quad \therefore \neg(F \vee \neg G)$

Since $\neg(F \vee \neg G)$ $\quad \therefore \neg F \wedge G$

Since $F \leftrightarrow (G \wedge \neg H)$ and $\neg F$ $\quad \therefore \neg(G \wedge \neg H)$

Since G and $\neg H$ $\quad \therefore G \wedge \neg H$

Since $G \wedge \neg H$ and $\neg(G \wedge \neg H)$ we have a contradiction

So H is true.

Since H is true, then $\neg\neg H$

Since $\neg\neg H$ we have $\neg G \vee \neg\neg H$

Since $\neg G \vee \neg\neg H$ we have $\neg(G \wedge \neg H)$

Since $F \leftrightarrow (G \wedge \neg H)$ and $\neg(G \wedge \neg H)$, we have $\neg F$

Since H and $\neg F$, we have $H \wedge \neg F$

Since $H \wedge \neg F$ we have $\neg(H \rightarrow F)$

$\square$

$$\{(F \vee \neg G) \rightarrow H, F \leftrightarrow (G \wedge \neg H)\} \vDash \neg(H \rightarrow F)$$

Here is a derivation

Proof.

1.	$S = \{(F \vee \neg G) \rightarrow H, F \leftrightarrow (G \wedge \neg H), \neg H\} \vDash \neg(H \rightarrow F)$	v1
2.	$S \vDash F \leftrightarrow (G \wedge \neg H)$	v1
3.	$S \vDash \neg H$	v1
4.	$S \vDash \neg(F \vee \neg G)$	v24
5.	$S \vDash \neg F \wedge \neg\neg G$	v45,E8
6.	$S \vDash \neg F \wedge G$	v11 on 5
7.	$S \vDash \neg F$	v31 on 2,6
8.	$S \vDash \neg\neg G$	12 on 5
9.	$S \vDash G$	v45,e2 on 8
10.	$S \vDash G \wedge \neg H$	v10 on 9,3
11.	$T = \{(F \wedge \neg G) \rightarrow H, F \leftrightarrow (F \wedge \neg H)\} \vDash H$	v5 on 10,7
12.	$T \vDash \neg\neg H$	
13.	$T \vDash \neg G \vee \neg\neg H$	
14.	$T \vDash \neg(G \wedge \neg H)$	
15.	$T \vDash F \leftrightarrow (G \wedge \neg H)$	
16.	$T \vDash \neg F$	
17.	$T \vDash H \wedge \neg F$	
18.	$T \vDash \neg(H \rightarrow F)$	

$\square$