Lecture 10, Sept. 30

Series

10.1 Definition. A series $\sum_{n=1}^{\infty} a_n$ is **positive** is for all $n \in \mathbb{N}$, if $S_k = \sum_{n=1}^{k} a_n$, then $S_{k+1} - S_k = a_{k+1} \geq 0$

10.2 Example. Harmonic Series Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

Let $S_k = \displaystyle\sum_{n=1}^{k} \frac{1}{n}$,

$$
\begin{aligned}
S_1 &= 1 = \frac{2}{2} \\
S_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\
S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{4}{2} \\
S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{2} \\
&\vdots \\
S_{2^k} &> \frac{2+k}{2}
\end{aligned}
$$

Since $\{\frac{2+k}{2}\}$ is not bounded, $\{S_k\}$ is not bounded.

10.3 Example. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 - n}$

Note.

$$
\begin{aligned}
\frac{1}{n^2 - n} &= \frac{1}{n(n-1)} \\
&= \frac{1}{n-1} - \frac{1}{n}
\end{aligned}
$$

Solution.

$$
\begin{aligned}
S_1 &= 1 - \frac{1}{2} = 1 - \frac{1}{2} \\
S_2 &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3} \\
S_3 &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4} \\
&\vdots \\
S_k &= 1 - \frac{1}{k}
\end{aligned}
$$

As $k \to \infty$, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 - n} = 1$

10.4 Example. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Note. For $n \geq 2$,

$$\frac{1}{n^2} < \frac{1}{n^2 - n}$$

$$\begin{aligned}
T_k = \sum_{n=1}^{k} \frac{1}{n^2} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \\
&< 1 + \frac{1}{2^2 - 2} + \frac{1}{3^2 - 2} + \cdots + \frac{1}{k^2 - k} \\
&< 1 + 1 \\
&= 2
\end{aligned}$$

Since $T_k \leq 2$ for all k, $\{T_k\}$ is bounded and by the Monotone Convergence Theorem is convergent with $1 \leq \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$.

In fact, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

10.5 Example. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$, does this converge?

Note that $\dfrac{1}{n!} < \dfrac{1}{2^n}$ for $n \geq k$.

In fact, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!} = e$

Note.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Arithmetic Rules for Sequences

10.6 Question. Assume $a_n \to 3$, $b_n \to 7$.

What can you say about

1) $\{4a_n\}$
2) $\{a_n b_n\}$
3) $\{a_n + b_n\}$

4) $\{\frac{a_n}{b_n}\}$

10.7 Theorem. Arithmetic Rules for Sequences *Let* $\{a_n\}, \{b_n\}$ *be such that* $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$. *Then*

1) $\lim_{n\to\infty} ca_n = cL$ *for all* $c \in \mathbb{R}$

2) $\lim_{n\to\infty} a_n + b_n = L + M$

3) $\lim_{n\to\infty} a_n b_n = LM$

4) $\displaystyle\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$ *if* $L \neq 0$

5) $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ *if* $M \neq 0$

Proof. 1) If $c = 0$ then $ca_n = 0$ for all n. Hence $\lim_{n\to\infty} ca_n = \lim_{n\to\infty} 0 = 0L = cL$ Suppose $c \neq 0$, Let $\epsilon > 0$. We want N so that if $n \geq N$, $|ca_n - cL| < \epsilon \Leftrightarrow |a_n - L| < \dfrac{\epsilon}{|c|}$

Choose N_0 such that if $n \geq N_0$ we have $|a_n - L| < \dfrac{\epsilon}{|c|}$

If $n \geq N_0$,

$$|ca_n - cL| \leq |a_n - L||c| < \frac{\epsilon}{|c|}|c| = \epsilon$$

$\square$