Lecture 12, Sept. 28

V13. How to prove an or statement

V14. $S \cup F \vDash H$ and $S \cup G \vDash H \iff S \cup (F \vee G) \vDash H$

V15. In words, from F we can conclude $F \vee G$

V16.

V17. In words, from $F \vee G$ and $\neg F$ we can conclude G

V18.

...

V25. In words, to prove $F \iff G$ we suppose F then prove G, and we suppose G and prove F

V26. $(F \iff G) \equiv (F \wedge G) \vee (\neg F \wedge \neg G)$

...

V33. $S \vDash t = t$. In words, we can always conclude that $t = t$ is true under any assumptions.

V34. From $s = t$ we can conclude $t = s$

V35. From $r = s$ and $s = t$ we can conclude $r = t$

V36. If $S \vDash s = t$ then $(S \vDash [F]_{x \mapsto t} \iff S \vDash [F]_{x \mapsto s})$. In words, if $\vDash s = t$, we can always replace any occurrence of the term s by the term t.

V37. If $S \vDash [F]_{x \mapsto y}$ and y is not free in $S \cup \{\forall x\ F\}$ then $S \vDash \forall x\ F$

If have not made any assumptions about x (earlier in out proof) then to prove $\forall x\ F$ we write "let x be arbitrary" then we prove F.

If we have not made any assumptions about y, then to prove $\forall x\ F$, we write "let y be arbitrary" then prove $[F]_{x \mapsto y}$

(This is related to the equivalence

$$\forall x\ F \equiv \forall y\ [F]_{x \mapsto y}$$

)

V38. If $S \vDash \forall x\ F$, then $S \vDash [F]_{x \mapsto t}$

V39.

V40. If $S \cup \{[F]_{x \mapsto t}\} \vDash G$ then $S \vDash \exists X\ F$. In words, to prove $\exists x\ F$ we choose any term t, and prove $[F]_{x \mapsto t}$.

V41. If y is not free in $S \cup \{\exists x\ F, G\}$ and if $S \cup [F]_{x \mapsto y} \vDash G$ then $S \cup \exists x\ F \vDash G$. In words, to prove that $\exists x\ F$ implies G, choose a variable y which we have not made assumptions about and which does not occur in G, we write "choose y so that $[F]_{x \mapsto y}$ is true", then prove G.

Note. In standard mathematical language,

$$\forall x \ \in A \ F$$

means

$$\forall x \ (x \in a \to F)$$

To prove $\forall x \ (x \in a \to F)$ we write "let x be arbitrary", then prove $x \in a \to F$ which we do by writing "suppose $x \in A$" then prove F.

Usually, instead of writing "let x be arbitrary" and "suppose $x \in A$ " we write "let $x \in A$ be arbitrary" or simply "let $x \in A$".

So to prove $\forall x \in A \ F$ we write "let $x \in A$" then prove F. Alternatively, write "let $y \in A$" then prove $[F]_{x \mapsto y}$.

12.1 Example. Prove that

$$\{F \to (G \wedge H), (F \wedge G) \vee H\} \vDash H$$

For all assignment $\alpha\colon \{P, Q, R, \dots\} \to \{0, 1\}$, if $\alpha(F \to (G \wedge H)) = 1$ and $\alpha((F \wedge G) \vee H) = 1$ then $\alpha(H) = 1$

Proof. Let α be an arbitrary assignment. Suppose that $F \to (G \wedge H)$ is true (under α), and $(F \wedge G) \vee H$ is true (under α).

Suppose, for a contradiction, that H is false.

$$\begin{array}{l}
(F \wedge G) \vee H, \neg H \quad \therefore F \wedge G \\
(F \wedge G) \quad \therefore F \\
F \to (G \wedge H), F \quad \therefore G \wedge H \\
G \wedge H \quad \therefore H \\
\neg H, H \quad \textit{gives the contradiction} \\
\therefore H
\end{array}$$

□