Lecture 15, Oct. 4

15.1 Definition. An **ordered n-tuple** with entries in a set A, is a function $a\colon \{1, 2, 3, \dots\} \to A$ where we write $a(k)$ as a_k.

We write $a = (a_1, a_2, \dots)$ to indicate that $a = \{1, 2, 3, \dots\} \to A$ is given by $a(k) = a_k$ for $k \in \{1, 2, 3, \dots, n\}$

The set of all such n-tuples is denoted by A^n

$$A^n = \{(a_1, a_2, \dots) \mid \ \textit{each } a_{\mathbb{Z}} \in A\}$$

15.2 Definition. A **sequence** with **entries** or **terms** in a set A is a function

$$a\colon \{1, 2, 3, \dots\} \to A$$

Where we write $a(k) = a_k$ or sometimes a function

$$a\colon \{m, m+1, m+2, \dots\} \to A$$

where $m \in \mathbb{Z}$.

We write $a = (a_k)_{k \geq m} = (a_m, a_{m+1}, \dots)$

or we write $a = \{a_k\}_{k \geq m} = \{a_m, a_{m+1}, \dots\}$

to indicate that $a = \{m, m+1, \dots\} \to A$ is given by $a(k) = a_k$

Remark. For sets A and B we define A^B to be the set of all functions

$$f\colon B \to A$$

Also the integer n is defined to be

$$n = \{0, 1, 2, \dots, n-1\}$$

So Actually

$$A^n = A^{\{0,1,2,\dots,n-1\}} = \{a\colon \{0, 1, \dots, n-1\} \to A\}$$

and we write elements in A^n as $(a_0, a_1, \dots a_{n-1})$

And the set of sequences with entries in A is the set $A^{\mathbb{N}} = \{a\colon \{0, 1, 2, \dots\} \to A\}$

15.3 Definition. We say that a sequence is defined in **closed-form** when we are given a formula for a_k in terms of k.

We say that a sequence is defined **recursively** when we are given a formula for a_n in terms of k and in terms of previous terms a_i in the sequence.

15.4 Example. Fibonacci Sequence

$$a_{n+2} = a_{n+1} + a_n$$

15.5 Example. When we write

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

we mean that $S_1 = 1$ and $S_n = S_{n-1} + \dfrac{1}{n^2}$

15.6 Example. When we write

$$P_n = \prod_{k=1}^{n} \frac{2k-1}{2k}$$

We mean that $P_1 = \dfrac{1}{2}$ and $P_n = P_{n-1} \cdot \dfrac{2n-1}{2n}$

15.7 Example. When we write $n!$, we mean that $0! = 1$ and $n! = (n-1)! \cdot n$ for $n \geq 1$

15.8 Example. In Set Theory, we define addition on $\mathbb{N}$, recursively as follows

$$0 = \emptyset, 1 = \{0\}, x + 1 = x \cup \{x\}$$

For $n \in \mathbb{N}$, $n + 0 = n$, $n + (m+1) = (n+m) + 1 = (n+m) \cup \{(n+m)\}$

15.9 Theorem. Mathematical Induction *Let $F(n)$ be a mathematical statement about an integer n. Let $m \in \mathbb{Z}$*

Suppose $F(m)$ is true. (that is $[F]_{n \mapsto m}$)

Suppose that for all $k \geq m$, if $F(k)$ is true then $F(k+1)$ is true.

Then $F(n)$ is true for all $n \geq m$.

15.10 Example. Define a_n recursively by $a_1 = 1$ and $a_{n+1} = \dfrac{n}{n+1} \cdot a_n + 1$. Find a closed-form formula for a_n

Solution. We have $a_1 = 1$, $a_2 = \dfrac{3}{2}$, $a_3 = \dfrac{4}{2}, \dots$

It appears that $a_n = \dfrac{n+1}{2}$

When $n = 1$, $\cdots$

Suppose $a_k = \dfrac{k+1}{2}$

When $n = k+1$ we have

$$\begin{aligned}
a_n = a_{k+1} &= \frac{k}{k+1} \cdot a_k + 1 \\
&= \frac{k}{k+1} \cdot \frac{k+1}{2} + 1 \\
&= \frac{k+2}{2} \\
&= \frac{(k+1)+1}{2} \\
&= \frac{n+1}{2}
\end{aligned}$$

By induction, $a_n = \dfrac{n+1}{2}$ forall $n \geq 1$

15.11 Exercise.

1. $$\sum_{k=1}^{n} k^3$$

2. $$\prod_{k=1}^{n} (1 - \frac{1}{k^2})$$