Lecture 18, Oct. 24

Midterm: 7:00-8:45

RCH 307 - A-J

RCH 306 - K-O

DWE 3522 - P-W

DWE 3522A - X-Z

Woman in Pure Math/Math Finance Lunch

Tuesday 12:30-1:20 MC5417

18.1 Definition. We say that L is the limit of $f(x)$ from above (from the right) if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \epsilon$. We write

$$\lim_{x \to a^+} f(x) = L$$

We say that L is the limit of $f(x)$ from below (from the left) if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $-\delta < x - a < 0$, then $|f(x) - L| < \epsilon$. We write

$$\lim_{x \to a^-} f(x) = L$$

Note. Both the Arithmetic Rules and Sequential Characterization hold for one-sided limits. As does the Squeeze Theorem.

$$lim_{x \to a^+} f(x) = L \text{ iff whenever } \{x_n\} \text{ is such that } x_n \to a, a < x_n \text{ we have } \lim_{x \to \infty} f(x_n) = L$$

18.2 Theorem. *The following are equivaent*

1. $\lim_{x \to a} f(x) = L$
2. $\lim_{x \to a^-} f(x) = L$ *and* $\lim_{x \to a^+} f(x) = L$

Proof.

1. Assume that $\lim_{x \to a} f(x) = L$, Let $\epsilon > 0$, then there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$. Hence if $0 < x - a < \delta$ then $|f(x) - L| < \epsilon$ and if $0 < a - x < \delta$ then $|f(x) - L| < \epsilon$. Thus $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.
2. Conversely, assume that $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$. Let $\epsilon > 0$. We can find $\delta_1 > 0$ such that if $0 < x - a < \delta_1$ then $|f(x) - L| < \delta$ and $\delta_2 > 0$ such that if $0 < a - x < \delta_1$ then $|f(x) - L| < \delta$. Let $\delta = min\{\delta_1, \delta_2\}$, hence if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

$\square$

18.3 Example.

$$lim_{x \to 0} \frac{|x|}{x}$$

18.4 Example.

$$lim_{x \to 0^+} \sqrt{x} = 0$$

18.5 Definition. A function $f(x)$ is **even** if $f(x) = f(-x)$ for all $x \in \mathbb{R}$ (graph is symmetric about $x = 0$)

Note. If $f(x)$ is even, (assume these limits exist)

$$\lim_{x \to a^+} f(x) = \lim_{x \to -a^-} f(x)$$

$$\lim_{x \to a^-} f(x) = \lim_{x \to -a^+} f(x)$$

In particular, $\lim_{x \to 0} f(x)$ exists iff $\lim_{x \to 0^+} f(x)$ exists iff $\lim_{x \to 0^-} f(x)$ exists.

18.6 Definition. A function $f(x)$ is **odd** if $f(x) = -f(-x)$ for all $x \in \mathbb{R}$ (graph is symmetric about $(0, 0)$)

Note. If $f(x)$ is odd, (assume these limits exist)

$$\lim_{x \to a^+} f(x) = -\lim_{x \to -a^-} f(x)$$

$$\lim_{x \to a^-} f(x) = -\lim_{x \to -a^+} f(x)$$

$\lim_{x \to 0} f(x)$ exists iff $\lim_{x \to 0^+} f(x) = 0$ or $\lim_{x \to 0^-} f(x) = 0$

18.7 Example. $\lim_{x \to 0} \sin x$ and $\lim_{x \to 0} \cos x$

18.8 Example.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$