Lecture 4, Sept. 19

Least Upper Bound Property

Upper Bound

4.1 Theorem. *Let $S \subset \mathbb{R}$ then $\alpha \in \mathbb{R}$ is an upper bound for S if $x \leq \alpha$ for all $x \in S$. We say that S is bounded above if S has an upper bound.*

We say that β is a lower bound for S if $\beta \leq x$ for all $x \in S$. We say that S is bounded below if S has a lower bound.

We say that S is bounded if it is bounded above and below.

4.2 Example. Let $S = \{x_1, x_2, \ldots, x_n\}$ be finite.

By relabeling, if necessary we can assume that

$$x_1 < x_2 < \cdots < x_n$$

Then $\beta = x_1$, β is a lower bound and $\alpha = x_n$ is an upper bound.

4.3 Theorem. *Every finite set is bounded.*

4.4 Example. Let $S = [0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ (finite interval)

5 is an upper bound. -1 is a lower bound.

1 is also an upper bound. Moreover if γ is any upper bound of S, then $1 \leq \gamma$

Least Upper Bound

4.5 Theorem. *We say that α is the least upper bound of a set $S \subset \mathbb{R}$ if*

1) *α is an uppper bound of S*
2) *if γ is an upper bound of S, then $\alpha \leq \gamma$*

We write

$$\alpha = lub(S)$$

(Sometimes α os called the supremum of S and is denoted by $\alpha = sup(S)$)

Back to the example $S = [0, 1)$. 0 is a lower bound and if γ is any lower bound, then $\gamma \leq 0$

Greatest Upper Bound

4.6 Theorem. *We say that β is the greatest lower bound of a set $S \subset \mathbb{R}$ if*

1) *β is an lower bound of S*
2) *if γ is an lower bound of S, then $\gamma \leq \beta$*

We write

$$\beta = glb(S)$$

(Sometimes β os called the infimum of S and is denoted by $\beta = inf(S)$)

4.7 Example. if $S = [0, 1)$, $lub(S) = 1$, $glb(S) = 0$.

Note. Is $\emptyset$ bounded (above or below)?

Note: 6 is an upper bound for $\emptyset$. If not, there exists an element in $\emptyset$ that is greater than 6. Similarly, 6 is a lower bound.

In fact, if $\gamma \in \mathbb{R}$ then γ is both an upper and a lower bound of $\emptyset$. $\emptyset$ is a bounded set.

4.8 Example. Let $S = \{x \in \mathbb{Q} \mid x^2 < 2\} \subset \mathbb{R}$

$\sqrt{2}$ is an upper bound and $-\sqrt{2}$ is a lower bound. And $lub(S) = \sqrt{2}$, $glb(S) = -\sqrt{2}$

4.9 Example. Let $S = \{x \in \mathbb{Q} \mid x^2 < 2\} \subset \mathbb{Q}$

S does not have a least upper bound or a greatest lower bound.

4.10 Question. If $S \subset R$ is bounded above, does it always have a least upper bound?

Least Upper Bound Property

4.11 Theorem. *If $S \subset R$ is non-empty and bounded above, then S has a least upper bound.*

Observation

1) $\emptyset$ does not have a lub
2) If we only have rational numbers in the world, then $S = \{x \mid x^2 < 2\}$ does not have a lub. In other words, Least Upper Bound Property fails for $\mathbb{Q}$

4.12 Question. is $\mathbb{N}$ bounded?

1) $\mathbb{N}$ is bounded below, $glb(S) = 1$