Lecture 9, Sept. 23

9.1 Definition. In the language of **first-order set theory**, we only use the one additional symbol

$$\in$$

(the membership or "is an element of" symbol), which is a binary relation symbol used with infix notation.

All mathematical statement can be expressed in **this language**

When we use this language, we normally take the universal class to be the class of all sets.

Example: Express each of the following statements about sets as formulas in **first-order set theory**

1. $u = v \backslash (x \cap y)$
2. $u \subseteq P(v \cup w)$
3. $u = 2$

Solution. 1. For sets u, v, x, y

$$\begin{aligned} u = v\backslash(x \cap y) &\iff \forall t(t \in u \leftrightarrow t \in v\backslash(x \cap y)) \\ &\iff \forall t(t \in u \leftrightarrow (t \in v \wedge \neg t \in (x \cap y))) \\ &\iff \forall t(t \in u \leftrightarrow (t \in v \wedge \neg(t \in x \wedge t \in y))) \end{aligned}$$

2.

$$\begin{aligned} u \subseteq P(v \cup w) &\iff \forall x\ (x \in u \to x \in P(v \cup w)) \\ &\iff \forall x\ (x \in u \to \forall y\ (y \in x \to y \in (v \cup w))) \\ &\iff \forall x\ (x \in u \to \forall y\ (y \in x \to (y \in v \vee y \in w))) \end{aligned}$$

3.

$$\begin{aligned} u = 2 &\iff u = \{\emptyset, \{\emptyset\}\} \\ &\iff \forall x\ (x \in u \leftrightarrow x \in \{\emptyset, \{\emptyset\}\}) \\ &\iff \forall x\ (x \in u \leftrightarrow (x = \emptyset \vee x = \{\emptyset\})) \\ &\iff \forall x\ (x \in u \leftrightarrow (\forall y\ \neg y \in x \vee \forall y\ y \in x \leftrightarrow y = \emptyset)) \\ &\iff \forall x\ (x \in u \leftrightarrow (\forall y\ \neg y \in x \vee \forall y\ y \in x \leftrightarrow (\forall z \neg z \in y))) \end{aligned}$$

The ZFC axioms can all be expressed as formulas in first order set theory.

1. Equality Axiom:
$$\forall u \forall v\ (u = v \leftrightarrow \forall x\ (x \in u \leftrightarrow x \in u))$$
2. Empty Set Axiom:
$$\exists u \forall x\ \neg x \in u$$

3. Pair Axiom:
$$\forall u \, \forall v \, \exists w \, \forall x \; (x \in w \leftrightarrow (x = u \vee x = v))$$
4. Union Axiom:
$$\forall u \, \exists w \, \forall x \; (x \in w \leftrightarrow \exists v \; (v \in u \cup x \in v))$$

Proof. When we do mathematical proofs, one of the things we allow ourselves to do is make use of some equivalences.

When F, G, H are formulas, s, t are terms, and x, y are variables, the following are equivalences which we call **basic equivalence**

1. $F \equiv F$
2. $\neg\neg F \equiv F$
3. $F \wedge F \equiv F$
4. $F \vee F \equiv F$
5. $F \wedge G \equiv G \wedge G$
6. $F \vee G \equiv G \vee F$
7. $(F \wedge G) \vee H \equiv F \wedge (G \vee H)$
8. $(F \vee G) \wedge H \equiv F \vee (G \wedge H)$
9. $F \wedge (G \vee H) \equiv (F \wedge G) \vee (F \wedge H)$
10. $F \vee (G \wedge H) \equiv (F \vee G) \wedge (F \vee H)$
11. $\neg(F \wedge G) \equiv \neg F \vee \neg G$
12. $\neg(F \vee G) \equiv \neg F \wedge \neg G$
13. $F \rightarrow G \equiv \neg G \rightarrow \neg F$
14. $F \rightarrow G \equiv \neg F \vee G$
15. $\neg(F \rightarrow G) \equiv F \wedge \neg G$
16. $F \leftrightarrow G \equiv (F \rightarrow G) \wedge (G \rightarrow F)$
17. $F \leftrightarrow G \equiv (\neg F \vee G) \wedge (\neg G \vee F)$
18. $F \leftrightarrow G \equiv (F \wedge G) \vee (\neg F \wedge G)$
19. $F \wedge (G \vee \neq G) \equiv F$
20. $F \vee (G \vee \neq G) \equiv (G \vee \neq G)$
21. $F \wedge (G \wedge \neg G) \equiv G \wedge \neg G$
22. $F \vee (G \wedge \neg G) \equiv F$

$\square$

Note. We can use basic equivalences one at a time, to derive other equivalences.

9.2 Example. Derive the equivalence:

$$(F \vee G) \to H \equiv (F \to H) \wedge (G \to H)$$

Solution.

$$\begin{aligned}
(F \vee G) \to H &\equiv \neg(F \vee G) \vee H \text{ (by the equivalence)} \\
&\equiv (\neg F \wedge \neg G) \vee H \text{ (by the de Morgan's)} \\
&\equiv H \vee (\neg F \wedge \neg G) \text{ (by the Commutativity)} \\
&\equiv (H \vee \neg F) \wedge (H \vee \neg G) \text{ (by the Distributivity)} \\
&\equiv (\neg F \vee H) \wedge (\neg G \vee H) \text{ (by the Commutativity)} \\
&\equiv (F \to H) \wedge (G \to H) \text{ (by the equivalence)}
\end{aligned}$$

9.3 Definition. Here are some more basic equivalences.

1. $s = t \equiv t = s$
2. $\forall x \, \forall y \; F \equiv \forall y \, \forall x \; F$
3. $\exists x \, \exists y \; F \equiv \exists y \, \exists x \; F$
4. $\neg \forall x \; F \equiv \exists x \; \neg F$
5. $\neg \exists x \; F \equiv \forall x \; \neg F$
6. $\forall x \; (F \wedge G) \equiv \forall x \; F \wedge \forall x \; G$
7. $\exists x \; (F \vee G) \equiv \exists x \; F \vee \exists x \; G$