Lecture 10, Sept. 26

10.1 Example.

$$\begin{aligned}\exists x\ (F \to G) &\equiv \exists X\ (\neg F \vee G)\\ &\equiv \exists x\ \neg F \vee \exists x\ G\\ &\equiv \neg\forall x\ F \vee \exists x\ G\\ &\equiv \forall x\ F \to \exists x\ G\end{aligned}$$

10.2 Definition. In a formula f, every occurrence of a variable symbol x (Except when the occurrence of x immediately follows a quantifier $\forall, \exists$) is either **free** or **bound**.

In the formulas $\forall x\ F$ and $\exists x\ F$, every free occurrence of x in F becomes **bound** by the initial quantifier, and every bound occurrence of x in F remains bound (by the same quantifier which binds it in F).

10.3 Example.

$$\forall y\ (x \times y = y \times x)$$

Both occurrence of x are free, and both occurrence of y are bound by the initial quantifier.

$$\forall x\ (\forall y\ (x \times y = y \times x) \to x \times a = a \times x)$$

10.4 Definition. An **interpretation** in a first-order language consists of the following: a choice of the universal set u, and a choice of meaning for each constant, function and relation symbol.

A formula is a meaningless string of symbols until we choose an interpretation. Once we choose an interpretation, a formula becomes a meaningful mathematical statement about its free variables.

The truth or falsehood of a formula may still depend on the value in u which are assigned to the free variable symbols in F.

An assignment (of values in u to the variable symbols) is a function $\alpha : \{\text{variable symbols}\} \to u$

10.5 Example. Consider the formula

$$\forall y\ (x \times y = y \times x)$$

when $u = \mathbb{R}$ (and $\times$ is multiplication) the formula becomes true (for any value assigned to x).

when $u = \mathbb{R}^3$ annd $\times$ is cross-product, the formula is true iff $x = 0$

when u is the set of all $n \times n$ matrices with entries in $\mathbb{R}$, and $\times$ denotes matrix multiplication, the formula is can be read as "the matrix x commutes with every matrix", and it is true iff $x = cI$ for some $c \in \mathbb{R}$

Notation. For a formula F, a variable symbol x and a term t, we write $[F]_{x \mapsto t}$ to denote the formula which is constructed from F by replacing x by t.

In an interpretation, the formula $[F]_{x \mapsto t}$ has the same meaning about t that f has about x.

Roughly speaking, $[F]_{x \mapsto t}$ is obtained from F by replacing each free occurrence of the symbol x by the term t. (but if a variable symbol in t would become bound by this replacement, we rename the variable first.)

10.6 Example. In $u = \mathbb{Z}$, $x \mid y$ means $\exists z\ y = x \times z$

$|\exists z\ y = x \times z|_{y \mapsto u} = \exists z\ u = x \times z$ means $x \mid u$

$|\exists z\ y = x \times z|_{y \mapsto x} = \exists z\ x = x \times z$ means $x \mid x$

$|\exists z\ y = x \times z|_{y \mapsto z} \neq \exists z\ z = x \times z$

$|\exists z\ y = x \times z|_{y \mapsto z} = |\exists u\ y = x \times u|_{y \mapsto z} = \exists u\ z = x \times u$

Here are some more basic equivalences:

E32 $\forall x\ F \equiv F$ if x is not free in F

E34 $\forall x\ F \equiv \forall y\ [F]_{x \mapsto y}$ if y is not free in F