Lecture 28, Nov. 11

Chain Rule Assume $f: I \to \mathbb{R}$, with I open and containing $x = a$, $g: J \to \mathbb{R}$, with J open and containing $f(a)$ with $f(I) \subset J$. Assume that f is differentiable at $x = a$ and g is differentiable at $y = f(a)$. Let $h(x): I \to \mathbb{R}$ be $h(x) = g \circ g(x) = g(f(x))$.

28.1 Question. Is $h(x)$ differentiable at $x = a$ and if so what is $h'(a)$?

We know that if $x \approxeq a$, then
$$f(x) \approxeq L_a^f(x)$$
and if $y \approxeq f(a)$ then
$$g(y) = L_{f(a)}^g(y).$$
If $x \approxeq a$, then $f(x) \approxeq f(a)$, hence $h(x) = g(f(x)) \approxeq g(L_a^f(x)) \approxeq L_{f(a)}^g(L_a^f(x))$.

The equation
$$\begin{aligned}
&L_{f(a)}^g \circ L_a^f(x) \\
=&L_{f(a)}^g(f(a) + f'(a)(x-a)) \\
=&g(f(a)) + g'(f(a))(f(a) + f'(a)(x-a) - f(a)) \\
=&g(f(a)) + g'(f(a))f'(a)(x-a) \\
=&h(a) + h'(a)(x-a) \\
=&L_a^h(x)
\end{aligned}$$
holds if an only if $h'(x) = g'(f(a))f'(a)$.

28.2 Theorem (Chain Rule)**.** *If $f: I \to \mathbb{R}$ is an open interval containing $x = a$, $g: J \to \mathbb{R}$ is an open interval containing $y = f(a)$, $f(I) \subset J$, J is differentiable at $x = a$, g is differentiable at $y = f(a)$, then if $h: I \to \mathbb{R}$ is given by $h(x) = g \circ f(x)$, then h is differentiable at $x = a$ with $h'(a) = g'(f(a))f'(a)$.*

False proof.
$$\begin{aligned}
h'(a) &= \lim_{x\to a} \frac{h(x) - g(a)}{x-a} \\
&= \lim_{x\to a} \frac{g(f(x)) - g(f(a))}{x-a} \\
&= \lim_{x\to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x-a} \\
&= \lim_{y\to f(a)} \frac{g(y) - g(f(a))}{y - f(a)} \cdot \lim_{x\to a} \frac{f(x) - f(a)}{x-a} \\
&=g'(f(a)f'(a)
\end{aligned}$$
This is false because $f(x)$ might equal $f(a)$, and thus you multiplied $\frac{0}{0}$. □

Real Proof. Let
$$\varphi(y) = \begin{cases} \frac{g(y)-g(f(a))}{y-f(a)} & \text{if } y \neq f(a) \\ g'(f(a)) & \text{if } y = f(a) \end{cases}$$

Note that $\varphi(y)$ is continuous.

Observe that $g(y) - g(f(a)) = \varphi(y)[y - f(a)]$ for all $y \in J$, even $y = f(a)$. Then now,

$$\begin{aligned} h'(a) &= \lim_{x \to a} \frac{h(x) - g(a)}{x - a} \\ &= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} \\ &= \lim_{x \to a} \frac{\varphi(f(x))[f(x) - f(a)]}{x - a} \\ &= \lim_{x \to a} \varphi(f(x)) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \varphi(f(a)) \cdot f'(a) \\ &= g'(f(a))f'(a) \end{aligned}$$

□

28.3 Example. Consider $h(x) = \cos x = \sin(x + \pi/2)$