Lecture 11, Oct. 3

WA2 now due Monday Oct. 17

EA2 due today

11.1 Theorem. Arithmetic Rules for Sequences *Let* $\{a_n\}$, $\{b_n\}$ *be such that* $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$.

Then

1) $\lim_{n\to\infty} ca_n = cL$ *for all* $c \in \mathbb{R}$
2) $\lim_{n\to\infty} a_n + b_n = L + M$
3) $\lim_{n\to\infty} a_n b_n = LM$
4) $\displaystyle\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$ *if* $L \neq 0$
5) $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ *if* $M \neq 0$
6) $\displaystyle\lim_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{L}$ *if* $L \geq 0$

Proof.

1) If $c = 0$ then $ca_n = 0$ for all n. Hence $\lim_{n\to\infty} ca_n = \lim_{n\to\infty} 0 = 0L = cL$ Suppose $c \neq 0$, Let $\epsilon > 0$. We want N so that if $n \geq N$, $|ca_n - cL| < \epsilon \Leftrightarrow |a_n - L| < \dfrac{\epsilon}{|c|}$

   Choose N_0 such that if $n \geq N_0$ we have $|a_n - L| < \dfrac{\epsilon}{|c|}$

   If $n \geq N_0$,
$$|ca_n - cL| \leq |a_n - L||c| < \frac{\epsilon}{|c|}|c| = \epsilon$$

2) Consider
$$\begin{aligned} |(a_n + b_n) - (L + M)| &= |a_n - L + b_n - M| \\ &\leq |a_n - L| + |b_n - M| \end{aligned}$$

   Let $\epsilon > 0$. Choose $N_1 \in \mathbb{N}$ so that
$$n \geq N_1 \to |a_n - L| < \frac{\epsilon}{2}$$
   Choose $N_2 \in \mathbb{N}$ so that
$$n \geq N_2 \to |b_n - M| < \frac{\epsilon}{2}$$
   Let $N_0 = max\{N_1, N_2\}$. If $n \geq N_0$
$$\begin{aligned} |(a_n + b_n) - (L + M)| &\leq |a_n - L| + |b_n - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

3) Consider $|a_n b_n - LM|$

$$\begin{aligned}
&|a_n b_n - LM| \\
=&|a_n b_n - b_n L + b_n L - LM| \\
=&|(a_n - L)b_n + L(b_n - M)| \\
\leq&|(a_n - L)b_n| + |L(b_n - M)|
\end{aligned}$$

By 1), we can find N_1 so that if $n \geq N_1$,

$$|L|\,|b_n - M| \leq \frac{\epsilon}{2}$$

Since $\{b_n\}$ is convergent it is bounded. So there exists $c > 0$ so that $|b_n| < c$

Then $|b_n|\,|a_n - L| < c\,|a_n - L|$

Choose N_2 so that if $n \geq N_2$

$$|a_n - L| < \frac{\epsilon}{2c}$$

If $N_0 = max\{N_1, N_2\}$ and $n \geq N_0$ then

$$|a_n b_n - LM| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

4)

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|a_n - L|}{|a_n|\,|L|}$$

Since $a_n \to L, L \neq 0$ we can find $N_1 \in \mathbb{N}$ so that if $n \geq N_1$, then

$$|a_n - L| < \frac{|L|}{2} \to |a_n| \geq \frac{|L|}{2}$$

If $n \geq N_1$ then

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|a_n - L|}{|a_n|\,|L|} \leq \frac{|a_n - L|}{\frac{|L|}{2}|L|} = \frac{|a_n - L|}{\frac{|L|^2}{2}}$$

Let $\epsilon > 0$. Choose N_2 so that if $n \geq N_2$

$$\frac{|a_n - L|}{\frac{|L|^2}{2}} < \epsilon$$

Let $N_0 = max\{N_1, N_2\}$ if $n \geq N_0$

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| < \epsilon$$

5) Follows from 3 and 4.

6) Homework

$\square$

Note. If $\lim_{n\to\infty} a_n = L, \lim_{n\to\infty} n_n = M$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M} \text{ if } M \neq 0$$

What happens if $M = 0$?

It depends on a_n.

11.2 Example. $a_n = b_n = \dfrac{1}{n}$

11.3 Example. $a_n = \dfrac{1}{n}, b_n = \dfrac{1}{n^2}$

11.4 Proposition. *Assume that* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists and that* $\lim_{n\to\infty} b_n = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

Proof.

$$\begin{aligned}
a_n &= (b_n)(\frac{a_n}{b_n}) \\
&= \lim_{n\to\infty} a_n \\
&= \lim_{n\to\infty} b_n \lim_{n\to\infty} \frac{a_n}{b_n} \\
&=0L \\
&=0
\end{aligned}$$

$\square$