# Lecture 25, Nov. 7

**25.1 Theorem** (Extreme Value Theorem)**.** *If $f(x)$ is continuous on $[a, b]$, then there exists $c, d \in [a, b]$ such that*

$$f(c) \leq f(x) \leq f(d)$$

*for all $x \in [a, b]$.*

**Uniform Continuity**

**25.2 Question.** Assume that $f(x)$ is continuous on some interval $I$. Let $\epsilon > 0$. Does there exists a single $\delta > 0$ such that for every $a \in I$, we have if $|x - a| < \delta$, $x \in I$, then $|f(x) - f(a)| < \delta$?

**25.3 Definition** (Uniform Conitnuity)**.** We say that $f(x)$ is uniformly continuous on $S \subset \mathbb{R}$ if for every $\epsilon$, there exists a $\delta > 0$ such that if $|x - y| < \delta$, $x, y \in S$, then $|f(x) - f(y)| < \delta$.

**25.4 Theorem** (Sequential Characterization for Uniform Continuity)**.** *Let $f : S \to \mathbb{R}$. Then the followings are equivalent*

1. *$f(x)$ is continuous on $S$*
2. *If $\{x_n\}, \{y_n\} \subset S$ with $\lim_{x\to\infty} |x_n - y_n| = 0$, then $\lim_{n\to\infty} |f(x_n) - f(y_n)| = 0$.*

*Proof.* Assume that $f(x)$ is uniformly continuous on $S$. Let $\epsilon > 0$ and let $\{x_n\}, \{y_n\} \subset S$ with $|x_n - y_n| \to 0$. Choose $\delta > 0$ so that if $x, y \in S$, $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. We can pick $N_0 \in \mathbb{N}$ so that if $n \geq N_0$, then $|x_n - y_n| < \delta$. It follows that if $n \geq N_0$, then $|f(x_n) - f(y_N)| < \epsilon$. Hence $\lim_{x\to\infty} |f(x_n) - f(y_n)| = 0$.

Conversely, assume that 1 fails ($f(x)$ is not uniformly continuous on $S$). Then there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ we can find $x_\delta, y_\delta \in S$ with $|x_\delta - y_\delta| < \delta$, but $|f(x_\delta) - f(y_\delta)| \geq \epsilon_0$. Let $\delta = 1/n$, and $x_\delta = x_n$, $y_\delta = y_n$. This gives us $\{x_n\}, \{y_n\} \subset S$, with $|x_n - y_n| < 1/n \to 0$, but $\lim_{x\to\infty} |f(x_n) - f(y_n)| \neq 0$ □

**25.5 Theorem.** *If $f(x)$ is continuous on $[a, b]$, then $f(x)$ is uniformly continuous on $[a, b]$.*

*Proof.* Assume that $f(x)$ is not uniformly continuous on $[a, b]$, then there exists $\epsilon_0$ and $\{x_n\}, \{y_n\} \subset S$ with $|x_n - y_n| \to 0$, but $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all $n$.

By the BWT $\{x_n\}$ has a sub-sequence $\{x_{n_k}\}$ with $x_{n_k} \to a \in S$. Since $|x_{n_k} - y_{n_k}| \to 0$, then $y_{n_k} \to a$. But then $\lim_{x\to\infty} |f(x_{n_k}) - f(y_{n_k})| = 0$, which is impossible. □