Lecture 22, Oct. 31

Anton's tutorial on Tuesady is cancelled.

Aside

Suppose $f : \mathbb{R} \to \mathbb{R}$.

$$D(f) = \{x_0 \in \mathbb{R} \mid f \text{ is discontinuous at } x_0\}$$

$$D_n(f) = \{x_0 \in \mathbb{R} \mid \forall \delta > 0\, \exists x, y \in (x_0 - \delta, x_0 + \delta)\ \ |f(x) - f(y)| \geq \frac{1}{n}\}$$

Then if $x_0 \in D_n(f)$ for some n, then $x_0 \in D(f)$.

Note.

$$D(f) = \bigcup_{n=1}^{\infty} D_n(f)$$

22.1 Definition. A set $A \subset \mathbb{R}$ is called F_σ if

$$A = \bigcup_{n=1}^{\infty} F_n$$

where each F_n is closed.

22.2 Example. Let $\mathbb{Q} = \{r_1, r_2, r_3, \cdots\}$

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\} \to F_\sigma$$

22.3 Definition. A set $A \subset \mathbb{R}$ is called G_δ if

$$A = \bigcap_{n=1}^{\infty} U_n$$

where each U_n is open.

Note. 1. $\{0\} = \bigcup_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$

2. A is G_δ iff A^c is F_σ

3. $D(f)$ is $F_\sigma \to D(f)^c$ is G_δ

Note. $\mathbb{Q} = \{r_1, r_2, r_3, \cdots\}$

$$U_k = \bigcup_{n=1}^{\infty}(r_n - \frac{1}{2^{n+k+1}}, r_n + \frac{1}{2^{n+k+1}}) \supset \mathbb{Q}$$

So is it true that

$$\bigcap_{k=1}^{\infty} U_k = \mathbb{Q}$$

Continuity on an Interval

22.4 Question. Is $f(x) = \sqrt{x}$ continuous at $x = 0$?

22.5 Definition (Continuity on an Interval)**.** We say that $f(x)$ is continuous on the open interval (a, b) if $f(x)$ is continuous at each $x_0 \in (a, b)$

We say that $f(x)$ is continuous on the closed interval $[a, b]$ if $f(x)$ is continuous at (a, b) and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

Similarly for $(a, b], (a, \infty), \cdots$

22.6 Theorem (Sequential Characterization for Continuity on [a,b])**.** *Let $f : [a, b] \to \mathbb{R}$, then the followings are equivalent:*

1. *f is continuous at $[a, b]$*
2. *if $\{x_n\} \subset [a, b]$ with $x_n \to x_0 \in [a, b]$ then $f(x_n) \to f(x_0)$*

Remark. Given $S \subset \mathbb{R}, S \neq \emptyset$, we say that $f : S \to \mathbb{R}$ is continuous on S is whenever $\{x_n\}$ is a sequence in S with $x_n \to x_0 \in S$ we have $f(x_n) \to f(x_0)$