Lecture 14, Oct. 7

14.1 Theorem. Bolzano-Weierstrass Theorem *Every bounded sequences has a convergent sub-sequence.*

14.2 Definition. We say that $\alpha \in \mathbb{R}$ is a **limit point** of $\{a_n\}$ if there exists a sub-sequence $\{a_{n_k}\}$ with $\lim_{n\to\infty} a_{n_k} = \alpha$

LET $LIM(\{a_n\}) = \{\alpha \in \mathbb{R} \mid \alpha$ is a limit point of $\{a_n\}$

14.3 Example. $a_n = (-1)^{n+1} \to \{1, -1, 1, -1, \dots\}$

$LIM(\{a_n\}) = \{1, -1\}$

14.4 Example. $a_n = n \to \{1, 2, 3, \dots\}$

$LIM(\{a_n\}) = \emptyset$

Fact If $\{a_n\}$ converges with $\lim_{n\to\infty} a_n = L$, then $LIM(\{a_n\}) = \{L\}$

14.5 Question. If $\{a_n\}$ is such that $LIM(\{a_n\})$ contains only one value α, does $\{a_n\}$ converges to α?

No. Counterexample:

$$\{a_n\} = \{1, \frac{1}{2}, 3, \frac{1}{4}, 5, \dots\}$$

14.6 Proposition. *α is a limit point of $\{a_n\}$ if for every $(\alpha - \epsilon, \alpha + \epsilon)$ contains infinite many terms of the sequence.*

Assume α is a limit point of $\{a_n\}$, then there exists a sub-squence $\{a_{n_k}\}$ with $a_{n_k} \to \alpha$. There exists $K_0 \in \mathbb{N}$ so that $k \geq K_0 \to |a_{n_k} - \alpha| < \epsilon \to a_{n_k} \in (\alpha - \epsilon, \alpha + \epsilon)$

Proof. Assume that $\forall \epsilon > 0$, $(\alpha - \epsilon, \alpha + \epsilon)$ contains infinitely many terms of $\{a_b\}$

For $\epsilon = 1$ we can find n_1 so that $a_{n_1} \in (\alpha - 1, \alpha + 1)$

$a_{n_2} \in (\alpha - \frac{1}{2}, \alpha + \frac{1}{2})$

Suppose we have $n_1 < n_2 < n_3 < \cdots < n_k$ with

$$a_{n_j} \in (\alpha - \frac{1}{j}, \alpha + \frac{1}{j})$$

Since $(\alpha - \frac{1}{k+1}, \alpha + \frac{1}{k+1})$ contains infinitely many $a_n s$. there is $n_{k+1} > n_k$ with $a_{n_{k+1}} \in (\alpha - \frac{1}{k+1}, \alpha + \frac{1}{k+1})$

We proceed recursively to get a sub-sequence $\{a_{n_k}\}$ with

$$a_{n_k} = (\alpha - \frac{1}{k}, \alpha + \frac{1}{k})$$

$$\alpha - \frac{1}{k} < a_{n_k} < \alpha + \frac{1}{k}$$

By the squeeze theorem, $a_{n_k} \to \alpha$

$\square$

14.7 Question.

1. Suppose $\{a_n\}$ is bounded and $LIM(\{a_n\}) = \{L\}$, does $\lim_{n\to\infty} L$?
2. Does there exists $\{a_n\}$ with $LIM(\{a_n\}) = \{R\}$
3. For which subsets S of R does there exists $\{a_n\}$ with $LIM(\{a_n\}) = S$?

Cauchy Sequence

14.8 Question. Is there an intrinsic way to characterize a convergent sequence?

Note. If $\lim_{n\to\infty} a_n = L$ and if $\epsilon > 0$ then we can find N_0 so that if $n \geq N_0$m

$$|a_n - L| < \frac{\epsilon}{2}$$

If $n, m \geq N_0$, then

$$\begin{aligned} |a_n - a_m| &= |(a_n - L) + (L - a_m)| \\ &\leq |a_n - L| + |L - a_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

14.9 Definition. A sequence $\{a_n\}$ is **Cauchy** is for every $\epsilon > 0$, then there exists $N_0 \in \mathbb{N}$ so that if $n, m \geq N_0$, then

$$|a_n - a_m| < \epsilon$$

14.10 Proposition. *Every convergent sequence is Cauchy*

14.11 Question. Does every Cauchy sequence Converges?

14.12 Lemma. *Every Cauchy Sequence is bounded.*

Proof. Let $\epsilon = 1$ and choose N_0 so that if $n, m \geq N_0$, then $|a_n - a_m| < \epsilon$

Hence, if $n \geq N_0$ then

$$|a_n - a_{N_0}| < 1 \to |a_n| \leq |a_{N_0}| + 1$$

$\square$

Let $M = max\{|a_1|, |a_q|, \ldots, |a_{N_0-1}|, |a_{N_0}| + 1\}$

14.13 Lemma. *Let $\{a_n\}$ be Cauchy. Assume that $\{a_{n_k}\}$ is such that $\lim_{k\to\infty} a_{n_k} = L$, then*

$$\lim_{n\to\infty} a_n = L$$

Proof. Let $\epsilon > 0$. We can find a N_0 so that if $n, m \geq N_0$, then

$$|a_n - a_m| < \frac{\epsilon}{2}$$

Let $n \geq N_0$

$$\begin{aligned} |a_n - L| &= |(a_n - a_{n_k}) + (a_{n_k} - L)| \\ &\leq |a_n - a_{n_k}| + |a_{n_k} - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

□

14.14 Theorem. Completeness Property for $\mathbb{R}$ *Every Cauchy Sequence Converges.*

Proof. If a_n is Cauchy, then a_n is bounded. By BWT, a_n has a convergent sub-sequence $\{a_{n_k}\}$. Hence a_n converges. (by Lemma 2.) □