Lecture 6, Sept. 23

Inequalities

6.1 Example. Find all $x \in \mathbb{R}$ such that
$$0 < |x-2| \leq 4$$

Solution. $[-2, 6]$ with $x \neq 2$

Three Basic Inequalities

1. $|x-a| < \delta$
2. $0 < |x-a| < \delta$
3. $|x-a| \leq \delta$

Solution. 1. $(a-\delta, a+\delta) = \{x \in R \mid a-\delta < x < a+\delta\}$

2. $(a-\delta, a+\delta)$ with $x \neq a = \{x \in \mathbb{R} \mid a-\delta < x < a+\delta, x \neq a\}$
3. $[a-\delta, a+\delta] = \{x \in R \mid a-\delta \leq x \leq a+\delta\}$

Sequence

6.2 Definition. A **sequence** is an infinite ordered list of real numbers.

Notation. $\{1, 2, 3, 4, \dots\}$ or $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

6.3 Definition. A **sequence** of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$

The element $f(n)$ is called the n-th term of the sequence. We often denote this by $f_(n) = a_n$

Notation. We can denote sequences in many ways

1. $f(n) = \frac{1}{n}$ for all $n \in \mathbb{N}$
2. Let $a_n = \frac{1}{n}$
3. $\{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$
4. $\{\frac{1}{n}\}$
5. Sometimes we define sequences recursively.
   $a_1 = 1$ and $a_{n+1} = \sqrt{3 + 2a_n}$ for all $n \geq 1$.

Graphing Sequence

Subsequence

6.4 Definition. Let $\{a_n\}$ be a sequence, and let $\{n_k\}$ be a sequence of natural numbers with $n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \dots$.

The sequence $b_k = a_{n_k} \to \{b_k\}_{k=1}^{\infty}$ is called **subsequence** of $\{a_n\}$. We often write this as
$$\{a_{n_1}, a_{n_2}, \dots, a_{n_k}, \dots\}$$

Important Subsequences Given $\{a_n\}$, let $n_0 \in \mathbb{N} \cup \{0\}$. Define

$$b_k = a_{n_0+k}$$

This sequence is called a tail of $\{a_n\}$

Limits of Sequences Consider $\{\frac{1}{n}\} = \{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$

Note. As n gets larger and larger, the terms of the sequence $\{\frac{1}{n}\}$ get closer and closer to 0. We would like to say that the sequence $\{\frac{1}{n}\}$ converges to 0 and call 0 the limit of $\{\frac{1}{n}\}$.

6.5 Definition. (Heuristic Definition of Convergence). We say that a sequence $\{a_n\}$ has a limit L if for every positive tolerance $\epsilon > 0$, the term a_n will approximate L with an error less than ϵ so long as the index n is large enough.