Lecture 21, Oct. 19

21.1 Definition. A **total order** on a set S is a binary relation $\leq$ on S such that

1. Totality: for all $a, b \in S$, either $a \leq b$ or $b \leq a$
2. Antisymmetry: for all $a, b \in S$, if $a \leq b$ and $b \leq a$, then $a = b$
3. Transitivity; for all $a, b, c \in S$, if $a \leq b$ and $b \leq c$ then $a \leq c$

21.2 Example. The usual order $\leq$ is a total order on each of the sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and indeed on any subset of $\mathbb{R}$.

$\subseteq$ is a partial order on $P(S)$. If we define $a \leq b$ for $a, b \in \mathbb{N}$ to mean $a \mid b$ then $\leq$ is a partial order on $\mathbb{N}$

21.3 Definition. Given a total order $\leq$ on S, for $a, b \in S$, we define $a < b$ to mean $(a \leq b$ and $a \neq b)$, $a \geq b$ to mean $b \leq a$, $a > b$ to mean $b < a$.

Remark. We could also define a total order on S to be a binary relation $<$ such that

1. for all $a, b \in S$ exactly one of the following holds:
$$a < b, a = b, b < a$$
2. for all $a, b, c \in S$ if $a < b$ and $b < c$ then $a < c$.

21.4 Definition. A **ordered field** is a field F with a total order $<$ such that

1. $<$ is compatible with $+$: for all $a, b, c \in F$
$$a < b \to a + c < b + c$$
2. $<$ is compatible with $\times$: for all $a, b \in F$,
$$0 < a \wedge 0 < b \to 0 < ab$$

21.5 Example. $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields. Also $\mathbb{Q}[\sqrt{2}]$ is an ordered field.

21.6 Theorem. Properties of Ordered Fields *Let F be an ordered fields, and let $a, b, c \in F$.*

1. *If $a > 0$ then $-a < 0$ and if $a < 0$ then $-a > 0$*
2. *If $a > 0$ and $b < c$ then $ab < ac$*
3. *If $a < 0$ and $b < c$ then $ab < ac$*
4. *If $a \neq 0$ then $a^2 > 0$. In particular, $1 > 0$*
5. *if $0 < a < b$, then $0 < 1/b < 1/a$*

Proof.

1. Suppose $a > 0$, then

$$\begin{aligned}
0 &< a \\
0 + (-a) &< a + (-a) \text{ since } < \text{ is compatible with } + \\
-a &< 0.
\end{aligned}$$

   Suppose $a < 0$, then...

2. Suppose $a > 0$ and $b < c$, then

$$\begin{aligned}
b &< c \\
b + (-b) &< c + (-b) \text{ since } < \text{ is compatible with } + \\
0 &< c - b \\
0 &< a(c - b) \text{ since } < \text{ is compatible with } \times \\
0 &< ac - ab \\
0 + ab &< (ac - ab) + ab \text{ since } < \text{ is compatible with } + \\
0 + ab &< ac + (-ab + ab) \\
0 + ab &< ac + (ab - ab) \\
0 + ab &< ac + 0 \\
ab &< ac
\end{aligned}$$

□

21.7 Example. When p is a prime numer we shall see that $\mathbb{Z}_p$ is a field. It is not possible to define an order which makes $\mathbb{Z}_p$ into an ordered field.

Proof. If $<$ was such an order then we would have

$$\begin{aligned}
1 &> 0 \\
-1 &< 0 \\
-1 = p - 1 = 1 + 1 + \cdots + 1 &> 0
\end{aligned}$$

By contradiction, such order does not exist. □

Similarly, it is not possible to define an order $<$ on $\mathbb{C}$ which makes $\mathbb{C}$ into an ordered field.

$$\begin{aligned}
1 &> 0 \\
-1 &< 0 \\
-1 = i^2 &> 0 \text{ by Property 21.6.4}
\end{aligned}$$

21.8 Definition. Let F be an ordered field. For $a \in F$ we define the **absolute value** of a to be

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases}$$

21.9 Theorem. Properties of Absolute Value *Let F be an ordered field. Let $a, b \in F$. Then*

1. *Positive Definiteness*
$$|a| \geq 0 \land (|a| = 0 \leftrightarrow a = 0)$$
2. *Symmetry*
$$|a - b| = |b - a|$$
3. *Multiplicative*
$$|ab| = |a|\,|b|$$
4. *Triangle Inequality*
$$||a| - |b|| \leq |a - b| \leq |a| + |b|$$
5. *Approximation: for $b \geq 0$ and $x \in F$*
$$|x - a| < b \leftrightarrow a - b < x < a + b$$

Order Properties in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$

21.10 Theorem. *In $\mathbb{Z}$,*

1. *for all $n \in \mathbb{Z}$*
$$n \in \mathbb{N} \leftrightarrow n \geq 0$$
2. *Discreteness: for all $n, kin\mathbb{Z}$,*
$$n \leq k \leftrightarrow n < k + 1$$
3. *Well Ordering Property: for every non-empty subset $S \subseteq \mathbb{Z}$, if S is bounded above in $\mathbb{Z}$, then S has a maximum element.*
4. *Well Ordering Property: for every non-empty subset $S \subseteq \mathbb{Z}$, if S is bounded below in $\mathbb{Z}$, then S has a minimum element.*

Remark. Well-Ordering is related to Induction.