Lecture 18, Oct. 14

18.1 Example. Given $n, m \in \mathbb{Z}^+$, find

$$\sum_{k=1}^{n} k^m = 1^m + 2^m + 3^m + \dots$$

Solution. For fixed $n \in \mathbb{Z}^+$, we can find a recursion formula for

$$S_m = \sum_{k=1}^{n} k^m$$

$$S_0 = \sum_{k=1}^{n} k^0 = n$$

$$S_1 = \sum_{k=1}^{n} k^1 = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

Find

$$\sum_{k=1}^{n} (k+1)^{m+1} - k^{m+1}$$

in 2 ways.

1.
$$\sum_{k=0}^{n} (k+1)^{m+1} - k^{m+1} = (n+1)^{m+1}$$

2.
$$\begin{aligned}
&\sum_{k=0}^{n} (k+1)^{m+1} - k^{m+1} \\
=&\sum_{k=0}^{n} ((k^{m+1} + \binom{m+1}{1} k^m + \binom{m+1}{2} k^{m-1} + \dots + \binom{m+1}{m} k + \binom{m+1}{m+1}) - k^{m+1}) \\
=&\binom{m+1}{1} \sum_{k=0}^{n} k^m + \binom{m+1}{2} \sum_{k=0}^{n} k^{m-1} + \dots + \binom{m+1}{m} \sum_{k=0}^{n} k + \binom{m+1}{m+1} \sum_{k=0}^{n} 1
\end{aligned}$$

$$(n+1)^{m+1} = \binom{m+1}{1} \sum_{k=0}^{n} k^m + \binom{m+1}{2} \sum_{k=0}^{n} k^{m-1} + \dots + \binom{m+1}{m} \sum_{k=0}^{n} k + (n+1)$$

Thus

$$S_m = \frac{1}{m+1}((n+1)^{m+1} - \binom{m+1}{2} S_{m-1} - \dots - \binom{m+1}{m} S_1 - S_0 - 1)$$

18.2 Theorem. *Let $a, b, p, q \in \mathbb{R}$ (or $\mathbb{C}$) with $q \neq 0$ and let $m \in \mathbb{Z}$ Let $(X_n)_{n \geq m}$ be the sequence*

$$x_m = a, x_{m+1} = b, x_n = px_{n-1} + qx_{n-2} \text{ for } n \geq m+2$$

Let $f(x) = x^2 - px - q$ ($f(x)$ is called the characteristic polynomial for the recursion formula)

Suppose that $f(x)$ factors as

$$f(x) = (x-u)(x-v)$$

with $u, v \in \mathbb{R}$ (or $\mathbb{C}$) with $u \neq v$

Then there exist $A, B \in \mathbb{R}$ or $\mathbb{C}$ such that

$$x_n = Au^n + Bv^n$$

for all $n \geq m$

Proof. exercise ☐

18.3 Example. Let $(x_n)_{n \geq 0}$ be defined by

$$x_0 = 4, x_1 = -1, \; x_n = 3x_{n-1} + 10x_{n-2}$$

for $n \geq 2$.

Find a closed form formula for x_n

Solution. Let $f(x) = x^2 - 3x - 10 = (x-5)(x+2)$.

By the Linear Recursion Theorem, there exists $A, B \in \mathbb{R}$ such that

$$x_n = A5^n + B(-2)^n$$

for all $n \geq 0$.

To get $x_0 = A5^0 + B(-2)^0$, we need

$$A + B = 4. \tag{1}$$

To get $x_1 = A5^1 + B(-2)^1$, we need

$$5A - 2B = -1. \tag{2}$$

Solve 1 and 2 to get

$$A = 1, \; B = 3$$

Then

$$x_n = 5^n + 3(-2)^n$$

for all $n \geq 0$.

18.4 Example. There are n points on a circle around a disc. Each of the $\binom{n}{2}$ pairs of points is joined by a line segment. Suppose that no three of these line segment have a common point of intersection inside the disc.

Into how many regions is the disc divided by the line segments?

Solution. HINT

Suppose that we have l lines, each of which intersects the circle twice and intersects with the disc in a line segment.

Suppose these l line segments intersect at p points inside the disc. Suppose that no three of these line segment have a common point of intersection inside the disc. Into how many regions is the disc divided by the line segments?