Lecture 8, Sept. 28

8.1 Theorem. *Assume that $\{a_n\}$ converges. then $\{a_n\}$ is bounded.*

Proof. Assume that

$$L = \lim_{n\to\infty} a_n$$

Let $\epsilon = 1$. Then there exists $N_0 \in \mathbb{N}$ so that if $n \geq N_0$ then $|a_n - L| < 1$

If $n \geq N_0$,then

$$\begin{aligned} |a_n| = |a_n - L + L| &\leq |a_n - l| + |L| \\ &< 1 + |L| \end{aligned}$$

Let

$$M = max|a_1|, |a_2|, \ldots, |a_{N_0-1}|, |L| + 1$$

.

Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. ☐

Question: Do all bounded sequences converge? No.

8.2 Definition.

1. We say that a sequence $\{a_n\}$ is increasing if $a_n < a_{\{}n+1\}$ for all $n \in \mathbb{N}$
2. We say that $\{a_n\}$ is non-decreasing if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$
3. We say that $\{a_n\}$ is decreasing if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$
4. We say that $\{a_n\}$ is non-increasing if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$

We say that $\{a_n\}$ is monotonic if $\{a_n\}$ satisfies one of the conditions.

Example:

1. $$\{a_n\} = \{\frac{1}{n}\}$$

   is decreasing, since

   $$\frac{1}{n+1} \leq \frac{1}{n}$$

   for all $n \in \mathbb{N}$

2. $$\{cos(n)\}$$

3. Let $a_1 = 1$,

   $$a_{n+1} = \sqrt{3 + 2a_n}$$

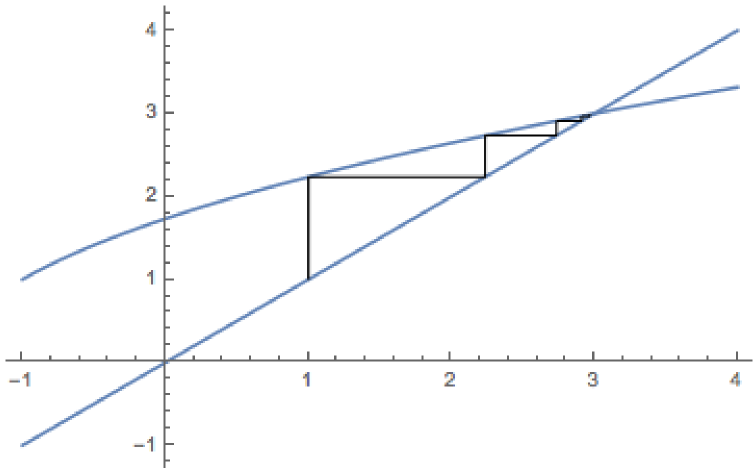

Figure 1: $y = \sqrt{3+2x}$ and $y = x$

8.3 Theorem. Monotone Convergence Theorem

If $\{a_n\}$ is monotonic and bounded, then $\{a_n\}$ converges.

Proof. Assume that $\{a_n\}$ is non-decreasing and bounded above. Let $L = lub(\{a_n\})$

Let $\epsilon > 0$, then $L - \epsilon$ is not an upper bound. Then there exists $N_0 \in \mathbb{N}$ so that $L - \epsilon < a_{N_0} \leq L$. If $n \geq N_0$, then $L - \epsilon < a_{N_0} \leq a_n \leq L$, so $|a_n - L| < \epsilon$. Hence $L = \lim_{n\to\infty} a_n$

Similarly, if $\{a_n\}$ is non-increasing then $L = \lim_{n\to\infty} a_n$ where $L = glb(\{a_n\})$ $\square$

8.4 Example. Let $a_1 = 1$,

$$a_{n+1} = \sqrt{3 + 2a_n}$$

We know that $0 \leq a_n < a_{n+1} \leq 3$ for all $n \in \mathbb{N}$. $\{a_n\}$ is increasing and bounded above. Hence $\{a_n\}$ converges.

8.5 Corollary. *A monotonic sequence $\{a_n\}$ converges iff it is bounded.*

8.6 Definition. We say a sequence **diverges to** ∞ if for every $M > 0$ we can find a a cutoff $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then $M \leq a_n$, we write $\lim_{n\to\infty} a_n = \infty$.