Lecture 27, Nov. 10

27.1 Theorem (Arithmetic Rules for Differentiation)**.** *Assume that $f(x)$, $g(x)$ are differentiable at $x = a$.*

1. *If $f(x) = c$ for all x, then $f'(a) = 0$*
2. *$(f+g)(x)$ is differentiable at $x = a$ with $(f+g)'(a) = f'(a) + g'(a)$*
3. *$(fg)(x)$ is differentiable at $x = a$ with $(fg)'(a) = f'(a)g(a) + g'(a)f(a)$*
4. *Let $h(x) = \frac{1}{f(x)}$. Then $h(x)$ is differentiable at $x = a$ if $f(a) \neq 0$ and*
$$h'(a) = \frac{-f'(a)}{f(a)}$$
5. *If $h(x) = \frac{f(x)}{g(x)}$ then $h(x)$ is differentiable at $x = a$, if $g(a) \neq 0$ and*
$$h'(a) = \frac{f'(a)g(a) - g'(a)f(a)}{g^2(a)}$$

Proof.

3 Consider
$$\lim_{x\to a} \frac{(fg)(x) - (fg)(a)}{x - a}$$

$$\begin{aligned}
&\lim_{x\to a} \frac{(fg)(x) - (fg)(a)}{x - a} \\
=&\lim_{x\to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x - a} \\
=&\lim_{x\to a} g(x)\frac{f(x) - f(a)}{x - a} + \lim_{x\to a} f(a)\frac{g(x) - g(a)}{x - a} \\
=&\lim_{x\to a} g(x) \lim_{x\to a} \frac{f(x) - f(a)}{x - a} + f(a)g'(a) \\
=&g(a)f'(a) + f(a)g'(a)
\end{aligned}$$

4 Consider
$$\lim_{x\to a} \frac{1/f(x) - 1/f(a)}{x - a}.$$

$$\begin{aligned}
&\lim_{x\to a} \frac{1/f(x) - 1/f(a)}{x - a} \\
=&\lim_{x\to a} \frac{f(a) - f(x)}{x - a} \cdot \frac{1}{f(a)f(x)} \\
=&\frac{-f'(a)}{f^2(a)}
\end{aligned}$$

5 Combine 3 and 4.

$\square$

Linear Approximation

Note. Assume that $f(x)$ is differentiable at $x = a$. Then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

If $x \approxeq a$, then

$$\begin{aligned} f'(a) &\approxeq \frac{f(x) - f(a)}{x - a} \\ \Rightarrow f'(a)(x-a) &\approxeq f(x) - f(a) \\ \Rightarrow f(x) &\approxeq f'(a)(x-a) + f(a) \end{aligned}$$

27.2 Definition. Let $f(x)$ be differentiable at $x = a$. We define the linear approximation to $f(x)$ at $x = a$ to be the function

$$L_a^f(x) = f(a) + f'(a)(x - a)$$

27.3 Theorem (Properties of Linear Approximation)**.** $L_a^f(x)$ *has the following properties*

1. $L_a^f(a) = f(a)$
2. $(L_a^f)'(x) = f'(a)$
3. *If* $h(x) = mx + b$ *and* $h(x)$ *satisfies 1) and 2) then* $h(x) = L_a^f(a)$
4. $L_a^f(a) \approxeq f(x)$ *if* $x \approxeq a$
5. *The graph of* $L_a^f(a)$ *is the tangent line to graph of* $f(x)$ *at* $x = a$

27.4 Example. Consider $f(x) = \sin x$.

$$L_0^{\sin x} = \sin 0 + \cos 0(x - 0) = x$$

27.5 Example. Consider $f(x) = e^x$, we have $f(0) = 1$ and $f'(0) = 1$. Then

$$L_0^{e^x} = f(0) + f'(0)(x - 0) = 1 + x$$

27.6 Example. If $f(x) = e^{-u^2}$,

$$e^{-u^2} \approxeq 1 - u^2$$

if u is small.