Lecture 12, Oct. 5

12.1 Example. Find $\dfrac{3n^2+2n}{5n^2+2}$

Solution.

$$\begin{aligned}
\lim_{n\to\infty}\frac{3n^2+2n}{5n^2+2} &= \lim_{n\to\infty}\frac{n^2}{n^2}\frac{3+\dfrac{2}{n}}{5+\dfrac{2}{n^2}} \\
&= \frac{\lim\limits_{n\to\infty}3+\lim\limits_{n\to\infty}\dfrac{2}{n}}{\lim\limits_{n\to\infty}5+\lim\limits_{n\to\infty}\dfrac{2}{n^2}} \\
&= \frac{3+0}{5+0} \\
&= \frac{3}{5}
\end{aligned}$$

Note. If $a_k b_j \neq 0$

$$\lim_{n\to\infty}\frac{a_0+a_1n+\cdots+a_kn^k}{b_0+b_1n+\cdots+b_jn^j} = \begin{cases} \dfrac{a_k}{b_j} & \text{if } k=j \\ 0 & \text{if } j>k \\ \infty & \text{if } j<k,\ a_kb_j>0 \\ -\infty & \text{if } j<k,\ a_kb_j<0 \end{cases}$$

12.2 Example. Find

$$\lim_{n\to\infty}\sqrt{n^2+n}-n$$

Solution.

$$\begin{aligned}
\lim_{n\to\infty}\sqrt{n^2+n}-n &= \lim_{n\to\infty}(\sqrt{n^2+n}-n)\cdot\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty}\frac{n}{\sqrt{1+\dfrac{1}{n}}+1} \\
&= \frac{n}{\sqrt{1+\lim\limits_{n\to\infty}\dfrac{1}{n}}+1} \\
&= \frac{1}{2}
\end{aligned}$$

12.3 Example. $a_1 = 1$ and $a_{n+1} = \dfrac{1}{1+a_n}$. Suppose that $\{a_n\}$ converges, find $\lim_{n\to\infty} a_n$

12.4 Proposition. *A sequence $\{a_n\}$ converges to L if and only if every sub-sequence $\{a_{n_k}\}$ converges to L*

Proof. Assume that $lim_{n \to \infty} a_n = L$. Let $\{a_{n_k}\}$ be a sub-sequence. Let $\epsilon > 0$, we can find a N_0 so that if $n \geq N_0$, then $|a_n - L| < \epsilon$.

Let $k_0 \geq N_0$, then $k \geq k_0 \Rightarrow n_k \geq n_{k_0} \geq N_0$

Hence $|a_{n_k} - L| < \epsilon$ $\square$

Solution. If

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1}$$

Then,

$$L = \frac{1}{1+L}$$
$$L^2 + L - 1 = 0$$
$$L = \frac{-1 \pm \sqrt{5}}{2}$$

12.5 Question. Does $\{a_n\}$ converge?

Solution. Claim that for any k,

$$a_{2k} < a_{2k+2} < a_{2k+1} < a_{2k-1}$$

Proof by induction.

$\{a_{2k-1}\}$ is decreasing and bounded below by 0

$\{a_{2k}\}$ is increasing and bounded above by 1.

Let $\lim_{n \to \infty} a_{2k} = M$ and $\lim_{n \to \infty} a_{2k-1} = L$.

Since $M = \dfrac{-1 + \sqrt{5}}{2}$ and $L = \dfrac{-1 + \sqrt{5}}{2}$, $M = L$

Thus, $\{a_n\}$ converges.

12.6 Example. FInd

$$\lim_{n \to \infty} \frac{cos(n)}{n}$$