MATH 237 Lecture 5

Zhongwe Zhao

September 19, 2016

Example 1: Let:

$$f(x,y)=\begin{cases}\dfrac{sin(x^2+2y^2)}{x^2+y^2} & (x,y)\neq(0,0)\\ k & (x,y)=(0,0)\end{cases}$$

Can we choose k to make $f(x,y)$ continuous at (0,0)?

Solution:

Find

$$\begin{aligned}&\lim_{(x,0)\to(0,0)} f(x,y)\\ =&\lim_{x\to 0}\frac{sin(x^2)}{x^2}\\ =&1\end{aligned}$$

$$\begin{aligned}&\lim_{(0,y)\to(0,0)} f(x,y)\\ =&\lim_{y\to 0}\frac{sin(2y^2)}{y^2}\\ =&2\end{aligned}$$

So it's impossible to choose k to make this function to be continuous.

Theorem. *Let f and g be continuous at (a,b), then*

1. *$f+g$ and fg are continuous at (a,b)*
2. *$\dfrac{f}{g}$ is continuous at (a,b) if $g(a,b)\neq 0$*

1 Continuity of a Composition

Let f be a function of one variable, let g be a function of two variables. if g is continuous at (a,b) and f is continuous at $g(a,b)$, then $(f\circ g)(a,b)=f(g(a,b))$ is continuous at (a,b).

Example 2: The function

$$\frac{y \sin x - \cos y}{x^2 + y^2}$$

is continuous for all (x, y) except possibly $(0, 0)$.

Example 3:

$$e^{x^2 + \sin(xy)}$$

is continuous $\forall (x, y)$ by continuity theorems.

Example 4:

$$\begin{aligned} &\lim_{(x,y)\to(0,0)} \frac{e^{x^2} + ln(2 + y^2 + x^4)}{(x-1)^2 + y^{100}} \\ =&\frac{1 + ln(2)}{1} \\ =&1 + ln(2) \end{aligned}$$

Example 5: Determine where

$$f(x, y) = \begin{cases} \dfrac{x^4 y^6}{x^6 + y^{12}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is continuous.

Solution: If $(x, y) \neq (0, 0)$, f is continuous by continuous theorem.

If $(x, y) = (0, 0)$, use definition.

Check whether

$$\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$$

$$LHS = \lim_{(x,y)\to(0,0)} \frac{x^4 y^6}{x^6 + y^{12}}$$

Proof. To show the limit exist and equals 0,

$$\begin{aligned} \left| \frac{x^4 y^6}{x^6 + y^{12}} - 0 \right| &= \frac{x^4 y^6}{x^6 + y^{12}} \\ &= \frac{(x^6)^{4/6} (y^{12})^{1/2}}{x^6 + y^{12}} \\ &\leq \frac{(x^6 + y^{12})^{4/6} (y^{12} + x^6)^{1/2}}{x^6 + y^{12}} \\ &= (x^6 + y^{12})^{1/6} \end{aligned}$$

By the squeeze theorem, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 y^6}{x^6 + y^{12}} = 0$. So f is continuous at $(0, 0)$ $\square$

Example 6:

$$f(x,y) = \begin{cases} \dfrac{e^{xy}-1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution: Try along $y = mx$

$$\begin{aligned} &\lim_{x\to 0} \frac{e^{mx^2}-1}{x^2+m^2x^2} \\ =&\lim_{x\to 0} \frac{2mxe^{mx^2}}{2x(1+m^2)} \\ =&\frac{m}{1+m^2} \end{aligned}$$

So f is continuous iff $(x,y) \neq (0,0)$