Lecture 32, Nov. 18

32.1 Theorem. *If $f\colon [a,b] \to \mathbb{R}$ is increasing, then TFAE*

1. *$f(x)$ is continuous on $[a,b]$*
2. *$f([a,b]) = [f(a), f(b)]$*

32.2 Corollary. *If $f\colon [a,b]$ is strictly monotonic with inverse $g\colon f([a,b]) \to [a,b]$ then f is continuous on $[a,b]$ if and only if g is continuous on $f([a,b])$.*

32.3 Theorem (Inverse Function Theorem)**.** *Assume that if $f\colon [a,b] \to \mathbb{R}$ is strictly monotonic with inverse $g\colon f([a,b]) \to \mathbb{R}$. If f is continuous on $[a,b]$, differentiable on $[a,b]$, and if $x_0 \in (a,b)$ with $f'(x_0) \neq 0$ with $y_0 = f(x_0)$, then g is differentiable at y_0 with*

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}.$$

Proof. Let $\{y_n\} \subset f([a,b])$ with $y_n \to y_0$, $y_n \neq y_0$. Let $x_n = g(y_n) \in [a,b]$. Since f and g are continuous, $g(y_n) \to g(y_0) \Rightarrow x_n \to x_0$. Then

$$\lim_{x\to\infty} \frac{g(y_n) - g(y_0)}{y_n - y_0} = \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n - f(x_0))} = \lim_{n\to\infty} \frac{1}{f'(x_0)}.$$

By the Sequenctial Characterization of limits $g'(x) = \lim_{n\to\infty} \frac{1}{f'(x_0)}$ $\square$

32.4 Example. $f(x) = x^3$ and $g(x) = x^{1/3}$.

$$f'(0) = 0$$

$$g'(x) = \begin{cases} \frac{1}{3x^{2/3}} & \text{if } x \neq 0 \\ \text{does not exist} & \text{if } x = 0 \end{cases}$$

32.5 Example (Inverse Trig Functions)**.**

1. $\arcsin x$

   $f(x) = \sin x$ on $[-\pi/2, \pi/2]$, $f(x)$ is strictly increasing $\Rightarrow$ invertible on $[-\pi/2, \pi/2]$.

   $$\sin([-\pi/2, \pi/2]) = [-1, 1].$$

   Define $g(x) = \arcsin(y)$ on $[-1,1]$ by $g(y) = x$ iff $\sin x = y$ for $x \in [-\pi/2, \pi/2]$

   If $g(y) = \arcsin y$. if $y_0 \in (-1,1)$,

   $$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{\cos x}.$$

   where $f(x) = \sin x$ and $x_0 = \arcsin y_0$ and $y_0 = \sin x_0$, $x_0 \in (-\pi/2, \pi/2)$. Since $\cos x_0 = \sqrt{1 - \sin^2 x_0} = \sqrt{1 - y_0^2}$,

   $$g'(y_0) = \frac{1}{\sqrt{1 - y_0^2}}.$$

   Note. $\sin(\arcsin x) = x$ holds for $x \in [-1,1]$ while $\arcsin(\sin x) = x$ Holds iff $x \in [-\pi/2, \pi/2]$

2. $\arctan x$

   For each $y \in \mathbb{R}$ define $g(y) = \arctan y$ by $g(y) = x$ iff $\tan x = y$ for $x \in (-\pi/2, \pi/2)$. That is,

$$\arctan y : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$$

   with $\tan(\arctan y) = y$ for $y \in \mathbb{R}$

   Note that

$$\frac{d}{dx} \tan x = \sec^2 x = \frac{1}{\cos^2 x}$$

   By the Inverse Function Theorem,

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \frac{1}{\sec^2(\arctan y)} = \frac{1}{1 + \tan^2(\arctan y)} = \frac{1}{1 + y^1}$$

3. $\arccos y$ $\cos(x)$ is 1-1 on $[0, \pi]$

   Note. $\cos([0, \pi]) = [-1, 1]$

   For each $y \in [-1, 1]$ define $g(y) = x$ iff $y = \cos x$ for $x \in [0, \pi]$

$$g'(y) = \frac{1}{-\sqrt{1 - y^2}}$$