Lecture 8, Sept. 21

Recap

Symbol Set

Term

Formula

First-Order Language

8.1 Definition. In the language of first-order number theory, we allow us to use the following additional symbols:

$$\{0, 1, +, \times, <\}$$

Unless otherwise stated, we do not allow ourselves to use any other additional symbols.

8.2 Example. Express each of the following statement as formulas in the language of first-order number theory.

a) x is a factor of y

b) x is a prime number

c) x is a power of 3

Solution. We take he universal set to be $\mathbb{Z}$.

a) $\exists z \in \mathbb{Z} \;\; y = x \times z$

b) $1 < x \wedge \forall y(\exists z \;\; x = y \times z \to ((y = 1 \vee y = x) \vee (y + 1 = 0 \vee y + x = 0)))$
$1 < x \wedge \forall y((1 < y \wedge \exists z \;\; x = z \times y) \to y = x)$

c) $(0 < x) \wedge$ the only prime factor of x is 3
$\iff (0 < x) \wedge \forall y \in \mathbb{Z}(($y is prime $\wedge$ y is a factor of x$) \to y = 3)$
$\iff (0 < x) \wedge \forall y \in \mathbb{Z}((1 < y \wedge \exists z \;\; x = y \times z) \to \exists z \;\; y = ((z + z) + z))$

$$x = -y \iff x + y = 0$$
$$x = y - z \iff x + z = y$$

Remark. The two minus signs in the two equations above are different.

8.3 Example. Express the following statements about a function $f : \mathbb{R} \to \mathbb{R}$ as formulas in first-order number theory after adding the function symbol f to the symbol set.

a) f is surjective (or onto)

b) f is bijective (or invertible)

c) $\lim_{x \to u} f(x) = v$

Solution. a) $\forall y \in \mathbb{R} \;\; \exists x \in \mathbb{R} \;\; y = f(x)$

b) $\forall y \in \mathbb{R} \;\; \exists! x \in \mathbb{R} \;\; y = f(x)$
$\iff \forall y \in \mathbb{R} \;\; (\exists x \in \mathbb{R}(y = f(x) \wedge \forall z(y = f(z) \rightarrow z = x)))$

c) $\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in \mathbb{R}(0 < |x - v| < \delta \rightarrow |f(x) - v| < \epsilon)$
$\iff \forall \epsilon(0 < \epsilon \rightarrow \exists \delta(0 < \delta \wedge \forall x((\neg x = u \wedge (u < x + \delta \wedge x < u + \delta)) \rightarrow (v < f(x) + \epsilon \wedge f(x) < v + \epsilon))))$