Lecture 14, Oct. 3

14.1 Example.

$$\models \forall x\ (\exists y\ \neg xRy \vee \exists y\ yRx)$$

Solution.

$$\begin{aligned} &\forall x\ (\exists y\ \neg xRy \vee \exists y\ yRx) \\ [E28] \equiv &\forall x\ (\neg\forall y\ xRy \vee \exists y\ yRx) \\ [E20] \equiv &\forall x\ (\forall y\ xRy \rightarrow \exists y\ yRx) \end{aligned}$$

Proof. Let u be an arbitrary non-empty set. Let R be an arbitrary binary relation on u (that is $R \subseteq u^2$)

Let $x \in u$ be arbitrary.

Suppose that $\forall y\ xRy$.

Then in particular we have xRx. [V38]

Since xRx it follows that $\exists y\ yRx$. [V40]

We have proven that $\forall y\ xRy \rightarrow \exists y\ yRx$. [V19]

Since x was arbitrary, we have proven that $\forall x\ (\forall y\ xRy \rightarrow \exists y\ yRx)$. [V37]

Since u and R are arbitrary, we have proven that $\models \forall x\ (\forall y\ xRy \rightarrow \exists y\ yRx)$. [V37, V19]

Since equivalence, we have proven that $\models \forall x\ (\exists y\ \neg xRy \vee \exists y\ yRx)$.

□

Here is a derivation

1	$\{\forall y\ xRy\} \models \forall y\ xRy$	V1
2	$\{\forall y\ xRy\} \models xRx$	V38 on 1
3	$\{\forall y\ xRy\} \models \exists y\ yRx$	V40 on 2
4	$\models (\exists y\ xRy \rightarrow \exists y\ yRx)$	V19 on 3
5	$\models (\neg\forall y\ xRy \vee \exists y\ yRx)$	V45, E20
6	$\models (\exists y\ \neg xRy \vee \exists y\ yRx)$	V45, E28
7	$\forall x\ (\exists y\ \neg xRy \vee \exists y\ yRx)$	V37 on 6

14.2 Example. For $a, b, c \in \mathbb{Z}$, show that if $a \mid b$ and $b \mid c$ then $a \mid c$

(We say a divides b, or a is a factor of b, of b is a multiple of a, and we write $a \mid b$, when $\exists x\ b = a \cdot x$)

Here is a proof in standard mathematical language.

Proof. Let $a, b, c \in \mathbb{Z}$ be arbitrary.

Suppose that $a \mid b$ and $b \mid c$.

Since $a \mid b$, choose $u \in \mathbb{Z}$ so that $b = a \cdot u$

Since $a \mid b$, choose $v \in \mathbb{Z}$ so that $c = b \cdot v$

Since $b = a \cdot u$ and $c = b \cdot v$

We have $c = (a \cdot u) \cdot v = a \cdot (u \cdot v)$

Thus $a \mid c$ (we have $\exists x \; c = a \cdot x$ choose $x = u \cdot v$)

$\square$

Here is a step-by-step proof to show that

$$\{\exists x \; b = a \times x, \exists x \; c = b \times x, \forall x \forall y \forall z \; ((x \times y) \times z) = (x \times (y \times z))\} \vDash \exists x \; c = a \times x$$

V37, v19. Let U be a non-empty set,

[V37, v19] Let $\times$ be a binary function on U.

[V9] Suppose $\exists x \; b = a \times x$,

[V9] Suppose $\exists x \; c = b \times x$,

[V9] Suppose $\forall x \forall y \forall z \; ((x \times y) \times z) = (x \times (y \times z))$

[V41] Since $\exists x \; b = a \times x$, we can choose $u \in U$ so that $b = a \times u$

[V41] Since $\exists x \; c = b \times x$, we can choose $v \in U$ so that $c = b \times v$

[V36]Since $b = a \times u$ and $c = b \times v$, we have $c = (a \times u) \times v$

[V38] Since $\forall x \forall y \forall z \; ((x \times y) \times z) = (x \times (y \times z))$ we have $\forall y \forall z \; ((a \times y) \times z) = (a \times (y \times z))$

[V38] Since $\forall y \forall z \; ((a \times y) \times z) = (a \times (y \times z))$ we have $\forall z \; ((a \times u) \times z) = (a \times (u \times z))$

[V38] Since $\forall z \; ((a \times u) \times z) = (a \times (u \times z))$ we have $((a \times u) \times v) = (a \times (u \times v))$.

[V35] Since $c = (a \times u) \times v$ and $((a \times u) \times v) = (a \times (u \times v))$, we have $c = a \times (u \times v)$

[V40] Since $c = a \times (u \times v)$ we have proven that $\exists x \; c = a \times x$

$\square$