Lecture 7, Sept. 20

7.1 Example. Let F and G be formulas

Determine whether

$$\{F \to G, F \vee G\} \models F \wedge G$$

Solution. We make a truth table

F	G	$F \to G$	$F \vee G$	$F \wedge G$
1	1	1	1	1
1	0	0	1	0
0	1	1	1	0
0	0	1	0	0

Remark. It appears from row 3 that the argument is not valid.

But in fact, the argument may or may not be valid, depending on the formulas F and G.

For example, if F is a tautology and G is any formula, the argument is valid.

Or if $F = G$ then the argument is valid.

First-Order Language

Symbol Set

7.2 Definition. A First-Order Language is determined by its symbol set. The symbol set includes symbols from the common symbol set

$$\{\neg, \wedge, \vee, \to, \leftrightarrow, =, \forall, \exists, (,), , \}$$

along with some variable symbols such as

$$x, y, z, u, v, w, \ldots$$

The symbol $=$ is read as "equals". The symbols $\forall$, $\exists$ are called quantifier symbols. The symbol $\forall$ is read as "for all" or "for every", and the symbol $\exists$ is read in "for some" or "there exists".

The symbol set can also include some additional symbols which can include

1. constant symbols
$$a, b, c, \emptyset, 0, i, e, \pi, \ldots$$
2. function symbols
$$f, g, h, \cup, \cap, +, \times, \ldots$$
3. relation symbols
$$P, Q, R, \in, \subset, \subseteq, <, >, =, \ldots$$

The variable and constant symbols are intended to represent elements in a certain set or class u called the universal set or the universal class. The universal set or class is often understood from the context.

Function

7.3 Definition. A unary function f from a set u is a function $f: u \to U$ (for every $x \in u$ there is a unique element $y = f(x) \in U$)

A binary function g on u is a function $g: u^2 \to U$ where $u^2 = u \times u$ (for every $x, y \in u$ there is a unique element $z = g(x, y) \in U$)

Some binary function symbols are used with infix notation, which means that we write $g(x, y)$ as xgy or as (xgy)

7.4 Example. $+$ is a binary function on $\mathbb{N}$ written with infix notation. So we write $+(x, y)$ as $x + y$ or as $(x + y)$.

Relation

7.5 Definition. A unary relation P on a set u is a subset $P \subseteq U$. For $x \in u$, we write $P(x)$ to indicate that $x \in P$

A binary relation R on u is a subset of U^2. We write $R(x, y)$ to indicate that $(x, y) \in R$

Sometimes a binary relation symbol R is used with indix notation which means that we write $R(x, y)$ as xRy.

7.6 Example. $<$ is a binary relation on $\mathbb{N}$, which means that $< \subseteq \mathbb{N}^2$ and it is used with infix notation, So we write $<(x, y)$ as $x < y$

Also, the symbol $=$ is a binary relation symbol written with infix notation.

Remark. $(P \wedge Q)$ can be written with infix notation as $\wedge PQ$, which is also called polish notation.

Term

7.7 Definition. In a first-order language, a term is a non-empty finite string of symbols from the symbol set which can be obtained by applying the following rules.

1. Every variable symbol is a term and every constant symbol is a term.
2. if f is a unary function symbol and t is a term, then the string $f(t)$ is a term
3. if g is a binary function symbol and s and t are terms, the the string $g(s, t)$ (or the string sgt) is a term.

7.8 Example. The following strings are terms.

- u
- $u \cap v$
- $u \cap (v \cap \emptyset)$
- x
- $x + 1$
- $g(x, f(y + 1))$

Each term represents an element in the universal set or class u

Formula

7.9 Definition. A formula is a non-empty finite string of symbols which can be obtained using the following rules.

1. if P is a unary relation symbol and t is a term then the string $P(t)$ is a formula. (in standard mathematical language we would write $P(t)$ as $t \in P$)
2. if R is a binary relation symbol and s and t are terms then the string $R(s, t)$ is a formula (or sRt)
3. if F is a formula, then so is the string $\neg F$
4. if F and G are formulas then so is each of the strings $F \wedge G$, $F \vee G$, $F \rightarrow G$, $F \leftrightarrow G$
5. if F is a formula and x is a variable symbol, then the string $\forall x F$ and $\exists x F$ are both formulas

Examples: Each of the following strings is formula

- $u \subseteq R$
- $\forall u \ \ \emptyset \in u$
- $f(x) < x + 1$
- $x = g(y, z + 1)$

A formula is a formal way of expressing a mathematical statement about element in u, and about functions and relations on u.

Remark. In standard mathematical language, we continually to add new notations which we allow ourselves to use.

7.10 Example. $\dfrac{x+1}{y}$ could be written as $(x + 1)/y$

$\displaystyle\sum_{k=1}^{n} \frac{1}{k}$ could be written as .... (a very long formula)