Lecture 29, Nov. 14

29.1 Definition. Assume that $f(x)$ is differentiable at each x_0 in an interval I. We define $f' : I \to R$ by

$$f'(x) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

. f' is called the derivative (function) of f on I.

29.2 Example. $f(x) = sinx$, $f'(x) = cosx$ on $\mathbb{R}$

Notation.

1. $y = f(x) \to y'$ will denote $f'(x)$
2. $\dfrac{dy}{dx} = f'(x)$
3. $\dfrac{d}{dy} f(x) = f'(x)$

If $f'(x)$ is differentiable at $x_0 \in \mathbb{R}$, then we call

$$(f')'(x_0) = \lim_{h \to 0} \frac{f'(x_0 + h) - f'(h)}{h}$$

the second derivative of f at $x = x_0$. We denote this by $f''(x_0)$.

In general if f is twice differentiable on I, we write $f''(x)$ to represent the second derivative.

$f'''(x) \to$ third derivative.

$f^{(n)}(x)$ denotes the n-th derivative.

29.3 Theorem (More on Linear Approximation)**.** *If $f(x)$ is differentiable at $x = a$, and if*

$$L_a(x) = f(a) + f'(a)(x - a)$$

then $L_a(x) \approxeq f(x)$ if $x \approxeq a$

29.4 Theorem (Error in Linear Approximation)**.**

$$Error = |f(x) - L_a(x)|$$

The error is effected by

1. *Distance of x to a.*
2. *The larger $|f''(x)|$ the larger the error may be.*

Both 1 and 2 hold in general most of the time but not always.

29.5 Theorem (Newton's Method)**.** *Pick a_1.*

Let

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$$

Remark.

1. If $f'(c) \neq 0$, then there exists $\delta > 0$ such that if $a_1 \in (c - \delta, c + \delta)$ then $a_n \to c$
2. When the method work, the convergence is generally very fast. In general, the convergence is "quadratic" in nature. Roughly speaking this means the number of decimal places of accuracy will at least double with each iteration.
3. It can fail.

29.6 Example (Heron's Method)**.** Solve $x^2 - a = 0$.

Solution. Pick a_1.

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} = a_n - \frac{a_n^2 - a}{2a_n} = \frac{1}{2}(a_n + \frac{a}{a_n})$$