Lecture 16, Oct. 19

Thursday $\rightarrow$ Lecture

Friday $\rightarrow$ Tutorial

Basic Fact about Limits For $\lim_{n\to a} f(x)$ to exist $f(x)$ must be defined in some open interval I containing $x = a$, except possibly at $x = a$.

Sequential Characterization of Limits

16.1 Theorem. *Let $f(x)$ be defined in an open interval I containing a, except possibly at $x = a$. Then the following are equivalent.*

1. $$\lim_{x\to a} f(x) = L$$

2. *Whenever $\{x_n\}$ is such that $x_n \to a$ ($x_n \neq a$) we have $f(x) \to L$*

Proof. Assume that $\lim_{x\to a} f(x) = L$. Let $\{x_n\}$ be such that $x_n \to a$, $x_n \neq a$. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. Since $x_n \to a$, we can find a $N_0 \in \mathbb{N}$ so that if $n \geq N_0$, then $0 < |x_n - a| < \delta \Rightarrow |f(x_n) - L| < \epsilon$

Conversely, (prove by contrapositive) assume that L is not the limit. Then there exists $e_0 > 0$ such that for any $\delta > 0$, there exists $x_\delta \in (a - \delta, a + \delta)$, $x_\delta \neq a$ and $|f(x_\delta) - L| \geq e_0$. In particular, for each $n \in \mathbb{N}$, there exists $x_n \in (a - \frac{1}{n}, a + \frac{1}{n})$, $x_n \neq a$, such that $|f(x) - L| \geq e_0$. Hence $x_n \to a$, $x_n \neq a$, but $\{f(x_n)\}$ does not converge to L. $\square$

16.2 Theorem. Arithmetic Rules for Limits *Assume that* $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$ *then*

1. $lim_{x\to a}(cf)(x) = cL$
2. $lim_{x\to a}(f + g)(x) = L + M$
3. $lim_{x\to a}(fg)(x) = L \cdot M$
4. $lim_{x\to a}(f/g)(x) = L/M$ *if* $M \neq 0$

16.3 Theorem. Squeeze Theorem for Limits *Assume that on some open interval I containing $x = a$ that*

$$f(x) \leq g(x) \leq h(x)$$

for all $x \in I$, except posibly at $x = a$. If $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ *then*

$$\lim_{x\to a} g(x) = L$$

Remark. 1. Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots$ then

$$\lim_{x\to a} p(x) = p(a)$$

2. Let $f(x) = p(x)/q(x)$ where $p(x), q(x)$ are polynomials, then

$$\lim_{x\to a} f(x) = \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} = \frac{p(a)}{q(a)} = f(a)$$

if $q(a) \neq 0$.

Note. If $\lim_{x\to a} f(x)/g(x) = L$ exists and $\lim_{x\to a} g(x) = 0$ then

$$\lim_{x\to a} f(x) = 0$$

For $f(x) = p(x)/q(x)$ if $q(a) = 0$ and $p(a) \neq 0$ then $\lim_{x\to a} f(x)$ does not exist.

If $f(x) = p(x)/q(x)$, $p(x) = q(x) = 0$,

$$p(x) = (x-a)^n p_1(x) \qquad p_1(a) \neq 0$$

$$q(x) = (x-a)^m q_1(x) \qquad q_1(a) \neq 0$$

$$\lim_{x\to a} \frac{p(x)}{q(x)} = \begin{cases} \frac{p_1(a)}{p_2(a)} & \text{if } n = m \\ 0 & \text{if } n > m \\ \text{does not exist} & \text{if } n < m \end{cases}$$

16.4 Example.

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

16.5 Example.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

Let $a \in \mathbb{R}$. What can we say about $\lim_{x\to a} f(x)$? Exists a sequence in $\mathbb{Q}$ that converge to 1, and exists a sequence in $\mathbb{R}\backslash\mathbb{Q}$ that converge to -1. Thus the limit does not exist.