Lecture 16, Oct. 5

16.1 Theorem. *Let $F(n)$ be a statement about an integer n. Let $m \in \mathbb{Z}$*

Suppose $F(m)$ is true

Suppose that for all $k \geq m$, if $F(k)$ is true then $F(k+1)$ is true

Then $F(n)$ is true for all $n \geq m$

Proof Method

Let $F(n)$ be a statement about an integer and let $m \in \mathbb{Z}$

To prove $F(n)$ is true for all $n \geq m$, we can do the following.

1. Prove that $F(n)$ is true
2. Let $k \geq m$ be arbitrary and suppose, inductively, that $F(k)$ is true
3. Prove $F(k+1)$ is true

Alternatively, suppose $F(k-1)$ prove $F(k)$

A slightly different proof method

To prove that $F(n)$ is true for all $n \geq m$ we can do the following:

1. Prove that $F(m)$ is true and that $F(m+1)$ is true
2. Let $k \geq m+2$ be arbitrary and suppose that $F(k-1)$ and $F(k-2)$ are true
3. Prove that $F(k)$ is true

Another Proof Method

we can prove that $F(n)$ is true for all $n \geq m$ as follows.

1. Let $n \geq m$ be arbitrary and suppose that $F(k)$ is true for all k with $m \leq k < n$
2. prove that $F(n)$ is true.

16.2 Theorem. Strong Mathematical Induction *Let $F(n)$ be a statement about an integer n and let $m \in \mathbb{Z}$*

Suppose that for all $n \geq m$, if $F(k)$ for all $k \in \mathbb{Z}$ with $m \leq k < n$, then $F(n)$ is true.

Then $F(n)$ is true for all $n \geq m$.

Proof. Let $G(n)$ be a statement "F(n) is true for all $k \in \mathbb{Z}$ with $m \leq k < n$"

Note that $G(m)$ is true vacuously. (since there is no value of $k \in \mathbb{Z}$ with $m \leq k < n$)

Let $n \geq m$ be arbitrary.

Suppose $G(n)$ is true, that is "$F(n)$ is true for all $k \in \mathbb{Z}$ with $m \leq k < n$"

Since $F(n)$ is true for all $k \in \mathbb{Z}$ with $m \leq k < n$, then $F(n)$ is true for all $k \in \mathbb{Z}$ with $m \leq k < n+1$. In other words, $G(n+1)$ is true.

Now let $n \geq m$ be arbitrary. Since $G(k)$ is true for all $k \geq m$, in particular $G(n+1)$. In other words, $F(k)$ is true for all k with $m \leq k < n+1$. In particular $F(n)$ is true

Since $n \geq m$ was arbitraty, $F(n)$ is true for all $n \geq m$. $\square$

16.3 Example. Let $(x_n)_{n \geq 0}$ be the sequence which is defined recursively by $x_0 = 2$, $x_1 = 2$ and $x_n = 2x_{n-1} + 3x_{n-2}$ for all $x \geq 2$

Find a closed formula for x_n

Solution. Observe that $x_n = 3^n + (-1)^n$

When $n = 0$, $x_0 = 2$ and $3^0 + (-1)^0 = 2$, so $x_n = 3^n + (-1)^n$ is true when $n = 0$

When $n = 1$, $x_1 = 2$ and $3^1 + (-1)^1 = 2$, so $x_n = 3^n + (-1)^n$ is true when $n = 1$

Let $n \geq 2$ be arbitrary.

Suppose that $x_{n-1} = 3^{n-1} + (-1)^{n-1}$ and $x_{n-2} = 3^{n-2} + (-1)^{n-2}$

$$\begin{aligned}
x_n &= 2x_{n-1} + 3x_{n-2} \\
&= 2(3^{n-1} + (-1)^{n-1}) + 3(3^{n-2} + (-1)^{n-2}) \\
&= 9^{n-2} + (3-2)(-1)^{n-2} \\
&= 3^n + (-1)^n
\end{aligned}$$

By induction, $x_n = 3^n + (-1)^n$ for all $n \geq 0$

Binomial Theorem

16.4 Definition. For $n, k \in \mathbb{N}$ with $0 \leq k \leq n$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$$