Lecture 11, Sept. 27

11.1 Definition. **Interpretation** is a choice of a non-empty set u, and constant, functions and relations for each constant, function and relation symbol.

An **Assignment** in u,

$$\alpha : \{\text{variale symbols}\} \to u$$

We write $\alpha(F) = 1$ when F is true in u under α.

We write $\alpha(F) = 0$ when F is false in u under α.

We say F is true in u when $\alpha(F) = 1$ for **every** assignment $\alpha \in u$

11.2 Example. the formula $x \times y = y \times x$ is true in $\mathbb{Z}$ but not true in $n \times n$ matrices.

11.3 Definition. For formulas F and G and a set of formulas S, we say that F is a **tautology** and we write $\vDash F$, when for every interpretation u and every assignment $\alpha \in u$, $\alpha(F) = 1$

11.4 Definition. We say that F and G are **equivalent**, and we write $F \equiv G$, when for every interpretation u, for every assignment $\alpha \in u$, $\alpha(F) = \alpha(G)$, we say that the argument "F there fore G" is valid, or that "S induces G", or that "G is a consequences of S", when for every interpretation u and for every assignment $\alpha \in u$, if $\alpha(F) = 1$ for every $F \in S$ then $\alpha(G) = 1$

11.5 Definition. Given a formula G and a set of formulas S, such that $S \vDash G$, a **derivation** for the valid argument $S \vDash G$ is a list of valid arguments

$$S_1 \vDash G_1, S_2 \vDash G_2, S_3 \vDash G_3, \dots$$

where for some index k we have $S_k = S$ and $G_k = G$, such that each valid argument in the list is obtained from previous valid arguments in the list by applying one of the basic validity rules.

1 Basic Validity Rules

Each basic validity rule is a formal and precise way of describing standard method of mathematical proof.

Rules V1, V2 and V3 are used in derivations because we make a careful distinction between **premises** and **conclusions**. In standard mathematical proofs we do not make a careful distinction.

Premise V1. If $F \in S$ then $S \vDash F$. In words, if we assume F, we can conclude F.

V2. If $S \vDash F$ and $S \subseteq \mathcal{T}$ then $\mathcal{T} \vDash F$. In words, if we can prove F without assuming G, then we can still prove F if we assume G.

Chain Rule V3. If $S \vDash F$ and $S \cup \{F\} \vDash G$ then $S \vDash G$. In words, if we can prove F, and by assuming F we can prove G, then we can prove G directly without assuming F.

Proof by Cases V4. If $S \cup \{F\} \vDash G$ and $S \cup \{\neg F\} \vDash G$ then $S \vDash G$. In words, in either case G is true.

Contradiction V5. If $S \cup \{\neg F\} \models G$ and $S \cup \{\neg F\} \models \neg G$ then $S \models F$. In words, to prove F by contradiction, we suppose, for a contradiction, that F is false, we choose a formula G, then we prove that G is true and we prove that G is false.

V6. If $S \cup \{F\} \models G$ and $S \cup \{F\} \models \neg G$ then $S \models \neg F$

V7. If $S \models F$ and $S \models \neg F$ then $S \models G$

Conjunction V8. $S \models F \wedge G \iff (S \models F$ and $S \models G)$

V9. If $S \cup \{F, G\} \models H$ then $S \cup \{F \wedge G\} \models H$

V10. ...

V11. ...

V12. ...

Disjunction V13. $S \models F \vee G \iff S \cup \{\neg F\} \models \iff S \cup \{\neg G\} \models F$

V14. ...