Lecture 7, Sept. 26

Writing Assignment 2 is due Friday Oct 14th.

Convergence of Sequences

7.1 Definition. Heuristic definition I We say that a sequence $\{a_n\}$ converges to a limit L if as n gets larger and larger the a_ns get closer and closer to L.

7.2 Definition. Heuristic definition II We say that a sequence $\{a_n\}$ converges to a limit L if for every positive tolerance $\epsilon > 0$, we have that the terms in $\{a_n\}$ approximate L with an error at most ϵ, provided that n is large enough.

7.3 Definition. Convergence of a Sequence We say that $\{a_n\}$ converges to a limit L if for every $\epsilon > 0$, there exists a cutoff $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then $|a_n - L| < \epsilon$

If no such L exists, we say that $\{a_n\}$ **diverges**.

7.4 Example. Consider $\{(-1)^{n+1}\} = \{1, -1, 1, -1, \dots\}$. Does this have a limit?

Proof. Let $\epsilon = 1$. Suppose $L = \lim_{n\to\infty} a_n$. Let N_0 be such that if $n \geq N_0$, then $|a - L| < 1$

Let $n_1 \geq N_0$ with n_0 even. Then

$$\begin{aligned} |-1-L| &= |a_n - L| < 1 \\ &\to L \in (-2, 0) \end{aligned}$$

Let $n_1 \geq N_0$ with n_0 odd. Then

$$\begin{aligned} |1-L| &= |a_n - L| < 1 \\ &\to L \in (0, 2) \end{aligned}$$

So

$$L \in (-2, 0) \cap (0, 2)$$

which is impossible.

Hence $\{a_n\}$ diverges. $\square$

Note. Suppose that $\lim_{n\to\infty} a_n = L$. Let $\epsilon > 0$. What can we say about the terms in $\{a_n\}$ that are in $(L - \epsilon, L + \epsilon)$?

For some N_0, if $n \geq N_0$, then $a_n \in (L - \epsilon, L + \epsilon)$. ie) $(L - \epsilon, L + \epsilon)$ contains a tail of the sequence.

7.5 Proposition. *Let $\{a_n\}$ be a sequence. Then the following are equivalent.*

1. $L = \lim_{n\to\infty} a_n$
2. *for every $\epsilon > 0$, $(L - \epsilon, L + \epsilon)$ contains a tail of $\{a_n\}$*
3. *for every $\epsilon > 0$, $(L - \epsilon, L + \epsilon)$ contains all but finitely many a_n*

4. *for open interval (a, b) with $L \in (a, b)$, we have (a, b) contains a tail of $\{a_n\}$*
5. *for open interval (a, b) with $L \in (a, b)$, the interval (a, b) contains all but finitely many a_n*

7.6 Question. Can $\{a_n\}$ have more than 1 limit?

7.7 Theorem. Uniqueness of Limit *Suppose that* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} a_n = M$, *then* $L = M$

Proof. Assume that $L < M$. Let $\epsilon = \frac{M-L}{2}$.

We can choose N_1 large enough so that if $n \geq N_1$, $a_n \in (L - \epsilon, L + \epsilon)$

We can also choose N_2 large enough so that if $n \geq N_2$, $a_n \in (M - \epsilon, M + \epsilon)$

Let $N_0 = max\{N_1, N_2\}$. Choose $n \geq N_0$. Then $a_n \in (L - \epsilon, L + \epsilon) \cap (M - \epsilon, M + \epsilon)$

But $(L - \epsilon, L + \epsilon) \cap (M - \epsilon, M + \epsilon) = \emptyset$

$\square$