Lecture 26, Nov. 9

26.1 Theorem. *If $f(x)$ is continuous on $[a, b]$, then f is uniformly continuous.*

Basic Facts about Uniform Continuity

1. If f is uniformly continuous on $S \subset \mathbb{R}$ and if $T \subseteq S$ then f is uniformly continuous on T.
2. If f is uniformly continuous on S and if $\{x_n\} \subset S$ is Cauchy then $\{f(x_n)\}$ is Cauchy.
3. If f is uniformly continuous on (a, b), then $\lim_{x\to a^+} f(x)$ exists and $\lim_{x\to b^-} f(x)$ exists.
4. f is uniformly continuous on (a, b) iff there exists $F\colon [a, b] \to R$ such that F is continuous on $[a, b]$ and $F(x) = f(x)$ for all $x \in (a, b)$.
5. if $f(x)$ is uniformly continuous on (a, b), then $f((a, b))$ is bounded.

Derivatives

26.2 Definition (Differentiable)**.** We say that f is differentiable at $x = a$ if

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists. In this case, we write

$$f'(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

and we call $f'(x)$ the derivative of f at $x = a$.

26.3 Definition (Tangent Line)**.** Assume that $f'(x)$ exists. Then the line with slope $f'(x)$ passing through $(a, f(a))$ is called the tangent line to $f(x)$ at $x = a$.

$$y = f(a) + f'(a(x - a))$$

26.4 Definition (Alternative Definition)**.**

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$

26.5 Example. $f(x) = \cos x$, find $f'(0)$

Solution.

$$\begin{aligned}
f'(0) &= \lim_{h\to 0} \frac{\cos h - \cos 0}{h} \\
&= \lim_{h\to 0} \frac{\cos h - 1}{h} \\
&= \lim_{h\to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos x + 1)} \\
&= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} \\
&= \lim_{h\to 0} \frac{-\sin h}{h} \frac{\sin h}{(\cos h + 1)} \\
&=0
\end{aligned}$$

26.6 Example. $f(x) = \sin x$, find $f'(a)$

Solution.

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{\sin(a+h) - \sin(a)}{h} \\ &= \lim_{h\to 0} \frac{\sin a \cos h + \cos a \sin h - \sin a}{h} \\ &= \lim_{h\to 0} \sin a \frac{\cos h - 1}{h} + \cos a \frac{\sin h}{h} \\ &= \cos a \end{aligned}$$

26.7 Theorem. *If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.*

Proof. Since $\lim_{x\to a}(f(x) - f(a))/(x - a)$ exists and $\lim_{x\to a} x - a = 0$, we have $\lim_{x\to a} f(x) - f(a) = 0 \iff \lim_{x\to a} f(x) = f(a)$. $\square$

26.8 Example.

$$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is $g(x)$ differentiable at $x = 0$?

26.9 Example.

$$h(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is $h(x)$ differentiable at $x = 0$?