Lecture 24, Nov. 4

24.1 Theorem (Intermediate Value Thm (IVT))**.** *If $f(x)$ is continuous on $[a, b]$ and $f(a)f(b) < 0$, then there exists $c \in (a, b)$ with $f(c) = 0$.*

24.2 Example. Show that

$$f(x) = x^2 + x - 3$$

has a root on $[0, 4]$.

Solution. Observation:

1. $f(x)$ is continuous.
2. $f(0) = -3 < 0$
3. $f(4) > 0$

By the theorem 24.1 there exists $c \in [0, 4]$ with $f(c) = 0$.

24.3 Question. How do we find c?

Solution. Binary Search Algorithm.

Set Up $f(x)$ is continuous. We want to solve $f(x) = 0$.

Step 1 Find $a < b$ with $f(a)f(b) < 0$.

Algorithm

1. $a_1 = a, b_1 = b$
2. If $|b - a| < 2\epsilon$, let $d = \frac{a+b}{2}$, stop
3. If $f(d) = 0$, stop
4. If $f(a)f(d) < 0$, let $a_1 = a_1$, $b_1 = d$, goto 1.
5. If $f(d)f(b) < 0$, let $a_1 = d$, $b_1 = b_1$, goto 2

24.4 Example. Show that there exists $c \in [0, \frac{\pi}{2}]$ with $\cos c = c$

Solution. Let $f(x) = \cos x - x$, which is continuous on $[0, \frac{\pi}{2}]$. Observe that $f(0) > 0$ and $f(\frac{\pi}{2}) < 0$. By the theorem 24.1 there exists c with $f(c) = 0 = \cos c - c$.

24.5 Theorem (Extreme Value Theorem)**.** *If $f(x)$ is continuous on $[a, b]$, then there exists $c, d \in [a, b]$ such that*

$$f(c) \leq f(x) \leq f(d)$$

for all $x \in [a, b]$.

Proof. First we show that $f(x)$ is bounded. Suppose it is not bounded. Then for each $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ with $f(x_n) \geq n$. By the BWT, $\{x_n\}$ has a convergent sub-sequence $\{x_{n_k}\}$ with $x_{n_k} \to x_0 \in [a, b]$. Since f is continuous, $f(x_{n_k}) \to f(x_0)$. But $f(x_{n_k}) \geq n_k \to \infty$, which is impossible. Thus $f([a, b])$ is bounded.

Let $M = lub(f([a, b]))$. For each $n \in \mathbb{N}$ choose $y_m \in [a, b]$ with $M - \frac{1}{n} < f(y_n) \leq M$. By the BWT, $\{y_n\}$ has a convergent sub-sequence $\{y_{n_k}\}$ with $y_{n_k} \to d \in [a, b]$. Hence $f(d) = \lim_{k\to\infty} f(x_{n_k}) = M$ $\square$