Lecture 5, Sept. 21

1) No office hours this afternoon

2) WA1$\rightarrow$ Due 2:30 PM Monday, Sept. 26. Submit in dropbox outside Math Tutorial Center.

Least Upper Bound Property If $S \subset \mathbb{R}$ is non-empty and bounded above, then S has a least upper bound.

Archimedean Property I

5.1 Theorem. *$\mathbb{N}$ is not bounded above.*

Proof. Suppose that $\mathbb{N}$ was bounded above. Then $\mathbb{N}$ has a least upper bound α.

Note that $\alpha - \frac{1}{2} < \alpha$. Hence $\alpha - \frac{1}{2}$ is not an upper bound for $\mathbb{N}$. Then there exists $n \in \mathbb{N}$ with $\alpha - \frac{1}{2} < n \leq \alpha$. But then $n + 1 \in \mathbb{N}$ and $n + 1 > \alpha$ which is impossible.

Therefore $\mathbb{N}$ must not be bounded above.

□

Note. Let $S \neq \emptyset \subset \mathbb{R}$ be bounded above. Let $\alpha = lub(S)$. if $\epsilon > 0$ then there exist $x_0 \in S$ with $\alpha - \epsilon < x_0 \leq \alpha$.

Archimedean Property II

5.2 Corollary. *Let $\epsilon > 0$, Then there exists $n \in \mathbb{N}$ such that*

$$0 < \frac{1}{n} < \epsilon$$

Proof. Take $\alpha = \frac{1}{\epsilon}$ in Archimedean Property I. □

Density of $\mathbb{R}$

5.3 Definition. A subset $S \subset \mathbb{R}$ is said to be dense if for every $\epsilon > 0$ and $x \in \mathbb{R}$,

$$S \cap (x - \epsilon, x + \epsilon) \neq \emptyset$$

or equivalently if $S \cap (a, b) \neq \emptyset$ for all $a < b$ in $\mathbb{R}$

5.4 Proposition. *$\mathbb{Q}$ and $\mathbb{R} \backslash \mathbb{Q} = \mathbb{Q}^c$ are dense in $\mathbb{R}$*

Absolute Values

5.5 Definition.

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

5.6 Example.

$$g(x) = \frac{|x|}{x}$$

Domain $= \{x \in \mathbb{R} \mid x \neq 0\}$

$$g(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Geometric Interpretation of $|x|$

- $|x|$ represents the distance from x to 0.
- $|x-a|$ represents the distance from x to a.

Note. Distance between $(0,0)$ and (x,y)

$$\sqrt{x^2+y^2}$$

Properties of $|x|$

1) $|x| \geq 0$ and $|x| = 0 \iff x = 0$
2) $|ax| = |a||x|$ for all $a \in \mathbb{R}, x \in \mathbb{R}$
3) Triangle Inequality

$$|x-z| + |z-y| \geq |x-y|$$

5.7 Theorem. Triangle Inequality *If $x, y, z \in \mathbb{R}$, then*

$$|x-z| + |z-y| \geq |x-y|$$

Proof. Use Geometric Interpretation. □

5.8 Theorem. Variants I *For all $x, y \in \mathbb{R}$,*

$$|x+y| \leq |x| + |y|$$

5.9 Theorem. Variants II *For all $x, y \in \mathbb{R}$,*

$$||x| - |y|| \leq |x-y|$$