Lecture 3, Sept. 13

Women in Math

Tue Sept. 13
4:30-6:00
DC 1301

ZFC Axioms

- Empty Set: there exist a set, denoted by $\emptyset$, with no elements.
- Equality: two sets are equal when they have the same elements. $A = B$ when for every set x, $x \in A \iff x \in B$
- Pair Axiom: if A and B are sets then so is $\{A, B\}$. In particular, taking $A = B$ shows that $\{A\}$ is a set.
- Union Axiom: if S is a set of sets then $\cup_S = \bigcup_{A \in S} A = \{x \mid x \in A \text{ for some } A \in S\}$. If A and B are sets, then so is $\{A, B\}$ hence so is $A \cup B = \cup_{\{A,B\}}$
- Power Set Axiom: if A is a set, then so is its Power Set $P(A)$. $P(A) = \{X \mid X \subseteq A\}$. In particular, $\emptyset \subseteq X$, $X \subseteq X$
- Axiom of Infinity: if we define

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \{0\} = \{\emptyset\} \\ 2 &= \{0, 1\} = \{\emptyset, \{\emptyset\}\} \\ 3 &= \{0, 1, 2\} = \{\emptyset, \{\emptyset, \{\emptyset\}\}\} \\ &\vdots \\ n + 1 &= n \cup \{n\} \end{aligned}$$

  Then $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is a set (called the set of natural numbers)

- Specification Axioms: if A is a set, and $F(x)$ is a mathematical statement about an unknown set x, then $\{x \in A \mid F(x) \text{ is true }\}$ is a set.

  Examples:

$$\{x \in \mathbb{N} \mid x \text{ is even }\} = \{0, 2, 4, 6, \ldots\}$$

$$A \cap B = \{x \in A \cup B \mid x \in A \text{ and } x \in B\}$$

- Replacement Axioms: if A is a set and $F(x, y)$ is a mathematical statement about unknown sets x and y with the property that for every $x \in A$ there is a unique set y such that the statement is true, and if we denote this unique set y by $y = F(x)$, then $\{F(x) \mid x \in A\}$ is a set.
- Axiom of Choice: if S is a set of non-empty sets then there exists a function $F : S \to U_S$ which is called a choice function for S such that

$$F(A) \in A \quad \forall A \in S$$

Things that are sets

3.1 Example.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x \in A \cup B \mid x \in A \text{ and } x \in B\}$$
$$A \backslash B = \{x \in A \mid x \notin B\}$$
$$A \times B = \{(x, y) \mid x \in A, x \in B\}$$
$$A^2 = A \times A$$

One way to define ordered pairs

$$(x, y) = \{\{x\}, \{x, y\}\}$$
$$x \in A, y \in B \therefore x, y \in A \cup B$$
$$\{x\}, \{x, y\} \in P(A \cup B)$$
$$(x, y) = \{\{x\}, \{x, y\}\} \subseteq P(A \cup B)$$
$$(x, y) \in P(P(A \cup B))$$
$$\therefore A \times B = \{(x, y) \in P(P(A \cup B)) \mid x \in A \text{ and } y \in B\}$$

function

When A and B are sets, a function from A yo B is a subset $F \subseteq A \times B$ with the property that for every $x \in A$ there exists a unique $y \in B$ such that $(x, y) \in F$

When F is a function from A to B we write

$$F \colon A \to B$$

and for $x \in A$ and $y \in B$ we write $y = F(x)$ to indicate that $(x, y) \in F \subseteq A \times B$

Sequence

A sequence $a_0, a_1, a_2, \ldots$ of natural numbers is a function $a \colon \mathbb{N} \to \mathbb{N}$ and we write $a(k)$ as a_k

Less than

the relation $<$ on $\mathbb{Z}$ is a subset $< \subseteq \mathbb{Z}^2$ and we write $x < y$ when $(x, y) \in <$

We can use the ZFC Axioms to define and construct

- $\mathbb{Z}$: the set of integers
- $\mathbb{Q}$: the set of rationals
- $\mathbb{R}$: the set of real numbers
- $+, \times$: operations
- $<, >$: relations