Lecture 9, Sept. 29

9.1 Definition. We sat that $\{a_n\}$ diverges to ∞ if for every $M \geq 0$ there exists $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then $a_n > M$

We write

$$\lim_{n \to \infty} a_n = \infty$$

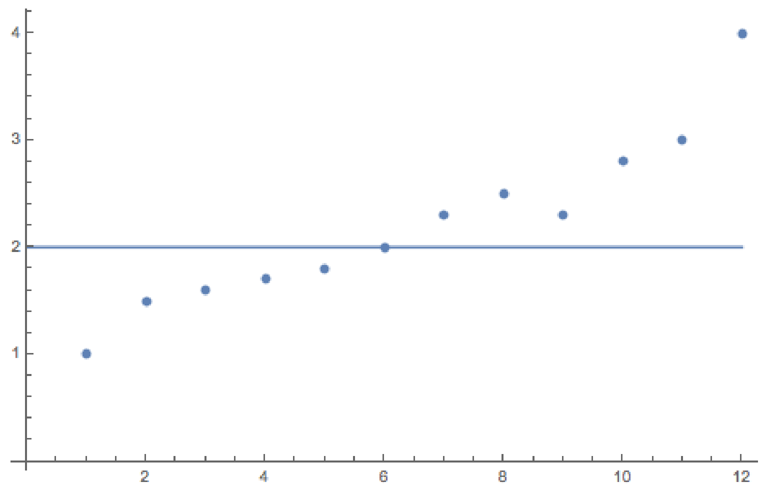

Figure 1: $\{a_n\}$ and $M = 2$

9.2 Question. Does every sequence $\{a_n\}$ that is <u>not</u> bounded above diverges to ∞?

No. $\{0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \dots\}$

Note. If $\{a_n\}$ is non-decreasing then either

1) $\{a_n\}$ is bounded and convergent

2) $\{a_n\}$ is unbounded and diverges to ∞

9.3 Question. If a sequence is not bounded above, does it have a sub-sequence that diverges to ∞?

Series

Given a Sequence $\{a_n\}$, what does it mean to sum all of the terms of the sequence? That is what does the formal sum mean

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

9.4 Example.

$$\sum_{n=1}^{\infty} (-1)^{n+1}$$

9.5 Example.

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

9.6 Definition. For each $k \in \mathbb{N}$, the kth partial sum is

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

We say that $\sum_{n=1}^{\infty} a_n$ converges if the sequence $\{S_k\}$ of partial sums converges. Otherwise we say the series diverges.

If the series converges we let

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{n=1}^{k} a_n$$

9.7 Example.

$$\sum_{n=1}^{\infty} (-1)^{n+1}$$

$$S_k = \begin{cases} 1 & \textit{if } k \textit{ is odd} \\ 0 & \textit{if } k \textit{ is even} \end{cases}$$

Thus S_k diverges

Geometric Series Let $r \in R$, consider

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots$$

$$S_k = \sum_{n=0}^{k} r^n = 1 + r + r^2 + r^3 + \cdots + r^k$$

$$S_k = \frac{1 - r^{k+1}}{1 - r} \ \textit{if } r \neq 1$$

Note. If $|r| < 1$ then $\lim_{k\to\infty} r^{k+1} = 0$

If $|r| > 1$ then $\lim_{k\to\infty} r^{k+1}$ does not exists

If $r = -1$ then $\lim_{k\to\infty} r^{k+1}$ does not exists.

If $r = 1$ then $S_k = k$ which diverges to infinity.

9.8 Example. $r = \frac{1}{2}$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - \dfrac{1}{2}} = 2$$