Lecture 22, Oct. 21

Order Properties in $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$

22.1 Theorem. The Completeness Property in $\mathbb{R}$ *Every non-empty set $S \subseteq \mathbb{R}$ which is bounded above has a* **supremum** *(or least upper bound) in $\mathbb{R}$. Every non-empty set $S \subseteq \mathbb{R}$ which is bounded below has a* **infimum** *(or greatest lower bound) in $\mathbb{R}$*

In $S \subseteq R$, we say S is bounded above in $\mathbb{R}$ when there exists $b \in \mathbb{R}$ such that $b \geq x$ for every $x \in S$. Such a number b is called an **upper bound** *for S in $\mathbb{R}$. A* **Supremum** *for S is a number $b \in \mathbb{R}$ such that $b \geq x$ for every $x \in S$ and for all $c \in \mathbb{R}$, if $c \geq x$ for every $x \in \S$, then $b \leq c$.*

22.2 Theorem. Density of $\mathbb{Q}$ in $\mathbb{R}$ *For all $a, b \in \mathbb{R}$, if $a < b$ then there exists $c \in \mathbb{Q}$ such that $a < c < B$.*

22.3 Theorem. Order Properties in $\mathbb{Z}$

1. **Natural numbers are non-negative**. $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$
2. **Discreteness** *for all $k, n \in \mathbb{Z}$, $k \leq n \leftrightarrow k < n+1$*
3. **Well Ordering Property of $\mathbb{Z}$ in $\mathbb{R}$**. *Every nonempty set $S \subseteq \mathbb{Z}$ which is bounded above in $\mathbb{R}$ has a maximum element in S. Every nonempty set $S \subseteq \mathbb{Z}$ which is bounded below in $\mathbb{R}$ has a minimum element in S. In particular, every nonempty set $S \subseteq \mathbb{N}$ has a minimum number.*
4. *For every $x \in \mathbb{R}$, there exists $a \in \mathbb{Z}$ such that $a \leq x$. For every $x \in \mathbb{R}$, there exists $b \in \mathbb{Z}$ such that $x \leq b$.*
5. **Floor and Ceiling Property** *For every $x \in \mathbb{R}$ there exists a unique $n \in \mathbb{Z}$ which we denoted by $n = \lfloor x \rfloor$, such that $n \leq x$ and $n + 1 > x$. For every $x \in \mathbb{R}$ there exists a unique $m \in \mathbb{Z}$ which we denoted by $n = \lceil x \rceil$, such that $x \leq m$ and $x > m - 1$*
6. **Monotone Sequence Property of $\mathbb{Z}$** *Let $m \in \mathbb{Z}$ and let $(x_n)_{n \geq m}$ be a sequence of integers (so each $x_n \in \mathbb{Z}$). If $x_{n+1} > x_n$ for all $n \geq m$, then for all $b \in \mathbb{R}$, there exists $n \geq m$ such that $x_n > b$. If $x_{n+1} < x_n$ for all $n \geq m$, then for all $b \in \mathbb{R}$, there exists $n \geq m$ such that $x_n < b$.*

Remark. If N has a total ordering $\leq$ and N has the property that every nonempty set $S \subseteq N$ has a minimum element, then we say that N is a well ordering set.

22.4 Exercise. 1. Show that for all $a \in \mathbb{Z}$, if $a \neq 0$ then $|a| \geq 1$

2. Show that the only units in $\mathbb{Z}$ are ± 1. Indeed show that for all $a, b \in \mathbb{Z}$, if $ab = 1$ then ($a = b = 1$ or $a = b = -1$)

Here ends Chapter 2: Rings Fields, Orders and Induction

Chapter 3: Factorization in $\mathbb{Z}$

22.5 Definition. For $a, b \in \mathbb{Z}$, we say a **divides** b, or a is a **factor** of b, or b is a **multiple** of a, and we write $a \mid b$, when

$$b = ak \text{ for some } k \in F$$

22.6 Theorem.

1. *$1 \mid a$ for all $a \in \mathbb{Z}$*

2. *$a \mid 1 \leftrightarrow a = \pm 1$*
3. *$0 \mid a \leftrightarrow a = 0$*
4. *$a \mid 0$ for all $a \in \mathbb{Z}$*
5. *$a \mid b \leftrightarrow |a| \mid |b|$*
6. *if $b \neq 0$ and $a \mid b$ then $|a| \leq |b|$*
7. *$a \mid a$*
8. *if $a \mid b$ and $b \mid a$ then $a = b$*
9. *if $a \mid b$ and $b \mid c$ then $a \mid c$*
10. *if $a \mid b$ and $a \mid c$ then*

$$\forall x, y \in \mathbb{Z}\ a \mid (bx + cy)$$

Proof.

6. Suppose $b \neq 0$ and $a \mid b$. Choose $k \in \mathbb{Z}$ so that $b = ak$. If $k = 0$, then $b = ak = a0 = 0$. But $b \neq 0$, so $k \neq 0$. Since $k \neq 0$ we have $|k| \geq 1$. Since $b = ak$, we have $|b| = |ak| = |a|\,|k| \geq |a|\,1 = |a|$

$\square$