Lecture 5, Sept. 16

Mathematical Statement

Given a formula F and an assignment α, (that is give the values of $\alpha(P), \alpha(Q), \alpha(R), \ldots$), we can calculate $\alpha(F)$ by making a derivation $F_1, F_2, F_3, \ldots, F_l$ for F then calculate the values $\alpha(F_1), \alpha(F_2), \ldots$ one at a time.

5.1 Example. Let F be the formula $F = (\neg(P \leftrightarrow R) \vee (Q \to \neg R))$ and let α be an assignment then with $\alpha(P) = 0$, $\alpha(Q) = 1$ and $\alpha(R) = 0$. Find $\alpha(F)$.

We make a derivation $F_1, F_2, F_3, \ldots, F_l$ for F and calculate the values $\alpha(F_k)$

P	Q	R	$P \leftrightarrow R$	$\neg(P \leftrightarrow R)$	$\neg R$	$Q \to \neg R$	F
0	1	0	1	0	1	1	0

Truth Table

5.2 Definition. For variable symbols $P_1, P_2, \ldots, P_n$, an assignment on $(P_1, P_2, \ldots, P_n)$ is a function

$$\alpha\colon \{P_1, P_2, \ldots, P_n\} \to \{0, 1\}$$

For a formula F which only involves the variable symbols in $\{P_1, P_2, \ldots, P_n\}$, a truth table for F on $(P_1, P_2, \ldots, P_n)$ is a table whose top header row is a derivation $F_1, F_2, F_3, \ldots, F_l$ for F with $F_i = P_i$ for $1 \leq i \leq n$, and under the header row there are 2^n rows which correspond to the 2^n assignments on $(P_1, P_2, \ldots, P_n)$. For each assignment $\alpha\colon \{P_1, P_2, \ldots, P_n\} \to \{0, 1\}$ there is a row of the form $\alpha(F_1), \alpha(F_2), \ldots, \alpha(F_n)$ and the rows are listed in order such that in the first n columns, the rows $\alpha(F_1), \alpha(F_2), \ldots, \alpha(F_n)$ (that is $\alpha(P_1), \alpha(P_2), \ldots, \alpha(P_n)$) list the 2^n binary numbers from $111\ldots1$ at the top, in order, down to $000\ldots0$ at the bottom.

5.3 Example. Make a truth table for the formula

$$F = P \leftrightarrow (Q \wedge \neg(R \to P))$$

P	Q	R	$R \to$P	$\neg(R \to P)$	$Q \wedge \neg(R \to P)$	F
1	1	1	1	0	0	0
1	1	0	1	0	0	0
1	0	1	1	0	0	0
1	0	0	1	0	0	0
0	1	1	0	1	1	0
0	1	0	1	0	0	1
0	0	1	0	1	0	1
0	0	0	1	0	0	1

Tautology

Let F and G be formula and let S be a set of formulas

5.4 Definition. We say that F is a tautology, and we write $\vDash F$, when $\alpha(F) = 1$ for every assignment α

We say that F is a contradiction when $\alpha(F) = 0$ for every assignment α, or equivalently when $\vDash \neg F$

We say that F is equivalent to G, and we write $F \equiv G$ when $\alpha(F) = \alpha(G)$ for every assignment α

We say that argument "S therefore G" is valid, or that "S induces G" or that "G is a consequence of S", when for every assignment α for which $\alpha(F) = 1$ for every $F \in S$ we have $\alpha(G) = 1$.

When $S = \{F_1, F_2, \ldots, F_n\}$ we have $S \vDash G$ is equivalent to $\{((F_1 \wedge F_2) \wedge \cdots \wedge F_n)\} \vDash G$ which is equivalent to $\vDash (((F_1 \wedge F_2) \wedge \cdots \wedge F_n) \rightarrow G)$