Lecture 17, Oct. 20

Seq Characteriation of Limits

17.1 Theorem. *Let $f(x)$ be defined in an open interval I containing a, except possibly at $x = a$. Then the following are equivalent.*

1. $\lim\limits_{x \to a} f(x) = L$
2. *Whenever $\{x_n\}$ is such that $x_n \to a$ ($x_n \neq a$) we have $f(x) \to L$*

17.2 Example. $\lim\limits_{x \to 0} \sin(\frac{1}{x})$ does not exist.

17.3 Example.

$$g(x) = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$\lim\limits_{x \to 0} g(x) = 0$. In other words, the limit exists (by using squeeze theorem.)

17.4 Example.

$$\lim_{x \to 0} x^2 \sin(\frac{1}{x}) = 0$$

17.5 Example.

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \\ 1 & \text{if } x = 0 \\ \frac{1}{m} & \text{if } x = \frac{k}{m} \in \mathbb{Q} \text{ with } gcd(k, m) = 1 \end{cases}$$

Suppose $\lim_{x \to a} f(x)$ exists. Then the limit is 0 (because for every irrational sequence that approaches a, all element in the irrational sequence is 0.)

17.6 Definition. We say that L is the limit of $f(x)$ from above (from the right) if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \epsilon$. We write

$$\lim_{x \to a^+} f(x) = L$$

We say that L is the limit of $f(x)$ from below (from the left) if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $-\delta < x - a < 0$, then $|f(x) - L| < \epsilon$. We write

$$\lim_{x \to a^-} f(x) = L$$