Lecture 13, Oct. 6

Squeeze Theorem

13.1 Example. Find
$$\lim_{n\to\infty} \frac{cos(n)}{n}$$

Observation:
$$|cos(n)| \leq 1$$
$$\frac{-1}{n} \leq \frac{cos(n)}{n} \leq \frac{1}{n}$$

13.2 Theorem. Squeeze Theorem *If $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are such that $a_n \leq b_n \leq c_n$ with $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$*

Proof. Let $\epsilon > 0$, then exists $N_0 \in \mathbb{N}$ so that if $n \geq N_0$ then $a_n \in (L-\epsilon, L+\epsilon)$ and $c_n \in (L-\epsilon, L+\epsilon)$

If $n \geq N_0$,
$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$
$$|b_n - L| < \epsilon$$

□

Solution. We know that
$$\frac{-1}{n} \leq \frac{cos(n)}{n} \leq \frac{1}{n}$$

since $|cos(n)| \leq 1$

Since $\lim_{n\to\infty} -\frac{1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$

Then
$$\lim_{n\to\infty} \frac{cos(n)}{n} = 0$$

13.3 Example.
$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$$

Note. If $\{a_n\}$ is bounded, then
$$\lim_{n\to\infty} \frac{a_n}{n} = 0$$

Bolzano-Weierstrass Theorem

Note. We know that convergent sequences are bounded. But bounded sequences do not have to converge.

Does every bounded sequences have a convergent sub-sequence?

Strategy Bounded + monotonic ⇒ convergent

Does every sequence have a monotonic sub-sequence

13.4 Definition. Given $\{a_n\}$ we call an index n_0 a **peak point** for $\{a_n\}$ if $a_n < a_{n_0}$ for all $n \geq n_0$

13.5 Lemma. Peak Point Lemma *Every sequence $\{a_n\}$ has a monotonic sub-sequence.*

Proof. Let $P = \{n \in \mathbb{N} \mid n \text{ is a peak point of } \{a_n\}\}$

Case 1. P is infinite.

Let $n_1 =$ least element of P

Let $n_2 =$ least element of P
$\{n_1\}$

...

This gives us a sequence recursively

$$n_1 < n_2 < \cdots < n_k < \cdots \in P$$

Since these are peak points,

$$a_{n_k} > a_{n_{k+1}}$$

Thus $\{a_{n_k}\}$ is decreasing.

Case 2. Let n_1 be the least index that is not a peak point. Since n_1 is not a peak point, we can choose $n_2 > n_1$ so that

$$a_{n_1} \leq a_{n_2}$$

Since n_2 is not a peak point, then we can choose $n_3 > n_2$ so that

$$a_{n_2} \leq a_{n_3}$$

We can proceed recursively, to find that

$$n_1 < n_2 < \cdots < n_k < \ldots$$

Where $a_{n_k} \leq a_{n_{k+1}}$

Thus $\{a_{n_k}\}$ is non-decreasing.

In either case we have a monotonic sub-sequence. □

13.6 Theorem. Bolzano-Weierstrass Theorem *Every bounded sequences has a convergent sub-sequence.*

Proof. Give $\{a_n\}$, by the Peak Point Lemma $\{a_n\}$ has a monotinic subsequence $\{a_{n_k}\}$, which is also bounded. By the MCT, $\{a_{n_k}\}$ is convergent. □

Note. BWT is equivalent to MCT which is equivalent to the LUBP.