Lecture 31, Nov. 17

Inverse Function Theorem

Note. If f is $1-1$, we get $f\colon X \to range(f) \subset Y = \{y \in Y \mid y = f(x) \text{ for some } x\}$. If f is $1-1$ and onto its range, we can define $g\colon range(f) \to x$ by $g(y) = x$ if and only if $f(x) = y$.

31.1 Definition. We say that f is invertible on $A \subset R$ if f is $1-1$ on A. In this case, we define the inverse of f on A by

$$g(y) = x \iff y = f(x) \text{ for } x \in A$$

Note. Geometrically the inverse function is the reflection of the original function through $y = x$.

31.2 Example. $f(x) = mx + b$ is always invertible if $m \neq 0$. The inverse function is

$$g(y) = \frac{1}{m}x - \frac{b}{m}$$

Observation. We have

$$L^g_{f(a)}(x) = \frac{1}{f'(a)}(x - f(a))$$

$$g'(f(a)) = \frac{1}{f'(a)}$$

31.3 Definition. We say that $f(x)$ is increasing (strictly increasing) on an interval I if whenever $x_1, x_2 \in I$ with $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$ $(f(x_1) < f(x_2))$.

Similarly we define "decreasing (strictly decreasing)".

We say that f is monotonic on I if one of these holds.

Basic Facts.

1. If $f(x)$ is strictly increasing or decreasing on I, then f is $1-1$ on I, and hence invertible on I.
2. If f is continuous on I and $1-1$ then f is either strictly increasing or strictly decreasing.
3. Assume that $f(x)$ is increasing on $[a, b]$. Let $c \in (a, b)$. Claim that $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists with $\lim_{x\to a^-} f(x) \leq \lim_{x\to a^+} f(x)$

31.4 Theorem. *Assume that $f(x)$ is increasing on $[a, b]$, then the following are equivalent*

1. *$f(x)$ is continuous on $[a, b]$*
2. *$f([a, b]) = [f(a), f(b)]$*