Lecture 28, Nov. 1

28.1 Theorem (Unique Prime Factorization)**.** *Every integer $n \geq 2$ can be expressed uniquely in the form*

$$n = \prod_{i=1}^{l} p_i = p_1 p_2 \cdots p_l$$

for some $l \in \mathbb{Z}^+$ and some primes $p_1, p_2, \cdots, p_l$ with $p_1 \leq p_2 \leq \cdots \leq p_l$.

Proof. First we show existence. Let $n \geq 2$. Suppose, inductively, that every integer k with $2 \leq k < n$ can be written (uniquely) in the required form. If n is prime then $n = p_1$ with $p_1 = n$.

Suppose n is composite, say $n = ab$ with $1 < a < n$ and $1 < b < n$. Since $2 \leq a < n$ and $2 \leq n < n$ we can write

$$a = \prod_{i=1}^{l} p_i$$

and

$$b = \prod_{j=1}^{m} q_j$$

with $l, m \in \mathbb{Z}$ and the q_j, p_i are primes.

Thus

$$\begin{aligned} n =& ab \\ =& p_1 p_2 \cdots p_l q_1 q_2 \cdots q_m \\ =& r_1 r_2 \cdots r_{l+m} \end{aligned}$$

where the $(l+m)-$tuple $(r_1, r_2, \cdots, r_{l+m})$ is obtained by rearranging the entries of the

$$\text{(l+m)-tuple } (p_1, p_2, \cdots, p_l, q_1, q_2, \cdots, q_m)$$

into non-decreasing order.

Next we prove uniqueness. We need to show that if $n = p_1 p_2 \cdots p_l$ and $n = q_1 q_2 \cdots q_m$ where $l, m \in \mathbb{Z}^+$ and the p_i and q_j are primes with $p_1 \leq p_2 \leq \cdots \leq p_l$ and $q_1 \leq q_2 \leq \cdots q_m$, then $l = m$ and $p_i = q_i$ for all i.

Suppose $n = p_1 p_2 \cdots p_l = q_1 q_2 \cdots q_m$ as above. Since $n = p_1 p_2 \cdots p_l$ we have $p_1 \mid n$. Since $n = q_1 q_2 \cdots q_m$ we have $p_1 \mid q_1 q_2 \cdots q_m$. It follows that $p_1 \mid q_k$ for some k with $1 \leq k \leq m$. Say $p_1 \mid q_k$. Since q_k is prime, its only positive divisors are 1 and q_k. Since $p_1 \neq 1$, so $p_1 = q_k$. Similarly, $q_1 = p_j$ for some j with $1 \leq j \leq l$. Since $p_1 = q_k \geq q_1 = p_j \geq p_1$, so we must have $p_1 = p_j = q_1$.

Since $p_1 p_2 \cdots p_l = q_1 q_2 \cdots q_m$ and $q_1 = p_1 \neq 0$, we have $p_2 p_3 \cdots p_l = q_2 q_3 \cdots q_m$. A similar argument shows that $p_2 = q_2$.

Suppose for a contradiction, that $l \neq m$, say $l < m$. By repeating the above argument, we eventually obtain

$$p_l = q_l \cdots q_m$$

then $p_l = q_l$ then $1 = q_{l+1} \cdots q_m$. But each $q_j \geq 2$ so $q_{l+1} \cdots q_m \geq 2$, so we have a desired contradiction, hence $m = l$.

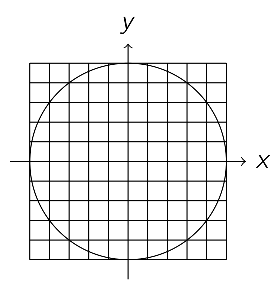
Thus repeating the above argument gives

$$p_1 = q_1, p_2 = q_2, \cdots, p_l = q_l = q_m. \qquad \square$$

28.2 Definition (Diophantine Equation)**.** A Diophantine Equation is a polynomial equation where the variables represent integers.

28.3 Example. Solve

$$x^2 + y^2 = 25$$



28.4 Example. Solve

$$x^2 + y^2 = n$$

in $\mathbb{Z}[i]$ where $i^2 = -1$

28.5 Example. A Linear Diophantine Equation is an equation of the form

$$ax + by = c$$

where $a, b, c \in \mathbb{Z}$ with $(a, b) \neq 0$

28.6 Example (Pell's Equation)**.** Solve

$$x^2 - dy^2 = \pm 1$$

28.7 Example (Pythagorean Triples)**.** Solve

$$x^2 + y^2 = z^2$$

$$x^2 + y^2 + z^2 = w^2$$

Stereographic Projection