MATH 147 Calculus (Advanced)

Lecture Notes

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List of Lectures

Lecture 1, Sept. 12

Mathematical tools $\LaTeX$

MikTex, Winshell

Basics on Sets and Functions

1.1 Definition. Basic Sets

- $\mathbb{N}$ = Natural numbers = $\{1, 2, 3, \dots\}$
- $\mathbb{Z}$ = Integers = $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- $\mathbb{Q} = \{\frac{m}{n} \mid n \in \mathbb{N}, m \in \mathbb{Z}, gcd(n, |m|) = 1\}$
- $\mathbb{R}$ = Real Numbers
- $\mathbb{R}\backslash\mathbb{Q} = \{x \in \mathbb{R} \mid x \text{ is not in } \mathbb{Q}\}$

Notation.

$S \subset X \to S$ is a subset of X

If $S, T \subset X$ then $S \cup T = \{x \in X \mid x \in S \text{ or } x \in T\}$

If $S, T \subset X$ then $S \cap T = \{x \in X \mid x \in S \text{ and } x \in T\}$

Given a collection $\{A_\alpha\}_{\alpha \in I}$ of subsets of X

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for some } \alpha \in I\}$$
$$\bigcap_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for all } \alpha \in I\}$$

$\emptyset$ = empty set, $\emptyset \subset X$

What if $I = \emptyset$, what is $\bigcup_{\alpha \in \emptyset} A_\alpha$

Define
$$\bigcup_{\alpha \in \emptyset} A_\alpha = \emptyset$$

Then
$$\bigcap_{\alpha \in \emptyset} A_\alpha = ??$$

Given $S, T \subset X$ we define

$$S\backslash T = \{x \in X \mid x \in S, x \text{ does not belong to } T\}$$

We denote $X\backslash T$ by T^c = compliment of T in $X = \{x \in X \mid x \text{ does not belong to } T\}$

Note.
$$(S \cup T)^c = S^c \cap T^c$$

De Morgans Law

1.2 Theorem.
$$(\bigcup_{\alpha \in I} A_\alpha)^c = \bigcap_{\alpha \in I} A_\alpha^c$$

Proof.
$$\begin{aligned} x \in (\bigcup_{\alpha \in I} A_\alpha)^c &\iff x \text{ is not a member of } \bigcup_{\alpha \in I} A_\alpha \\ &\iff x \text{ is not in } A_\alpha \quad \forall \alpha \in I \\ &\iff x \in A_\alpha^c \quad \forall \alpha \in I \\ &\iff x \in \bigcap_{\alpha \in I} A_\alpha^c \end{aligned}$$

□

Note. From this we really should have

$$\begin{aligned} \bigcap_{\alpha \in \emptyset} A_\alpha &= (\bigcup_{\alpha \in \emptyset} A_\alpha^c)^c \\ &= \emptyset^c \\ &= X \end{aligned}$$

Power Set

1.3 Definition. Given X, the Power Set of X is the set of all subset of X

Notation.
$$\begin{aligned} P(X) &= \text{ power set of } X \\ &= \{S \mid S \subset X\} \end{aligned}$$

Note. We can observe that
$$\emptyset, X \in P(X)$$

Lecture 2, Sept. 14

New Section 12:30-1:20 CPH 3604
Tutorial Moved to DC 1350 Th 4:30-5:20

Greek Letters

- α - alpha
- β - beta
- δ - delta
- ϵ - epsilon
- γ - gamma

Properties of $\mathbb{N}$

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

Mathematical Induction

2.1 Axiom. Assume $S \in \mathbb{N}$ such that

1. $1 \in S$
2. If $k \in S$, then $k + 1 \in S$

Then $S = \mathbb{N}$

Proof by Induction

1. Establish for each $n \in \mathbb{N}$ a statement $P(n)$ to be proved.
   Example. Let $P(n)$ be the statement that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, show this is true for all $n \in \mathbb{N}$.
   Let $S = \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$, show $S = \mathbb{N}$
2. Base Case: show that $P(1)$ is true. ie): $1 \in S$
3. Inductive Step: Assume that $P(k)$ is true for some k (Inductive Hypothesis). Use this to show that $P(k+1)$ is also true. ie): $k \in S \Rightarrow k + 1 \in S$

By the Principle of Mathematical Induction, $S = \mathbb{N}$

2.2 Example. Prove that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof. Step.1 Let $P(n)$ be the statement that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Step.2 Let $n = 1$ then $P(1) = 1 = \frac{1(1+1}{2}$. Hence $P(1)$ is true.

Step.3 Assume that $P(k)$ is rue for some k

$$P(k)\frac{k(k+1}{2}$$

Step.4

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1)\\ &= \frac{k(k+1)}{2} + (k+1)\\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

Hence $P(k+1)$ is true

Step.5 By Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$ □

2.3 Example. Prove that $3^n + 4^n$ is divisible by 7 for every odd n

Proof. Let $P(k)$ be the statement that $3^{2k-1} + 4^{2k-1}$ is divisible by 7.

Base case: $k = 1$, $P(1)$ is true.

Inductive Step: Assume $P(j)$ is true.

$$\begin{aligned}&3^{2(j+1)-1} + 4^{2(j+1)-1}\\ =&9(3^{2j-1}) + 16(4^{2j-1})\\ =&9(3^{2j-1} + 4^{2j-1}) + 7(4^{2j-1})\end{aligned}$$

Hence $P(j+1)$ is true.

By Principle of Mathematical Induction, $P(k)$ is true for all n

□

Lecture 3, Sept. 16

Well Ordering Property

3.1 Theorem. *If $S \in \mathbb{N}$ and $S \neq \emptyset$, then S contains a least element.*

The following are equivalent

1. Principle of Mathematical Induction
2. Strong Induction
3. Well Ordering Principle

Note. A function f such that $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ can be defined by $f((m, n)) = 7^n 13^m$

Properties of $\mathbb{R}$

Interval

3.2 Theorem. *A set $I \in \mathbb{R}$ is an interval if for each $x, y \in I$ with $x \leq y$ and $z \in I$ with $x \leq y \leq z$, we have $z \in I$*

3.3 Question. 1. Is $\emptyset$ an interval? Yes

2. Is $\{3\}$ an interval? Yes

Other Intervals

1. $[a, b] = x \in \mathbb{R} \mid a \leq x \leq b \to$ Closed Interval
2. $(a, b) = x \in \mathbb{R} \mid a < x < b \to$ Open Interval
3. $[a, b) = x \in \mathbb{R} \mid a \leq x < b \to$ Half Open Half Closed Interval
4. $[a, \infty) = x \in \mathbb{R} \mid a \leq x \to$ Closed Ray
5. $(\infty, b] = x \in \mathbb{R} \mid x \leq b \to$ Closed Ray
6. $(0, \infty)$
7. $(-\infty, b)$
8. $(-\infty, \infty) = \mathbb{R}$

Lecture 4, Sept. 19

Least Upper Bound Property

Upper Bound

4.1 Theorem. *Let $S \subset \mathbb{R}$ then $\alpha \in \mathbb{R}$ is an upper bound for S if $x \leq \alpha$ for all $x \in S$. We say that S is bounded above if S has an upper bound.*

We say that β is a lower bound for S if $\beta \leq x$ for all $x \in S$. We say that S is bounded below if S has a lower bound.

We say that S is bounded if it is bounded above and below.

4.2 Example. Let $S = \{x_1, x_2, \ldots, x_n\}$ be finite.

By relabeling, if necessary we can assume that

$$x_1 < x_2 < \cdots < x_n$$

Then $\beta = x_1$, β is a lower bound and $\alpha = x_n$ is an upper bound.

4.3 Theorem. *Every finite set is bounded.*

4.4 Example. Let $S = [0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ (finite interval)

5 is an upper bound. -1 is a lower bound.

1 is also an upper bound. Moreover if γ is any upper bound of S, then $1 \leq \gamma$

Least Upper Bound

4.5 Theorem. *We say that α is the least upper bound of a set $S \subset \mathbb{R}$ if*

1) *α is an uppper bound of S*
2) *if γ is an upper bound of S, then $\alpha \leq \gamma$*

We write

$$\alpha = lub(S)$$

(Sometimes α os called the supremum of S and is denoted by $\alpha = sup(S)$)

Back to the example $S = [0, 1)$. 0 is a lower bound and if γ is any lower bound, then $\gamma \leq 0$

Greatest Upper Bound

4.6 Theorem. *We say that β is the greatest lower bound of a set $S \subset \mathbb{R}$ if*

1) *β is an lower bound of S*
2) *if γ is an lower bound of S, then $\gamma \leq \beta$*

We write

$$\beta = glb(S)$$

(Sometimes β os called the infimum of S and is denoted by $\beta = inf(S)$)

4.7 Example. if $S = [0, 1)$, $lub(S) = 1$, $glb(S) = 0$.

Note. Is $\emptyset$ bounded (above or below)?

Note: 6 is an upper bound for $\emptyset$. If not, there exists an element in $\emptyset$ that is greater than 6. Similarly, 6 is a lower bound.

In fact, if $\gamma \in \mathbb{R}$ then γ is both an upper and a lower bound of $\emptyset$. $\emptyset$ is a bounded set.

4.8 Example. Let $S = \{x \in \mathbb{Q} \mid x^2 < 2\} \subset \mathbb{R}$

$\sqrt{2}$ is an upper bound and $-\sqrt{2}$ is a lower bound. And $lub(S) = \sqrt{2}$, $glb(S) = -\sqrt{2}$

4.9 Example. Let $S = \{x \in \mathbb{Q} \mid x^2 < 2\} \subset \mathbb{Q}$

S does not have a least upper bound or a greatest lower bound.

4.10 Question. If $S \subset R$ is bounded above, does it always have a least upper bound?

Least Upper Bound Property

4.11 Theorem. *If $S \subset R$ is non-empty and bounded above, then S has a least upper bound.*

Observation

1) $\emptyset$ does not have a lub
2) If we only have rational numbers in the world, then $S = \{x \mid x^2 < 2\}$ does not have a lub. In other words, Least Upper Bound Property fails for $\mathbb{Q}$

4.12 Question. is $\mathbb{N}$ bounded?

1) $\mathbb{N}$ is bounded below, $glb(S) = 1$

Lecture 5, Sept. 21

1) No office hours this afternoon
2) WA1→ Due 2:30 PM Monday, Sept. 26. Submit in dropbox outside Math Tutorial Center.

Least Upper Bound Property If $S \subset \mathbb{R}$ is non-empty and bounded above, then S has a least upper bound.

Archimedean Property I

5.1 Theorem. *$\mathbb{N}$ is not bounded above.*

Proof. Suppose that $\mathbb{N}$ was bounded above. Then $\mathbb{N}$ has a least upper bound α.

Note that $\alpha - \frac{1}{2} < \alpha$. Hence $\alpha - \frac{1}{2}$ is not an upper bound for $\mathbb{N}$. Then there exists $n \in \mathbb{N}$ with $\alpha - \frac{1}{2} < n \leq \alpha$. But then $n+1 \in \mathbb{N}$ and $n+1 > \alpha$ which is impossible.

Therefore $\mathbb{N}$ must not be bounded above.

$\square$

Note. Let $S \neq \emptyset \subset \mathbb{R}$ be bounded above. Let $\alpha = lub(S)$. if $\epsilon > 0$ then there exist $x_0 \in S$ with $\alpha - \epsilon < x_0 \leq \alpha$.

Archimedean Property II

5.2 Corollary. *Let $\epsilon > 0$, Then there exists $n \in \mathbb{N}$ such that*

$$0 < \frac{1}{n} < \epsilon$$

Proof. Take $\alpha = \frac{1}{\epsilon}$ in Archimedean Property I. $\square$

Density of $\mathbb{R}$

5.3 Definition. A subset $S \subset \mathbb{R}$ is said to be dense if for every $\epsilon > 0$ and $x \in \mathbb{R}$,

$$S \cap (x - \epsilon, x + \epsilon) \neq \emptyset$$

or equivalently if $S \cap (a, b) \neq \emptyset$ for all $a < b$ in $\mathbb{R}$

5.4 Proposition. *$\mathbb{Q}$ and $\mathbb{R} \backslash \mathbb{Q} = \mathbb{Q}^c$ are dense in $\mathbb{R}$*

Absolute Values

5.5 Definition.

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

5.6 Example.

$$g(x) = \frac{|x|}{x}$$

Domain $= \{x \in \mathbb{R} \mid x \neq 0\}$

$$g(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Geometric Interpretation of $|x|$

- $|x|$ represents the distance from x to 0.
- $|x - a|$ represents the distance from x to a.

Note. Distance between $(0, 0)$ and (x, y)

$$\sqrt{x^2 + y^2}$$

Properties of $|x|$

1) $|x| \geq 0$ and $|x| = 0 \iff x = 0$
2) $|ax| = |a||x|$ for all $a \in \mathbb{R}, x \in \mathbb{R}$
3) Triangle Inequality

$$|x - z| + |z - y| \geq |x - y|$$

5.7 Theorem. Triangle Inequality *If* $x, y, z \in \mathbb{R}$, *then*

$$|x - z| + |z - y| \geq |x - y|$$

Proof. Use Geometric Interpretation. □

5.8 Theorem. Variants I *For all* $x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y|$$

5.9 Theorem. Variants II *For all* $x, y \in \mathbb{R}$,

$$||x| - |y|| \leq |x - y|$$

Lecture 6, Sept. 23

Inequalities

6.1 Example. Find all $x \in \mathbb{R}$ such that

$$0 < |x - 2| \leq 4$$

Solution. $[-2, 6]$ with $x \neq 2$

Three Basic Inequalities

1. $|x - a| < \delta$
2. $0 < |x - a| < \delta$
3. $|x - a| \leq \delta$

Solution. 1. $(a - \delta, a + \delta) = \{x \in R \mid a - \delta < x < a + \delta\}$

2. $(a - \delta, a + \delta)$ with $x \neq a = \{x \in \mathbb{R} \mid a - \delta < x < a + \delta, x \neq a\}$
3. $[a - \delta, a + \delta] = \{x \in R \mid a - \delta \leq x \leq a + \delta\}$

Sequence

6.2 Definition. A **sequence** is an infinite ordered list of real numbers.

Notation. $\{1, 2, 3, 4, \dots\}$ or $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

6.3 Definition. A **sequence** of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$

The element $f(n)$ is called the n-th term of the sequence. We often denote this by $f_(n) = a_n$

Notation. We can denote sequences in many ways

1. $f(n) = \frac{1}{n}$ for all $n \in \mathbb{N}$
2. Let $a_n = \frac{1}{n}$
3. $\{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$
4. $\{\frac{1}{n}\}$
5. Sometimes we define sequences recursively.
   $a_1 = 1$ and $a_{n+1} = \sqrt{3 + 2a_n}$ for all $n \geq 1$.

Graphing Sequence

Subsequence

6.4 Definition. Let $\{a_n\}$ be a sequence, and let $\{n_k\}$ be a sequence of natural numbers with $n_1 < n_2 < n_3 < \dots < n_k < n_{k+1} < \dots$.

The sequence $b_k = a_{n_k} \to \{b_k\}_{k=1}^{\infty}$ is called **subsequence** of $\{a_n\}$. We often write this as

$$\{a_{n_1}, a_{n_2}, \dots, a_{n_k}, \dots\}$$

Important Subsequences Given $\{a_n\}$, let $n_0 \in \mathbb{N} \cup \{0\}$. Define

$$b_k = a_{n_0+k}$$

This sequence is called a tail of $\{a_n\}$

Limits of Sequences Consider $\{\frac{1}{n}\} = \{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$

Note. As n gets larger and larger, the terms of the sequence $\{\frac{1}{n}\}$ get closer and closer to 0. We would like to say that the sequence $\{\frac{1}{n}\}$ converges to 0 and call 0 the limit of $\{\frac{1}{n}\}$.

6.5 Definition. (Heuristic Definition of Convergence). We say that a sequence $\{a_n\}$ has a limit L if for every positive tolerance $\epsilon > 0$, the term a_n will approximate L with an error less than ϵ so long as the index n is large enough.

Lecture 7, Sept. 26

Writing Assignment 2 is due Friday Oct 14th.

Convergence of Sequences

7.1 Definition. Heuristic definition I We say that a sequence $\{a_n\}$ converges to a limit L if as n gets larger and larger the a_ns get closer and closer to L.

7.2 Definition. Heuristic definition II We say that a sequence $\{a_n\}$ converges to a limit L if for every positive tolerance $\epsilon > 0$, we have that the terms in $\{a_n\}$ approximate L with an error at most ϵ, provided that n is large enough.

7.3 Definition. Convergence of a Sequence We say that $\{a_n\}$ converges to a limit L if for every $\epsilon > 0$, there exists a cutoff $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then $|a_n - L| < \epsilon$

If no such L exists, we say that $\{a_n\}$ **diverges**.

7.4 Example. Consider $\{(-1)^{n+1}\} = \{1, -1, 1, -1, \dots\}$. Does this have a limit?

Proof. Let $\epsilon = 1$. Suppose $L = \lim_{n\to\infty} a_n$. Let N_0 be such that if $n \geq N_0$, then $|a - L| < 1$

Let $n_1 \geq N_0$ with n_0 even. Then

$$\begin{aligned} |-1-L| &= |a_n - L| < 1 \\ &\to L \in (-2, 0) \end{aligned}$$

Let $n_1 \geq N_0$ with n_0 odd. Then

$$\begin{aligned} |1-L| &= |a_n - L| < 1 \\ &\to L \in (0, 2) \end{aligned}$$

So

$$L \in (-2, 0) \cap (0, 2)$$

which is impossible.

Hence $\{a_n\}$ diverges. $\square$

Note. Suppose that $\lim_{n\to\infty} a_n = L$. Let $\epsilon > 0$. What can we say about the terms in $\{a_n\}$ that are in $(L - \epsilon, L + \epsilon)$?

For some N_0, if $n \geq N_0$, then $a_n \in (L - \epsilon, L + \epsilon)$. ie) $(L - \epsilon, L + \epsilon)$ contains a tail of the sequence.

7.5 Proposition. *Let $\{a_n\}$ be a sequence. Then the following are equivalent.*

1. $L = \lim_{n\to\infty} a_n$
2. *for every $\epsilon > 0$, $(L - \epsilon, L + \epsilon)$ contains a tail of $\{a_n\}$*
3. *for every $\epsilon > 0$, $(L - \epsilon, L + \epsilon)$ contains all but finitely many a_n*

4. *for open interval* (a, b) *with* $L \in (a, b)$*, we have* (a, b) *contains a tail of* $\{a_n\}$

5. *for open interval* (a, b) *with* $L \in (a, b)$*, the interval* (a, b) *contains all but finitely many* a_n

7.6 Question. Can $\{a_n\}$ have more than 1 limit?

7.7 Theorem. Uniqueness of Limit *Suppose that* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} a_n = M$*, then* $L = M$

Proof. Assume that $L < M$. Let $\epsilon = \frac{M-L}{2}$.

We can choose N_1 large enough so that if $n \geq N_1$, $a_n \in (L - \epsilon, L + \epsilon)$

We can also choose N_2 large enough so that if $n \geq N_2$, $a_n \in (M - \epsilon, M + \epsilon)$

Let $N_0 = max\{N_1, N_2\}$. Choose $n \geq N_0$. Then $a_n \in (L - \epsilon, L + \epsilon) \cap (M - \epsilon, M + \epsilon)$

But $(L - \epsilon, L + \epsilon) \cap (M - \epsilon, M + \epsilon) = \emptyset$

$\square$

Lecture 8, Sept. 28

8.1 Theorem. *Assume that $\{a_n\}$ converges. then $\{a_n\}$ is bounded.*

Proof. Assume that

$$L = \lim_{n\to\infty} a_n$$

Let $\epsilon = 1$. Then there exists $N_0 \in \mathbb{N}$ so that if $n \geq N_0$ then $|a_n - L| < 1$

If $n \geq N_0$,then

$$\begin{aligned}|a_n| = |a_n - L + L| &\leq |a_n - l| + |L| \\ &< 1 + |L|\end{aligned}$$

Let

$$M = max|a_1|, |a_2|, \dots, |a_{N_0 - 1}|, |L| + 1$$

.

Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. □

Question: Do all bounded sequences converge? No.

8.2 Definition. 1. We say that a sequence $\{a_n\}$ is increasing if $a_n < a_{\{}n + 1\}$ for all $n \in \mathbb{N}$

2. We say that $\{a_n\}$ is non-decreasing if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$
3. We say that $\{a_n\}$ is decreasing if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$
4. We say that $\{a_n\}$ is non-increasing if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$

We say that $\{a_n\}$ is monotonic if $\{a_n\}$ satisfies one of the conditions.

Example:

1. $$\{a_n\} = \{\frac{1}{n}\}$$
   is decreasing, since
   $$\frac{1}{n+1} \leq \frac{1}{n}$$
   for all $n \in \mathbb{N}$
2. $$\{cos(n)\}$$
3. Let $a_1 = 1$,
   $$a_{n+1} = \sqrt{3 + 2a_n}$$

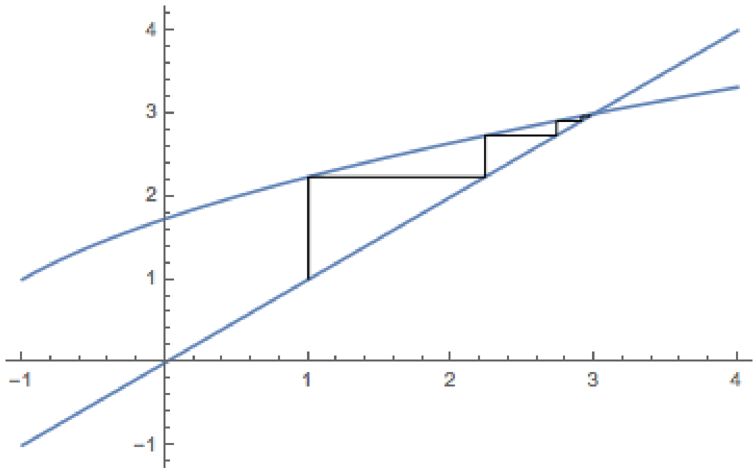

Figure 1: $y = \sqrt{3+2x}$ and $y = x$

8.3 Theorem. Monotone Convergence Theorem

If $\{a_n\}$ is monotonic and bounded, then $\{a_n\}$ converges.

Proof. Assume that $\{a_n\}$ is non-decreasing and bounded above. Let $L = lub(\{a_n\})$

Let $\epsilon > 0$, then $L - \epsilon$ is not an upper bound. Then there exists $N_0 \in \mathbb{N}$ so that $L - \epsilon < a_{N_0} \leq L$. If $n \geq N_0$, then $L - \epsilon < a_{N_0} \leq a_n \leq L$, so $|a_n - L| < \epsilon$. Hence $L = \lim_{n\to\infty} a_n$

Similarly, if $\{a_n\}$ is non-increasing then $L = \lim_{n\to\infty} a_n$ where $L = glb(\{a_n\})$ $\square$

8.4 Example. Let $a_1 = 1$,

$$a_{n+1} = \sqrt{3 + 2a_n}$$

We know that $0 \leq a_n < a_{n+1} \leq 3$ for all $n \in \mathbb{N}$. $\{a_n\}$ is increasing and bounded above. Hence $\{a_n\}$ converges.

8.5 Corollary. *A monotonic sequence $\{a_n\}$ converges iff it is bounded.*

8.6 Definition. We say a sequence **diverges to** ∞ if for every $M > 0$ we can find a a cutoff $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then $M \leq a_n$, we write $\lim_{n\to\infty} a_n = \infty$.

Lecture 9, Sept. 29

9.1 Definition. We sat that $\{a_n\}$ diverges to ∞ if for every $M \geq 0$ there exists $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then $a_n > M$

We write

$$\lim_{n\to\infty} a_n = \infty$$

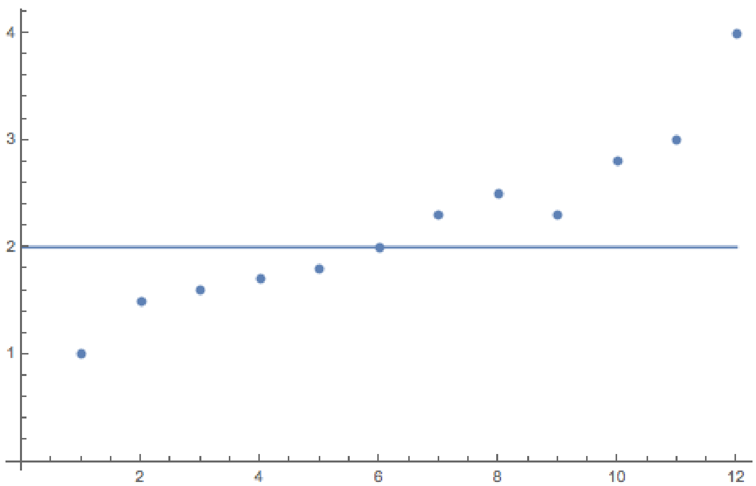

Figure 2: $\{a_n\}$ and $M = 2$

9.2 Question. Does every sequence $\{a_n\}$ that is <u>not</u> bounded above diverges to ∞?

No. $\{0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \dots\}$

Note. If $\{a_n\}$ is non-decreasing then either

1) $\{a_n\}$ is bounded and convergent

2) $\{a_n\}$ is unbounded and diverges to ∞

9.3 Question. If a sequence is not bounded above, does it have a sub-sequence that diverges to ∞?

Series

Given a Sequence $\{a_n\}$, what does it mean to sum all of the terms of the sequence? That is what does the formal sum mean

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

9.4 Example.

$$\sum_{n=1}^{\infty} (-1)^{n+1}$$

9.5 Example.

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

9.6 Definition. For each $k \in \mathbb{N}$, the kth partial sum is

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

We say that $\sum_{n=1}^{\infty} a_n$ converges if the sequence $\{S_k\}$ of partial sums converges. Otherwise we say the series diverges.

If the series converges we let

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{n=1}^{k} a_n$$

9.7 Example.

$$\sum_{n=1}^{\infty} (-1)^{n+1}$$

$$S_k = \begin{cases} 1 & \textit{if k is odd} \\ 0 & \textit{if k is even} \end{cases}$$

Thus S_k diverges

Geometric Series Let $r \in R$, consider

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots$$

$$S_k = \sum_{n=0}^{k} r^n = 1 + r + r^2 + r^3 + \cdots + r^k$$

$$S_k = \frac{1 - r^{k+1}}{1 - r} \ \textit{if } r \neq 1$$

Note. If $|r| < 1$ then $\lim_{k\to\infty} r^{k+1} = 0$

If $|r| > 1$ then $\lim_{k\to\infty} r^{k+1}$ does not exists

If $r = -1$ then $\lim_{k\to\infty} r^{k+1}$ does not exists.

If $r = 1$ then $S_k = k$ which diverges to infinity.

9.8 Example. $r = \frac{1}{2}$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - \frac{1}{2}} = 2$$

Lecture 10, Sept. 30

Series

10.1 Definition. A series $\sum_{n=1}^{\infty} a_n$ is **positive** is for all $n \in \mathbb{N}$, if $S_k = \sum_{n=1}^{k} a_n$, then $S_{k+1} - S_k = a_{k+1} \geq 0$

10.2 Example. Harmonic Series Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

Let $S_k = \displaystyle\sum_{n=1}^{k} \frac{1}{n}$,

$$\begin{aligned}
S_1 &= 1 = \frac{2}{2} \\
S_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\
S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{4}{2} \\
S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{2} \\
&\vdots \\
S_{2^k} &> \frac{2+k}{2}
\end{aligned}$$

Since $\{\frac{2+k}{2}\}$ is not bounded, $\{S_k\}$ is not bounded.

10.3 Example. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 - n}$

Note.

$$\begin{aligned}
\frac{1}{n^2 - n} &= \frac{1}{n(n-1)} \\
&= \frac{1}{n-1} - \frac{1}{n}
\end{aligned}$$

Solution.

$$\begin{aligned}
S_1 &= 1 - \frac{1}{2} = 1 - \frac{1}{2} \\
S_2 &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3} \\
S_3 &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4} \\
&\vdots \\
S_k &= 1 - \frac{1}{k}
\end{aligned}$$

As $k \to \infty$, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 - n} = 1$

10.4 Example. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Note. For $n \geq 2$,

$$\frac{1}{n^2} < \frac{1}{n^2 - n}$$

$$\begin{aligned}
T_k = \sum_{n=1}^{k} \frac{1}{n^2} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \\
&< 1 + \frac{1}{2^2 - 2} + \frac{1}{3^2 - 2} + \cdots + \frac{1}{k^2 - k} \\
&< 1 + 1 \\
&= 2
\end{aligned}$$

Since $T_k \leq 2$ for all k, $\{T_k\}$ is bounded and by the Monotone Convergence Theorem is convergent with $1 \leq \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$.

In fact, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

10.5 Example. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$, does this converge?

Note that $\dfrac{1}{n!} < \dfrac{1}{2^n}$ for $n \geq k$.

In fact, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!} = e$

Note.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Arithmetic Rules for Sequences

10.6 Question. Assume $a_n \to 3$, $b_n \to 7$.

What can you say about

1) $\{4a_n\}$
2) $\{a_n b_n\}$
3) $\{a_n + b_n\}$

4) $\{\frac{a_n}{b_n}\}$

10.7 Theorem. Arithmetic Rules for Sequences *Let* $\{a_n\}$, $\{b_n\}$ *be such that* $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$. *Then*

1) $\lim_{n\to\infty} ca_n = cL$ *for all* $c \in \mathbb{R}$

2) $\lim_{n\to\infty} a_n + b_n = L + M$

3) $\lim_{n\to\infty} a_n b_n = LM$

4) $\displaystyle\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$ *if* $L \neq 0$

5) $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ *if* $M \neq 0$

Proof. 1) If $c = 0$ then $ca_n = 0$ for all n. Hence $\lim_{n\to\infty} ca_n = \lim_{n\to\infty} 0 = 0L = cL$ Suppose $c \neq 0$, Let $\epsilon > 0$. We want N so that if $n \geq N$, $|ca_n - cL| < \epsilon \Leftrightarrow |a_n - L| < \dfrac{\epsilon}{|c|}$

Choose N_0 such that if $n \geq N_0$ we have $|a_n - L| < \dfrac{\epsilon}{|c|}$

If $n \geq N_0$,

$$|ca_n - cL| \leq |a_n - L||c| < \frac{\epsilon}{|c|}|c| = \epsilon$$

□

Lecture 11, Oct. 3

WA2 now due Monday Oct. 17

EA2 due today

11.1 Theorem. Arithmetic Rules for Sequences *Let* $\{a_n\}$, $\{b_n\}$ *be such that* $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$. *Then*

1) $\lim_{n\to\infty} ca_n = cL$ *for all* $c \in \mathbb{R}$
2) $\lim_{n\to\infty} a_n + b_n = L + M$
3) $\lim_{n\to\infty} a_n b_n = LM$
4) $\displaystyle\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$ *if* $L \neq 0$
5) $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ *if* $M \neq 0$
6) $\displaystyle\lim_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{L}$ *if* $L \geq 0$

Proof.

1) If $c = 0$ then $ca_n = 0$ for all n. Hence $\lim_{n\to\infty} ca_n = \lim_{n\to\infty} 0 = 0L = cL$ Suppose $c \neq 0$, Let $\epsilon > 0$. We want N so that if $n \geq N$, $|ca_n - cL| < \epsilon \Leftrightarrow |a_n - L| < \dfrac{\epsilon}{|c|}$

   Choose N_0 such that if $n \geq N_0$ we have $|a_n - L| < \dfrac{\epsilon}{|c|}$

   If $n \geq N_0$,
$$|ca_n - cL| \leq |a_n - L||c| < \frac{\epsilon}{|c|}|c| = \epsilon$$

2) Consider
$$\begin{aligned}|(a_n + b_n) - (L + M)| &= |a_n - L + b_n - M| \\ &\leq |a_n - L| + |b_n - M|\end{aligned}$$

   Let $\epsilon > 0$. Choose $N_1 \in \mathbb{N}$ so that
$$n \geq N_1 \to |a_n - L| < \frac{\epsilon}{2}$$

   Choose $N_2 \in \mathbb{N}$ so that
$$n \geq N_2 \to |b_n - M| < \frac{\epsilon}{2}$$

   Let $N_0 = max\{N_1, N_2\}$. If $n \geq N_0$
$$\begin{aligned}|(a_n + b_n) - (L + M)| &\leq |a_n - L| + |b_n - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

3) Consider $|a_n b_n - LM|$

$$\begin{aligned} & |a_n b_n - LM| \\ = & |a_n b_n - b_n L + b_n L - LM| \\ = & |(a_n - L) b_n + L(b_n - M)| \\ \leq & |(a_n - L) b_n| + |L(b_n - M)| \end{aligned}$$

By 1), we can find N_1 so that if $n \geq N_1$,

$$|L|\,|b_n - M| \leq \frac{\epsilon}{2}$$

Since $\{b_n\}$ is convergent it is bounded. So there exists $c > 0$ so that $|b_n| < c$

Then $|b_n|\,|a_n - L| < c\,|a_n - L|$

Choose N_2 so that if $n \geq N_2$

$$|a_n - L| < \frac{\epsilon}{2c}$$

If $N_0 = max\{N_1, N_2\}$ and $n \geq N_0$ then

$$|a_n b_n - LM| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

4)

$$\left| \frac{1}{a_n} - \frac{1}{L} \right| = \frac{|a_n - L|}{|a_n|\,|L|}$$

Since $a_n \to L$, $L \neq 0$ we can find $N_1 \in \mathbb{N}$ so that if $n \geq N_1$, then

$$|a_n - L| < \frac{|L|}{2} \to |a_n| \geq \frac{|L|}{2}$$

If $n \geq N_1$ then

$$\left| \frac{1}{a_n} - \frac{1}{L} \right| = \frac{|a_n - L|}{|a_n|\,|L|} \leq \frac{|a_n - L|}{\frac{|L|}{2}|L|} = \frac{|a_n - L|}{\frac{|L|^2}{2}}$$

Let $\epsilon > 0$. Choose N_2 so that if $n \geq N_2$

$$\frac{|a_n - L|}{\frac{|L|^2}{2}} < \epsilon$$

Let $N_0 = max\{N_1, N_2\}$ if $n \geq N_0$

$$\left| \frac{1}{a_n} - \frac{1}{L} \right| < \epsilon$$

5) Follows from 3 and 4.

6) Homework

$\square$

Note. If $\lim_{n\to\infty} a_n = L, \lim_{n\to\infty} n_n = M$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M} \textit{ if } M \neq 0$$

What happens if $M = 0$?

It depends on a_n.

11.2 Example. $a_n = b_n = \dfrac{1}{n}$

11.3 Example. $a_n = \dfrac{1}{n}, b_n = \dfrac{1}{n^2}$

11.4 Proposition. *Assume that* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists and that* $\lim_{n\to\infty} b_n = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

Proof.

$$\begin{aligned} a_n &= (b_n)(\frac{a_n}{b_n}) \\ &= \lim_{n\to\infty} a_n \\ &= \lim_{n\to\infty} b_n \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= 0L \\ &= 0 \end{aligned}$$

$\square$

Lecture 12, Oct. 5

12.1 Example. Find $\dfrac{3n^2+2n}{5n^2+2}$

Solution.

$$
\begin{aligned}
\lim_{n\to\infty}\frac{3n^2+2n}{5n^2+2} &= \lim_{n\to\infty}\frac{n^2}{n^2}\frac{3+\dfrac{2}{n}}{5+\dfrac{2}{n^2}} \\
&= \frac{\lim\limits_{n\to\infty}3+\lim\limits_{n\to\infty}\dfrac{2}{n}}{\lim\limits_{n\to\infty}5+\lim\limits_{n\to\infty}\dfrac{2}{n^2}} \\
&= \frac{3+0}{5+0} \\
&= \frac{3}{5}
\end{aligned}
$$

Note. If $a_k b_j \neq 0$

$$
\lim_{n\to\infty}\frac{a_0+a_1n+\cdots+a_kn^k}{b_0+b_1n+\cdots+b_jn^j} = \begin{cases} \dfrac{a_k}{b_j} & \text{if } k=j \\ 0 & \text{if } j>k \\ \infty & \text{if } j<k,\ a_kb_j>0 \\ -\infty & \text{if } j<k,\ a_kb_j<0 \end{cases}
$$

12.2 Example. Find

$$
\lim_{n\to\infty}\sqrt{n^2+n}-n
$$

Solution.

$$
\begin{aligned}
\lim_{n\to\infty}\sqrt{n^2+n}-n &= \lim_{n\to\infty}(\sqrt{n^2+n}-n)\cdot\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty}\frac{n}{\sqrt{1+\dfrac{1}{n}}+1} \\
&= \frac{n}{\sqrt{1+\lim\limits_{n\to\infty}\dfrac{1}{n}}+1} \\
&= \frac{1}{2}
\end{aligned}
$$

12.3 Example. $a_1=1$ and $a_{n+1}=\dfrac{1}{1+a_n}$. Suppose that $\{a_n\}$ converges, find $\lim_{n\to\infty}a_n$

12.4 Proposition. *A sequence $\{a_n\}$ converges to L if and only if every sub-sequence $\{a_{n_k}\}$ converges to L*

Proof. Assume that $lim_{n\to\infty} a_n = L$. Let $\{a_{n_k}\}$ be a sub-sequence. Let $\epsilon > 0$, we can find a N_0 so that if $n \geq N_0$, then $|a_n - L| < \epsilon$.

Let $k_0 \geq N_0$, then $k \geq k_0 \Rightarrow n_k \geq n_{k_0} \geq N_0$

Hence $|a_{n_k} - L| < \epsilon$ □

Solution. If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$$

Then,

$$L = \frac{1}{1+L}$$
$$L^2 + L - 1 = 0$$
$$L = \frac{-1 \pm \sqrt{5}}{2}$$

12.5 Question. Does $\{a_n\}$ converge?

Solution. Claim that for any k,

$$a_{2k} < a_{2k+2} < a_{2k+1} < a_{2k-1}$$

Proof by induction.

$\{a_{2k-1}\}$ is decreasing and bounded below by 0

$\{a_{2k}\}$ is increasing and bounded above by 1.

Let $\lim_{n\to\infty} a_{2k} = M$ and $\lim_{n\to\infty} a_{2k-1} = L$.

Since $M = \dfrac{-1+\sqrt{5}}{2}$ and $L = \dfrac{-1+\sqrt{5}}{2}$, $M = l$

Thus, $\{a_n\}$ converges.

12.6 Example. FInd

$$\lim_{n\to\infty} \frac{cos(n)}{n}$$

Lecture 13, Oct. 6

Squeeze Theorem

13.1 Example. Find

$$\lim_{n\to\infty} \frac{cos(n)}{n}$$

Observation:

$$|cos(n)| \leq 1$$

$$\frac{-1}{n} \leq \frac{cos(n)}{n} \leq \frac{1}{n}$$

13.2 Theorem. Squeeze Theorem *If* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *are such that* $a_n \leq b_n \leq c_n$ *with* $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$*, then* $\lim_{n\to\infty} b_n = L$

Proof. Let $\epsilon > 0$, then exists $N_0 \in \mathbb{N}$ so that if $n \geq N_0$ then $a_n \in (L-\epsilon, L+\epsilon)$ and $c_n \in (L-\epsilon, L+\epsilon)$

If $n \geq N_0$,

$$L - \epsilon < a_n \leq b_n \geq c_n < L + \epsilon$$

$$|b_n - L| < \epsilon$$

$\square$

Solution. We know that

$$\frac{-1}{n} \leq \frac{cos(n)}{n} \leq \frac{1}{n}$$

since $|cos(n)| \leq 1$

Since $\lim_{n\to\infty} -\frac{1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$

Then

$$\lim_{n\to\infty} \frac{cos(n)}{n} = 0$$

13.3 Example.

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$$

Note. If $\{a_n\}$ is bounded, then

$$\lim_{n\to\infty} \frac{a_n}{n} = 0$$

Bolzano-Weierstrass Theorem

Note. We know that convergent sequences are bounded. But bounded sequences do not have to converge.

Does every bounded sequences have a convergent sub-sequence?

Strategy Bounded + monotonic $\Rightarrow$ convergent

Does every sequence have a monotonic sub-sequence

13.4 Definition. Given $\{a_n\}$ we call an index n_0 a **peak point** for $\{a_n\}$ if $a_n < a_{n_0}$ for all $n \geq n_0$

13.5 Lemma. Peak Point Lemma *Every sequence $\{a_n\}$ has a monotonic sub-sequence.*

Proof. Let $P = \{n \in \mathbb{N} \mid n \textit{ is a peak point of } \{a_n\}\}$

Case 1. P is infinite.

Let $n_1 =$ least element of P

Let $n_2 =$ least element of P
$\{n_1\}$

...

This gives us a sequence recursively

$$n_1 < n_2 < \cdots < n_k < \cdots \in P$$

Since these are peak points,

$$a_{n_k} > a_{n_{k+1}}$$

Thus $\{a_{n_k}\}$ is decreasing.

Case 2. Let n_1 be the least index that is not a peak point. Since n_1 is not a peak point, we can choose $n_2 > n_1$ so that

$$a_{n_1} \leq a_{n_2}$$

Since $n_@$ is not a peak point, then we can choose $n_3 > n_2$ so that

$$a_{n_2} \leq a_{n_3}$$

We can proceed recursively, to find that

$$n_1 < n_2 < \cdots < n_k < \dots$$

Where $a_{n_k} \leq a_{n_{k+1}}$

Thus $\{a_{n_k}\}$ is non-decreasing.

In either case we have a monotonic sub-sequence. □

13.6 Theorem. Bolzano-Weierstrass Theorem *Every bounded sequences has a convergent sub-sequence.*

Proof. Give $\{a_n\}$, by the Peak Point Lemma $\{a_n\}$ has a monotinic subsequence $\{a_{n_k}\}$, which is also bounded. Bu the MCT, $\{a_{n_k}\}$ is convergent. □

Note. BWT is equivalent to MCT which is equivalent to the LUBP.

Lecture 14, Oct. 7

14.1 Theorem. Bolzano-Weierstrass Theorem *Every bounded sequences has a convergent sub-sequence.*

14.2 Definition. We say that $\alpha \in \mathbb{R}$ is a **limit point** of $\{a_n\}$ if there exists a sub-sequence $\{a_{n_k}\}$ with $\lim_{n\to\infty} a_{n_k} = \alpha$

LET $LIM(\{a_n\}) = \{\alpha \in \mathbb{R} \mid \alpha \text{ is a limit point of } \{a_n\}$

14.3 Example. $a_n = (-1)^{n+1} \to \{1, -1, 1, -1, \dots\}$

$LIM(\{a_n\}) = \{1, -1\}$

14.4 Example. $a_n = n \to \{1, 2, 3, \dots\}$

$LIM(\{a_n\}) = \emptyset$

Fact If $\{a_n\}$ converges with $\lim_{n\to\infty} a_n = L$, then $LIM(\{a_n\}) = \{L\}$

14.5 Question. If $\{a_n\}$ is such that $LIM(\{a_n\})$ contains only one value α, does $\{a_n\}$ converges to α?

No. Counterexample:

$$\{a_n\} = \{1, \frac{1}{2}, 3, \frac{1}{4}, 5, \dots\}$$

14.6 Proposition. *α is a limit point of $\{a_n\}$ if for every $(\alpha - \epsilon, \alpha + \epsilon)$ contains infinite many terms of the sequence.*

Assume α is a limit point of $\{a_n\}$, then there exists a sub-squence $\{a_{n_k}\}$ with $a_{n_k} \to \alpha$. There exists $K_0 \in \mathbb{N}$ so that $k \geq K_0 \to |a_{n_k} - \alpha| < \epsilon \to a_{n_k} \in (\alpha - \epsilon, \alpha + \epsilon)$

Proof. Assume that $\forall \epsilon > 0$, $(\alpha - \epsilon, \alpha + \epsilon)$ contains infinitely many terms of $\{a_b\}$

For $\epsilon = 1$ we can find n_1 so that $a_{n_1} \in (\alpha - 1, \alpha + 1)$

$a_{n_2} \in (\alpha - \frac{1}{2}, \alpha + \frac{1}{2})$

Suppose we have $n_1 < n_2 < n_3 < \cdots < n_k$ with

$$a_{n_j} \in (\alpha - \frac{1}{j}, \alpha + \frac{1}{j})$$

Since $(\alpha - \frac{1}{k+1}, \alpha + \frac{1}{k+1})$ contains infinitely many $a_n s$. there is $n_{k+1} > n_k$ with $a_{n_{k+1}} \in (\alpha - \frac{1}{k+1}, \alpha + \frac{1}{k+1})$

We proceed recursively to get a sub-sequence $\{a_{n_k}\}$ with

$$a_{n_k} = (\alpha - \frac{1}{k}, \alpha + \frac{1}{k})$$

$$\alpha - \frac{1}{k} < a_{n_k} < \alpha + \frac{1}{k}$$

By the squeeze theorem, $a_{n_k} \to \alpha$

$\square$

14.7 Question.

1. Suppose $\{a_n\}$ is bounded and $LIM(\{a_n\}) = \{L\}$, does $\lim_{n\to\infty} L$?
2. Does there exists $\{a_n\}$ with $LIM(\{a_n\}) = \{R\}$
3. For which subsets S of R does there exists $\{a_n\}$ with $LIM(\{a_n\}) = S$?

Cauchy Sequence

14.8 Question. Is there an intrinsic way to characterize a convergent sequence?

Note. If $\lim_{n\to\infty} a_n = L$ and if $\epsilon > 0$ then we can find N_0 so that if $n \geq N_0$m

$$|a_n - L| < \frac{\epsilon}{2}$$

If $n, m \geq N_0$, then

$$\begin{aligned} |a_n - a_m| &= |(a_n - L) + (L - a_m)| \\ &\leq |a_n - L| + |L - a_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

14.9 Definition. A sequence $\{a_n\}$ is **Cauchy** is for every $\epsilon > 0$, then there exists $N_0 \in \mathbb{N}$ so that if $n, m \geq N_0$, then

$$|a_n - a_m| < \epsilon$$

14.10 Proposition. *Every convergent sequence is Cauchy*

14.11 Question. Does every Cauchy sequence Converges?

14.12 Lemma. *Every Cauchy Sequence is bounded.*

Proof. Let $\epsilon = 1$ and choose N_0 so that if $n, m \geq N_0$, then $|a_n - a_m| < \epsilon$

Hence, if $n \geq N_0$ then

$$|a_n - a_{N_0}| < 1 \to |a_n| \leq |a_{N_0}| + 1$$

$\square$

Let $M = max\{|a_1|, |a_q|, \ldots, |a_{N_0-1}|, |a_{N_0}| + 1\}$

14.13 Lemma. *Let $\{a_n\}$ be Cauchy. Assume that $\{a_{n_k}\}$ is such that $\lim_{k\to\infty} a_{n_k} = L$, then*

$$\lim_{n\to\infty} a_n = L$$

Proof. Let $\epsilon > 0$. We can find a N_0 so that if $n, m \geq N_0$, then

$$|a_n - a_m| < \frac{\epsilon}{2}$$

Let $n \geq N_0$

$$\begin{aligned}
|a_n - L| &= |(a_n - a_{n_k}) + (a_{n_k} - L)| \\
&\leq |a_n - a_{n_k}| + |a_{n_k} - L| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

□

14.14 Theorem. Completeness Property for $\mathbb{R}$ *Every Cauchy Sequence Converges.*

Proof. If a_n is Cauchy, then a_n is bounded. By BWT, a_n has a convergent sub-sequence $\{a_{n_k}\}$. Hence a_n converges. (by Lemma 2.) □