Module 2: How Fast is it? From Average to Instantaneous Rate of Change

In the previous module, we explored the concept of **average rate of change**, which tells us how quickly a quantity changes over a given interval of time or space. However, in many real-world situations, it is crucial to determine the rate of change at one precise moment rather than averaged over a period. For example, when a car applies emergency braking, engineers must know the exact deceleration (instantaneous rate of change of velocity) at the moment the brakes engage to design effective safety systems. Similarly, environmental scientists measuring pollution along a river need the instantaneous concentration gradient (rate of change of pollutant concentration per unit distance) at specific locations to pinpoint sources of contamination accurately. This module will introduce the concept of **instantaneous rate of change**, showing how it can be derived by taking the limit of the average rate of change as the interval approaches zero.

Example 1: Instantaneous Velocity of a Ball Thrown Vertically Upward

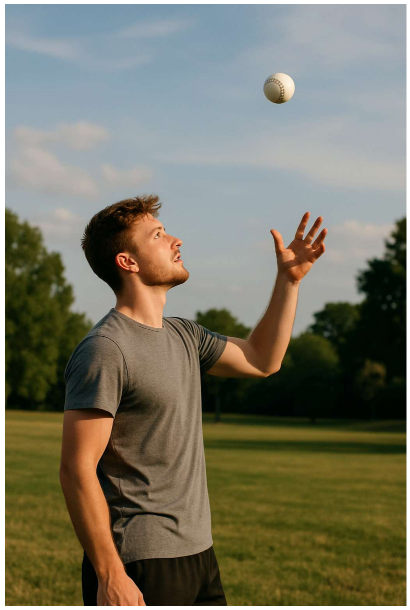
Suppose you throw a ball straight upward, and its height above the ground (in meters) at any time t (in seconds) is described by the function:

$$y(t) = 20t - 4.9t^2$$

We know velocity describes how quickly the position of an object changes with time. But how can we determine the velocity at the exact instant $t = 1.5\,\mathrm{s}$? Recall from earlier that velocity is defined simply as the ratio of the change in position to the change in time:

$$v = \frac{\text{change in position}}{\text{change in time}}$$

Take a moment to think about how you might use this idea to find the velocity of the ball at exactly $t = 1.5\,\mathrm{s}$. What challenge do we face when trying to apply this definition at a single point in time?



Throwing a ball into the air. Image generated by AI.

Solution: Finding the Velocity at Exactly 1.5 s

Step 1: Formulating our Strategy

We want the velocity at exactly $t = 1.5\,\mathrm{s}$. Previously, we calculated the **average velocity** (average rate of change) over an interval of time. To find the exact velocity at a single instant, we'll calculate the average velocity over a very small time interval starting at $t = 1.5\,\mathrm{s}$, then take the limit as that interval shrinks to zero.

Step 2: Applying the Strategy Mathematically

Recall the formula we learned for average rate of change:

$$\text{Average rate of change} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Applying this formula to our height function $y(t) = 20t - 4.9t^2$, the average velocity between $t = 1.5\,\mathrm{s}$ and $t = 1.5 + \Delta t$ is:

$$v_{\text{avg}} = \frac{y(1.5 + \Delta t) - y(1.5)}{\Delta t}$$

Step 3: Simplifying the Expression

Substituting the given function $y(t) = 20t - 4.9t^2$ into the expression, we have:

$$v_{\text{avg}} = \frac{\left[20(1.5 + \Delta t) - 4.9(1.5 + \Delta t)^2\right] - \left[20(1.5) - 4.9(1.5)^2\right]}{\Delta t}$$

Expanding and combining terms, this simplifies to:

$$v_{\text{avg}} = \frac{5.3\,\Delta t - 4.9(\Delta t)^2}{\Delta t}$$

We further simplify by factoring out and canceling the common term Δt:

$$v_{\text{avg}} = 5.3 - 4.9\,\Delta t$$

Step 4: Taking the Limit

Now, taking the limit as Δt approaches zero, we get:

$$v(1.5) = \lim_{\Delta t \to 0} (5.3 - 4.9\,\Delta t) = 5.3\,\mathrm{m/s}$$

Interpretation: The instantaneous velocity at $t = 1.5\,\mathrm{s}$ is $5.3\,\mathrm{m/s}$. Since the velocity is positive, the ball is still moving upward at this precise moment.

Rate of change beyond time

This approach of shrinking the interval to zero applies not just to time-based rates of change, but also to any situation where a quantity changes with respect to another variable, such as distance or length. For example, imagine a factory producing a long metal sheet where the thickness of the sheet changes gradually along its length. Engineers need to know precisely how fast the thickness is changing at specific points along the sheet to ensure it meets design specifications and safety standards. A rapid change in thickness could lead to weak spots or structural issues.

In the next example, we'll challenge you to calculate such a **rate of change with respect to length**—specifically, how quickly the thickness of a metal sheet is changing at a particular position along its length.

Example 2: Rate of Change of Material Thickness Along a Sheet

A manufacturing process produces a metal sheet where the thickness of the sheet varies along its length. The thickness T (measured in millimeters) at a position x centimeters from one end of the sheet is given by:

$$T(x) = 10 - 0.05x^2$$

In this case, the thickness decreases as you move further along the sheet. Suppose you need to find out **how quickly the thickness is changing** at the position $x = 6\,\mathrm{cm}$. This information is critical because engineers must ensure that the thickness does not decrease too sharply in any section of the sheet to maintain structural integrity.

How would you apply the method of average rate of change, and then shrink the interval, to find the **exact rate of change of thickness** at $x = 6\,\mathrm{cm}$?

A metal sheet of varying thickness. Image generated by AI.

Solution: Finding the Rate of Change of Thickness at $x = 6\,\text{cm}$

Step 1: Formulating our Strategy

We want to find the rate at which the sheet's thickness is changing at exactly $x = 6\,\text{cm}$. Previously, we calculated the **average rate of change** over an interval. To get the exact (instantaneous) rate at a single point, we'll calculate the average rate of change over a very small interval starting at $x = 6\,\text{cm}$, then take the limit as that interval shrinks to zero.

Step 2: Applying the Strategy Mathematically

Recall the formula for average rate of change:

$$\text{Average rate of change} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Applying this to our thickness function $T(x) = 10 - 0.05x^2$, the average rate of change of thickness between $x = 6\,\text{cm}$ and $x = 6 + \Delta x\,\text{cm}$ is:

$$R_{\text{avg}} = \frac{T(6 + \Delta x) - T(6)}{\Delta x}$$

Step 3: Simplifying the Expression

Substitute the thickness function into the formula:

$$R_{\text{avg}} = \frac{[10 - 0.05(6 + \Delta x)^2] - [10 - 0.05(6)^2]}{\Delta x}$$

Expanding and simplifying the numerator gives:

$$R_{\text{avg}} = \frac{-0.6\Delta x - 0.05(\Delta x)^2}{\Delta x}$$

Factoring and canceling Δx:

$$R_{\text{avg}} = -0.6 - 0.05\Delta x$$

Step 4: Taking the Limit

Now, take the limit as Δx approaches zero:

$$R = \lim_{\Delta x \to 0} (-0.6 - 0.05\Delta x) = -0.6\,\text{mm/cm}$$

Interpretation: The negative value means that the thickness is decreasing at this location. At exactly $x = 6\,\text{cm}$, the thickness of the sheet is getting thinner at a rate of $0.6\,\text{mm}$ for each centimeter along the sheet.