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A class of compact boundary value methods applied to semi-linear reaction–diffusion equations☆

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ABSTRACT

This paper deals with a class of compact boundary value methods (CBVMs) for solving semi-linear reaction–diffusion equations (SLREs). The presented CBVMs are constructed by combining a fourth-order compact difference method (CDM) with the p-order boundary value methods (BVMs), where the former is for the spatial discretization and the latter for temporal discretization. It is proven under some suitable conditions that the CBVMs are locally stable and uniquely solvable and have fourth-order accuracy in space and p-order accuracy in time. The computational effectiveness and accuracy of CBVMs are further testified by applying the methods to the Fisher equation. Besides these research, we also extend the CBVMs to solve the two-component coupled system of SLREs. The numerical experiment shows that the extended CBVMs are effective and can arrive at the high-precision.

1. Introduction

Consider the following initial-boundary value problem of SLREs:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = a\dfrac{\partial^2 u}{\partial x^2}(x,t) + g(u(x,t)), \quad (x,t) \in [0,l] \times [t_0, T], \\ u(x,t_0) = \varphi(x), \quad x \in [0,l], \\ u(0,t) = \psi_1(t), \quad u(l,t) = \psi_2(t), \quad t \in [t_0, T], \end{cases} \tag{1.1}$$

where x, t denote the spatial and temporal variables, respectively, $a > 0$ is the diffusion coefficient, $\varphi : [0, l] \to \mathbb{R}$, $\psi_1 : [t_0, T] \to \mathbb{R}$, $\psi_2 : [t_0, T] \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are some given sufficiently smooth mappings. This type of problem plays an important role in modeling real phenomena arising in physics, chemistry, biology and many other scientific fields (see e.g. [1–4]).

Generally speaking, it is difficult to obtain the exact solution of an initial-boundary value problem of SLREs. Hence ones turn to develop various numerical methods to solve the problem. Up to now, a lot of numerical methods for SLREs have been presented. For example, finite difference methods, implicit-explicit predictor-corrector methods, linearized compact multi-splitting methods, compact finite difference methods, spectral methods and continous/discontinous Galerkin finite element

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methods have been introduced and studied in Refs. [4–12], respectively. More related research also refer to the references therein.

Besides the above full discretization methods, in recent years, many researchers considered the so-called method of lines (MOLs) for various partial differential equations (PDEs), where the spatial and temporal discretization methods were combined using. In particular, BVMs and the block BVMs (BBVMs) (cf. [13–24]) are often used as the discretization approximation in time. With this idea, Sun and Zhang [25] introduced the high-order CBVMs for one-dimensional heat equations, Dehghan and Mohebbi [26] studied the high-order CBVMs for unsteady convection-diffusion equations and Liu et al. [27] combined Galerkin–Chebyshev spectral method with BBVMs to derive a class of high-effective MOLs for two-dimensional semi-linear parabolic equations.

The above research show that BVMs are a kind of good candidates in temporal discretization. However, they were not applied to the problem of SLREs. In view of this, in the present paper, we give an investigation to this topic. The paper is organized as follows. In Section 2, by combining a fourth-order compact difference method with BVMs, a class of CBVMs are derived for SLREs (1.1). In Section 3, the local stability and unique solvability of the induced CBVMs are studied and the corresponding criteria are established. In Section 4, the error analysis is performed for the CBVMs and a convergence result is obtained. In Section 5, the CBVMs are applied to the Fisher equation, whose numerical results show that the CBVMs are effective and can arrive at high-precision. In Section 6, the CBVMs are extended successfully to solve the two-component coupled system of SLREs. Finally, in Section 7, a concluding remark is presented to summarize the whole paper and propose some related open problems for the future research.

2. A class of compact boundary value methods

For solving the initial-boundary value problems (1.1), in this section, we will present a class of numerical methods by combining CDMs with BVMs, where the former is for the discretization of spatial variable and the latter for the discretization of temporal variable.

Let $0 = x_0 < x_1 < \cdots < x_m = l$ be a uniform mesh with $x_i = x_0 + ih,\ i = 0, 1, \ldots, m,\ h = l/m$, and $\mathcal{W} = \{\omega_i |\ 0 \le i \le m\}$ be the grid function space defined on $\Omega_m = \{x_i |\ 0 \le i \le m\}$. Write

$$\delta_x^2\omega_i = \frac{1}{h^2}(\omega_{i+1} - 2\omega_i + \omega_{i-1}), \quad \mathcal{K}\omega_i = \frac{1}{12}(\omega_{i+1} + 10\omega_i + \omega_{i-1}), \quad i = 1, \ldots, m-1. \tag{2.1}$$

On the plane $x = x_i$, the Eqs. (1.1) become

$$\frac{\partial u}{\partial t}(x_i, t) = a\frac{\partial^2 u}{\partial x^2}(x_i, t) + g(u(x_i, t)), \quad i = 1, \ldots, m-1.$$

Acting the operator $\mathcal{K}$ on both sides of the above equations yields

$$\mathcal{K}\frac{\partial u}{\partial t}(x_i, t) = a\mathcal{K}\frac{\partial^2 u}{\partial x^2}(x_i, t) + \mathcal{K}g(u(x_i, t)), \quad i = 1, \ldots, m-1. \tag{2.2}$$

For giving a spatial discretization scheme, we introduce a result from Numerov [28] (see also Sun [29]).

Lemma 2.1. *(cf. [28, 29]) Suppose $v(x) \in C^6([x_{i-1}, x_{i+1}])$. Then*

$$\frac{1}{12}[v''(x_{i+1}) + 10v''(x_i) + v''(x_{i-1})] - \frac{1}{h^2}[v(x_{i+1}) - 2v(x_i) + v(x_{i-1})] = \frac{h^4}{240}v^{(6)}(\zeta_i), \quad \zeta_i \in (x_{i-1}, x_{i+1}).$$

It follows from (2.1), (2.2) and Lemma 2.1 that there exist a series of bounded functions $r_i(t)$ $(i = 1, \ldots, m-1)$ on the interval $[t_0, T]$ such that

$$\mathcal{K}\frac{\partial u}{\partial t}(x_i, t) = a\delta_x^2 u(x_i, t) + \mathcal{K}g(u(x_i, t)) + h^4 r_i(t), \quad i = 1, \ldots, m-1. \tag{2.3}$$

A combination of boundary conditions in (1.1) and (2.3) generates that

$$Ky'(t) = aLy(t) + Kf(y(t)) + q(t) + h^4 r(t), \quad t \in [t_0, T], \tag{2.4}$$

where

$$y(t) = (u(x_1, t), u(x_2, t), \ldots, u(x_{m-1}, t))^T, \quad f(y(t)) = (g(u(x_1, t)), g(u(x_2, t)), \ldots, g(u(x_{m-1}, t)))^T,$$

$$K = \frac{1}{12}(10I_{m-1} + S), \quad L = \frac{1}{h^2}(-2I_{m-1} + S), \quad r(t) = (r_1(t), r_2(t), \ldots, r_{m-1}(t))^T,$$

$$q(t)=-\frac{1}{12}\begin{pmatrix}\psi_1'(t)\\0\\\vdots\\0\\\psi_2'(t)\end{pmatrix}+\frac{a}{h^2}\begin{pmatrix}\psi_1(t)\\0\\\vdots\\0\\\psi_2(t)\end{pmatrix}+\frac{1}{12}\begin{pmatrix}g(\psi_1(t))\\0\\\vdots\\0\\g(\psi_2(t))\end{pmatrix},$$

in which I_{m-1} denotes the $(m-1)\times(m-1)$ identity matrix and

$$S=\begin{pmatrix}0&1&&&\\1&0&1&&\\&\ddots&\ddots&\ddots&\\&&1&0&1\\&&&1&0\end{pmatrix}\in\mathbb{R}^{(m-1)\times(m-1)}.$$

Drop the remainder term $h^4r(t)$ in (2.4) and denote the approximation of $y(t)$ by $\bar{y}(t)$, we obtain the following semi-discretization scheme in space:

$$K\bar{y}'(t)=aL\bar{y}(t)+Kf(\bar{y}(t))+q(t),\quad t\in[t_0,T],\tag{2.5}$$

where the initial approximation $\bar{y}(t_0)$ is taken as

$$y_0:=\bar{y}(t_0)=y(t_0)=(\varphi(x_1),\varphi(x_2),\ldots,\varphi(x_{m-1}))^T.\tag{2.6}$$

Next, we consider the discretization scheme in time. Let $t_j=t_0+j\tau$ $(j=0,1,\ldots,N)$ and $\tau=(T-t_0)/N$. Applying the k-step BVMs with k_1 initial conditions and $k_2(=k-k_1)$ final conditions (see e.g. [13–15]) to (2.5) and (2.6) yields the following schemes:

$$\sum_{n=0}^{k}\alpha_n^{(j)}K\bar{y}_n=\tau\sum_{n=0}^{k}\beta_n^{(j)}(aL\bar{y}_n+Kf_n+q_n),\quad j=1,\ldots,k_1-1,\tag{2.7}$$

$$\sum_{n=-k_1}^{k_2}\alpha_{n+k_1}K\bar{y}_{j+n}=\tau\sum_{n=-k_1}^{k_2}\beta_{n+k_1}\left(aL\bar{y}_{j+n}+Kf_{j+n}+q_{j+n}\right),\ \ j=k_1,\ldots,N-k_2,\tag{2.8}$$

$$\sum_{n=0}^{k}\alpha_{k-n}^{(j)}K\bar{y}_{N-n}=\tau\sum_{n=0}^{k}\beta_{k-n}^{(j)}(aL\bar{y}_{N-n}+Kf_{N-n}+q_{N-n}),\ \ j=N-k_2+1,\ldots,N,\tag{2.9}$$

where $\bar{y}_j\approx\bar{y}(t_j)$, $f_j=f(\bar{y}_j)$, $q_j=q(t_j)$ and $\alpha_i^{(j)}$, $\beta_i^{(j)}$, α_i and β_i are some real coefficients such that schemes (2.7)–(2.9) have the same local order. When introducing the following notations:

$$\bar{Y}=\left(\bar{y}_1^T,\bar{y}_2^T,\ldots,\bar{y}_N^T\right)^T,\quad F(\bar{Y})=\left(f_1^T,f_2^T,\ldots,f_N^T\right)^T,\quad Q=\left(q_1^T,q_2^T,\ldots,q_N^T\right)^T,$$

$$b=-a_0\otimes(Ky_0)+\tau ab_0\otimes(Ly_0)+\tau b_0\otimes(Kf_0)+\tau(B\otimes I_{m-1})Q+\tau b_0\otimes q_0,$$

schemes (2.7)–(2.9) can be written in a compact form:

$$(A\otimes K)\bar{Y}=\tau a(B\otimes L)\bar{Y}+\tau(B\otimes K)F(\bar{Y})+b,\tag{2.10}$$

where $\otimes$ denotes the Kronecker product, $A^e:=[a_0\,|\,A]$ is given by

$$A^e:=\left(\begin{array}{c|cccccc}\alpha_0^{(1)}&\alpha_1^{(1)}&\ldots&\alpha_k^{(1)}&&&\\\vdots&\vdots&&\vdots&&&\\\alpha_0^{(k_1-1)}&\alpha_1^{(k_1-1)}&\ldots&\alpha_k^{(k_1-1)}&&&\\\alpha_0&\alpha_1&\ldots&\alpha_k&&&\\&\alpha_0&\alpha_1&\ldots&\alpha_k&&\\&&\ddots&\ddots&&\ddots&\\&&&\alpha_0&\alpha_1&\ldots&\alpha_k\\&&&\alpha_0^{(N-k_2+1)}&\alpha_1^{(N-k_2+1)}&\ldots&\alpha_k^{(N-k_2+1)}\\&&&\vdots&\vdots&&\vdots\\&&&\alpha_0^{(N)}&\alpha_1^{(N)}&\ldots&\alpha_k^{(N)}\end{array}\right)\in\mathbb{R}^{N\times(N+1)},\tag{2.11}$$

and $B^e:=[b_0\,|\,B]$ is defined similarly by replacing α_i and $\alpha_i^{(j)}$ with β_i and $\beta_i^{(j)}$, respectively. So far, we have derived a class of numerical methods for problems (1.1). In the following, we will call this class of methods as *compact boundary value methods*, or CBVMs for short.

3. Local stability and unique solvability of the CBVMs

In this section, we will deal with local stability and unique solvability of the CBVMs (2.10). For this, in the following, we introduce some notations and basic lemmas. For any given vectors $u=(u_1,u_2,\dots,u_{m-1})^T$ and $v=(v_1,v_2,\dots,v_{m-1})^T$ in $\mathbb{R}^{m-1}$, we define an inner product $\langle\cdot,\ \cdot\rangle_h$ and its induced norm $\|\cdot\|_h$ as follows:

$$\langle u,v\rangle_h = h\sum_{i=1}^{m-1}u_iv_i, \quad \|u\|_h=\sqrt{h\sum_{i=1}^{m-1}|u_i|^2}, \tag{3.1}$$

and by which we introduce the corresponding matrix norm $\|\cdot\|_h$ on $\mathbb{R}^{m-1}$:

$$\|M\|_h = \max_{\|u\|_h=1}\|Mu\|_h, \quad \forall M\in\mathbb{R}^{(m-1)\times(m-1)}. \tag{3.2}$$

It is remarkable that

$$\|M\|_h=\|M\|_2, \quad \forall M\in\mathbb{R}^{(m-1)\times(m-1)}, \tag{3.3}$$

since, when setting $\hat{u}=\sqrt{h}u$ ($\forall u\in\mathbb{R}^{m-1}$), it holds that

$$\max_{\|u\|_h=1}\|Mu\|_h = \max_{\sqrt{h}\|u\|_2=1}\sqrt{h}\|Mu\|_2 = \max_{\|\hat{u}\|_2=1}\|M\hat{u}\|_2.$$

Moreover, in the following, we always assume that there exists a constant μ such that function f satisfies that

$$\langle f(u)-f(v),u-v\rangle_h \le \mu\|u-v\|_h^2. \tag{3.4}$$

On the basis of (3.1), we define an inner product $\langle\cdot,\ \cdot\rangle_{h,\,\tau}$ and its induced norm $\|\cdot\|_{h,\,\tau}$ on $\mathbb{R}^{N(m-1)}$:

$$\langle U,V\rangle_{h,\tau} = \tau\sum_{j=1}^{N}\langle U_j,V_j\rangle_h, \quad \|U\|_{h,\tau}=\sqrt{\tau\sum_{j=1}^{N}\|U_j\|_h^2}, \tag{3.5}$$

where $U=\left(U_1^T,U_2^T,\dots,U_N^T\right)^T$, $V=\left(V_1^T,V_2^T,\dots,V_N^T\right)^T\in\mathbb{R}^{N(m-1)}$ are two any given vectors with $U_i,\ V_i\in\mathbb{R}^{m-1}$. Based on the vector norm in (3.5), we also introduce a matrix norm $\|\cdot\|_{h,\,\tau}$ on $\mathbb{R}^{[N(m-1)]\times[N(m-1)]}$:

$$\|\mathcal{M}\|_{h,\tau} = \max_{\|U\|_{h,\tau}=1}\|\mathcal{M}U\|_{h,\tau}, \quad \forall\mathcal{M}\in\mathbb{R}^{[N(m-1)]\times[N(m-1)]}. \tag{3.6}$$

In terms of the following equalities:

$$\max_{\|U\|_{h,\tau}=1}\|\mathcal{M}U\|_{h,\tau} = \max_{\sqrt{h\tau}\|U\|_2=1}\sqrt{h\tau}\|\mathcal{M}U\|_2 = \max_{\|\hat{U}\|_2=1}\|\mathcal{M}\hat{U}\|_2, \quad \forall U\in\mathbb{R}^{N(m-1)} \text{ and } \hat{U}=\sqrt{h\tau}U,$$

we have that

$$\|\mathcal{M}\|_{h,\tau}=\|\mathcal{M}\|_2, \quad \forall\mathcal{M}\in\mathbb{R}^{[N(m-1)]\times[N(m-1)]}. \tag{3.7}$$

Denote by $\tilde{Y}=\left(\tilde{y}_1^T,\tilde{y}_2^T,\dots,\tilde{y}_N^T\right)^T$ the solution of the following perturbed equation with the local perturbation $\delta\in\mathbb{R}^{N(m-1)}$:

$$(A\otimes K)\tilde{Y} = \tau a(B\otimes L)\tilde{Y}+\tau(B\otimes K)F(\tilde{Y})+b+\delta, \tag{3.8}$$

and write that $\Delta V=\tilde{Y}-\bar{Y}$ and $\Delta W=F(\tilde{Y})-F(\bar{Y})$. Then, subtracting (2.10) from (3.8) gives that

$$(A\otimes K)\Delta V = \tau a(B\otimes L)\Delta V+\tau(B\otimes K)\Delta W+\delta. \tag{3.9}$$

Since matrix K is real symmetric positive definite, its inverse matrix K^{-1} exists and is also real symmetric positive definite. Multiplying both sides of (3.9) by $I_N\otimes K^{-1}$ yields that

$$(A\otimes I_{m-1})\Delta V = \tau a[B\otimes(K^{-1}L)]\Delta V+\tau(B\otimes I_{m-1})\Delta W+\eta, \quad \text{where } \eta=(I_N\otimes K^{-1})\delta. \tag{3.10}$$

Based on the above arguments and a similar proof to the Lemma 3.1 in Iavernaro and Mazzia [15], a relation between the perturbations ΔV and ΔW can be derived as follows.

Lemma 3.1. *Suppose that condition (3.4) is fulfilled and $D=\mathrm{diag}(d_1,d_2,\dots,d_N)$ an $N\times N$ positive diagonal matrix. Then the vectors $\Delta V,\Delta W$ in (3.9) satisfy that*

$$\langle\Delta V,(D\otimes I_{m-1})\Delta W\rangle_{h,\tau} \le \mu\langle\Delta V,(D\otimes I_{m-1})\Delta V\rangle_{h,\tau}. \tag{3.11}$$

Moreover, the following Lemma from Li et al. [8] will also paly an important role in our subsequent analysis.

Lemma 3.2. (cf. [8]) *Suppose ξ is an eigenvector of the matrix S with corresponding eigenvalue $\lambda_i^S := 2\cos(i\pi/m)$ ($i=1,2,\dots,m-1$). Then ξ is also an eigenvector of the matrix $K^{-1}L$ with corresponding eigenvalue $\lambda_i^{K^{-1}L} := 12(-2+\lambda_i^S)/[h^2(10+\lambda_i^S)]$ ($i=1,2,\dots,m-1$), where matrices S, K and L are indicated in last section.*

According to Lemma 3.2, we know that $\lambda_i^{K^{-1}L} < 0$ and it holds for $m > 1$ that

$$-\lambda_i^{K^{-1}L} = \frac{12(2-\lambda_i^S)}{h^2(10+\lambda_i^S)} = \frac{24\sin^2(\frac{i\pi}{2m})}{h^2[5+\cos(\frac{i\pi}{m})]} \geq \frac{4}{h^2}\sin^2\left(\frac{\pi}{2m}\right) \geq \frac{4}{h^2}\left(\frac{\pi}{4m}\right)^2 = \frac{\pi^2}{4l^2}, \quad i = 1,2,\ldots,m-1,$$

where we have used inequality $\sin\vartheta \geq \vartheta/2\,(0 \leq \vartheta \leq \pi/3)$ and equality $mh = l$.

In the subsequent analysis, besides the above arguments, we also need introducing a hypothesis:

$\mathcal{H}$: *There exist constants* $\tau_0, h_0 > 0$, *two* $N\times N$ *positive diagonal matrices* D *and* $\tilde{D}$ *and a positive bounded function* $\mathcal{S}(\tau, h)$ *on* $(0,\ \tau_0]\times(0,\ h_0]$ *such that*

$$\lambda_{min}\left(\frac{\mathcal{A}\mathcal{B}^T+\mathcal{B}\mathcal{A}^T}{2} - \frac{\tau a(\mathcal{C}\mathcal{B}^T+\mathcal{B}\mathcal{C}^T)}{2} - \tau\mu\mathcal{B}\mathcal{B}^T\right) \geq \mathcal{S}(\tau,h), \quad \forall(\tau,h)\in(0,\tau_0]\times(0,h_0],$$

where $\mathcal{A} = (DA\tilde{D})\otimes I_{m-1},\ \mathcal{B} = (DB\tilde{D})\otimes I_{m-1},\ \mathcal{C} = (DB\tilde{D})\otimes(K^{-1}L)$, *and* $\lambda_{min}(\cdot)$ *denotes the minimum eigenvalue of a matrix.*

It is remarkable that the conditions analogous to the above hypothesis have also been used in Refs. [15,21,30], where BVMs for ordinary differential equations, Volterra integral and integro-differential equations and fractional differential equations were concerned, respectively. This type of condition was firstly proposed by Iavernaro and Mazzia [15]. Here, we will apply hypothesis $\mathcal{H}$ to derive the corresponding theoretical results of CBVMs (2.10). In the following, we first study the local stability of the methods. A CBVM (2.10) is called *locally stable* if there exists a constant $c > 0$ such that $\|\Delta V\|_{h,\,\tau} \leq c\|\delta\|_{h,\,\tau}$. A criterion for the local stability is stated as follows.

Theorem 3.3. *Suppose that condition (3.4) and hypothesis* $\mathcal{H}$ *are fulfilled. Then the CBVM (2.10) is locally stable with*

$$\|\Delta V\|_{h,\tau} \leq \left[\frac{3\|\tilde{D}\|_2\|\mathcal{B}^T\|_2\|D\|_2}{2\mathcal{S}(\tau,h)}\right]\|\delta\|_{h,\tau}, \quad \forall(\tau,h)\in(0,\tau_0]\times(0,h_0]. \tag{3.12}$$

Proof. Since by hypothesis $\mathcal{H}$ we know that $\mathcal{S}(\tau,h)$ is a positive bounded function on $(0,\ \tau_0]\times(0,\ h_0]$, it suffices to prove the inequality (3.12). Let $\hat{V}$ be a vector satisfying $\mathcal{B}^T\hat{V} = (\tilde{D}^{-1}\otimes I_{m-1})\Delta V$. Then by (3.10) we have that

$$\mathcal{A}\mathcal{B}^T\hat{V} = \tau a\mathcal{C}\mathcal{B}^T\hat{V} + \tau\mathcal{B}(\tilde{D}^{-1}\otimes I_{m-1})\Delta W + (D\otimes I_{m-1})\eta. \tag{3.13}$$

Taking the inner product with $\hat{V}$ on both sides of (3.13) leads to

$$\langle\hat{V},\mathcal{A}\mathcal{B}^T\hat{V}\rangle_{h,\tau} = \tau a\langle\hat{V},\mathcal{C}\mathcal{B}^T\hat{V}\rangle_{h,\tau} + \tau\langle\hat{V},\mathcal{B}(\tilde{D}^{-1}\otimes I_{m-1})\Delta W\rangle_{h,\tau} + \langle\hat{V},(D\otimes I_{m-1})\eta\rangle_{h,\tau}. \tag{3.14}$$

From the usual properties of inner product and Kronecker product, as well as Lemma 3.1, it holds that

$$\langle\hat{V},\mathcal{B}(\tilde{D}^{-1}\otimes I_{m-1})\Delta W\rangle_{h,\tau} = \langle\Delta V,(\tilde{D}^{-2}\otimes I_{m-1})\Delta W\rangle_{h,\tau} \leq \mu\langle\Delta V,(\tilde{D}^{-2}\otimes I_{m-1})\Delta V\rangle_{h,\tau} = \mu\langle\hat{V},\mathcal{B}\mathcal{B}^T\hat{V}\rangle_{h,\tau} \tag{3.15}$$

Moreover, we note the fact that

$$\langle\hat{V},\mathcal{A}\mathcal{B}^T\hat{V}\rangle_{h,\tau} = \left\langle\hat{V},\left(\frac{\mathcal{A}\mathcal{B}^T+\mathcal{B}\mathcal{A}^T}{2}\right)\hat{V}\right\rangle_{h,\tau}, \quad \langle\hat{V},\mathcal{C}\mathcal{B}^T\hat{V}\rangle_{h,\tau} = \left\langle\hat{V},\left(\frac{\mathcal{C}\mathcal{B}^T+\mathcal{B}\mathcal{C}^T}{2}\right)\hat{V}\right\rangle_{h,\tau}. \tag{3.16}$$

Combining (3.14)–(3.16) and applying the Cauchy–Schwartz inequality to the last term of (3.14) yield that

$$\left\langle\hat{V},\left(\frac{\mathcal{A}\mathcal{B}^T+\mathcal{B}\mathcal{A}^T}{2} - \frac{\tau a(\mathcal{C}\mathcal{B}^T+\mathcal{B}\mathcal{C}^T)}{2} - \tau\mu\mathcal{B}\mathcal{B}^T\right)\hat{V}\right\rangle_{h,\tau} \leq \|\hat{V}\|_{h,\tau}\|D\|_2\|\eta\|_{h,\tau}, \tag{3.17}$$

where the following identities have been used:

$$\|D\otimes I_{m-1}\|_{h,\tau} = \|D\otimes I_{m-1}\|_2 = \|D\|_2\|I_{m-1}\|_2 = \|D\|_2.$$

Also, a well-known property of inner product and the condition $\mathcal{H}$ imply that

$$\begin{aligned}&\left\langle\hat{V},\left(\frac{\mathcal{A}\mathcal{B}^T+\mathcal{B}\mathcal{A}^T}{2} - \frac{\tau a(\mathcal{C}\mathcal{B}^T+\mathcal{B}\mathcal{C}^T)}{2} - \tau\mu\mathcal{B}\mathcal{B}^T\right)\hat{V}\right\rangle_{h,\tau}\\ &\geq \lambda_{min}\left(\frac{\mathcal{A}\mathcal{B}^T+\mathcal{B}\mathcal{A}^T}{2} - \frac{\tau a(\mathcal{C}\mathcal{B}^T+\mathcal{B}\mathcal{C}^T)}{2} - \tau\mu\mathcal{B}\mathcal{B}^T\right)\|\hat{V}\|_{h,\tau}^2\\ &\geq \mathcal{S}(\tau,h)\|\hat{V}\|_{h,\tau}^2, \quad \forall(\tau,h)\in(0,\tau_0]\times(0,h_0].\end{aligned} \tag{3.18}$$

A combination of (3.17) and (3.18) gives that

$$\mathcal{S}(\tau,h)\|\hat{V}\|_{h,\tau} \leq \|D\|_2\|\eta\|_{h,\tau}, \quad \forall(\tau,h)\in(0,\tau_0]\times(0,h_0]. \tag{3.19}$$

Moreover, by

$$\lambda_i^K = \frac{10}{12} + \frac{2}{12}\cos\left(\frac{i\pi}{m}\right) = \frac{2}{3} + \frac{1}{6}\left[1 + \cos\left(\frac{i\pi}{m}\right)\right] = \frac{2}{3} + \frac{1}{3}\cos^2\left(\frac{i\pi}{2m}\right) \geq \frac{2}{3}, \quad i = 1, \dots, m-1,$$

we have that

$$\|K^{-1}\|_2 = \sqrt{\rho[(K^{-1})^T K^{-1}]} = \sqrt{\rho(K^{-2})} = \rho(K^{-1}) = \frac{1}{\lambda_{min}(K)} \leq \frac{3}{2}, \tag{3.20}$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix. Therefore, inequality (3.12) is deduced by (3.19) and (3.20) and the following equalities:

$$\Delta V = (\tilde{D} \otimes I_{m-1})\mathcal{B}^T \hat{V}, \quad \eta = (I_N \otimes K^{-1})\delta.$$

This completes the proof. □

On the basis of Theorem 3.3, a unique solvability theorem of CBVMs (2.10) can be stated as follows.

Theorem 3.4. *Suppose that condition (3.4) and hypothesis $\mathcal{H}$ are fulfilled. Then the CBVM (2.10) has a unique solution.*

Proof. Multiplying both sides of (2.10) by $I_N \otimes K^{-1}$ generates that

$$(A \otimes I_{m-1})\bar{Y} = \tau a[B \otimes (K^{-1}L)]\bar{Y} + \tau(B \otimes I_{m-1})F(\bar{Y}) + \hat{b}, \quad \text{where } \hat{b} = (I_N \otimes K^{-1})b. \tag{3.21}$$

Let $\breve{Y}$ be a vector satisfying $(\tilde{D}^{-1} \otimes I_{m-1})\bar{Y} = \mathcal{B}^T\breve{Y}$. Then (3.21) is equivalent to the following equation:

$$\mathcal{A}\mathcal{B}^T\breve{Y} = \tau a\mathcal{C}\mathcal{B}^T\breve{Y} + \tau\mathcal{B}(\tilde{D}^{-1} \otimes I_{m-1})F((\tilde{D} \otimes I_{m-1})\mathcal{B}^T\breve{Y}) + (D \otimes I_{m-1})\hat{b}. \tag{3.22}$$

Define mapping $\Psi : \mathbb{R}^{N(m-1)} \to \mathbb{R}^{N(m-1)}$ as follows:

$$\Psi(\breve{Y}) := \mathcal{A}\mathcal{B}^T\breve{Y} - \tau a\mathcal{C}\mathcal{B}^T\breve{Y} - \tau\mathcal{B}(\tilde{D}^{-1} \otimes I_{m-1})F((\tilde{D} \otimes I_{m-1})\mathcal{B}^T\breve{Y}) - (D \otimes I_{m-1})\hat{b}, \quad \forall\, \breve{Y} \in \mathbb{R}^{N(m-1)}.$$

In terms of the uniform monotonicity theorem (cf. [31, p. 167]), to prove the existence of (2.10)'s solution, it suffices to prove that there exists a constant $c_1 > 0$ such that

$$\left\langle \Psi(\breve{Y}) - \Psi(\breve{Z}), \breve{Y} - \breve{Z} \right\rangle_{h,\tau} \geq c_1 \left\| \breve{Y} - \breve{Z} \right\|_{h,\tau}^2, \quad \forall\, \breve{Y},\ \breve{Z} \in \mathbb{R}^{N(m-1)}. \tag{3.23}$$

In fact, by some common properties of inner product, Lemma 3.1 and the hypothesis $\mathcal{H}$, for all $\breve{Y},\ \breve{Z} \in \mathbb{R}^{N(m-1)}$ and $(\tau, h) \in (0, \tau_0] \times (0, h_0]$, we have that

$$\begin{aligned}
&\left\langle \Psi(\breve{Y}) - \Psi(\breve{Z}), \breve{Y} - \breve{Z} \right\rangle_{h,\tau} \\
&= \left\langle (\mathcal{A}\mathcal{B}^T - \tau a\mathcal{C}\mathcal{B}^T)(\breve{Y} - \breve{Z}), \breve{Y} - \breve{Z} \right\rangle_{h,\tau} - \tau\left\langle \mathcal{B}(\tilde{D}^{-1} \otimes I_{m-1})[F((\tilde{D} \otimes I_{m-1})\mathcal{B}^T\breve{Y}) - F((\tilde{D} \otimes I_{m-1})\mathcal{B}^T\breve{Z})], \breve{Y} - \breve{Z} \right\rangle_{h,\tau} \\
&= \left\langle (\mathcal{A}\mathcal{B}^T - \tau a\mathcal{C}\mathcal{B}^T)(\breve{Y} - \breve{Z}), \breve{Y} - \breve{Z} \right\rangle_{h,\tau} - \tau\left\langle (\tilde{D}^{-2} \otimes I_{m-1})[F((\tilde{D} \otimes I_{m-1})\mathcal{B}^T\breve{Y}) - F((\tilde{D} \otimes I_{m-1})\mathcal{B}^T\breve{Z})], (\tilde{D} \otimes I_{m-1})\mathcal{B}^T(\breve{Y} - \breve{Z}) \right\rangle_{h,\tau} \\
&\geq \left\langle (\mathcal{A}\mathcal{B}^T - \tau a\mathcal{C}\mathcal{B}^T)(\breve{Y} - \breve{Z}), \breve{Y} - \breve{Z} \right\rangle_{h,\tau} - \tau\mu\left\langle (\tilde{D}^{-1} \otimes I_{m-1})\mathcal{B}^T(\breve{Y} - \breve{Z}), (\tilde{D} \otimes I_{m-1})\mathcal{B}^T(\breve{Y} - \breve{Z}) \right\rangle_{h,\tau} \\
&= \left\langle (\mathcal{A}\mathcal{B}^T - \tau a\mathcal{C}\mathcal{B}^T)(\breve{Y} - \breve{Z}), \breve{Y} - \breve{Z} \right\rangle_{h,\tau} - \tau\mu\left\langle \mathcal{B}\mathcal{B}^T(\breve{Y} - \breve{Z}), (\breve{Y} - \breve{Z}) \right\rangle_{h,\tau} \\
&= \left\langle \left(\frac{\mathcal{A}\mathcal{B}^T + \mathcal{B}\mathcal{A}^T}{2} - \frac{\tau a(\mathcal{C}\mathcal{B}^T + \mathcal{B}\mathcal{C}^T)}{2} - \tau\mu\mathcal{B}\mathcal{B}^T \right)(\breve{Y} - \breve{Z}), \breve{Y} - \breve{Z} \right\rangle_{h,\tau} \geq \mathcal{S}(\tau, h)\|\breve{Y} - \breve{Z}\|_{h,\tau}^2.
\end{aligned}$$

Also, we note that $\mathcal{S}(\tau, h)$ is a positive bounded function on $(0, \tau_0] \times (0, h_0]$, thus (3.23) is true. This shows the existence of (2.10)'s solution.

Next, we continue to prove the uniqueness of (2.10)'s solution. Assume that Eq. (2.10) has two solutions $\bar{Y}$ and $\tilde{Y}$, then $\Delta V := \tilde{Y} - \bar{Y}$ satisfies Eq. (3.9) with $\delta = 0$. Therefore, it follows from Theorem 3.3 that $\Delta V = 0$. This implies the uniqueness of (2.10)'s solution. Hence the proof is completed. □

4. Error analysis of the CBVMs

In order to perform the error analysis of CBVMs (2.10), we first introduce several notations as follows:

$$Y = \left(y_1^T, y_2^T, \dots, y_N^T\right)^T, \quad F(Y) = \left(f(y_1)^T, f(y_2)^T, \dots, f(y_N)^T\right)^T, \quad R = \left(r_1^T, r_2^T, \dots, r_N^T\right)^T,$$

where $y_j = y(t_j)$ and $r_j = r(t_j)$. Assume the CBVMs (2.10) are consistent of order p in time, then, by (2.4) and (2.10), there exists a vector $\Gamma = \left(\gamma_1^T, \gamma_2^T, \dots, \gamma_N^T\right)^T$ such that

$$(A \otimes K)Y = \tau a(B \otimes L)Y + \tau(B \otimes K)F(Y) + b + \mathcal{R}(h, \tau), \tag{4.1}$$

where $\mathcal{R}(h, \tau)$ is the local truncation error of method (2.10) and given by

$$\mathcal{R}(h, \tau) = \tau h^4(B \otimes I_{m-1})R + \tau h^4 b_0 \otimes r_0 + \tau^{p+1}\Gamma,$$

which satisfies the following estimation:

$$\|\mathcal{R}(h,\tau)\|_{h,\tau} = \left\|\tau h^4(B\otimes I_{m-1})R + \tau h^4 b_0\otimes r_0 + \tau^{p+1}\Gamma\right\|_{h,\tau} \le c_2(\tau h^4 + \tau^{p+1}), \tag{4.2}$$

with $c_2 = \max\left\{2\|(B\otimes I_{m-1})R\|_{h,\tau},\ 2\|b_0\otimes r_0\|_{h,\tau},\ \|\Gamma\|_{h,\tau}\right\}$.

An estimation of global error $\mathcal{E} := Y - \bar{Y}$ of CBVMs (2.10) can be stated as follows.

Theorem 4.1. *Suppose that condition (3.4) and hypothesis $\mathcal{H}$ are fulfilled and the CBVM (2.10) is consistent of order p in time. Then the global error $\mathcal{E}$ of this method satisfies*

$$\|\mathcal{E}\|_{h,\tau} = \mathcal{O}(h^4 + \tau^p). \tag{4.3}$$

Proof. Subtracting (2.10) from (4.1) yields

$$(A\otimes K)\mathcal{E} = \tau a(B\otimes L)\mathcal{E} + \tau(B\otimes K)[F(Y) - F(\bar{Y})] + \mathcal{R}(h,\tau). \tag{4.4}$$

A direct application of Theorem 3.3 to (4.4) gives that

$$\|\mathcal{E}\|_{h,\tau} \le \left[\frac{3\|\tilde{D}\|_2\|\mathcal{B}^T\|_2\|D\|_2}{2\mathcal{S}(\tau,h)}\right]\|\mathcal{R}(h,\tau)\|_{h,\tau}, \quad \forall(\tau,h)\in(0,\tau_0]\times(0,h_0]. \tag{4.5}$$

Since by hypothesis $\mathcal{H}$ it holds that $\mathcal{S}(\tau,h)$ is a positive bounded function on $(0,\ \tau_0]\times(0,\ h_0]$, there exist constants $c_3 > 0$, $\tau_1\in(0,\ \tau_0]$ and $h_1\in(0,\ h_0]$ such that

$$\mathcal{S}(\tau,h) \ge c_3\tau, \quad \forall(\tau,h)\in(0,\tau_1]\times(0,h_1]. \tag{4.6}$$

Therefore, a combination of (4.2), (4.5) and (4.6) infers (4.3). This completes the proof. □

The above theorem shows that the CBVMs (2.10) have convergence orders 4 and p in space and time, respectively, under the suitable conditions.

5. A numerical example

In this section, for verification of the effectiveness and accuracy of CBVMs (2.10), we present a numerical example. Applying the fourth-order ETR$_2$, fifth-order GAM and sixth-order TOM (cf. [13]) to problems (2.5), respectively, we can obtain the corresponding CBVMs. For convenience, we write these methods as CETR$_2$-4, CGAM-5 and CTOM-6, respectively. Moreover, we will use the following formulas:

$$\mathrm{err}(h,\tau) = \|\mathcal{E}\|_{h,\tau}; \quad p = \log_2\left[\frac{\mathrm{err}(h,\tau)}{\mathrm{err}(h,\tau/2)}\right], \text{ when } h>0 \text{ is small enough,}$$

and

$$p^* = \log_2\left[\frac{\mathrm{err}(h,\tau)}{\mathrm{err}(h/2,\tau)}\right], \quad \text{when } \tau>0 \text{ is small enough,}$$

to compute the CBVMs' global errors, temporal and spatial convergence orders, respectively.

Example 5.1. Consider Fisher equation (see e.g. [5])

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + u(x,t)[1-u(x,t)], \quad (x,t)\in[0,1]\times[0,2], \tag{5.1}$$

whose initial and boundary values are determined by the equation's exact solution

$$u(x,t) = \left[1+\exp\left(-\frac{5}{6}t + \frac{\sqrt{6}}{6}x\right)\right]^{-2}.$$

Applying CETR$_2$-4, CGAM-5 and CTOM-6 with the appropriate stepsizes τ and h to Eq. (5.1), we can obtain a series of effective numerical solutions. In the actual computation, the resulting nonlinear algebraic system by (2.10) is solved by Newton iterative method, where the initial iterative vector is taken as $e\otimes y_0$ with $e=(1,1,\ldots,1)^T\in\mathbb{R}^N$, $y_0 = \left((1+\exp(\sqrt{6}x_1/6))^{-2}, (1+\exp(\sqrt{6}x_2/6))^{-2}, \ldots, (1+\exp(\sqrt{6}x_{m-1}/6))^{-2}\right)^T$, $x_i = ih,\ i=1,\ldots,m-1,\ m=1/h$, and the iterative tolerance is chosen to be 10^{-8}. For simplicity, we only reveal the numerical solution computed by CTOM-6 with $\tau=\frac{1}{20}$ and $h=\frac{1}{200}$ in Fig. 1(a). Comparing this numerical solution with the exact solution in Fig. 1(b), we can see that this numerical solution is a very effective approximation. Next, in order to present a further insight into the effectiveness and accuracy of CBVMs (2.10), we perform two groups of numerical experiments. Firstly, we take stepsizes $h=\frac{1}{200}$ and $\tau=\frac{2}{5\times 2^j}$ $(j=0,1,2,3)$ and apply the above three CBVMs to Eq. (5.1), respectively. The global errors and temporal convergence orders of the methods are displayed in Table 1, which shows that the numerical solutions can arrive at the high-precision and their convergence orders in time are approximately equivalent to the temporal consistence orders of the corresponding CBVMs. This is also shown in Fig. 2. Secondly, we take stepsizes $h=\frac{1}{5\times 2^j}$ $(j=0,1,2,3)$ and $\tau=\frac{1}{160}$ and

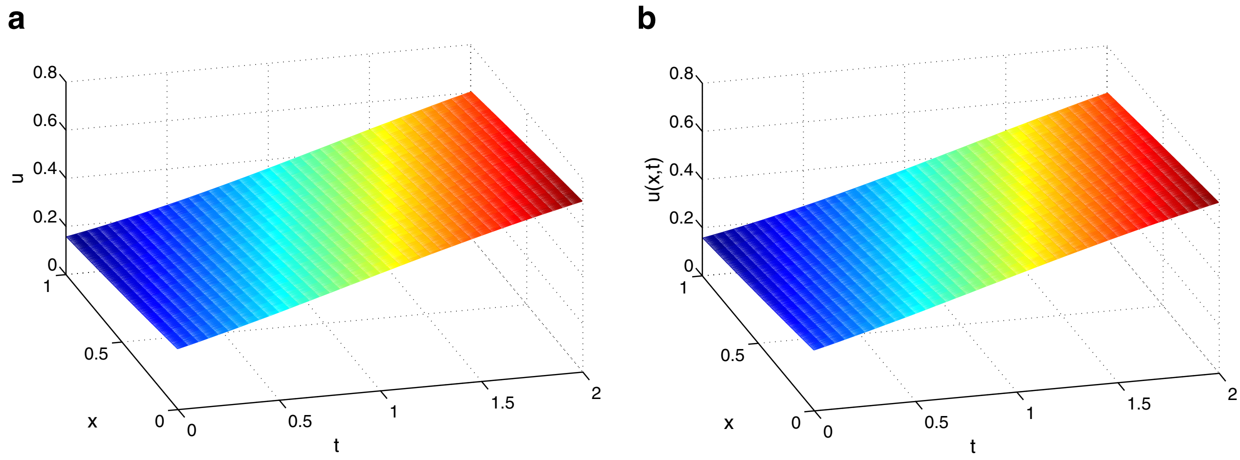

Fig. 1. (a) Numerical solution of Eq. (5.1) by CTOM-6 with $\tau = \frac{1}{20}$ and $h = \frac{1}{200}$; (b) Exact solution of Eq. (5.1).

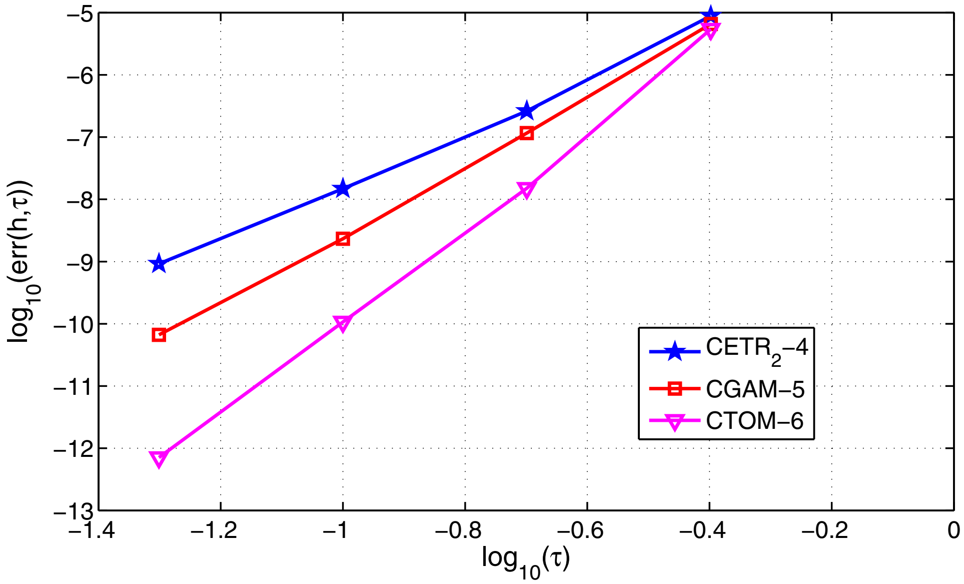

Fig. 2. Global errors versus temporal stepsizes in Log-Log scale of CBVMs for Eq. (5.1).

Table 1
Global errors and temporal convergence orders of CBVMs with $h = \frac{1}{200}$ and $\tau = \frac{2}{5\times 2^j}$ $(j = 0, 1, 2, 3)$ for (5.1).

	$CETR_2$-4		CGAM-5		CTOM-6	
τ	err(h, τ)	p	err(h, τ)	p	err(h, τ)	p
2/5	8.8490e−06	–	6.5792e−06	–	5.4241e−06	–
1/5	2.6145e−07	5.0809	1.1579e−07	5.8283	1.5071e−08	8.4915
1/10	1.4848e−08	4.1382	2.3217e−09	5.6402	1.0684e−10	7.1403
1/20	9.2122e−10	4.0106	6.6425e−11	5.1273	7.0943e−13	7.2345

apply the above three CBVMs to Eq. (5.1), respectively. The global errors and spatial convergence orders of the methods are displayed in Table 2, which shows that the spatial convergence orders of the CBVMs are about four and, again, reveals high-precision of the CBVMs. Moreover, these numerical results also further verify the theoretical precisions stated in Theorem 4.1.

Table 2
Global errors and spatial convergence orders of CBVMs with $h = \frac{1}{5\times 2^j}$ $(j = 0, 1, 2, 3)$ and $\tau = \frac{1}{160}$ for (5.1).

	CETR$_2$-4		CGAM-5		CTOM-6	
h	err(h, τ)	p^*	err(h, τ)	p^*	err(h, τ)	p^*
1/5	2.7408e−09	–	2.7408e−09	–	2.7408e−09	–
1/10	1.7122e−10	4.0007	1.7121e−10	4.0008	1.7121e−10	4.0008
1/20	1.0706e−11	3.9993	1.0696e−11	4.0007	1.0698e−11	4.0004
1/40	7.1076e−13	3.9129	6.6643e−13	4.0044	6.6854e−13	4.0001

6. An extension of CBVMs for the coupled system of SLREs

In previous sections, we have dealt with the CBVMs for the one-component system of SLREs. In fact, the CBVMs also can be extended to solve the multi-component coupled system of SLREs. As an example, we consider extending the CBVMs to solve the following two-component coupled system of SLREs:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = a_1 \dfrac{\partial^2 u}{\partial x^2}(x,t) + g_1(u(x,t), v(x,t)), & (x,t) \in [0,l] \times [t_0, T], \\ \dfrac{\partial v}{\partial t}(x,t) = a_2 \dfrac{\partial^2 v}{\partial x^2}(x,t) + g_2(u(x,t), v(x,t)), & (x,t) \in [0,l] \times [t_0, T], \\ u(x,t_0) = \varphi_1(x), \quad v(x,t_0) = \varphi_2(x), \quad x \in [0,l], \\ u(0,t) = \psi_1(t), \quad u(l,t) = \psi_2(t), \quad v(0,t) = \psi_3(t), \quad v(l,t) = \psi_4(t), \quad t \in [t_0, T], \end{cases} \tag{6.1}$$

where $a_i > 0$ $(i = 1, 2)$ are the diffusion coefficients, $\varphi_i : [0, l] \to \mathbb{R}$ $(i = 1, 2)$, $\psi_i : [t_0, T] \to \mathbb{R}$ $(i = 1, 2, 3, 4)$ and $g_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are some given sufficiently smooth mappings.

Write

$$y(t) = (u(x_1,t), u(x_2,t), \dots, u(x_{m-1},t))^T, \quad z(t) = (v(x_1,t), v(x_2,t), \dots, v(x_{m-1},t))^T,$$

$$y_0 = (\varphi_1(x_1), \varphi_1(x_2), \dots, \varphi_1(x_{m-1}))^T, \quad z_0 = (\varphi_2(x_1), \varphi_2(x_2), \dots, \varphi_2(x_{m-1}))^T,$$

$$\bar{y}_i(t) \approx u(x_i,t), \quad \bar{z}_i(t) \approx v(x_i,t), \quad i = 1, 2, \dots, m-1,$$

$$\bar{y}(t) = (\bar{y}_1(t), \bar{y}_2(t), \dots, \bar{y}_{m-1}(t))^T, \quad \bar{z}(t) = (\bar{z}_1(t), \bar{z}_2(t), \dots, \bar{z}_{m-1}(t))^T,$$

$$f_i(\bar{y}(t), \bar{z}(t)) = (g_i(\bar{y}_1(t), \bar{z}_1(t)), g_i(\bar{y}_2(t), \bar{z}_2(t)), \dots, g_i(\bar{y}_{m-1}(t), \bar{z}_{m-1}(t)))^T, \quad i = 1, 2,$$

$$q_1(t) = -\frac{1}{12}\begin{pmatrix} \psi_1'(t) \\ 0 \\ \vdots \\ 0 \\ \psi_2'(t) \end{pmatrix} + \frac{a_1}{h^2}\begin{pmatrix} \psi_1(t) \\ 0 \\ \vdots \\ 0 \\ \psi_2(t) \end{pmatrix} + \frac{1}{12}\begin{pmatrix} g_1(\psi_1(t), \psi_3(t)) \\ 0 \\ \vdots \\ 0 \\ g_1(\psi_2(t), \psi_4(t)) \end{pmatrix} \in \mathbb{R}^{m-1},$$

$$q_2(t) = -\frac{1}{12}\begin{pmatrix} \psi_3'(t) \\ 0 \\ \vdots \\ 0 \\ \psi_4'(t) \end{pmatrix} + \frac{a_2}{h^2}\begin{pmatrix} \psi_3(t) \\ 0 \\ \vdots \\ 0 \\ \psi_4(t) \end{pmatrix} + \frac{1}{12}\begin{pmatrix} g_2(\psi_1(t), \psi_3(t)) \\ 0 \\ \vdots \\ 0 \\ g_2(\psi_2(t), \psi_4(t)) \end{pmatrix} \in \mathbb{R}^{m-1}.$$

Then, by a similar derivation to (2.5), we can obtain the following semi-discretization scheme for (6.1):

$$\begin{cases} K\bar{y}'(t) = a_1 L\bar{y}(t) + Kf_1(\bar{y}(t), \bar{z}(t)) + q_1(t), & t \in [t_0, T], \\ K\bar{z}'(t) = a_2 L\bar{z}(t) + Kf_2(\bar{y}(t), \bar{z}(t)) + q_2(t), & t \in [t_0, T], \\ \bar{y}(t_0) = y_0, \quad \bar{z}(t_0) = z_0. \end{cases} \tag{6.2}$$

Further, we introduce the following notations:

$$\bar{Y} = \left(\bar{y}_1^T, \bar{y}_2^T, \dots, \bar{y}_N^T\right)^T, \quad \bar{Z} = \left(\bar{z}_1^T, \bar{z}_2^T, \dots, \bar{z}_N^T\right)^T, \quad \bar{y}_j \approx \bar{y}(t_j), \quad \bar{z}_j \approx \bar{z}(t_j), \quad j = 1, 2, \dots, N,$$

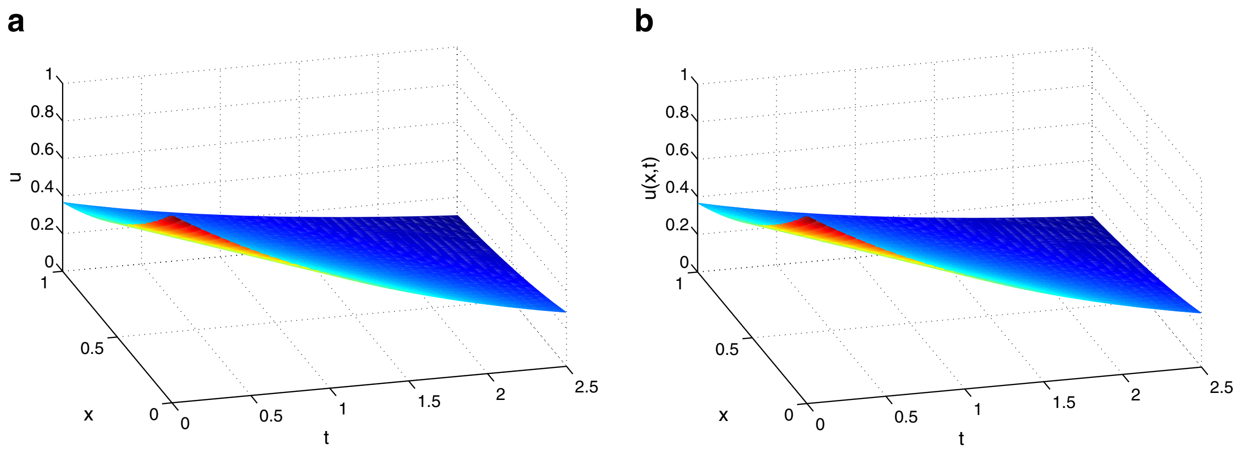

Fig. 3. (a) Numerical solution u of Eq. (6.5) by ECTOM-6 with $\tau=\frac{1}{16}, h=\frac{1}{200}$; (b) Exact solution $u(x,t)$ of (6.5). .

$$F_i(\bar{Y},\bar{Z})=\left(f_i(\bar{y}_1,\bar{z}_1)^T,f_i(\bar{y}_2,\bar{z}_2)^T,\dots,f_i(\bar{y}_N,\bar{z}_N)^T\right)^T,\quad Q_i=\left(q_i(t_1)^T,q_i(t_2)^T,\dots,q_i(t_N)^T\right)^T,\quad i=1,2,$$

$$b_1=-a_0\otimes(Ky_0)+\tau a_1b_0\otimes(Ly_0)+\tau b_0\otimes[Kf_1(y_0,z_0)]+\tau(B\otimes I_{m-1})Q_1+\tau b_0\otimes q_1(t_0),$$
$$b_2=-a_0\otimes(Kz_0)+\tau a_2b_0\otimes(Lz_0)+\tau b_0\otimes[Kf_2(y_0,z_0)]+\tau(B\otimes I_{m-1})Q_2+\tau b_0\otimes q_2(t_0).$$

Extending the BVMs (2.7)–(2.9) for (6.2) yields that

$$\begin{cases}(A\otimes K)\bar{Y}=\tau a_1(B\otimes L)\bar{Y}+\tau(B\otimes K)F_1(\bar{Y},\bar{Z})+b_1,\\(A\otimes K)\bar{Z}=\tau a_2(B\otimes L)\bar{Z}+\tau(B\otimes K)F_2(\bar{Y},\bar{Z})+b_2.\end{cases}\tag{6.3}$$

So far, we have obtained a class of extended CBVMs for (6.1). For short, in the following, we will refer to this class of methods as ECBVMs.

For the computational implementation of ECBVMs (6.3), we may adopt the following iterative algorithm:

$$\begin{cases}(A\otimes K)\bar{Y}^{(s+1)}=\tau a_1(B\otimes L)\bar{Y}^{(s+1)}+\tau(B\otimes K)F_1(\bar{Y}^{(s+1)},\bar{Z}^{(s)})+b_1,\\(A\otimes K)\bar{Z}^{(s+1)}=\tau a_2(B\otimes L)\bar{Z}^{(s+1)}+\tau(B\otimes K)F_2(\bar{Y}^{(s+1)},\bar{Z}^{(s+1)})+b_2,\end{cases}\tag{6.4}$$

where $\bar{Y}^{(s)}$, $\bar{Z}^{(s)}$ $(s=0,1,\dots)$ denote the sth iterative values of $\bar{Y}$ and $\bar{Z}$, respectively, the initial iterative vectors $\bar{Y}^{(0)}$ and $\bar{Z}^{(0)}$ are taken respectively as $e\otimes y_0$ and $e\otimes z_0$, and conditions $\|\bar{Y}^{(s+1)}-\bar{Y}^{(s)}\|_{h,\tau}\le\varepsilon_1$ and $\|\bar{Z}^{(s+1)}-\bar{Z}^{(s)}\|_{h,\tau}\le\varepsilon_2$ are used as stopping criteria of the algorithm, where ε_i $(i=1,2)$ denote the iteration tolerances. Moreover, each nonlinear equation of (6.4) is solved by Newton iterative method, where iterative initial vectors are taken as $e\otimes y_0$ and $e\otimes z_0$, respectively, and iterative tolerances are both chosen to be ε_0. In what follows, for simplicity, we always take $\varepsilon_0=\varepsilon_1=\varepsilon_2=10^{-8}$.

Example 6.1. Consider the following Brusselator system (cf. [32])

$$\begin{cases}\dfrac{\partial u}{\partial t}(x,t)=\dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(x,t)+u^2(x,t)v(x,t)-2u(x,t),\quad(x,t)\in[0,1]\times[0,2.5],\\\dfrac{\partial v}{\partial t}(x,t)=\dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2}(x,t)+u(x,t)-u^2(x,t)v(x,t),\quad(x,t)\in[0,1]\times[0,2.5],\end{cases}\tag{6.5}$$

whose initial and boundary values are determined by the above equations' exact solutions

$$u(x,t)=\exp(-0.5t-x),\qquad v(x,t)=\exp(0.5t+x).$$

For solving (6.5), we extend CETR$_2$-4, CGAM-5 and CTOM-6 as the corresponding ECBVMs and write the extended ECBVMs as ECETR$_2$-4, ECGAM-5 and ECTOM-6, respectively. In this way, when applying these methods with iterative scheme (6.4) to system (6.5), we can obtain a series of efficient numerical solutions. As an example, in Figs. 3(a) and 4(a), we plot the numerical solutions of (6.5) by ECTOM-6 with $\tau=\frac{1}{16}$ and $h=\frac{1}{200}$. Comparing this group of numerical solutions with (6.5)'s exact solutions $u(x,t)$ and $v(x,t)$ displayed respectively in Figs. 3(b) and 4(b), we can find that the numerical solutions are some approximations with high-precision.

In order to give an exacter insight into accuracies of the above methods, we introduce the notations:

$$Y=(y(t_1)^T,y(t_2)^T,\dots,y(t_N)^T)^T,\quad Z=(z(t_1)^T,z(t_2)^T,\dots,z(t_N)^T)^T,\quad\hat{\mathcal{E}}=((Y-\bar{Y})^T,(Z-\bar{Z})^T)^T,$$

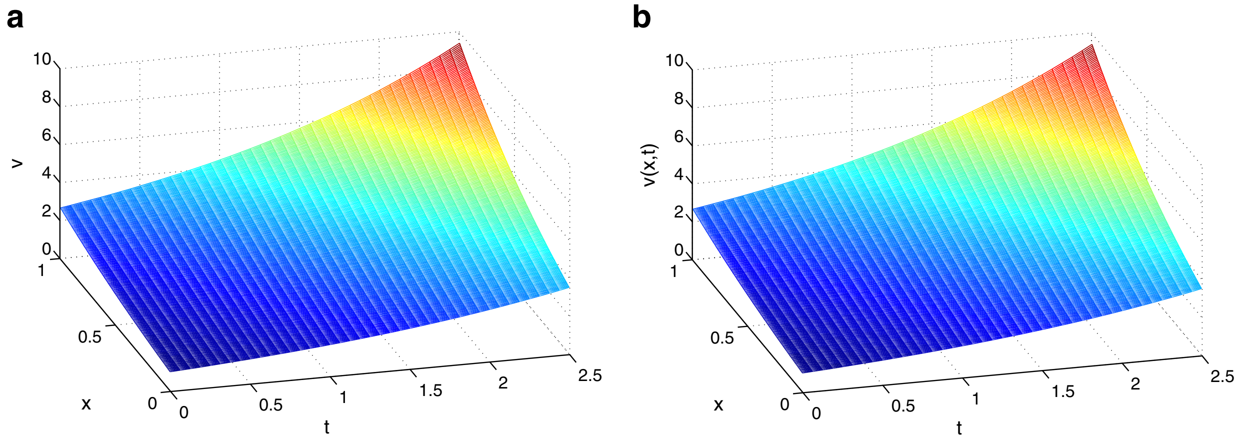

Fig. 4. (a) Numerical solution v of Eq. (6.5) by ECTOM-6 with $\tau=\frac{1}{16}, h=\frac{1}{200}$; (b) Exact solution $v(x,t)$ of (6.5).

Table 3
Global errors and temporal convergence orders of ECBVMs with $h=\frac{1}{200}$ and $\tau=\frac{1}{2^j}$ $(j=1,2,3,4)$ for (6.5).

	ECETR$_2$-4		ECGAM-5		ECTOM-6	
τ	$err_1(h,\tau)$	p_1	$err_1(h,\tau)$	p_1	$err_1(h,\tau)$	p_1
1/2	4.1951e−05	–	8.1751e−06	–	1.7210e−06	–
1/4	1.4130e−06	4.8919	1.5143e−07	5.7546	1.1511e−08	7.2240
1/8	6.1223e−08	4.5285	3.4347e−09	5.4623	7.8808e−11	7.1905
1/16	3.6180e−09	4.0808	1.0088e−10	5.0895	1.0509e−12	6.2287

Table 4
Global errors and spatial convergence orders of ECBVMs with $h=\frac{1}{5\times 2^j}$ $(j=0,1,2,3)$ and $\tau=\frac{1}{100}$ for (6.5).

	ECETR$_2$-4		ECGAM-5		ECTOM-6	
h	$err_1(h,\tau)$	p_1^*	$err_1(h,\tau)$	p_1^*	$err_1(h,\tau)$	p_1^*
1/5	2.9912e−06	–	2.9912e−06	–	2.9912e−06	–
1/10	1.8737e−07	3.9968	1.8737e−07	3.9968	1.8737e−07	3.9968
1/20	1.1717e−08	3.9992	1.1715e−08	3.9995	1.1715e−08	3.9995
1/40	7.3458e−10	3.9956	7.3222e−10	3.9999	7.3223e−10	3.9999

and use the formulas:

$$err_1(h,\tau)=\|\hat{\mathcal{E}}\|_{h,\tau};\quad p_1=\log_2\left[\frac{err_1(h,\tau)}{err_1(h,\tau/2)}\right],\quad \text{when } h>0 \text{ is small enough,}$$

and

$$p_1^*=\log_2\left[\frac{err_1(h,\tau)}{err_1(h/2,\tau)}\right],\quad \text{when } \tau>0 \text{ is small enough,}$$

to denote the global errors and temporal and spatial convergence orders of the methods, respectively. For testing the temporal convergence orders of the methods, we take stepsizes $h=\frac{1}{200}$ and $\tau=\frac{1}{2^j}$ $(j=1,2,3,4)$ and then apply the above three ECBVMs to (6.5). The global errors and temporal convergence orders of the methods are listed in Table 3, which shows that the methods can arrive at high-precision and the temporal convergence orders are consistent to the corresponding consistence orders of the methods. This is also reflected in Fig. 5. Next, we test the spatial convergence orders of the methods. Taking $h=\frac{1}{5\times 2^j}$ $(j=0,1,2,3)$ and $\tau=\frac{1}{100}$ and then applying the above methods to (6.5), respectively. The obtained numerical results are displayed in Table 4, which verifies the effectiveness of the methods again and shows that the three ECBVMs have the common spatial convergence order 4. Namely, their spatial convergence orders are in accord with the corresponding local order in space.

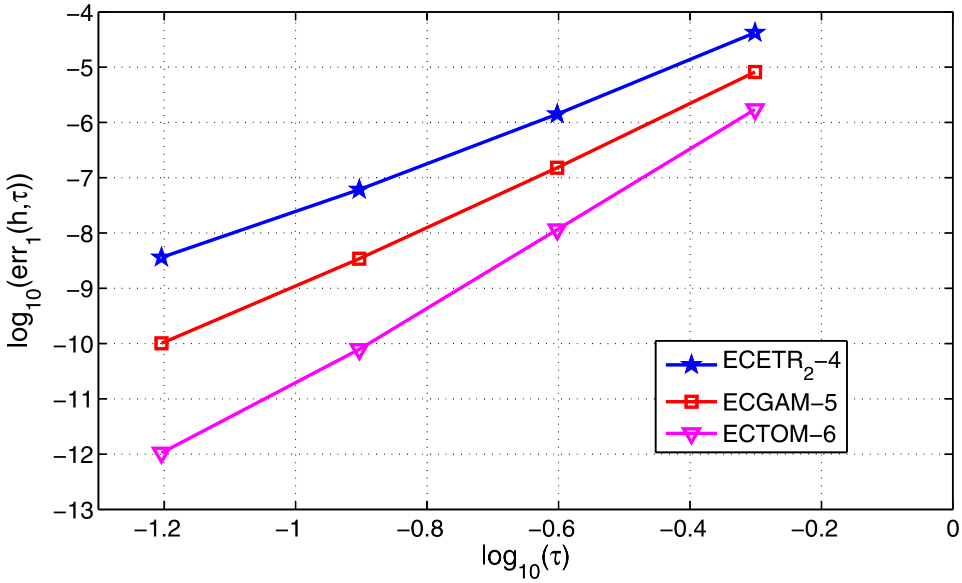

Fig. 5. Global errors versus temporal stepsizes in Log-Log scale of ECBVMs for (6.5).

7. Concluding remark

In this paper, we have dealt with a class of CBVMs for solving the one-component system of SLREs. The CBVMs were obtained by combining a fourth-order compact difference method with BVMs. Under some suitable conditions, the criteria on local stability, unique solvability and convergence of the methods were derived. Numerical experiment further verified the computational effectiveness and high-precision of the CBVMs. Moreover, we also extended the CBVMs to solve the two-component coupled system of SLREs. As to the related numerical analysis of the ECBVMs, at present, it still keeps open due to the complexity of the multi-component coupled system of SLREs and the lack of some analytical arguments. However, no doubt, this issue is important and interesting. Hence, in the future, we will continue to work for this topic.

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