Advanced Artificial Intelligence

Assignment 1

2.6.4

(i) $A \to B, A \vdash B$

Proof. The proof is as follow.

$$\begin{array}{lll} (1) & A \to B,\ A \vdash A \to B & \text{(from } (\in)) \\ (2) & A \to B,\ A \vdash A & \text{(from } (\in)) \\ (3) & A \to B,\ A \vdash B & \text{(from } (\to -),\ (1),\ (2)) \end{array}$$

□

(ii) $A \vdash B \to A$

Proof. The proof is as follow.

$$\begin{array}{lll} (1) & A,\ B \vdash A & \text{(from } (\in)) \\ (2) & A \vdash B \to A & \text{(from } (\to +),\ (1)) \end{array}$$

□

(iv) $A \to (B \to C),\ A \to B \vdash A \to C$

Proof. The proof is as follow.

$$
\begin{array}{lll}
(1) & A \to B,\ A \vdash B & (\text{from } 2.6.4(\text{i}),\ A \Rightarrow A,\ B \Rightarrow B) \\
(2) & A \to (B \to C),\ A \vdash B \to C & (\text{from } 2.6.4(\text{i}),\ A \Rightarrow A,\ B \Rightarrow (B \to C)) \\
(3) & A \to (B \to C),\ A \to B,\ A \vdash B & (\text{from } (+),\ (1)) \\
(4) & A \to (B \to C),\ A \to B,\ A \vdash B \to C & (\text{from } (+),\ (2)) \\
(5) & A \to (B \to C),\ A \to B,\ A \vdash C & (\text{from } (\to -),\ (3),\ (4))
\end{array}
$$

□

2.6.9

(i) $A \vdash A \vee B,\ B \vee A$

Proof. The proof is as follow.

$$
\begin{array}{lll}
(1) & A \vdash A & (\text{from } (\in)) \\
(2) & A \vdash A \vee B, B \vee A & (\text{from } (\vee +),\ (1))
\end{array}
$$

□

(ii) $A \vee B \vdash\dashv B \vee A$

Proof. The proof for $A \vee B \vdash B \vee A$ is as follow.

$$
\begin{array}{lll}
(1) & A \vdash B \vee A & (\text{from } 2.6.9(\text{i}),\ A \Rightarrow A,\ B \Rightarrow B, \text{term } 2) \\
(2) & B \vdash B \vee A & (\text{from } 2.6.9(\text{i}),\ A \Rightarrow B,\ B \Rightarrow A, \text{term } 1) \\
(3) & A \vee B \vdash B \vee A & (\text{from } (\vee -),\ (1),\ (2))
\end{array}
$$

When we do substitution $A \Rightarrow B,\ B \Rightarrow A$, we will have $B \vee A \vdash A \vee B$, therefore we have proved the other side. □

(iii) $A \vee (B \vee C) \vdash\dashv (A \vee B) \vee C$

Proof. Before proving the quality, we will first prove a lemma:

(transitivity): if $A \vdash B,\ \ B \vdash C$, then $A \vdash C$.

The proof is as follow.

$$
\begin{array}{lrl}
(1) & A \vdash B & \text{(given)} \\
(2) & B \vdash C & \text{(given)} \\
(3) & A, B \vdash C & \text{(from } (+),\ (2)) \\
(4) & A \vdash B \to C & \text{(from } (\to +),\ (3)) \\
(5) & A \vdash C & \text{(from } (\to -),\ (1),\ (4))
\end{array}
$$

Then, the proof for $A \vee (B \vee C) \vdash (A \vee B) \vee C$ can be generated as follow.

$$
\begin{array}{lrl}
(1) & A \vee B \vdash (A \vee B) \vee C & \text{(from 2.6.9(i), } A \Rightarrow (A \vee B),\ B \Rightarrow C) \\
(2) & C \vdash (A \vee B) \vee C & \text{(from 2.6.9(i), } A \Rightarrow C,\ B \Rightarrow (A \vee B)) \\
(3) & A \vdash A \vee B & \text{(from 2.6.9(i), } A \Rightarrow A,\ B \Rightarrow B) \\
(4) & B \vdash A \vee B & \text{(from 2.6.9(i), } A \Rightarrow B,\ B \Rightarrow A) \\
(5) & A \vdash (A \vee B) \vee C & \text{(from (transitivity), (1), (3))} \\
(6) & B \vdash (A \vee B) \vee C & \text{(from (transitivity), (1), (4))} \\
(7) & B \vee C \vdash (A \vee B) \vee C & \text{(from } (\vee -),\ (6),\ (2)) \\
(8) & A \vee (B \vee C) \vdash (A \vee B) \vee C & \text{(from } (\vee -),\ (3),\ (7))
\end{array}
$$

And the proof for $(A \vee B) \vee C \vdash A \vee (B \vee C)$ can be generated as follow.

$$
\begin{array}{lrl}
(1) & B \vee C \vdash A \vee (B \vee C) & \text{(from 2.6.9(i), } A \Rightarrow (B \vee C),\ B \Rightarrow A) \\
(2) & A \vdash A \vee (B \vee C) & \text{(from 2.6.9(i), } A \Rightarrow A,\ B \Rightarrow (B \vee C)) \\
(3) & B \vdash B \vee C & \text{(from 2.6.9(i), } A \Rightarrow B,\ B \Rightarrow C) \\
(4) & C \vdash B \vee C & \text{(from 2.6.9(i), } A \Rightarrow C,\ B \Rightarrow B) \\
(5) & B \vdash A \vee (B \vee C) & \text{(from (transitivity), (1), (3))} \\
(6) & C \vdash A \vee (B \vee C) & \text{(from (transitivity), (1), (4))} \\
(7) & A \vee B \vdash A \vee (B \vee C) & \text{(from } (\vee -),\ (2),\ (5)) \\
(8) & (A \vee B) \vee C \vdash A \vee (B \vee C) & \text{(from } (\vee -),\ (7),\ (4))
\end{array}
$$

□

(iv) $A \vee B \dashv\vdash \neg A \to B$

Proof. The proof for $A \vee B \vdash \neg A \to B$ is as follow.

$$
\begin{array}{lrl}
(1) & A,\ \neg A,\ \neg B\ \vdash A & \text{(from } (\in)) \\
(2) & A,\ \neg A,\ \neg B\ \vdash \neg A & \text{(from } (\in)) \\
(3) & A,\ \neg A\ \vdash B & \text{(from } (\neg -),\ (1),\ (2)) \\
(4) & A\ \vdash \neg A \to B & \text{(from } (\to +), (3)) \\
(5) & B\ \vdash \neg A \to B & \text{(from 2.6.4 (ii))} \\
(6) & A \vee B\ \vdash \neg A \to B & \text{(from } (\vee -),\ (4),\ (5))
\end{array}
$$

Before proving $\neg A \to B \vdash A \vee B$, we will first prove a lemma:

$$\text{(contraposition): if } A \vdash B\text{, then } \neg B \vdash \neg A.$$

The proof is as follow.

$$
\begin{array}{lrl}
(1) & A\ \vdash B & \text{(given)} \\
(2) & \varnothing\ \vdash A \to B & \text{(from } (\to +),\ (1)) \\
(3) & A,\ \neg B\ \vdash A \to B & \text{(from } (+),\ (2)) \\
(4) & A,\ \neg B\ \vdash A & \text{(from } (\in)) \\
(5) & A,\ \neg B\ \vdash \neg B & \text{(from } (\in)) \\
(6) & A,\ \neg B\ \vdash B & \text{(from } (\to -),\ (3),\ (4)) \\
(7) & \neg B\ \vdash \neg A & \text{(from } (\neg -),\ (5),\ (6))
\end{array}
$$

Then, the proof for $\neg A \to B \vdash A \vee B$ can be generated as follow.

$$
\begin{array}{lrl}
(1) & A\ \vdash A \vee B & \text{(from 2.6.9(i), } A \Rightarrow A,\ B \Rightarrow B) \\
(2) & \neg(A \vee B)\ \vdash \neg A & \text{(from (contraposition), (1))} \\
(3) & \neg(A \vee B)\ \vdash \neg B & \text{(similar to (2))} \\
(4) & \neg A \to B,\ \neg(A \vee B)\ \vdash \neg A & \text{(from } (+),\ (2)) \\
(5) & \neg A \to B,\ \neg(A \vee B)\ \vdash \neg B & \text{(from } (+),\ (3)) \\
(6) & \neg A \to B,\ \neg(A \vee B)\ \vdash \neg A \to B & \text{(from } (\in)) \\
(7) & \neg A \to B,\ \neg(A \vee B)\ \vdash B & \text{(from } (\to -),\ (4),\ (6)) \\
(8) & \neg A \to B\ \vdash A \vee B & \text{(from } (\neg -),\ (5),\ (7))
\end{array}
$$

The line (2) and (3) can be picked out as a theorem, which we will use many times afterwards:

$$(\neg\vee)\text{: } \neg(A \vee B) \vdash \neg A,\ \neg B.$$

□

(v) $A \to B \vdash\dashv \neg A \vee B$

Proof. The proof for $\neg A \vee B \vdash A \to B$ is as follow.

$$
\begin{array}{lrl}
(1) & A,\ \neg A,\ \neg B\ \vdash A & \text{(from } (\in)) \\
(2) & A,\ \neg A,\ \neg B\ \vdash \neg A & \text{(from } (\in)) \\
(3) & A,\ \neg A\ \vdash B & \text{(from } (\neg -),\ (1),\ (2)) \\
(4) & \neg A\ \vdash A \to B & \text{(from } (\to +),\ (3)) \\
(5) & B\ \vdash A \to B & \text{(from 2.6.4 (ii))} \\
(6) & \neg A \vee B\ \vdash A \to B & \text{(from } (\vee -),\ (4),\ (5))
\end{array}
$$

Before proving $A \to B \vdash \neg A \vee B$, we will first prove a lemma:

$$(\neg\neg):\ A \vdash\dashv \neg\neg A.$$

The proof is as follow.

$$
\begin{array}{lrl}
(1) & \neg\neg A,\ \neg A\ \vdash \neg A & \text{(from } \in) \\
(2) & \neg\neg A,\ \neg A\ \vdash \neg\neg A & \text{(from } \in) \\
(3) & \neg\neg A\ \vdash A & \text{(from } \neg -,\ (1),\ (2)) \\
(4) & \neg\neg\neg A,\ A\ \vdash A & \text{(from } \in) \\
(5) & \neg\neg\neg A,\ A\ \vdash \neg\neg\neg A & \text{(from } \in) \\
(6) & \neg\neg\neg A\ \vdash \neg A & \text{(from (3), } A \Rightarrow \neg A) \\
(7) & \neg\neg\neg A,\ A\ \vdash \neg A & \text{(from (transitivity), (5), (6))} \\
(8) & A\ \vdash \neg\neg A & \text{(from } \neg -,\ (4),\ (7))
\end{array}
$$

The proof for $A \to B \vdash \neg A \vee B$ can be generated as follow.

$$
\begin{array}{rrll}
(1) & \neg A & \vdash \neg A \vee B & \text{(from 2.6.9(i), } A \Rightarrow \neg A,\ B \Rightarrow B) \\
(2) & \neg(\neg A \vee B) & \vdash \neg\neg A & \text{(from (contraposition), (1); or } (\neg\vee)) \\
(3) & \neg(\neg A \vee B) & \vdash \neg B & \text{(similar to (2))} \\
(4) & A \to B,\ \neg(\neg A \vee B) & \vdash \neg\neg A & \text{(from (+), (2))} \\
(5) & A \to B,\ \neg(\neg A \vee B) & \vdash \neg B & \text{(from (+), (3))} \\
(6) & \neg\neg A & \vdash A & \text{(from } (\neg\neg)) \\
(7) & A \to B,\ \neg(\neg A \vee B) & \vdash A & \text{(from (transitivity), (4), (6))} \\
(8) & A \to B,\ \neg(\neg A \vee B) & \vdash A \to B & \text{(from } (\in)) \\
(9) & A \to B,\ \neg(\neg A \vee B) & \vdash B & \text{(from } (\to -),\ (6),\ (8)) \\
(10) & \neg A \to B & \vdash A \vee B & \text{(from } (\neg -),\ (5),\ (9))
\end{array}
$$

□

(vi) $\neg(A \vee B) \vdash\dashv \neg A \wedge \neg B$

Proof. The proof for $\neg(A \vee B) \vdash \neg A \wedge \neg B$ is as follow.

$$
\begin{array}{rlll}
(1) & \neg(A \vee B) & \vdash \neg A & \text{(from } (\neg\vee)) \\
(2) & \neg(A \vee B) & \vdash \neg B & \text{(from } (\neg\vee)) \\
(3) & \neg(A \vee B) & \vdash \neg A \wedge \neg B & \text{(from } (\wedge+),\ (1),\ (2))
\end{array}
$$

The proof for $\neg A \wedge \neg B \vdash \neg(A \vee B)$ is as follow.

$$
\begin{array}{rrll}
(1) & \neg\neg(A \vee B) & \vdash A \vee B & \text{(from } (\neg\neg)) \\
(2) & \neg A,\ \neg B,\ \neg\neg(A \vee B) & \vdash A \vee B & \text{(from (+), (1))} \\
(3) & A \vee B & \vdash \neg A \to B & \text{(from 2.6.9(v))} \\
(4) & \neg A,\ \neg B,\ \neg\neg(A \vee B) & \vdash \neg A \to B & \text{(from (transitivity), (2), (3))} \\
(5) & \neg A,\ \neg B,\ \neg\neg(A \vee B) & \vdash \neg A & \text{(from } (\in)) \\
(6) & \neg A,\ \neg B,\ \neg\neg(A \vee B) & \vdash B & \text{(from } (\to -),\ (4),\ (5)) \\
(7) & \neg A,\ \neg B,\ \neg\neg(A \vee B) & \vdash \neg B & \text{(from } (\in)) \\
(8) & \neg A,\ \neg B & \vdash \neg(A \vee B) & \text{(from } (\neg -),\ (6),\ (7)) \\
(9) & \neg A & \vdash \neg B \to \neg(A \vee B) & \text{(from } (\to +),\ (8)) \\
(10) & \varnothing & \vdash \neg A \to (\neg B \to \neg(A \vee B)) & \text{(from } (\to +),\ (9)) \\
(11) & \neg A \wedge \neg B & \vdash \neg A \to (\neg B \to \neg(A \vee B)) & \text{(from (+), (10))} \\
(12) & \neg A \wedge \neg B & \vdash \neg A \wedge \neg B & \text{(from (Ref))} \\
(13) & \neg A \wedge \neg B & \vdash \neg A & \text{(from } (\wedge -),\ (12))
\end{array}
$$

$$
\begin{array}{lll}
(14) & \neg A \wedge \neg B \ \vdash \neg B & \text{(from } (\wedge-),\ (12)) \\
(15) & \neg A \wedge \neg B \ \vdash \neg B \rightarrow \neg(A \vee B) & \text{(from } (\rightarrow -),\ (11),\ (13)) \\
(16) & \neg A \wedge \neg B \ \vdash \neg(A \vee B) & \text{(from } (\rightarrow -),\ (14),\ (15))
\end{array}
$$

□

(vii) $\neg(A \wedge B) \vdash\dashv \neg A \vee \neg B$

Proof. Similar to (vi), the proof for $\neg(A \wedge B) \vdash \neg A \vee \neg B$ is as follow.

$$
\begin{array}{lrll}
(1) & \neg(\neg A \vee \neg B) & \vdash \neg\neg A & \text{(from } (\neg\vee)) \\
(2) & \neg\neg A & \vdash A & \text{(from } (\neg\neg)) \\
(3) & \neg(\neg A \vee \neg B) & \vdash A & \text{(from (transitivity), (1), (2))} \\
(4) & \neg(\neg A \vee \neg B) & \vdash B & \text{(similar to (3))} \\
(5) & \neg(\neg A \vee \neg B) & \vdash A \wedge B & \text{(from } (\wedge+),\ (3),\ (4)) \\
(6) & \neg(A \wedge B) & \vdash \neg\neg(\neg A \vee \neg B) & \text{(from (contraposition), (5))} \\
(7) & \neg\neg(\neg A \vee \neg B) & \vdash (\neg A \vee \neg B) & \text{(from } (\neg\neg)) \\
(8) & \neg(A \wedge B) & \vdash (\neg A \vee \neg B) & \text{(from (transitivity), (6), (7))}
\end{array}
$$

The proof for $\neg A \vee \neg B \vdash \neg(A \wedge B)$ is as follow.

$$
\begin{array}{lrll}
(1) & \neg\neg(\neg A \vee \neg B) & \vdash \neg A \vee \neg B & \text{(from } (\neg\neg)) \\
(2) & A,\ B,\ \neg\neg(\neg A \vee \neg B) & \vdash \neg A \vee \neg B & \text{(from } (+),\ (1)\ ) \\
(3) & \neg A \vee \neg B & \vdash \neg\neg A \to \neg B & \text{(from 2.6.9(v))} \\
(4) & A,\ B,\ \neg\neg(\neg A \vee \neg B) & \vdash \neg\neg A \to \neg B & \text{(from (transitivity), (2), (3))} \\
(5) & A,\ B,\ \neg\neg(\neg A \vee \neg B) & \vdash A & \text{(from } (\in)) \\
(6) & A & \vdash \neg\neg A & \text{(from } (\neg\neg)) \\
(7) & A,\ B,\ \neg\neg(\neg A \vee \neg B) & \vdash \neg\neg A & \text{(from (transitivity), (5), (6))} \\
(8) & A,\ B,\ \neg\neg(\neg A \vee \neg B) & \vdash \neg B & \text{(from } (\to -),\ (4),\ (7)) \\
(9) & A,\ B,\ \neg\neg(\neg A \vee \neg B) & \vdash B & \text{(from } (\in)) \\
(10) & A,\ B & \vdash \neg(\neg A \vee \neg B) & \text{(from } (\neg -),\ (8),\ (9)) \\
(11) & A & \vdash B \to \neg(\neg A \vee \neg B) & \text{(from } (\to +),\ (10)) \\
(12) & \varnothing & \vdash A \to (B \to \neg(\neg A \vee \neg B)) & \text{(from } (\to +),\ (11)) \\
(13) & A \wedge B & \vdash A \to (B \to \neg(\neg A \vee \neg B)) & \text{(from } (+),\ (12)) \\
(14) & A \wedge B & \vdash A \wedge B & \text{(from (Ref))} \\
(15) & A \wedge B & \vdash A & \text{(from } (\wedge -),\ (14)) \\
(16) & A \wedge B & \vdash B & \text{(from } (\wedge -),\ (14)) \\
(17) & A \wedge B & \vdash B \to \neg(\neg A \vee \neg B) & \text{(from } (\to -),\ (13),\ (15)) \\
(18) & A \wedge B & \vdash \neg(\neg A \vee \neg B) & \text{(from } (\to -),\ (16),\ (17)) \\
(19) & \neg\neg(\neg A \vee \neg B) & \vdash \neg(A \wedge B) & \text{(from (contraposition), (18))} \\
(20) & \neg A \vee \neg B & \vdash \neg\neg(\neg A \vee \neg B) & \text{(from } (\neg\neg)) \\
(21) & \neg A \vee \neg B & \vdash \neg(A \wedge B) & \text{(from (transitivity), (19), (20))}
\end{array}
$$

□

(viii) $\varnothing \vdash A \vee \neg A$

Proof. The proof is as follow.

$$
\begin{array}{lrll}
(1) & \neg(A \vee \neg A) & \vdash \neg A & \text{(from } (\neg\vee)) \\
(2) & \neg(A \vee \neg A) & \vdash \neg\neg A & \text{(from } (\neg\vee)) \\
(3) & \varnothing & \vdash A \vee \neg A & \text{(from } (\neg -),\ (1),\ (2))
\end{array}
$$

□