Math 407: Probability Theory
5/10/2013 - Final exam (11am - 1pm)

Name:

USC ID:

Signature:

1. Write your name and ID number in the spaces above.
2. Show all your work and circle your final answer. Simplify as much as you can your answer. ANSWERS WITHOUT EXPLANATION WILL NOT RECEIVE CREDIT.
3. No books, no cell phone or other notes are permitted.
4. Calculators ARE permitted
5. There are 4 Problems, total grade is on 50 points. Please note that Problem 4 is on two pages.
6. Open your exam only when you are instructed to do so.

Problem 1:	/ ?? points	Problem 5:	/ ?? points
Problem 2:	/ ?? points	Problem 6:	/ ?? points
Problem 3:	/ ?? points	Problem 7:	/ ?? points
Problem 4:	/ ?? points	Problem 8:	/ ?? points

Total: / 100 points

Problem 1:

There are 4 black balls, and 4 white ones in an urn. You pick 4 balls.

1. Compute the probability mass function of the number of black balls you get.
2. You win \$2 per black ball you get, and an extra \$20 if you get all 4 black balls. What is your expected gain?

Problem 2:

There is an epidemy of A-flu: a person has a probability $1/100$ to have the disease. The authorities decide to test the population, but the test is not completely reliable: the probability of giving negative for a sick person is .01, and the probability of giving positive if the person is not sick is .02.

1. You are tested positive. What is the probability that you actually have the disease?
2. You decide to take another test. It is positive again. What is the probability that you have the disease?

Problem 3:

A ladder of length one is put against a wall. The distance between the wall and the bottom of the ladder is X, a uniform random variable in $[0,1]$. We denote Θ the angle between the vertical and the ladder.

1. Make a drawing to explain why $\sin\Theta = X$.
2. The ladder is stable if the angle is less than $\pi/6$. What is the probability that the ladder is stable?
3. Find the c.d.f. of Θ, $F_\Theta(t)$, for $t \in [0, \pi/2]$.
4. Deduce the p.d.f. of Θ.

Problem 4: You have two components, whose lifetimes are X and Y, two independent random variables with rate λ. You put the two components in series, meaning that your system works as long as both components work. Then, your system stop working at some random time T, with $T = \min(X, Y)$.

1. Compute $P(X > t)$, for any $t \geq 0$.
2. Compute the p.d.f. of T (Hint: compute $P(T > t) = P(X > t, Y > t)$).
3. Your remaining component has a remaining lifetime U (that is $U = \max(X, Y) - T$). Compute the joint p.d.f. of U and T, $F_{U,T}(u, t)$ for $t, u \geq 0$. (Hint: compute $P(U \leq u, T > t) = P(t < X \leq t + u, t < Y \leq t + u)$)
4. Are U and T independent?

Problem 5: The cost of an accident (in thousands of dollars) is X, an exponential R.V., of rate 2.

1. What is the average cost of an accident?
2. The paperwork has also a cost: for an accident costing X, additional $\sqrt{X}$ is spent on paperwork. What is the average additional paperwork cost of an accident?
3. A client has a chance $1/4$ to get an accident the coming year. If the client has no accident, 100 are spent in paperwork, and if the client has an accident, $100 + X + \sqrt{X}$ are spent. What is the average amount of money spent per client in one year?

Problem 6:

Let X and Y be two independent random variable, with respective p.d.f. $f_X(x) = xe^{-x}$ for $x \geq 0$, and $f_Y(y) = 2e^{-2y}$ for $y \geq 0$.

1. Write the joint probability density function of X, Y, $f_{X,Y}(x, y)$.
2. Compute the probability $P(X - Y > 2)$.
3. We consider a disk of diameter $|X - Y|$. What is the expected area of the disk?

Problem 7:

You send a signal s (an integer number) to one of your friend. The signal received is $s + X$, where X is a normal r.v. with mean 0 and Variance 1/2.

1. What is the probability that the signal received is in $[s - 1/2, s + 1/2]$, so that your friend won't misunderstand it.
2. The friend sends you back the signal. What you receive is then $s + X + Y$, where Y is another normal r.v. with mean 0 and Variance 1, independent of X.
3. What is the distribution of the signal that you receive?
4. What is the probability that you don't receive the same signal that you sent?

Problem 8:

We have m men, m women, and we make pairs at random (a man can be with another man, and a woman with another woman). There are m pairs.

1. What is the probability that there are only couples of opposite sex?
2. Let X be the number of couples of opposite sex. Compute $E[X]$. (Hint: use X_i, the indicator function that pair i is of opposite sex)
3. Compute $Var(X)$.

Problem 9:

You have N components. Let $X = \sum_{i=1}^{N} X_i$, where the X_i are independent exponential random variables, of parameter 1.

1. Show that the moment generating function of X_1 is $E[e^{tX_1}] = \frac{1}{1-t}$.
2. Compute the moment generating function of X, $M(t) = E[e^{tX}]$.
3. Show that, for any $a > 0$, $P(X > a) \leq e^{-at}M(t)$. (This is known as Chernoff's bound). HINT: Use that $e^{tX} \geq I_a e^{at}$ for every $t \geq 0$, where I_a is the indicator function of the event $X > a$.
4. The previous question shows that $P(X > aN) \leq \exp(-N(at + \ln(1-t)))$ for all $t \in [0, 1]$. Optimize this bound in t (Hint: $h(t) = at + \ln(1-t)$ is maximal for $t = 1 - 1/a$).
5. Give an upper bound of $P(X > 200)$.
6. Use the Central Limit Theorem to get the same result.

Problem 10:

You flip 100 times a biased coin, whose probability to get H is p, which is unknown

You call $\bar{X}$ the estimated parameter that you get, that is $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$, where X_i is the indicator function of having H.

1. Show that $E[\bar{X}] = p$.
2. Show that $Var(\bar{X}) = p(1-p)/N$
3. Approximate is the probability that $\bar{X} \notin [p - \sqrt{p(1-p)}/5, p + \sqrt{p(1-p)}/5]$. (Hint: Use Central Limit Theorem).
4. How confident are you (give a percentage) that the parameter is in the interval $[\bar{X} - 1/10, \bar{X} + 1/10]$? (Hint: remark that $p(1-p) \geq 1/4$)

Problem 11: Let X be a Poisson r.v. of parameter λ.

1. Compute $E[X]$.
2. Show that the moment generating function of X, $M_X(t) = E[e^{tX}]$, is equal to $\exp(\lambda(e^t - 1))$.
3. Compute the moment generating function of $\sum_{i=1}^{N} X_i$, where the X_i's are n independent Poisson R.V. of parameter λ.
4. Can you interpret that?

Some formulas, means and variances:

Uniform in $[a, b]$: mean $\frac{a+b}{2}$, variance $\frac{(b-a)^2}{12}$

Exponential(λ): density $\lambda e^{-\lambda x}$, mean $\frac{1}{\lambda}$, variance $\frac{1}{\lambda^2}$

Gamma(r, λ): density $\lambda \frac{(\lambda x)^{r-1}}{\Gamma(r)} e^{-\lambda x}$, mean $\frac{r}{\lambda}$, variance $\frac{r}{\lambda^2}$

Poisson(λ): probability $P(n) = e^{-\lambda} \frac{\lambda^n}{n!}$, mean λ, variance λ

Geometric(p): probability $P(n) = (1-p)^{n-1} p$, mean $\frac{1}{p}$, variance $\frac{1-p}{p^2}$

Negative binomial (number of failures before rth success):

probability $P(n) = \binom{n+r-1}{r-1} p^r (1-p)^n$, mean $\frac{r(1-p)}{p}$, variance $\frac{r(1-p)}{p^2}$

Standard normal: density $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, mean 0, variance 1