随机过程第 1 周作业

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1. 设随机变量X服从参数为 1 的指数分布，随机变量$Y\sim N(0,1)$，且X与Y独立。试求随机变量$Z=\sqrt{2X}|Y|$的分布密度函数。

解：设$U=|Y|$，易知X和U的概率密度函数分别为

$$f_X(x)=\begin{cases} e^{-x} & x\geq 0 \\ 0 & otherwise \end{cases}$$

$$f_U(u)=\begin{cases} \frac{\sqrt{2}}{\sqrt{\pi}}e^{-\frac{u^2}{2}} & x\geq 0 \\ 0 & otherwise \end{cases}$$

设$V=U$，则有$\begin{cases} Z=\sqrt{2X}U \\ V=U \end{cases}$，其逆映射为$\begin{cases} X=\frac{Z^2}{2V^2} \\ U=V \end{cases}$。$J=\begin{vmatrix} \frac{\partial X}{\partial Z} & \frac{\partial X}{\partial V} \\ \frac{\partial U}{\partial Z} & \frac{\partial U}{\partial V} \end{vmatrix}=\begin{vmatrix} \frac{Z}{V^2} & \\ 0 & 1 \end{vmatrix}=\frac{Z}{V^2}$

则随机变量Z,V的联合概率密度函数为

$$f_{Z,V}(z,v)=f_{X,U}(x,u)\cdot|J|=f_X\left(\frac{z^2}{2v^2}\right)f_U(v)\frac{z}{v^2}=\frac{\sqrt{2}z}{\sqrt{\pi}v^2}e^{-\left(\frac{z^2}{2v^2}+\frac{v^2}{2}\right)}$$

则随机变量Z的边缘概率密度函数为

$$f_Z(z)=\int_0^{+\infty}f_{Z,V}(z,v)dv=\frac{\sqrt{2}z}{\sqrt{\pi}}\int_0^{+\infty}\frac{1}{v^2}e^{-\left(\frac{z^2}{2v^2}+\frac{v^2}{2}\right)}dv$$

设$t=\frac{1}{v}$，则

$$f_Z(z)=\frac{\sqrt{2}z}{\sqrt{\pi}}\int_0^{+\infty}\frac{1}{v^2}e^{-\left(\frac{z^2}{2v^2}+\frac{v^2}{2}\right)}dv=\frac{\sqrt{2}z}{\sqrt{\pi}}\int_0^{+\infty}e^{-\left(\frac{1}{2t^2}+\frac{z^2t^2}{2}\right)}dt$$

设$m=\sqrt{z}t$，则

$$f_Z(z)=\frac{\sqrt{2}z}{\sqrt{\pi}}\int_0^{+\infty}e^{-\frac{z}{2}\left(\frac{1}{m^2}+m^2\right)}dm=\frac{\sqrt{2}z}{\sqrt{\pi}}e^{-z}\int_0^{+\infty}e^{-\frac{z}{2}\left(\frac{1}{m}-m\right)^2}dm$$

设$I=\int_0^{+\infty}e^{-\frac{z}{2}\left(\frac{1}{m}-m\right)^2}dm$，则

$$\begin{aligned} I&=\int_0^{+\infty}e^{-\frac{z}{2}\left(\frac{1}{m}-m\right)^2}dm=\int_0^{+\infty}\frac{m^2}{m^2+1}e^{-\frac{z}{2}\left(\frac{1}{m}-m\right)^2}d\left(m-\frac{1}{m}\right) \\ &=\int_0^{+\infty}e^{-\frac{z}{2}\left(\frac{1}{m}-m\right)^2}d\left(m-\frac{1}{m}\right)-\int_0^{+\infty}\frac{1}{m^2+1}e^{-\frac{z}{2}\left(\frac{1}{m}-m\right)^2}d\left(m-\frac{1}{m}\right) \\ &=\frac{\sqrt{2\pi}}{\sqrt{z}}-\int_0^{+\infty}\frac{1}{m^2}e^{-\frac{z}{2}\left(\frac{1}{m}-m\right)^2}dm=\frac{\sqrt{2\pi}}{\sqrt{z}}-\int_0^{+\infty}e^{-\frac{z}{2}\left(\frac{1}{r}-r\right)^2}dr\quad\left(r=\frac{1}{m}\right) \\ &=\frac{\sqrt{2\pi}}{\sqrt{z}}-I \end{aligned}$$

则

$$I=\frac{\sqrt{\pi}}{\sqrt{2z}}$$

则

$$f_Z(z)=\begin{cases} \frac{\sqrt{2}z}{\sqrt{\pi}}e^{-z}I=e^{-z}, z\geq 0 \\ 0 \end{cases}$$

2. 设随机变量$\boldsymbol{X_1, X_2}$独立同分布，服从参数为$\boldsymbol{\lambda > 0}$的指数分布。试证明随机变量$\boldsymbol{\frac{X_1}{X_1+X_2} \sim U[0,1]}$。

解：设$\begin{cases} Y_1 = \frac{X_1}{X_1+X_2} \\ Y_2 = X_2 \end{cases}$，其逆映射为$\begin{cases} X_1 = \frac{Y_1 Y_2}{1-Y_1} \\ X_2 = Y_2 \end{cases}$。$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} \frac{Y_2}{(1-Y_1)^2} & \\ 0 & 1 \end{vmatrix} = \frac{Y_2}{(1-Y_1)^2}$

当$y_1, y_2 \in [0,1]$时，有

$$f_{Y_1,Y_2}(y_1,y_2) = |J| f_{X_1,X_2}(x_1,x_2) = \frac{y_2}{(1-y_1)^2} f_{X_1}\left(\frac{y_1 y_2}{1-y_1}\right) f_{X_2}(y_2) = \frac{\lambda^2 y_2}{(1-y_1)^2} e^{-\frac{\lambda y_2}{1-y_1}}$$

则当$y_1 \in [0,1]$随机变量Y_1的边缘概率密度函数为

$$f_{Y_1}(y_1) = \int_0^{\infty} f_{Y_1,Y_2}(y_1,y_2) dy_2 = \frac{\lambda^2}{(1-y_1)^2} \int_0^{\infty} y_2 e^{-\frac{\lambda y_2}{1-y_1}} dy_2$$

设$I = \int_0^{\infty} y_2 e^{-\frac{\lambda y_2}{1-y_1}} dy_2$，则

$$I = \int_0^{\infty} y_2 e^{-\frac{\lambda y_2}{1-y_1}} dy_2 = \frac{y_1 - 1}{\lambda} \int_0^{\infty} y_2 d e^{-\frac{\lambda y_2}{1-y_1}} = \frac{y_1-1}{\lambda}\left(y_2 e^{-\frac{\lambda y_2}{1-y_1}} \Big|_{y_2=0}^{y_2=\infty} - \int_0^{\infty} e^{-\frac{\lambda y_2}{1-y_1}} dy_2 \right)$$

$$= \frac{1-y_1}{\lambda} \int_0^{\infty} e^{-\frac{\lambda y_2}{1-y_1}} dy_2 = -\left(\frac{1-y_1}{\lambda}\right)^2 e^{-\frac{\lambda y_2}{1-y_1}} \Big|_{y_2=0}^{y_2=\infty} = \left(\frac{1-y_1}{\lambda}\right)^2$$

$$f_{Y_1}(y_1) = \frac{\lambda^2}{(1-y_1)^2} I = 1$$

易知当$y_1 \notin [0,1]$时，$f_{Y_1}(y_1) = 0$。

综上$f_{Y_1}(y_1) = \begin{cases} 1 & y_1 \in [0,1] \\ 0 & otherwise \end{cases}$，即随机变量$Y_1 = \frac{X_1}{X_1+X_2} \sim U[0,1]$。

3. 设随机向量$\boldsymbol{(X,Y)}$的两个分量相互独立，且均服从标准正态分布$\boldsymbol{N(0,1)}$。

a) 分别写出随机变量$\boldsymbol{X+Y}$和$\boldsymbol{X-Y}$的分布密度。

解：设$U = X + Y$，由卷积公式有

$$f_U(u) = \int_{-\infty}^{+\infty} f_{X(x)} f_Y(u-x) dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} e^{-\frac{(u-x)^2}{2}} dx = \frac{1}{2\sqrt{\pi}} e^{-\frac{u^2}{4}}$$

即$U \sim N(0,2)$。

同理可证，$V = X - Y \sim U(0,2)$，即$f_V(v) = \frac{1}{2\sqrt{\pi}} e^{-\frac{v^2}{4}}$

b) 试问：$\boldsymbol{X+Y}$与$\boldsymbol{X-Y}$是否独立？说明理由。

解：设$\begin{cases} U = X + Y \\ V = X - Y \end{cases}$，其逆映射为$\begin{cases} X = \frac{1}{2}(U+V) \\ Y = \frac{1}{2}(U-V) \end{cases}$。$J = \begin{vmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

随机变量(U,V)联合概率密度函数为

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right) = \frac{1}{4\pi} e^{-\frac{(u^2+v^2)}{4}} = f_U(u) f_V(v)$$

故随机变量$X+Y, X-Y$独立。