

$$x^{(R+1)} = x^{(R)} + M^{-1} r^{(R)}$$

\Rightarrow if $r^{(R)} = 0$, then $x^{(R)} = x$
we have converged to the true
solution

Remark: Jacobi preconditioning is cheap,
but makes the iterative scheme converge
slower than Gauss-Seidel

Polynomial methods are useful since we have proven that $\exists! p \in K_n[X]$ such that $A^{-1} = p(A)$ (cayley-Hamilton)

Let's go back to the basic iterative scheme

$$\begin{aligned} x &= x^{(0)} + e^{(0)} \\ &= x^{(0)} + A^{-1} r^{(0)} \\ &= x^{(0)} + p(A) r^{(0)} \end{aligned}$$

Krylov methods are based on the following vectorial space :

$$K_m(A, r^{(0)}) = \text{span}(r^{(0)}, Ar^{(0)}, A^2r^{(0)}, \dots, A^{m-1}r^{(0)})$$

These are space of increasing dimensions,

$$\dim K_m \leq \dim K_{m+1} \leq n$$

They are also nested :

$$K_m \subset K_{m+1}$$

We can thus show that $\exists q \in [0,1]$
such that $A^{-1}r^{(0)} \in \mathcal{X}_q(A, r^{(0)})$

We introduce the
condition to define
methods:

Petrov-Galerkin
projection polynomial

$$\# x^{(m)} \in x^{(0)} + \mathcal{X}_m(A, r^{(0)})$$

$$\# \forall v \in \mathcal{X}_m(A, r^{(0)}) \quad (\beta e^{(m)})^T v = 0$$

* Arnoldi iterations to generate $\mathcal{X}_m(A, r^{(0)})$

$$\begin{aligned}
 \mathcal{X}_m(A, r^{(0)}) &= \text{Span} \left(r^{(0)}, Ar^{(0)}, \dots, \dots, A^{m-1}r^{(0)} \right) \\
 &\stackrel{\rightarrow}{=} \text{Span} \left(q_1, Aq_1, \dots, \dots, A^{m-1}q_1 \right) \\
 &= \text{Span} \left(q_1, q_1 + \alpha q_2, \dots, A^{m-2}(Aq_1) \right) \\
 &= \text{Span} \left(q_1, q_2, \dots, A^{m-2}q_2 \right) \\
 &= \text{Span} \left(q_1, q_2, q_3, \dots, q_{m-1}, Aq_{m-1} \right)
 \end{aligned}$$

$$q_1 = \frac{r^{(0)}}{\|r^{(0)}\|}$$

How to define q_m ? We want to generate an orthonormal basis of \mathcal{X}_m , we can apply the Gram-Schmidt algorithm to generate the proper basis vector:

$$\tilde{q}_m = Aq_{m-1} - \sum_{i=1}^{m-1} (q_i^T A q_{m-1}) q_i$$

Keep in mind that (q_1, \dots, q_{m-1}) is an orthonormal basis

So by definition, \tilde{q}_m is also orthogonal to all \tilde{q}_i , $i \in [1; m-1]$

We need to make \tilde{q}_m of norm 1

$$\tilde{q}_m = \frac{\tilde{q}_m}{\|\tilde{q}_m\|}$$

$$q_m^T \tilde{q}_m = q_m^T A q_{m-1} -$$

~~$$\sum_{i=1}^{m-1} q_m^T \left(q_i^T A q_{m-1} \right) q_i$$~~

Since q_m is orthogonal to all q_i

This process is known as the Arnoldi process which yields the Arnoldi relationship :

$$A Q_m = Q_{m+1} H_m$$

A : coefficient matrix

Q_m : matrix which column i is q_i

H_m : Hessenberg matrix defined as

$$(H_m)_{ij} = q_i^T * q_j$$

$$H_m = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \dots \\ h_{21} & h_{22} & & \\ 0 & h_{32} & h_{33} & \\ 0 & 0 & h_{43} & \\ 0 & & & \end{bmatrix}$$

(upper triangular matrix + first sub-diagonal)

The Arnoldi process, coupled with the Petrov-Galerkin condition, is the basis of the GMRES (Generalized Minimal RESidual method) 4. Saad 1986

It's a very general method, but there is no guarantee with respect to convergence

In practice, we still need to use a preconditioner M^{-1} , this slightly modifies the subspace condition

$$x^{(m)} \in x^{(0)} + P_M(A, r^{(0)})$$

* if we apply M^{-1} on the left:

$$M^{-1}Ax = M^{-1}f$$

$$r^{(0)} = f - Ax^{(0)}$$

$$\tilde{r}^{(0)} = M^{-1}f - M^{-1}Ax^{(0)} = M^{-1}(f - Ax^{(0)}) \\ = M^{-1}r^{(0)}$$

The subspace condition becomes

$$x^{(m)} \in x^{(0)} + \mathcal{K}_m(M^{-1}A, M^{-1}r^{(0)})$$

or we can also precondition on
the right:

$$\begin{aligned} & AM^{-1}y = f \\ & r^{(0)} = f - AM^{-1}y^{(0)} \\ & = f - Ax^{(0)} = r^{(0)} \end{aligned}$$

with $M^{-1}y = x$

with $M^{-1}y^{(0)} = x^{(0)}$

The subspace condition becomes:

$$y^{(m)} \in y^{(0)} + f_m(AH^{-1}, r^{(0)})$$

$$H^{-1}y^{(m)} \in H^{-1}y^{(0)} + H^{-1}f_m(AH^{-1}, r^{(0)})$$

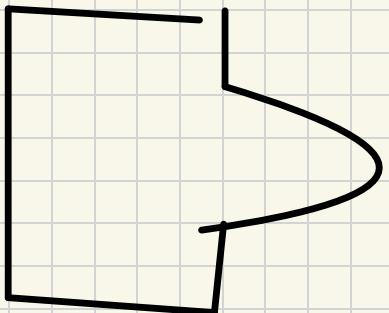
$$x^{(m)} \in x^{(0)} + H^{-1}f_m(AH^{-1}, r^{(0)})$$

With Jacobi or Gauss-Seidel, we can slightly reduce iteration count, but we still need better preconditioner

* Domain decomposition preconditioners

H. Schwarz 1870

Ω

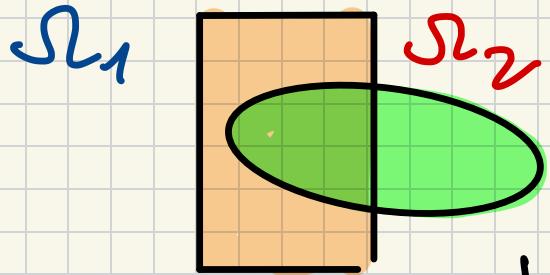


Poisson equation on Ω

$$-\Delta u = f \quad \Omega$$
$$u = 0 \quad \partial\Omega$$

$$-\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}$$

The main idea is to decompose Ω into two subdomains which are not complex geometries : rectangle and ellipse



Subproblem

Why? Because they could then use Fourier transform to solve the Poisson equation on each

We will now treat subproblems on Ω_1 and Ω_2

$$-\Delta u_1^{(n+1)} = f|_{\Omega_1}, \quad \Omega_1$$

$$u_1^{(n+1)} = 0 \quad \partial\Omega_1 \cap \partial\Omega$$

$$u_1^{(n+1)} = u_2^{(n)} \quad \partial\Omega_1 \setminus \partial\Omega$$

$$-\Delta u_2^{(n+1)} = f|_{\Omega_2}, \quad \Omega_2$$

$$u_2^{(n+1)} = 0 \quad \partial\Omega_2 \cap \partial\Omega$$

$$u_2^{(n+1)} = u_1^{(n+1)} \quad \partial\Omega_2 \setminus \partial\Omega$$

Just as for Gauss-Seidel with RB numbering, we update first the blue subdomain Ω_1 , and then move on to the red subdomain Ω_2

We now want to prove, as Schwarz did, that this iterative procedure indeed converges to the proper solution of the Poisson equation on Σ .

$$\Omega = [0; 1] \quad -\Delta u = f \Rightarrow \frac{-d^2 u}{dx^2} = f$$

$$u(0) = u(1) = 0$$

We introduce the error, defined as :

$$e_i^{(n)} = u_i^{(n)} - u|_{\Sigma_i} \quad \text{for both subdomains}$$

Let's see that in Schwarz method

$$-\frac{d^2 v_1^{(n+1)}}{dx^2} = f \quad [0; L_1] \quad -\frac{d^2 v_2^{(n+1)}}{dx^2} = f \quad [L_2; 1]$$

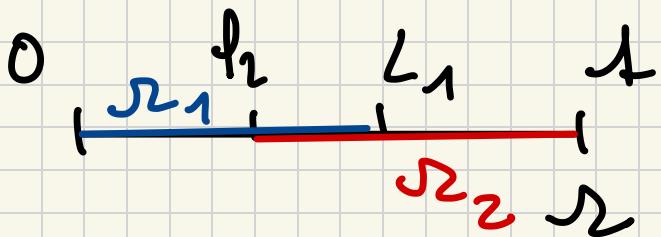
$$v_1^{(n+1)}(0) = 0$$

$$v_1^{(n+1)}(L_1) = v_2^{(n)}(L_1)$$

$$v_2^{(n+1)}(1) = 0$$

$$v_2^{(n+1)}(L_2) = v_1^{(n+1)}(L_2)$$

The $\frac{d^2}{dx^2}$ operator is linear,
so we can compute



$$-\frac{d^2 e_1^{(n+1)}}{ds^2} = -\frac{d^2 \omega_1^{(n+1)}}{ds^2} + \frac{d^2 \omega_1 \Omega_{21}}{ds^2} \Omega_{21}$$

$$= 0 \quad \Im \omega_1$$

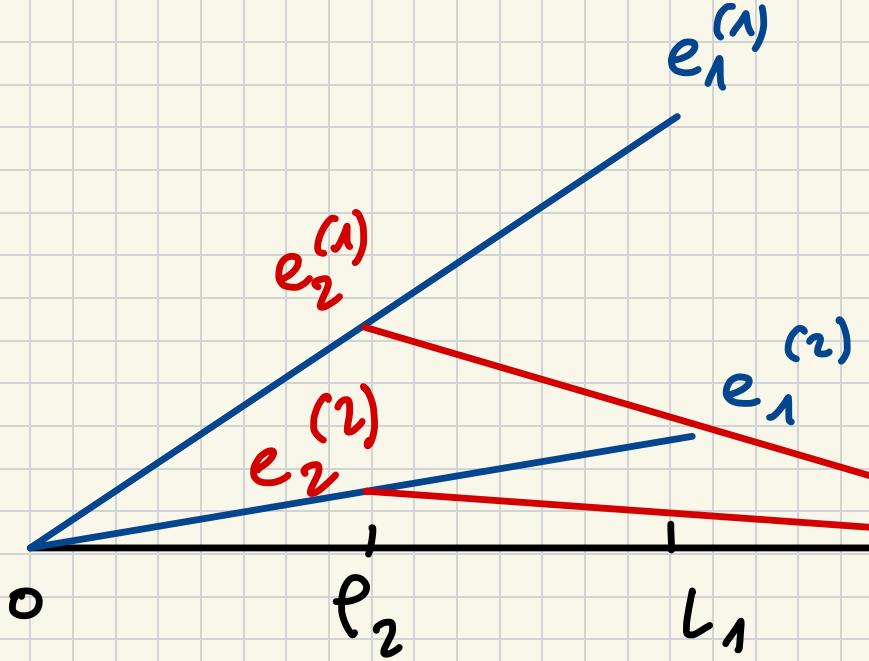
$$e_1^{(n+1)}(0) = 0$$

$$e_1^{(n+1)}(L_1) = e_1^{(n)}(L_1)$$

$$-\frac{d^2 e_2^{(n+1)}}{ds^2} = 0 \quad \Im \omega_2$$

$$e_2^{(n+1)}(1) = 0$$

$$e_2^{(n+1)}(L_2) = e_1^{(n+1)}(L_2)$$



We start with R_1
and suppose that we
have $v_2^{(0)} \rightarrow e_2^{(0)}$

We can "see"
 $e_i^{(n)} : i \in \{1, 2\}$

$1 \rightarrow 0$ when
 n increases

$$e_1^{(n+1)}(x) = \alpha x + \beta$$

Thanks to the BC
in 0, we have

$$e_1^{(n+1)}(x) = \alpha x$$

then, in L_1

$$e_1^{(n+1)}(x) = \frac{e_2^{(n)}(L_1)}{L_1} X$$

$$e_2^{(n+1)}(x) = \alpha x + \beta$$

$$e_2^{(n+1)}(x) = \gamma (1 - x)$$

$$e_2^{(n+1)}(x) = \frac{e_1^{(n+1)}(P_2)}{1 - \ell_2} (1 - x)$$

Let's investigate the contraction factor

$$\ell = \frac{e_2^{(n+1)}(L_1)}{e_2^{(n)}(L_1)}$$

$$\begin{aligned} e_2^{(n+1)}(L_1) &= e_1^{(n+1)}(P_2) \frac{(1-L_1)}{(1-P_2)} \\ &= \frac{1-L_1}{1-P_2} \cdot \frac{e_1^{(n)}(L_1)}{L_1} \cdot P_2 \end{aligned}$$

$$\ell = \frac{1-L_1}{L_1} \cdot \frac{P_2}{1-P_2}$$

We introduce $\delta = L_1 - \rho_2$

$$\ell = \frac{1 - L_1}{L_1} \cdot \frac{\rho_2}{1 - \rho_2}$$

$$= \frac{\rho_2}{1 - \rho_2} \cdot \frac{1 - \delta - \rho_2}{\delta + \rho_2}$$

$$= \frac{1 - \frac{\rho_2}{1 - \rho_2}}{1 + \frac{\delta}{\rho_2}}$$

$$= \frac{1 - \frac{\rho_2}{1 - \rho_2}}{1 + \frac{\delta}{\rho_2}}$$

We observe that :

* $\delta = 0 \Rightarrow p = 1 \Rightarrow$ no convergence

* the greater δ , the lower p
(faster convergence)

The initial additive Schwarz method
is not very scalable, but a slight
modification introduced by P.-L. Lions
in 1988

$$-\Delta v_1^{(n+1)} = f \quad \Omega_1$$

$$v_1^{(n+1)} = 0 \quad \partial\Omega_1 \cap \partial\Omega$$

$$v_1^{(n+1)} = v_2^{(n)} \quad \partial\Omega_1 \setminus \partial\Omega$$

$$-\Delta v_2^{(n+1)} = f \quad \Omega_2$$

$$v_2^{(n+1)} = 0 \quad \partial\Omega_2 \cap \partial\Omega$$

$$v_2^{(n+1)} = v_1^{(n)} \quad \frac{\partial\Omega_2}{\partial\Omega}$$

Given $(v_1^{(0)}, v_2^{(0)})$
compute each iteration

concurrently

use current row
 $(v_1^{(n+1)}, v_2^{(n+1)})$

These methods can be applied at
the algebraic level.

$$Ax = f$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

* Block Jacobi method

$$M = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

$x^{(0)}$ is given

$$x^{(n+1)} = x^{(n)} + M^{-1} r^{(n)}$$

$$r^{(n)} = f - Ax^{(n)}$$

$$\begin{bmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \end{bmatrix} \approx \begin{bmatrix} x_1^{(n)} \\ x_2^{(n)} \end{bmatrix} + \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} r_1^{(n)} \\ r_2^{(n)} \end{bmatrix}$$

The cost of computing the factorization

of A_{ii} is much lower than for the

factorization of A because of the polynomial
Gaussian Cholesky factorization

In practice, we can introduce restriction operators, R_i that map global degrees of freedom to local ones.

In a dual fashion, the dual operators R_i^T are prolongation operators.

Here is one example:

$$R_i \in \mathbb{R}^{m_i \times m}$$

$$R_1^T R_1 x = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad R_1 = \begin{bmatrix} I_{11} & 0_{12} \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 0_{21} & I_{22} \end{bmatrix}$$

$$R_1 x = x_1$$

$$R_2 x = x_2$$

$$R_1^T x_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$R_2^T x_2 = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

$$x^{(n+1)} = x^{(n)} + \sum_{i=1}^2 R_i^T A_{ii}^{-1} R_i r^{(n)}$$

$$A_{ii} = R_i^T A R_i^T$$

this can be extended to an arbitrary number of subdomains N :

$$x^{(n+1)} = x^{(n)} + \sum_{i=1}^N R_i^T A_{ii}^{-1} R_i r^{(n)}$$

* Block Gauss-Seidel

$$M = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad M^{-1}$$

$$x^{(n+1)} = x^{(n)} + M^{-1} r^{(n)}$$

$$Mx^{(n+1)} = x^{(n)} + r^{(n)}$$

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1^{(n)} \\ x_2^{(n)} \end{bmatrix} + \begin{bmatrix} r_1^{(n)} \\ r_2^{(n)} \end{bmatrix}$$

$$A_{11} x_1^{(n+1)} = A_{11} x_1^{(n)} + r_1^{(n)}$$

$$x_1^{(n+1)} = x_1^{(n)} + A_{11}^{-1} r_1^{(n)}$$

$$A_{21} x_1^{(n+1)} + A_{22} x_2^{(n+1)} = A_{21} x_1^{(n)} + A_{22} x_2^{(n)} + r_2^{(n)}$$

$$\begin{aligned} A_{22} x_2^{(n+1)} &= A_{22} x_2^{(n)} + \cancel{A_{21} x_1^{(n)}} + f_2 - \cancel{A_{21} x_1^{(n)}} - \cancel{A_{22} x_2^{(n)}} \\ &\quad - A_{21} x_1^{(n+1)} \end{aligned}$$

$$= A_{22} x_2^{(n)} + \left(f_2 - A_{21} x_1^{(n+1)} - A_{22} x_2^{(n)} \right)$$

$$x_2^{(n+1)} = x_2^{(n)} + A_{22}^{-1} \left(f_2 - A_{21} x_1^{(n+1)} - A_{22} x_2^{(n)} \right)$$

$$x^{(n+\frac{1}{2})} = \begin{bmatrix} x_1^{(n+1)} \\ x_2^{(n)} \end{bmatrix}$$

We can write the iterative process
in condensed form :

$$x^{(n+\frac{1}{2})} = x^{(n)} + R_1^T A_{11}^{-1} R_1 r^{(n)}$$

$$x^{(n+1)} = x^{(n+\frac{1}{2})} + R_2^T A_{22}^{-1} R_2 r^{(n+\frac{1}{2})}$$

$$x^{(n+\frac{1}{2})}$$

$$x^{(n+\frac{2}{2})}$$

$$x^{(n+\frac{3}{2})}$$

.

$$\dots - - - x^{(n+1)}$$

There is a link between :

- * Block Gauss-Seidel and the initial Schwarz method
- * Block Jacobi and the P.L. Lions variant of Schwarz method