Iterative methods for linear algebra

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2/2 *Advanced preconditioning*

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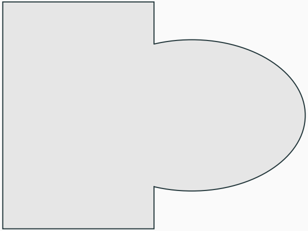
Introduction

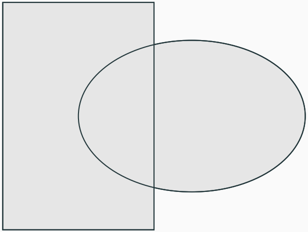
Motivation

How to solve Poisson equation on “complex” domains?



Motivation

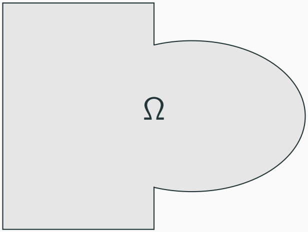
How to solve Poisson equation on “complex” domains?



“Simpler” domains and Fourier transforms [Schwarz 1870]

Overlapping Schwarz methods

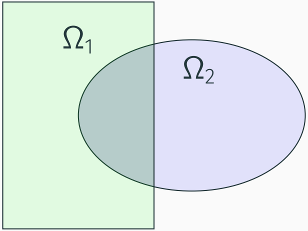
Continuous formulation



Given $\mathcal{L}$, a source term f, and Dirichlet BC g,

$$\begin{aligned} \mathcal{L}u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \end{aligned}$$

Continuous formulation



Given $\mathcal{L}$, a source term f, and Dirichlet BC g + $u_2^{(0)}$,

$$
\begin{aligned}
\mathcal{L}u_1^{(n+1)} &= f|_{\Omega_1} && \text{in } \Omega_1 & \qquad \mathcal{L}u_2^{(n+1)} &= f|_{\Omega_2} && \text{in } \Omega_2 \\
u_1^{(n+1)} &= g && \text{on } \partial\Omega_1 \setminus \Omega_2 & \qquad u_2^{(n+1)} &= g && \text{on } \partial\Omega_2 \setminus \Omega_1 \\
u_1^{(n+1)} &= u_2^{(n)} && \text{on } \partial\Omega_1 \cap \Omega_2 & \qquad u_2^{(n+1)} &= u_1^{(n+1)} && \text{on } \partial\Omega_2 \cap \Omega_1
\end{aligned}
$$

Error estimates

The errors $\{e_i^{(n+1)} = u_i^{(n+1)} - u|_{\Omega_i}\}_{i\in[\![1;2]\!]}$ verify:

$$\begin{aligned} \mathcal{L}e_1^{(n+1)} &= 0|_{\Omega_1} \text{ in } \Omega_1 & \mathcal{L}e_2^{(n+1)} &= 0|_{\Omega_2} \text{ in } \Omega_2 \\ e_1^{(n+1)} &= 0 \quad \text{on } \partial\Omega_1 \setminus \Omega_2 & e_2^{(n+1)} &= 0 \quad \text{on } \partial\Omega_2 \setminus \Omega_1 \\ e_1^{(n+1)} &= e_2^{(n)} \text{ on } \partial\Omega_1 \cap \Omega_2 & e_2^{(n+1)} &= e_1^{(n+1)} \text{ on } \partial\Omega_2 \cap \Omega_1 \end{aligned}$$

Error estimates

The errors $\{e_i^{(n+1)} = u_i^{(n+1)} - u|_{\Omega_i}\}_{i\in[\![1;2]\!]}$ verify:

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Convergence rates

- 1D, $\mathcal{L} = -\dfrac{\partial^2}{\partial x^2} \implies \rho(\delta) = \left|\dfrac{e_2^{(n+1)}(L_1)}{e_2^{(n)}(L_1)}\right| = \dfrac{1-\delta/(L-l_2)}{1+\delta/l_2}$
- 2D, $\mathcal{L} = (\nu - \Delta) \implies \rho(k,\delta,\nu) = \left|\dfrac{\hat{e}_1^{(n+1)}(0)}{\hat{e}_1^{(n)}(0)}\right| = e^{-2\delta\sqrt{\nu+k^2}}$

Remarks about the convergence rates

- both methods converge $\iff \delta > 0$

Remarks about the convergence rates

- both methods converge $\iff \delta > 0$
- for $k_1 > k_2$, $\rho(k_2, \delta, \nu) \gg \rho(k_1, \delta, \nu)$ (smoothing property)

Given $\left(u_1^{(0)}, u_2^{(0)}\right)$, [Lions 1988]:

$$
\begin{aligned}
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u_1^{(n+1)} &= g \quad \text{ on } \partial\Omega_1 \setminus \Omega_2 & u_2^{(n+1)} &= g \quad \text{ on } \partial\Omega_2 \setminus \Omega_1 \\
u_1^{(n+1)} &= u_2^{(n)} \text{ on } \partial\Omega_1 \cap \Omega_2 & u_2^{(n+1)} &= u_1^{(n)} \text{ on } \partial\Omega_2 \cap \Omega_1
\end{aligned}
$$

// variant of the alternating Schwarz method

Given $\left(u_1^{(0)}, u_2^{(0)}\right)$, [Lions 1988]:

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Slower convergence but more suited to parallel computing

Connection with the block Gauss–Seidel method

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $M = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$

given x_0

$$x_{k+1} = x_k + M^{-1} r_k$$

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given x_0

$$x_{k+1} = x_k + M^{-1} r_k$$

In order to explicit the workload distribution:

given x_0

$$x_{k+1/2} = x_k + \left(R_1^T (R_1 A R_1^T)^{-1} R_1\right) r_k$$

$$x_{k+1} = x_{k+1/2} + \left(R_2^T (R_2 A R_2^T)^{-1} R_2\right) r_{k+1/2}$$

with appropriate restriction operators $\{R_i\}_{i \in [\![1;2]\!]}$

Iteration matrix/local projections

Let $\{B_i = R_i^T(R_iAR_i^T)^{-1}R_i\}_{i\in[\![1;2]\!]}$

$$x_{k+1} = x_k + (B_1 + B_2 - B_2AB_1)r_k$$

Iteration matrix/local projections

Let $\{B_i = R_i^T(R_iAR_i^T)^{-1}R_i\}_{i\in[\![1;2]\!]}$

$$\begin{aligned} x_{k+1} &= x_k + (B_1 + B_2 - B_2AB_1)r_k \\ e_{k+1} &= (I - M^{-1}A)e_k \\ &= (I - B_2A)(I - B_1A)e_k \end{aligned}$$

A-orthogonal projection on span(R_i^T): $P_i = B_iA$

Iteration matrix/local projections

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Moreover, $B_i r = P_i e$

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A-orthogonal projection on $\mathrm{span}(R_i^T)$: $P_i = B_i A$

Moreover, $B_i r = P_i e$

A-orthogonal projection on $\mathrm{span}(R_i^T)^{\perp_A}$: $I - P_i$

Connection with the block Jacobi method

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $M = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$

given x_0

$$x_{k+1} = x_k + M^{-1} r_k$$

Connection with the block Jacobi method

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $M = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$

given x_0

$$x_{k+1} = x_k + M^{-1} r_k$$

In order to explicit the parallelism:

given x_0

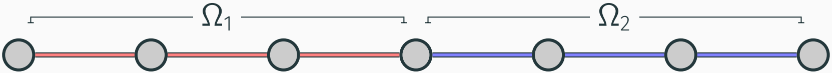
$$x_{k+1} = x_k + \left(R_1^T (R_1 A R_1^T)^{-1} R_1 + R_2^T (R_2 A R_2^T)^{-1} R_2 \right) r_k$$

1D domain

Algebraic overlap

1D domain

Initial decomposition



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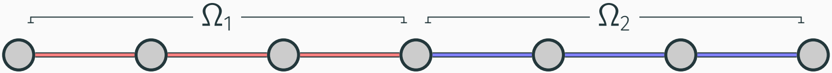
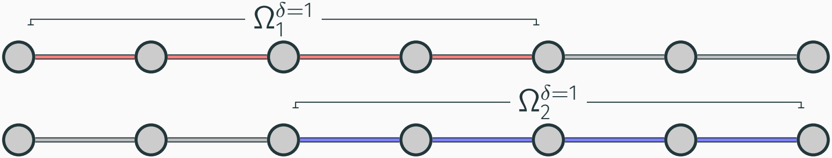


Overlapping decomposition



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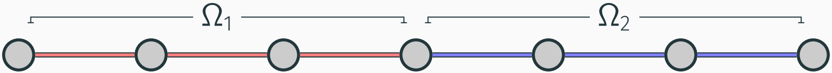

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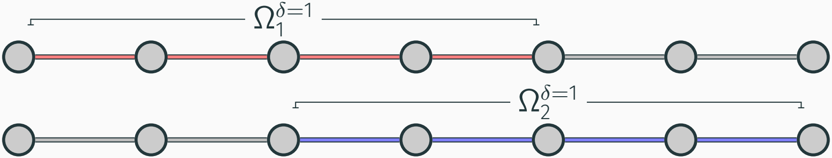

Larger restriction operators $\{R_i^\delta\}_{i \in [\![1;2]\!]}$

Alternating Schwarz method

Just as for the block GS method, with N subdomains:

given x_0

$$x_{k+1/N} = x_k + \left(R_1^{\delta\,T}(R_1^\delta A R_1^{\delta\,T})^{-1} R_1^\delta\right) r_k$$

$$\vdots$$

$$x_{k+1} = x_{k+(N-1)/N} + \left(R_N^{\delta\,T}(R_N^\delta A R_N^{\delta\,T})^{-1} R_N^\delta\right) r_{k+(N-1)/N}$$

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Red–Black coloring of subdomains for better concurrency

Additive Schwarz method

Just as for the block Jacobi method, with N subdomains:

given x_0

$$
\begin{aligned}
x_{k+1} &= x_k \\
x_{k+1} &\mathrel{+}= \left({R_1^\delta}^T (R_1^\delta A {R_1^\delta}^T)^{-1} R_1^\delta \right) r_k \\
&\vdots \\
x_{k+1} &\mathrel{+}= \left({R_N^\delta}^T (R_N^\delta A {R_N^\delta}^T)^{-1} R_N^\delta \right) r_k
\end{aligned}
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&\quad\vdots \\
&x_{k+1} \mathrel{+}= \left({R_N^\delta}^T (R_N^\delta A {R_N^\delta}^T)^{-1} R_N^\delta \right) r_k
\end{aligned}
$$

This fixed-point algorithm does not converge

The error in the overlap does not decrease

Restricted additive Schwarz method

Partition of unity

A set of diagonal matrices $\{D_i\}_{i \in [\![1;N]\!]}$:

$$\sum_{i=1}^{N} R_i^T D_i R_i = I$$

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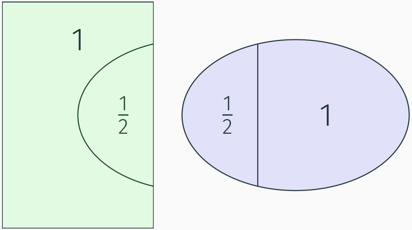
Multiplicity scaling



Restricted additive Schwarz method

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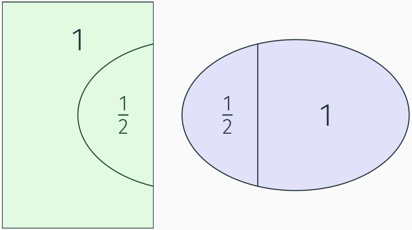
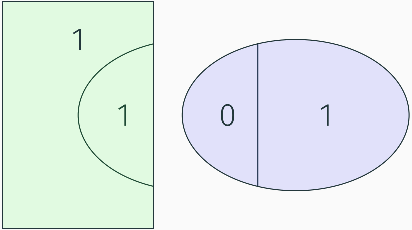


Master–slave



Restricted additive Schwarz method

Partition of unity

A set of diagonal matrices $\{D_i\}_{i \in [\![1;N]\!]}$:

$$\sum_{i=1}^{N} R_i^T D_i R_i = I$$

$$\begin{aligned}
&\text{given } x_0 \\
&\qquad x_{k+1} = x_k \\
&\qquad x_{k+1} \mathrel{+}= \left(R_1^T D_1 (R_1 A R_1^T)^{-1} R_1\right) r_k \\
&\qquad\qquad \vdots \\
&\qquad x_{k+1} \mathrel{+}= \left(R_N^T D_N (R_N A R_N^T)^{-1} R_N\right) r_k
\end{aligned}$$

Introduced by [Cai and Sarkis 1999]

Schwarz methods as preconditioners

Fixed-point algorithms as preconditioners, e.g.:

$$M_{\text{ASM}}^{-1} = \sum_{i=1}^{N} R_i^T (R_i A R_i^T)^{-1} R_i$$

$$M_{\text{RAS}}^{-1} = \sum_{i=1}^{N} R_i^T D_i (R_i A R_i^T)^{-1} R_i$$

Schwarz methods as preconditioners

Fixed-point algorithms as preconditioners, e.g.:

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$$M_{\text{RAS}}^{-1} = \sum_{i=1}^{N} R_i^T D_i (R_i A R_i^T)^{-1} R_i$$

Better convergence inside a Krylov method

M_{ASM}^{-1} does actually converge

Optimized boundary conditions

Instead of considering Dirichlet BC, use Robin BC

Let $\left\{\mathcal{B}_i = \frac{\partial}{\partial n_i} + \alpha\right\}_{i\in[\![1;2]\!]}$, for $\alpha \in \mathbb{R}^{+\star}$

$$
\begin{aligned}
\mathcal{L}u_1^{(n+1)} &= f|_{\Omega_1} && \text{in } \Omega_1 & \qquad \mathcal{L}u_2^{(n+1)} &= f|_{\Omega_2} && \text{in } \Omega_2 \\
u_1^{(n+1)} &= g && \text{on } \partial\Omega_1 \setminus \Omega_2 & \qquad u_2^{(n+1)} &= g && \text{on } \partial\Omega_2 \setminus \Omega_1 \\
\mathcal{B}_1 u_1^{(n+1)} &= \mathcal{B}_1 u_2^{(n)} && \text{on } \partial\Omega_1 \cap \Omega_2 & \qquad \mathcal{B}_2 u_2^{(n+1)} &= \mathcal{B}_2 u_1^{(n+1)} && \text{on } \partial\Omega_2 \cap \Omega_1
\end{aligned}
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\mathcal{B}_1 u_1^{(n+1)} &= \mathcal{B}_1 u_2^{(n)} && \text{on } \partial\Omega_1 \cap \Omega_2
\end{aligned}
\qquad
\begin{aligned}
\mathcal{L}u_2^{(n+1)} &= f|_{\Omega_2} && \text{in } \Omega_2 \\
u_2^{(n+1)} &= g && \text{on } \partial\Omega_2 \setminus \Omega_1 \\
\mathcal{B}_2 u_2^{(n+1)} &= \mathcal{B}_2 u_1^{(n+1)} && \text{on } \partial\Omega_2 \cap \Omega_1
\end{aligned}$$

Convergence rate

- $\mathcal{L} = (\nu - \Delta) \implies \rho(k, \delta, \nu, \alpha) = \left|\frac{\sqrt{\nu + k^2} - \alpha}{\sqrt{\nu + k^2} + \alpha}\right| e^{-\delta\sqrt{\nu + k^2}}$

Optimal boundary conditions

Given general BC operators $\{\mathcal{B}_i\}_{i\in[\![1;2]\!]}$

$$\begin{aligned}
\mathcal{L}u_i^{(n+1)} &= f|_{\Omega_i} && \text{in } \Omega_i \\
u_i^{(n+1)} &= g && \text{on } \partial\Omega_i \setminus \Omega_j \\
\nabla u_i^{(n+1)} \cdot n_i + \mathcal{B}_i u_i^{(n+1)} &= -\nabla u_j^{(n)} \cdot n_j + \mathcal{B}_i u_j^{(n)} && \text{on } \partial\Omega_i \cap \Omega_j
\end{aligned}$$

Optimal boundary conditions

Given general BC operators $\{\mathcal{B}_i\}_{i\in[\![1;2]\!]}$

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Use the Dirichlet-to-Neumann operators $\{\mathrm{DtN}_i\}_{i\in[\![1;2]\!]}$

$$\begin{aligned} \mathrm{DtN}_i \colon \partial\Omega_i \setminus \Omega &\to \mathbb{R} \\ g &\mapsto \nabla v \cdot n_i \end{aligned}$$

where v satisfies

$$\begin{aligned} \mathcal{L}v &= 0 \text{ in } \Omega_i \setminus \Omega_j \\ v &= 0 \text{ on } \partial\Omega_i \cap \partial\Omega \\ v &= g \text{ on } \partial\Omega_i \cap \partial\Omega_j \end{aligned}$$

Coarse space operators

Motivation

No exchange of global information

$$\kappa(M^{-1}A) \leqslant C\frac{1}{H^2}\left(1 + \frac{H}{\delta}\right)$$

- level of overlap δ
- characteristic size of a subdomain H

[Le Tallec 1994; Toselli and Widlund 2005]

Abstract setting

Let Z be a tall and skinny matrix

$$\begin{aligned} \text{search for } & \min_v ||A(y + Zv) - f||_{A^{-1}} \\ \iff & \min_v v^T Z^T A Z v - 2 v^T Z^T (f - Ay) + \lambda \\ \implies & v = \left(Z^T A Z\right)^{-1} Z^T (f - Ay) \end{aligned}$$

Abstract setting

Let Z be a tall and skinny matrix

$$\text{search for } \min_v ||A(y+Zv)-f||_{A^{-1}}$$

$$\iff \min_v v^T Z^T AZv - 2v^T Z^T(f-Ay) + \lambda$$

$$\implies v = \left(Z^T AZ\right)^{-1} Z^T(f-Ay)$$

Define a two-level additive Schwarz method as:

$$M^{-1} = Z\left(Z^T AZ\right)^{-1} Z^T + \sum_{i=1}^{N} R_i(R_i^T AR_i)^{-1} R_i^T$$

Nicolaides coarse space

Smallest coarse space possible [Nicolaides 1987]

$$Z = \begin{bmatrix} R_1^T D_1 R_1 & \cdots & R_N^T D_N R_N \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Nicolaides coarse space

Smallest coarse space possible [Nicolaides 1987]

$$Z = \begin{bmatrix} R_1^T D_1 R_1 & \cdots & R_N^T D_N R_N \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Galerkin operator $E = Z^T A Z$ of order $N \ll n$

Much better conditioning $\kappa(M^{-1}A) \leqslant C\left(1 + \frac{H}{\delta}\right)$

DtN coarse space

For this coarse space, $\mathcal{L} = \eta - \nabla \cdot (\kappa \nabla)$

- errors are harmonic inside subdomains
- fast decay $\implies$ large eigenvalue of the DtN operator

Solve local eigenvalue problems $\mathrm{DtN}_i(u) = \lambda \kappa u$

Substructuring methods

Algebraic decomposition

Given two subdomains, assume that:

$$A = \begin{bmatrix} A_{11} & 0 & A_{1\Gamma} \\ 0 & A_{22} & A_{2\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & A_{\Gamma\Gamma} \end{bmatrix} \qquad f = \begin{bmatrix} f_1 \\ f_2 \\ f_\Gamma \end{bmatrix}$$

Now, eliminate x_1 and x_2 from $Ax = f$:

$$\begin{aligned} \left(A_{\Gamma\Gamma} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma}\right) x_\Gamma &= f_\Gamma - A_{\Gamma 1}A_{11}^{-1}f_1 - A_{\Gamma 2}A_{22}^{-1}f_2 \\ &= g_b = g_b^{(1)} + g_b^{(2)} \end{aligned}$$

The Schur complement S_p is defined as:

$$\begin{aligned} S_p &= A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} + A_{\Gamma\Gamma}^{(2)} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma} \\ &= S_p^{(1)} + S_p^{(2)} \end{aligned}$$

Solving the condensed system

Appropriate preconditioner for $S_p x_b = g_b$?

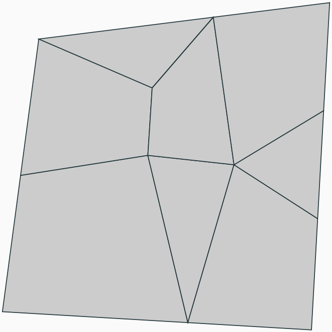
If $S_p^{(1)} = S_p^{(2)}$, choose $M^{-1} = \frac{1}{4}\left({S_p^{(1)}}^{-1} + {S_p^{(2)}}^{-1}\right)$

$$
\begin{aligned}
A^{(1)} &= \begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ A_{\Gamma 1}{A_{11}}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S_p^{(1)} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{1\Gamma} \\ 0 & I \end{bmatrix}
\end{aligned}
$$

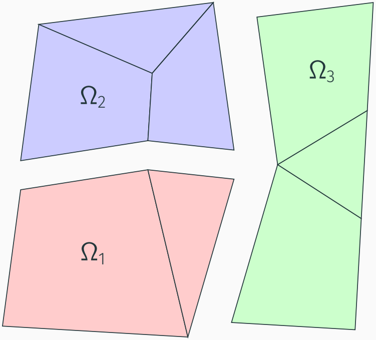
Thus,

$$
{S_p^{(1)}}^{-1}\lambda = \begin{bmatrix} 0 & I \end{bmatrix} {A^{(1)}}^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix}^T \lambda
$$

Back to domain decomposition

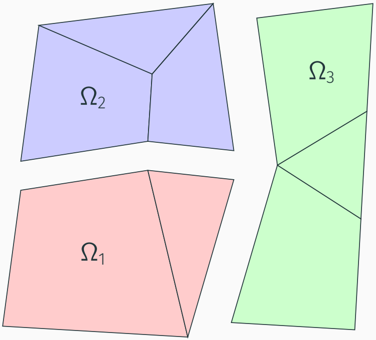


[Gosselet and Rey 2006]

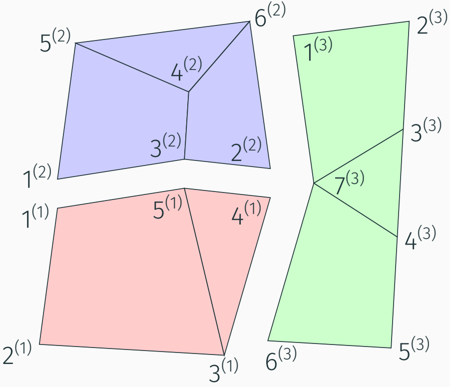


Subdomain tearing

Back to domain decomposition



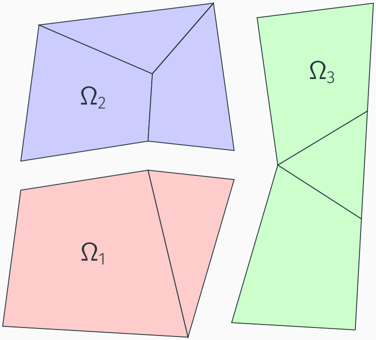
[Gosselet and Rey 2006]

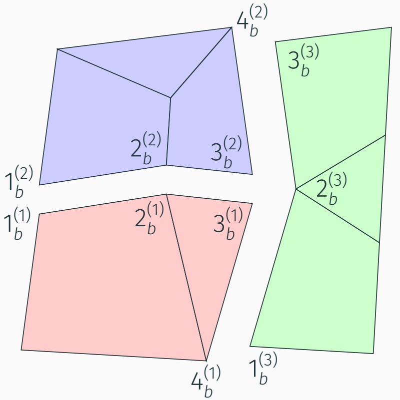


Local numbering

$$A^{(k)} = \begin{bmatrix} A_{ii} & A_{ib} \\ A_{bi} & A_{bb} \end{bmatrix}$$

Back to domain decomposition



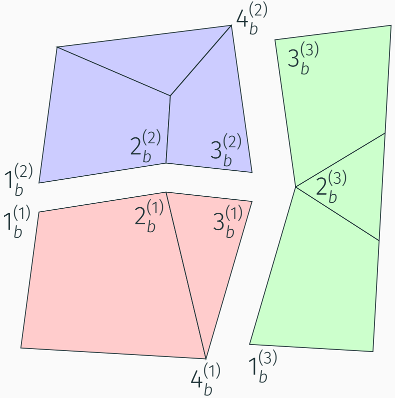


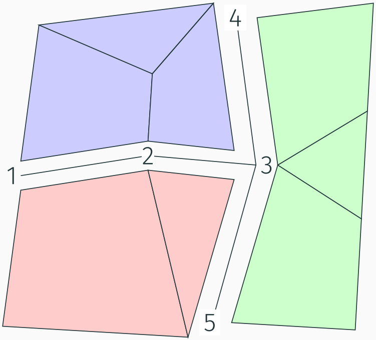
[Gosselet and Rey 2006]

Elimination of interior d.o.f.

$$S_p^{(k)} = A_{bb} - A_{bi}A_{ii}^{-1}A_{ib}$$

Back to domain decomposition

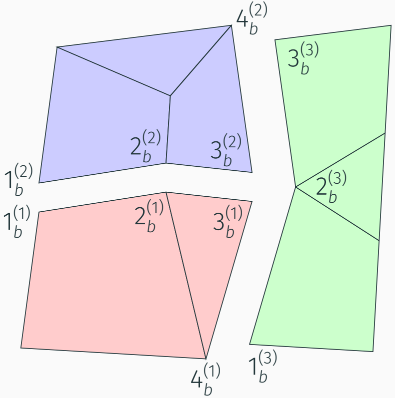


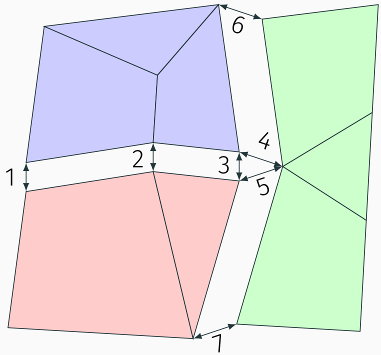


Jump operators: $\{B^{(i)}\}_{i=1}^{3}$

Primal constraints
[Mandel 1993]

Back to domain decomposition





Jump operators: $\{\underline{B}^{(i)}\}_{i=1}^{3}$

Dual constraints
[Farhat and Roux 1991]

Condensed system

The new system reads:

$$\forall k \in [\![1; N]\!], S_p^{(k)} x_b^{(k)} = g_b^{(k)} + \lambda_b^{(k)}$$

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$$\sum_{k=1}^{N} B^{(k)} \lambda_b^{(k)} = 0$$

Condensed system

The new system reads:

$$
\begin{aligned}
\forall k \in [\![1; N]\!], S_p^{(k)} x_b^{(k)} &= g_b^{(k)} + \lambda_b^{(k)} \\
{R_b^{(k)}}^T \lambda_b^{(k)} &= 0 \\
\sum_{k=1}^{N} \underline{B}^{(k)} x_b^{(k)} &= 0 \\
\sum_{k=1}^{N} B^{(k)} \lambda_b^{(k)} &= 0
\end{aligned}
$$

Primal methods

Define a unique displacement $x_b \implies x_b^{(k)} = {B^{(k)}}^T x_b$

Eliminate the reactions:

$$\sum_{k=1}^{N} B^{(k)} S_p^{(k)} {B^{(k)}}^T x_b = \sum_{k=1}^{N} B^{(k)} g^{(k)}$$

Weighted sum of pseudo-inverses as a preconditioner:

$$M^{-1} = \sum_{k=1}^{N} B^{(k)} D_p^{(k)} {S_p^{(k)}}^{\dagger} D_p^{(k)} {B^{(k)}}^T$$

The preconditioner should be applied to vectors in $\mathrm{Im}(S_p)$

Balancing Domain Decomposition

A vector r_b is balanced if:

$$\sum_{k=1}^{N} R_b^{(k)^T} D_p^{(k)} B^{(k)^T} r_b = 0$$

Let

$$\mathsf{R}_b = \left[B^{(1)} D_p^{(1)} R_b^{(1)} \quad \cdots \quad B^{(N)} D_p^{(N)} R_b^{(N)} \right]$$

$$\mathsf{S}_p = \sum_{k=1}^{N} B^{(k)} S_p^{(k)} B^{(k)^T}$$

Then, $P = I - \mathsf{R}_b \left(\mathsf{R}_b^T \mathsf{S}_p \mathsf{R}_b \right)^{-1} \mathsf{R}_b^T \mathsf{S}_p \implies \mathsf{R}_b^T \mathsf{S}_p P = 0$

Define a unique reaction $\lambda_b \implies \lambda_b^{(k)} = \underline{B}^{(k)^T} \lambda_b$

Eliminate the displacements:

$$\forall k \in [\![1; N]\!], x_b^{(k)} = S_d^{(k)}(g_b^{(k)} + \underline{B}^{(k)^T}\lambda_b) + R_b^{(k)}\alpha^{(k)}$$

$$0 = R_b^{(k)^T}(g_b^{(k)} + \underline{B}^{(k)^T}\lambda_b)$$

With similar notations:

$$\begin{bmatrix} \mathsf{S}_d & \underline{\mathsf{R}}_b \\ \underline{\mathsf{R}}_b^T & 0 \end{bmatrix} \begin{bmatrix} \lambda_b \\ \alpha \end{bmatrix} = \begin{bmatrix} -\mathbf{b}_d \\ -\mathbf{g}_b \end{bmatrix}$$

Implementation aspects

Distributed parallelism

Domain decomposition widely use with FE

Easy way to speedup the assembly phase

Message Passing Interface

- distributed numbering of unknowns
- ghost elements for assembled operators
- connectivity with neighboring subdomains

Matrix–vector products

- for overlapping Schwarz methods:

$$\forall k \in [\![1; N]\!], R_k A x = R_k \sum_{i=1}^{N} A R_i^T D_i R_i x$$
$$= \sum_{i=1}^{N} R_k A R_i^T D_i x_i$$

- for substructuring methods:

$$\forall k \in [\![1; N]\!], {B^{(k)}}^T \mathsf{S}_p x_b = {B^{(k)}}^T \sum_{j=1}^{N} B^{(j)} S_p^{(j)} {B^{(j)}}^T x_b$$

Dot products

Same strategy using the appropriate scaling:

$$\begin{aligned}(u, v) &= u^T v \\ &= u^T \sum_{k=1}^{N} R_k^T D_k R_k v \\ &= \sum_{k=1}^{N} u_k^T D_k v_k\end{aligned}$$

Coarse space corrections

Implies global information exchanges:

- local computations
- all-to-few gathering
- either compute or idle
- few-to-all scattering
- local computations

Links with multigrid methods

Convergence results

The damped Jacobi preconditioner reads $M^{-1} = \omega D^{-1}$

Error propagation:

$$\begin{aligned} e_{k+1} &= (I - M^{-1}A)e_k \\ &= (I - M^{-1}A)^{k+1}e_0 \end{aligned}$$

On a uniform 1D grid:

$$(I - M^{-1}A) = I - \frac{\omega}{2}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \\ & & -1 & 2 \end{bmatrix},$$

whose eigenvalues are $\left\{\mu_k = 1 - 2\omega\sin^2\left(\frac{k\pi}{2n}\right)\right\}_{k\in[\![1;n]\!]}$

Why use multiple grids?

Low frequencies of the error on a coarser grid:

- more oscillatory
- cheaper to relax
- better condition number of the operator

+ aliasing for the high frequencies

A two-grid method

Given x_0,

- smooth ν times to relax $Ax_k = f$ with the initial guess x_k
- compute the residual $r_k = f - Ax_k$
- restrict the residual to the coarser grid R_k
- solve $A_C E_{k+1} = R_k$
- interpolate the error back on the fine grid e_{k+1}
- smooth μ times with the initial guess $x_k + e_{k+1}$

Restriction and prolongation operators

- restriction R^h_{2h}: canonical injection, full-weighting $R^h_{2h,\text{fw}}$
- prolongation P^{2h}_h: interpolation
- coarse operator $A_C = A_{2h}$

On a regular grid,

$$
\begin{aligned}
2R^h_{2h,\text{fw}} &= {P^{2h}_h}^T \\
A_{2h} &= R^h_{2h,\text{fw}} A_h P^{2h}_h
\end{aligned}
$$

Iteration matrix of the two-grid method

- coarse grid correction $e_{k+1} = P_h^{2h}\left(R_{2h}^h A_h P_h^{2h}\right)^{-1} R_{2h}^h A_h e_k$
- coarse grid operator

$$
\begin{aligned}
\tilde{x}_{k+1} &= \tilde{x}_k + e_{k+1} \\
&= \left(I - P_h^{2h}\left(R_{2h}^h A_h P_h^{2h}\right)^{-1} R_{2h}^h A_h\right)\tilde{x}_k \\
&\quad + P_h^{2h}\left(R_{2h}^h A_h P_h^{2h}\right)^{-1} R_{2h}^h f
\end{aligned}
$$

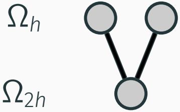
- complete iteration

$$
x_{k+1} = S^\mu\left(I - P_h^{2h}\left(R_{2h}^h A_h P_h^{2h}\right)^{-1} R_{2h}^h A_h\right) S^\nu x_k + g_k
$$

Recursion to create different cycles

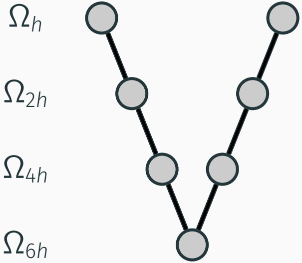
Solve the coarse problem using a two-grid method

Two-grid cycle



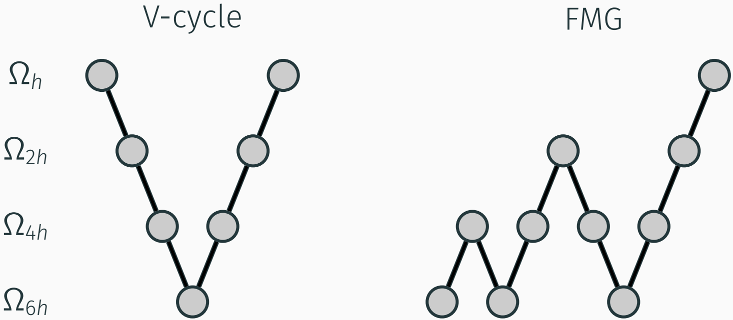
Solve the coarse problem using a two-grid method

V-cycle



Recursion to create different cycles

Solve the coarse problem using a two-grid method



Remarks

Multigrid methods may be used as preconditioners
Use the cycles to solve $M^{-1}z = r$, with 0 as an initial guess

Algebraic multigrid (AMG) is a black-box solver
Build the restriction + prolongation operators on the fly