Supplementary Material for "Optimal Personalized Incentive Design for Joint Mobility and Vehicle-to-Grid Services Based on a Stackelberg-Nash Game"

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In this supplementary material, we give the proof sketch for Lemma 1 and Proposition 1 in the paper "Optimal Personalized Incentive Design for Joint Mobility and Vehicle-to-Grid Services Based on a Stackelberg-Nash Game"

The detailed proof procedure is presented in the following.

Lemma 1. *Given any fixed $\boldsymbol{p}$, the corresponding lower-level Nash game possesses a unique NE.*

Proof. As shown in [1, Th. 2.3.3(B)], strong monotonicity of the pseudo-gradient field $F(\boldsymbol{p},\cdot)$ ensures both the existence and uniqueness of the Nash Equilibrium (NE). In the particular case where F is affine, this requirement becomes equivalent to the condition that the Jacobian matrix $\mathbf{J}F(\boldsymbol{p},\cdot)$ is positive definite [1, Th. 2.3.2(C)]. After an appropriate reordering of variables, $\mathbf{J}F(\boldsymbol{p},\cdot)$ takes a block-diagonal form $M=\mathrm{diag}(M_v,M_w)$, where $M_v:=\mathbf{J}_vF(\boldsymbol{p},\cdot)$ and $M_w:=\mathbf{J}_wF(\boldsymbol{p},\cdot)$. The strict convexity of f_i^{t} and f_i^{dev} ensures that $M_v\succ 0$ [2, Lemma 1]. In addition, for all $i\in\mathcal{N}$, the fact that $f_i^{\mathrm{e}}+f_i^{\mathrm{r}}$ are affine functions solely dependent on w implies that M_w is also positive definite. Consequently, M inherits the positive definiteness of its diagonal blocks, which verifies the lemma. □

Proposition 1. *Suppose that the sequence $\{\alpha^k\}_{k\in\mathbb{N}}$ is irreducible, non-negative, and satisfies the square-summability condition, i.e., $\sum_{k=0}^{\infty}(\alpha^k)^2<\infty$. Furthermore, consider another sequence $\{\sigma^k\}_{k\in\mathbb{N}}$ composed of non-negative real values, such that the associated weighted infinite series $\sum_{k=0}^{\infty}\alpha^k\sigma^k$ is convergent. Under the assumption that the step size parameter γ is chosen to be sufficiently small to ensure stability and descent of the overall process—one can conclude that every limit point (also referred to as an accumulation point) of the sequence ${\boldsymbol{p}^k}_{k\in\mathbb{N}}$ generated by the algorithm is a composite critical point of Stackelberg-Nash game.*

Proof. The applicability of [3, Th. 2] requires that Assumption 1 and Standing Assumptions 1–4 be satisfied. Because each f_i is a convex quadratic and Ω_i is polyhedral, Assumption 1 holds; moreover, Lemma 1 establishes $F(\boldsymbol{p},\cdot)$ is strongly monotone, and its uniform strong monotonicity and Lipschitz continuity directly result from the fact that it is affine in $(\boldsymbol{p},\boldsymbol{x})$. As F is affine, it satisfies Standing Assumptions 2–3, and since φ is semi-algebraic and Ω_{CSO} is convex and compact, Standing Assumption 4 is also met.

Assumption 1 and standing assumptions 1-4 are as follows:

Assumption 1: For each agent $i\in\mathcal{N}$, the cost function is

$$f_i(x,y)=\frac{1}{2}y_i^\top Q_iy_i+\left(\sum_{j\in\mathcal{N}}E_{i,j}y_j+E_{i,0}x+e_i\right)^\top y_i, \tag{1}$$

where $Q_i\in\mathbb{S}_+^{n_i}$, $E_{i,j}\in\mathbb{R}^{n_i\times n_j}$, $E_{i,0}\in\mathbb{R}^{n_i\times m}$, $e_i\in\mathbb{R}^{n_i}$.

Standing Assumption 1: For all $i\in\mathcal{N}$, and any fixed $x\in\mathcal{X}$, $y_{-i}\in\mathbb{R}^{n-n_i}$, the function $f_i(x,\cdot,y_{-i})$ is convex and continuously differentiable; the set $\mathcal{Y}_i(x)$ is of the form

$$\mathcal{Y}_i(x):=\{y_i\in\mathbb{R}^{n_i}\,|\,A_iy_i\le b_i+G_ix,\ \ C_iy_i=d_i+H_ix\}, \tag{2}$$

where $A_i\in\mathbb{R}^{p_i\times n_i}$ and $C_i\in\mathbb{R}^{r_i\times n_i}$ are full row rank, and $G_i\in\mathbb{R}^{p_i\times m}$, $b_i\in\mathbb{R}^{p_i}$, $H_i\in\mathbb{R}^{r_i\times m}$, $d_i\in\mathbb{R}^{r_i}$; $\mathcal{Y}(x)$ is nonempty and satisfies Slater's constraint qualification.

Standing Assumption 2: For any fixed $x\in\mathcal{X}$, $F(x,\cdot)$ is μ-strongly monotone and L_F-Lipschitz continuous.

Standing Assumption 3: The mapping F is definable[1], continuously differentiable, and there exist constants L_{JF1},L_{JF2} such

[1] *Definable* functions form a vast class of functions that encompasses virtually every objective function employed in optimization or machine learning, e.g., semialgebraic functions, exponentials, and logarithms. See [4, Def. 1.5] for a formal definition and [5, App. A.2] for a practitioner's perspective. If the f_i's are definable, so is F, while the inverse is not true [6, Remark. 8].

that the partial Jacobians of F satisfy

$$\|\mathbf{J}_1 F(x,y) - \mathbf{J}_1 F(x,y')\| \leq L_{JF1}\|y-y'\|, \tag{3}$$
$$\|\mathbf{J}_2 F(x,y) - \mathbf{J}_2 F(x,y')\| \leq L_{JF2}\|y-y'\|, \tag{4}$$

for any $y, y' \in \mathcal{Y}(x)$.

Standing Assumption 4: The function φ is definable and continuously differentiable in (x, y); its partial gradients $\nabla_1\varphi, \nabla_2\varphi$ are $L_{\varphi 1}$-, $L_{\varphi 2}$-Lipschitz continuous, respectively. The feasible set $\mathcal{X}$ is nonempty, convex, and compact.

$\square$

References

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