Machine Learning Exercise Sheet 1

Math Refresher

Group_369

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Problem 7

Let $\phi\left(x_1, x_2\right) = x_1 x_2$, we can observe that for all crosses $\phi\left(x_1, x_2\right) \leq 0$ and for all circles $\phi\left(x_1, x_2\right) \geq 0$, which means it is linearly separable. We can seperate the crosses and circles with a single hyperplane $\phi\left(x_1, x_2\right) = 0$.

Problem 8

On the boundry Γ, the $\boldsymbol{x}$ must realize

$$p\left(y=1 \mid \boldsymbol{x}\right) = p\left(y=0 \mid \boldsymbol{x}\right)$$

It is equivalent to

$$\log \frac{p\left(y=1 \mid \boldsymbol{x}\right)}{p\left(y=0 \mid \boldsymbol{x}\right)} = 0$$

We expand

$$\begin{aligned}
\log \frac{p\left(y=1 \mid \boldsymbol{x}\right)}{p\left(y=0 \mid \boldsymbol{x}\right)} &= \frac{\frac{1}{(2\pi)^{\frac{D}{2}} |\Sigma_1|^{\frac{1}{2}}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu_1})^T \Sigma_1^{-1} (x-\mu_1)} \cdot \pi_1}{\frac{1}{(2\pi)^{\frac{D}{2}} |\Sigma_0|^{\frac{1}{2}}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu_0})^T \Sigma_0^{-1} (x-\mu_0)} \cdot \pi_0} \\
&= \frac{1}{2} \boldsymbol{x}^T \left(\Sigma_0^{-1} - \Sigma_1^{-1}\right) \boldsymbol{x} + \boldsymbol{x}^T \left(\Sigma_1^{-1} \boldsymbol{\mu_1} - \boldsymbol{\Sigma_0}^{-1} \boldsymbol{\mu}_0\right) \\
&\quad - \frac{1}{2} \boldsymbol{\mu}_1^T \boldsymbol{\Sigma}_1^{-1} \boldsymbol{\mu}_1 + \frac{1}{2} \boldsymbol{\mu}_0^T \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\mu}_0 + \log \frac{\pi_1}{\pi_0} + \frac{1}{2} \log \frac{|\boldsymbol{\Sigma}_0|}{|\boldsymbol{\Sigma}_1|} \\
&= \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + c
\end{aligned}$$

where we define

$$
\begin{aligned}
\boldsymbol{A} &= \frac{1}{2}\left(\boldsymbol{\Sigma}_0^{-1} - \boldsymbol{\Sigma}_1^{-1}\right) \\
\boldsymbol{b} &= \Sigma_1^{-1}\boldsymbol{\mu_1} - \boldsymbol{\Sigma_0}^{-1}\boldsymbol{\mu}_0 \\
c &= -\frac{1}{2}\boldsymbol{\mu}_1^T\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1 + \frac{1}{2}\boldsymbol{\mu}_0^T\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0 + \log\frac{\pi_1}{\pi_0} + \frac{1}{2}\log\frac{|\boldsymbol{\Sigma}_0|}{|\boldsymbol{\Sigma}_1|}
\end{aligned}
$$