Hint: While this problem can be done using the Euler method, it is probably advisable, in order to conserve computer time, to use the Runge-Kutta algorithm.

3.26. Continue the previous problem, and construct the phase-space plots as in Figures 3.16 and 3.17 in the different regimes.

3.27. Show that the Poincaré sections in Figure 3.17 are independent of the initial conditions. For example, compare the attractor found for $x(0) = 1$, $y(0) = z(0) = 0$ with that found for $x(0) = 0$, $y(0) = z(0) = 1$.

***3.28.** Estimate qualitatively the Lyapunov exponent for a few trajectories of the Lorenz model near the transition to chaos at $r = 24.74\ldots$. Try to observe this exponent change from negative in the nonchaotic regime, to positive in the chaotic regime.

***3.29.** Explore the intermittency route to chaos for $r \geq 163$ in more detail. Begin by calculating z as a function of time for different values of r. Try $r = 163$ (which should be in the nonchaotic regime), and several larger values up to $r = 165$ or so. For the larger values of r you should observe chaotic "hiccups" like those found in Figure 3.18. Next calculate the average time between these hiccups and study how it diverges as the transition to chaos is approached While the idea here is easy to explain, writing a program to detect hiccups is a bit tricky. One way to accomplish this is to construct a histogram of times between adjacent maxima in $z(t)$. In the oscillatory (nonchaotic) regime these times will all be the same. An odd value signals a hiccup.

3.7 THE BILLIARD PROBLEM

So far we have considered two different chaotic systems, and you are probably willing to believe that there are many more. To get an appreciation for the different kinds of behavior that can be found, and also the common threads that run through this behavior, we will consider one more chaotic model in this chapter. Here we consider the problem of a ball moving without friction on a horizontal table. We imagine that there are walls at the edges of the table that reflect the ball perfectly and that there is no frictional force between the ball and the table. We can think of this as a billiard ball that moves without friction on a perfect billiard table.[26] The ball is given some initial velocity, and the problem is to calculate and understand the resulting trajectory. This is known as the stadium billiard problem.

Except for the collisions with the walls, the motion of the billiard is quite simple. Between collisions the velocity is constant so we have

$$\begin{aligned} \frac{dx}{dt} &= v_x \,, \qquad (3.30) \\ \frac{dy}{dt} &= v_y \,, \end{aligned}$$

where v_x and v_y change only through collisions with the walls. These equations can be solved using our usual Euler algorithm. Note that since the velocity is constant (except during the collisions), the Euler solution gives an exact description of the

[26]We will ignore any complications associated with the angular momentum of the ball, so it is better to think of this as a particle sliding on a frictionless sheet of ice. It would thus be more accurate to term this the "hockey puck" problem, but the name billiard is already firmly attached to the model.

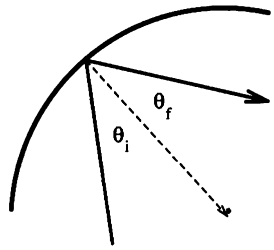

FIGURE 3.19: Geometry for perfect reflection of the billiard from a wall The angle of incidence is equal to the angle of reflection, $\theta_i = \theta_f$

motion across the table. The most difficult part of the calculation is the treatment of the collisions. Since we have assumed that they are perfectly elastic, the reflections will be mirrorlike, which means that the angle of incidence will be equal to the angle of reflection. These angles are defined in terms of the incoming and outgoing velocity vectors, and the vector normal to the wall at the location of the collision. This geometry is shown in Figure 3.19, where we have drawn a curved wall; our arguments apply just as well to a straight wall.

A numerical solution for the billiard's motion thus consists of two parts. First, when the billiard is away from the wall its motion is described by (3.30). These equations of motion are used to integrate forward in time, calculating x and y as functions of time. After each time step we must check to see if there has been a collision with one of the walls, that is, if the newly calculated position puts the billiard off the table. When this happens the program must backtrack to locate the position where the collision occurred. There are several ways to do this. One way is to back the billiard up to the position at the previous time step and then use a much smaller time step [for example, a factor of 100 smaller than the time step used to initially integrate (3.30)], so as to move the billiard in much smaller steps. When the billiard then goes off the table (again), we take the location after that iteration to be the point of collision.[27] After locating the collision point, we need to do a little vector manipulation. The initial velocity vector $\vec{v}_i \equiv (v_x, v_y)$ is already known, since v_x and v_y are known. We must next obtain the unit vector normal to the wall at the point of collision, $\hat{n}$. It is then useful to calculate the

[27]This approach will always yield a collision point that is off the table by a small amount. We can imagine other ways to locate the collision point, but in most cases they will never locate the collision point exactly. We have found that the approach described here yields results that are essentially identical to other methods for locating the collision point. A different method will be considered in the exercises.

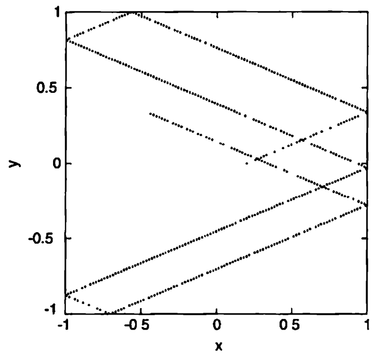

FIGURE 3.20: Trajectory of a billiard on a square table. Only the first few reflections are shown. The billiard started at $x = 0.2$, $y = 0$ with a speed of unity and with the velocity directed toward the upper right of the table The time step was 0.01.

components of $\vec{v}_i$ parallel and perpendicular to the wall. These are just

$$\vec{v}_{i,\perp} = (\vec{v}_i \cdot \hat{n})\,\hat{n} , \tag{3.31}$$
$$\vec{v}_{i,\parallel} = \vec{v}_i - \vec{v}_{i,\perp} .$$

Once we have the components of $\vec{v}_i$ we can reflect the billiard. A mirrorlike reflection reverses the perpendicular component of velocity, but leaves the parallel component unchanged (we'll leave it to the reader to show that this makes $\theta_f = \theta_i$ in Figure 3.19). Hence, the velocity after reflection from the wall is

$$\vec{v}_{f,\perp} = - v_{i,\perp} , \tag{3.32}$$
$$\vec{v}_{f,\parallel} = v_{i,\parallel} .$$

Some results are given in Figure 3.20, which show the first few bounces for a billiard on a square table. When (if) you write your own program for this problem, a graphical display of the trajectory is extremely useful in finding, and fixing, any errors. Two strong tests of the program are that the reflections should indeed be mirrorlike (usually referred to as *specular*), and that the energy, which is all kinetic for this problem, should be conserved. The trajectory on a square table is, as we might infer from Figure 3.20, very regular; it has a very simple predictable pattern. This is confirmed in Figure 3.21, which shows such a trajectory over a much longer time period.

Another way to graphically capture the regularity of the trajectory is with yet another phase-space plot. In Figure 3.21 we show a plot of v_x versus x, but

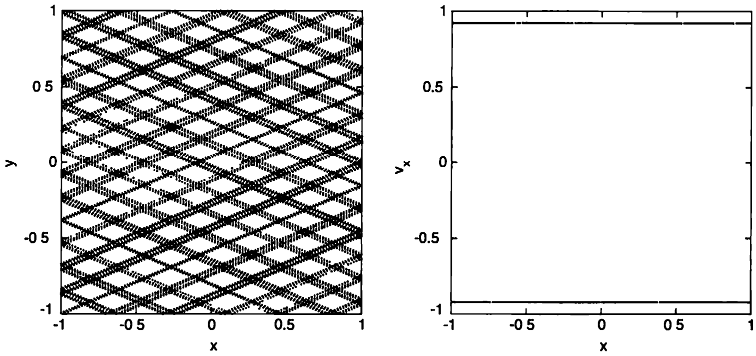

FIGURE 3.21: Left. trajectory of a billiard on a square table. This is a continuation of the trajectory shown in Figure 3.20. Right: corresponding Poincaré section derived from the trajectory shown on the left. Note that the phase-space plot was obtained from a much longer run of the program than was used for the trajectory plot. There are a few gaps visible in the phase-space plot; these would be filled in if the program were run for a longer period of time.

here we have not plotted every point of the phase-space trajectory. Rather, we have constructed another type of Poincaré section by plotting the points only when the billiard crosses the $y = 0$ axis.[28] We find two horizontal lines here; the billiard trajectories are all parallel to one of two different directions, so only two different values of v_x occur. Since the billiard can cross the $y = 0$ axis anywhere, the values of x in this plot vary continuously from -1 to $+1$.

The behavior of the billiard gets more interesting when we consider other table shapes. There are many possibilities; here we will consider only one, the so-called stadium shape, which can be described as follows. Imagine a circular table of radius $r = 1$, as shown on the left side of Figure 3.22. Now cut the table along the x axis, and pull the two semicircular halves apart (along y), a distance $2\alpha r$. Then fill in these two open sections with straight segments. Thus $\alpha = 0$ yields a circular table, while nonzero values of α give a table with a more traditional stadium shape. Figure 3.22 compares trajectories for a circular table with those for a table with $\alpha = 0.01$. While the trajectories depend on the initial conditions (the initial values of x, y, and $\vec{v}$), the results for the circular table are always highly symmetric. On the other hand, the trajectory for the $\alpha = 0.01$ stadium is much more complicated and is definitely not symmetric, except for very special initial conditions (such conditions were not used in Figure 3.22).[29] This should remind you of chaotic motion.

[28] This should remind you of how we dealt with the Lorenz model.

[29] Examples of such special, nonchaotic initial conditions are $x(0) = y(0) = 0$, with $\vec{v}$ parallel to either the x or y axis.

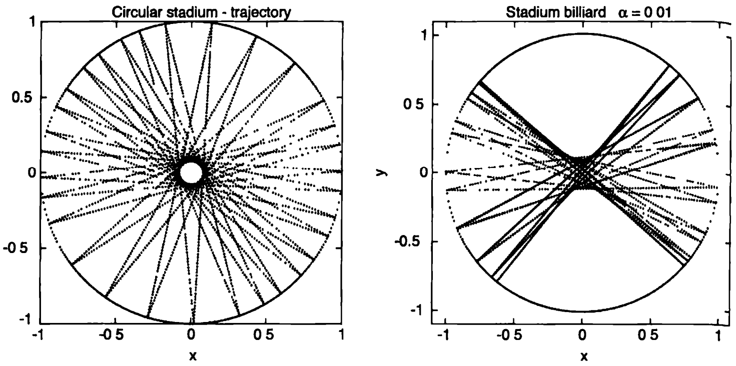

FIGURE 3.22: Left: trajectory of a billiard on a circular table, right. trajectory of a billiard on a stadium-shaped table with $\alpha = 0\ 01$.

The corresponding phase-space plots of v_x versus x (constructed as in Figure 3.21) are shown in Figure 3.23. The very ordered pattern for the circular table confirms our impression from the trajectories, that this is a nonchaotic system. However, for the $\alpha = 0.01$ stadium the phase-space plot is somewhat reminiscent of the chaotic attractor we found for the pendulum problem; it is indeed chaotic. Two more phase-space plots for other stadium shapes are shown in Figure 3.24, and both are seen to be chaotic.

A hallmark of a chaotic system is an extreme sensitivity to initial conditions. This property is also found in the billiard problem, as can be seen if we calculate the trajectories of two billiards with slightly different initial conditions. An example is shown in Figure 3.25 where we plot the distance between two billiards as a function of time. The billiards were on a chaotic table ($\alpha = 0.01$) and were given the same initial velocities, but were started a distance 1×10^{-5} apart (recall that the table has a radius of approximately 1 unit). The billiard separation shows a very sharp dip after about every one time unit. These dips occur when the billiards collide with the walls, as this causes their trajectories to cross. The overall separation is seen to increase very rapidly with time (note the logarithmic scale). The divergence of these trajectories can be described by a Lyapunov exponent, as we found for the pendulum.[30]

A remarkable feature of our results for the billiard problem is that the chaotic behavior is evident even for very small values of α. It turns out that the stadium billiard is chaotic for *any* nonzero value of α. In fact, only tables with very high

[30] To calculate the Lyapunov exponent quantitatively we would would have to average the behavior over different initial conditions, so as to smooth out the irregularities in Figure 3.25

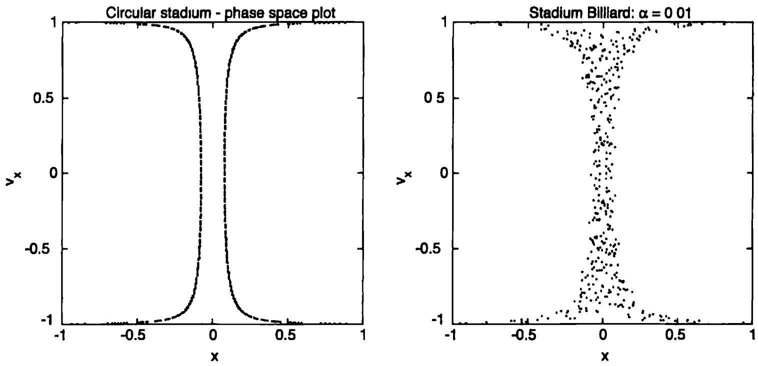

FIGURE 3.23: Phase-space plots for the trajectories shown in Figure 3.22. Left: for a circular-shaped table; right: for a stadium-shaped table with $\alpha = 0.01$ These were constructed by plotting points only when $y = 0$.

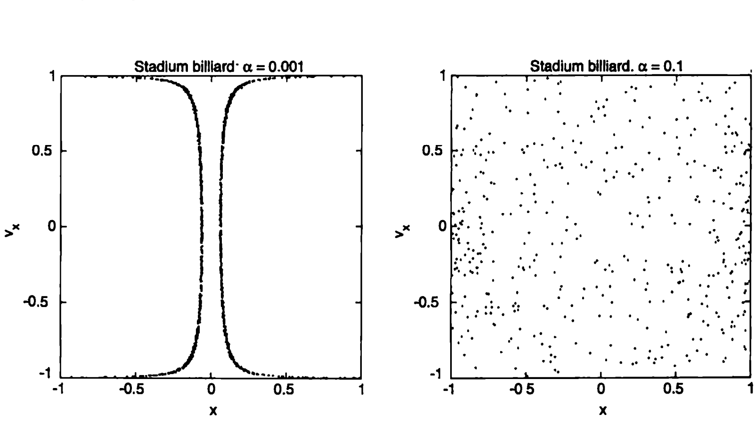

FIGURE 3.24: Phase-space plots for two more stadium-shaped tables. Left: for a table with $\alpha = 0.001$; right· table with $\alpha = 0.1$.

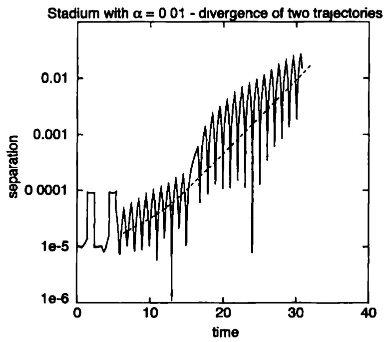

FIGURE 3.25: Divergence of the trajectories of billiards started at slightly different initial conditions, for a stadium-shaped table with $\alpha = 0.01$. The dashed line is drawn to emphasize the rapid overall increase of the separation with time The initial separation of the billiards was 1×10^{-5}.

symmetry are nonchaotic. The billiard problem may be relevant for describing the motion of gas molecules in a container. Our results suggest that for any realistically shaped container (i.e., any shape that is not extremely symmetric, such as the perfectly circular table) such motion is likely to be chaotic and thus unpredictable. This finding will be relevant to our discussion of entropy and the approach to equilibrium in Chapter 7.

EXERCISES

3.30. Investigate the Lyapunov exponent of the stadium billiard for several values of α. You can do this qualitatively by examining the behavior for only one set of initial conditions for each value of α you consider, or more quantitatively by averaging over a range of initial conditions for each value of α

***3.31.** Study the behavior for other types of tables. One interesting possibility is a square table with a circular interior wall located either in the center, or slightly off-center. Another possibility is an elliptical table.

***3.32.** The key part of a program for the billiard program is the treatment of collisions with the wall of the stadium, and one way of doing this was described above. Another way is to use the exact solution of (3.30) to compute the trajectory and then solve analytically (using the equation that specifies the perimeter of the stadium) for the location of the collision. Write a program that uses this method and compare your results with those given in this section.

3.8 BEHAVIOR IN THE FREQUENCY DOMAIN: CHAOS AND NOISE

Our intuitive ideas concerning what it means to be chaotic usually include some connection with terms such as *random*, *unpredictable*, and *noisy*. We have already