Supplementary Material for "Analyzing Power and Dynamic Traffic Flows in Coupled Power and Transportation Networks"

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In the supplementary material, we give the proof for the equivalence between the optimal control problem (14)-(20) and the dynamic user-optimal state in the paper "Analyzing Power and Dynamic Traffic Flows in Coupled Power and Transportation Networks".

I. Equivalence Analysis for Problem (14)-(20)

The proof follows a similar procedure given in [S1]. To demonstrate the equivalence between the optimal control problem (14)-(20) and user-optimal traffic assignment, we need to show that the flow pattern obtained by problem (14)-(20) satisfies the dynamic user-optimal conditions.

First, the Hamiltonian is formulated as

$$\begin{aligned}
\mathcal{H} \;=\; & \sum_{a}\left\{\int_{0}^{u_a(t)} g_a(w)dw\right\} + \sum_{w}\sum_{ar}\gamma_{ar,w}(t)\left[u_{ar,w}(t)-v_{ar,w}(t)\right] \\
& + \sum_{\mathrm{o}}\sum_{\mathrm{d}\neq\mathrm{o}}\sum_{r}\kappa_{r,w}(t)e_{r,w}(t) + \sum_{w}\sum_{j\neq\mathrm{o,d}}\sum_{r}\sigma_{jr,w}(t)\left[\sum_{a\in\mathcal{D}(j)}v_{ar,w}(t)-\sum_{a\in\mathcal{C}(j)}u_{ar,w}(t)\right] \\
& + \sum_{\mathrm{d}}\sum_{\mathrm{o}\neq\mathrm{d}}\sigma_{\mathrm{o},w}(t)\left[f_w(t)-\sum_{a\in\mathcal{C}(\mathrm{o})}\sum_{r}u_{ar,w}(t)\right] + \sum_{\mathrm{o}}\sum_{\mathrm{d}\neq\mathrm{o}}\sigma_{\mathrm{d},w}(t)\left[\sum_{a\in\mathcal{D}(\mathrm{d})}\sum_{r}v_{ar,w}(t)-e_w(t)\right] \\
& + \sum_{r,w}\sum_{j\neq\mathrm{o}}\sum_{a\in\mathcal{D}(j)}\mu_{ar,w}(t)\left\{x_{ar,w}(t)+\sum_{b\in\tilde{r}}x_{br,w}(t)+E_{r,w}(t)-\sum_{b\in\tilde{r}}x_{br,w}\left[t+\bar{t}_a(t)\right]-E_{r,w}\left[t+\bar{t}_a(t)\right]\right\}
\end{aligned} \tag{1}$$

where $\gamma_{ar,w}(t)$ and $\kappa_{r,w}(t)$ are Lagrange multipliers associated with arc state equations (1) and destination node state equations (7), respectively, $\sigma_{jr,w}(t)$, $\sigma_{\mathrm{o},w}(t)$, $\sigma_{\mathrm{d},w}(t)$ are Lagrange multipliers associated with the node flow conservation constraints for intermediate nodes (4), origin nodes (5) and destination nodes (6), respectively, and $\mu_{ar,w}(t)$ are Lagrange multipliers associated with flow propagation constraints (11).

For each arc a with vehicles flowing from node m to node n, the first-order necessary conditions for the optimal control problem (14)-(20) are

$$\frac{\partial\mathcal{H}}{\partial u_{ar,w}(t)} \;=\; g_a[u_a(t)]+\gamma_{ar,w}(t)-\sigma_{\mathrm{m}r,w}(t) \;\geq\; 0, \quad \forall\mathrm{m}; a\in\mathcal{C}(\mathrm{m}), r, w \tag{2}$$

$$u_{ar,w}(t)\frac{\partial\mathcal{H}}{\partial u_{ar,w}(t)} \;=\; 0, \quad \forall a, r, w \tag{3}$$

$$\frac{\partial\mathcal{H}}{\partial v_{ar,w}(t)} \;=\; -\gamma_{ar,w}(t)+\sigma_{\mathrm{n}r,w}(t) \;\geq\; 0, \quad \forall\mathrm{n}; a\in\mathcal{D}(\mathrm{n}), r, w \tag{4}$$

$$
\begin{aligned}
v_{ar,w}(t)\frac{\partial \mathcal{H}}{\partial v_{ar,w}(t)} &= 0, \quad \forall a,r,w && (5)\\
\frac{\partial \mathcal{H}}{\partial e_{r,w}(t)} \ \geq \ 0, \quad e_{r,w}(t)\frac{\partial \mathcal{H}}{\partial e_{r,w}(t)} &= 0, \quad \forall r,w && (6)\\
\frac{d\gamma_{ar,w}(t)}{dt} &= -\frac{\partial \mathcal{H}}{\partial x_{ar,w}(t)}, \quad \forall a,r,w && (7)\\
\frac{d\kappa_{r,w}(t)}{dt} &= -\frac{\partial \mathcal{H}}{\partial E_{r,w}(t)}, \quad \forall r,w && (8)\\
&\text{Non-negativity constraints: (9)-(10)} && (9)
\end{aligned}
$$

Note that $\sigma_{\mathrm{o}r,w}(t) = \sigma_{\mathrm{o},w}(t)$ when m is an origin, i.e., $\mathrm{m} = \mathrm{o}$.

By adding up (2) and (4), for each arc a pointing from nodes m to n, we obtain the following:

$$
\frac{\partial \mathcal{H}}{\partial u_{ar,w}(t)} + \frac{\partial \mathcal{H}}{\partial v_{ar,w}(t)} \ = \ t_a(t) - \sigma_{\mathrm{m}r,w}(t) + \sigma_{\mathrm{n}r,w}(t) \ \geq \ 0, \quad \forall a \in \mathcal{C}(\mathrm{m}) \cap \mathcal{D}(\mathrm{n}); r, w \tag{10}
$$

For route r between origin o and destination d, let node k denote node o or any intermediate node on this route. Let $\tilde{r}$ denote a subroute from node k to node d of route r $(k, 1, 2, \cdots, n, \mathrm{d})$. The instantaneous travel time for route $\tilde{r}$ is given by

$$
t_{\tilde{r},k\mathrm{d}}(t) \ = \ \sum_{a \in \tilde{r},k\mathrm{d}} t_a(t), \quad \forall\, k \in r, w \tag{11}
$$

Consider a set of routes r from $\mathrm{o}{\to}k{\to}\mathrm{d}$ and the corresponding set of subroutes $\tilde{r}$. The flow conservation constraints for node k are given by

$$
\sum_{a \in \mathcal{D}(k)} v_{ar,w}(t) \ = \ \sum_{a \in \mathcal{C}(k)} u_{ar,w}(t), \quad \forall\, k, r, w; k \neq \mathrm{o}, \mathrm{d} \tag{12}
$$

The fourth term in the Hamiltonian function is revised as

$$
\sum_{w} \sum_{k \neq \mathrm{o},\mathrm{d}} \sigma_{k,w}(t) \sum_{r} \left[\sum_{a \in \mathcal{D}(k)} v_{ar,w}(t) - \sum_{a \in \mathcal{C}(k)} u_{ar,w}(t) \right] \tag{13}
$$

since $\sigma_{kr,w}(t) = \sigma_{k,w}(t)$ for the set of subroutes $\tilde{r}$.

Note that the first-order necessary conditions (2)-(10) hold for the set of subroutes $\tilde{r}$. Therefore, based on (3), (5) and (10), the route cost for the set of subroutes equals

$$
\begin{aligned}
t_{\tilde{r},k\mathrm{d}}(t) = [\sigma_{k,w}(t) - \sigma_{1,w}(t)] + [\sigma_{1r,w}(t) - \sigma_{2r,w}(t)] + \cdots + \left[\sigma_{(n-1)r,w}(t) - \sigma_{nr,w}(t)\right] \\
+ [\sigma_{nr,w}(t) - \sigma_{\mathrm{d},w}(t)] \ = \ \sigma_{k,w}(t) - \sigma_{\mathrm{d},w}(t)
\end{aligned} \tag{14}
$$

Therefore, routes that are being utilized from nodes k to d at time t have identical travel cost $\sigma_{k,w}(t) - \sigma_{\mathrm{d},w}(t)$. Moreover, since the objective function is convex with respect to control variables, the optimal solution is unique, and the proof is complete.

References

[S1] B. Ran and D. Boyce, *Dynamic Urban Transportation Network Models: Theory and Implications for Intelligent Vehicle-Highway Systems*. Verlag Berlin Heidelberg: Springer, 1994.