Supplementary Material for "Power-traffic Network Equilibrium Incorporating Behavioral Theory: A Potential Game Perspective"

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In the supplementary material, we give the detailed proof for Proposition 3 that the game Ξ is a potential game with potential function $\Phi = -F_{\mathrm{T}} - F_{\mathrm{E}} + \sum_{a \in \mathcal{A}^{\mathrm{cs}}} \lambda_j p^{\mathrm{ev}} v_a$ in the paper "Power-Traffic Network Equilibrium Incorporating Behavioral Theory: A Potential Game Perspective".

The detailed proof procedure is presented in the following.

To begin with, we restate Proposition 1 (in the original manuscript) that characterizes the Wardrop equilibrium in the transportation network [R1].

Proposition 1. A Wardrop equilibrium is achieved when the allocation of traffic flow conforms to the following equation:

$$c_{rw} + \frac{1}{\theta} \ln f_{rw} \ = \ c_{jw} + \frac{1}{\theta} \ln f_{jw}, \quad \forall r, j \in \mathcal{R}_w, w \in \mathcal{W} \tag{R1}$$

Proof. The proof follows immediately from the route choice probability (R2):

$$P_{rw} \ = \ \frac{\exp\left(-\theta c_{rw}\right)}{\sum_{j \in \mathcal{R}_w} \exp\left(-\theta c_{jw}\right)}, \quad \forall r \in \mathcal{R}_w, w \in \mathcal{W} \tag{R2}$$

The brief idea is to demonstrate that (R1) implies the route choice probability (R2). To see this, we rearrange (R1) as follows:

$$\exp\left(\theta c_{rw}\right) \cdot \exp\left(-\theta c_{jw}\right) \cdot f_{rw} \ = \ f_{jw} \tag{R3}$$

We sum over all routes connecting O-D pair w by subscript j on both sides of (R3), which yields

$$\exp\left(\theta c_{rw}\right) \cdot \sum_j \exp\left(-\theta c_{jw}\right) \cdot f_{rw} \ = \ \sum_j f_{jw} \tag{R4}$$

Since the right hand side of (R4) satisfies $\sum_j f_{jw} = d_w$, equation (R1) gives the route choice probability (R2), i.e., a Wardrop equilibrium flow. $\square$

Equation (R1) shows that the term $c_{rw} + \frac{1}{\theta} \ln f_{rw}$ is identical for all routes between O-D pair w, where the second term is added for representing the stochastic behavior [R2]. Recall that Wardrop's first principle states that no driver can reduce his/her expected cost by unilaterally switching routes [R3]. We presume that this value is the minimal expected cost. The following proposition provides proof for this conjecture.

Proposition 2 (Modified from Lemma 2 of [R4])**.** Assume that both travel and queuing time are strictly increasing with traffic flow. Given the electricity price $\boldsymbol{\lambda}^*$, a Wardrop equilibrium flow pattern $(\mathbf{f}^*, \boldsymbol{\lambda}^*)$ with fixed O-D demand is equivalent to the following variational inequality problem, i.e., find a feasible route flow vector $\mathbf{f}$, such that

$$\left(\mathbf{c}\left(\mathbf{f}^*, \boldsymbol{\lambda}^*\right) + \frac{1}{\theta} \ln \mathbf{f}^*\right)^T \left(\mathbf{f} - \mathbf{f}^*\right) \;\geq\; 0 \tag{R5}$$

where $\mathbf{c} := (c_{rw}, r \in \mathcal{R}_w, w \in \mathcal{W})$ is the vector of actual route costs.

Proof. Letting $\nabla J = \mathbf{c}\left(\mathbf{f}, \boldsymbol{\lambda}^*\right) + (1/\theta)\mathbf{f}$, it suffices to show that $\mathbf{f}^*$ is a solution of the VI problem (R5) if it solves the following optimization problem [R5]:

$$\begin{aligned}
\min_{f_{rw}} \quad & F_{\mathrm{T}} \;:=\; \sum_{a \in \mathcal{A}} \int_0^{v_a(f)} c_a\left(\omega, \lambda_j\right) d\omega + \frac{1}{\theta} \sum_{w \in \mathcal{W}} \sum_{r \in \mathcal{R}_w} f_{rw}\left(\ln f_{rw} - 1\right) && \text{(R6)} \\
\text{s.t.} \quad & \sum_{r \in \mathcal{R}_w} f_{rw} \;=\; d_w, \quad \forall w \in \mathcal{W} && \text{(R7)} \\
& \sum_{w \in \mathcal{W}} \sum_{r \in \mathcal{R}_w} f_{rw} \delta_{ar} \;=\; v_a, \quad \forall a \in \mathcal{A} && \text{(R8)} \\
& f_{rw} \;\geq\; 0, \quad \forall r \in \mathcal{R}_w, w \in \mathcal{W} && \text{(R9)}
\end{aligned}$$

where constraints (R7) require that the traffic flow on routes connecting O-D pair w must equal the O-D demand, constraints (R8) require that the flow on arc a is the sum of the route flows that use arc a, $\delta_{ar} = 1$ if arc a belongs to route r and $\delta_{ar} = 0$ otherwise, and constraints (R9) enforces the non-negativity condition on route flows.

Note that the derivative of the first term of F_{T} yields

$$\frac{\partial}{\partial f_{rw}} \left(\sum_{a \in \mathcal{A}} \int_0^{v_a(f)} c_a\left(\omega, \lambda_j\right)\, d\omega \right) \;=\; \sum_{a \in \mathcal{A}} c_a \frac{\partial v_a(f)}{\partial f_{rw}} \;=\; \sum_{a \in \mathcal{A}} c_a \delta_{ar} \;=\; c_{rw}$$

by virtue of (R8).

Our next step is to show that the solution of the problem (R6)-(R9) is a Wardrop equilibrium. The Lagrangian with respect to the constraints (R7) can be formulated as

$$\mathcal{L}_{\mathrm{T}} \;=\; F_{\mathrm{T}} + \sum_{w \in \mathcal{W}} \mu_w \left(\sum_{r \in \mathcal{R}_w} f_{r,w} - d_w \right) \tag{R10}$$

where μ_w denotes the Lagrange multiplier associated with constraints (R7). Since the problem (R6)-(R9) is convex, a necessary and sufficient condition for the optimal solution is given by

$$\frac{\partial \mathcal{L}_{\mathrm{T}}}{\partial f_{rw}} = c_{rw} + \frac{1}{\theta} \ln f_{rw} + \mu_w = 0, \quad \forall r \in \mathcal{R}_w, w \in \mathcal{W} \tag{R11}$$

Observe that μ_w is equal for all routes between O-D pair w, which yields (R1), and the proof is complete. □

The Wardrop traffic model is a potential game with potential F_{T} that decreases whenever a traveler shifts from one route to another with lower perceived costs [R6, R7]. Interested readers may refer to [R8–R10] for proof. The following proposition shows that the game Ξ is also a potential game.

Proposition 3. The game Ξ is a potential game with potential function Φ. The solution of the optimization problem

$$\min_{\mathbf{f},\mathbf{p}^g} \quad \Phi := F_{\mathrm{T}} + F_{\mathrm{E}} - \sum_{a \in \mathcal{A}^{\mathrm{cs}}} \lambda_j p^{\mathrm{ev}} v_a \tag{R12}$$

$$\text{s.t.} \quad \text{Transportation constriants: (R7)-(R9)} \tag{R13}$$

$$p_j = P_{ij} - R_{ij} L_{ij} - \sum_{k:(j,k) \in \mathcal{B}} P_{jk}, \quad \forall j \in \mathcal{M} \tag{R14}$$

$$q_j = Q_{ij} - X_{ij} L_{ij} - \sum_{k:(j,k) \in \mathcal{B}} Q_{jk}, \quad \forall j \in \mathcal{M} \tag{R15}$$

$$\nu_j = \nu_i - 2\left(R_{ij} P_{ij} + X_{ij} Q_{ij}\right) + \left(R_{ij}^2 + X_{ij}^2\right) L_{ij}, \quad \forall (i,j) \in \mathcal{B} \tag{R16}$$

$$L_{ij} \geq \frac{P_{ij}^2 + Q_{ij}^2}{\nu_i}, \quad \forall (i,j) \in \mathcal{B} \tag{R17}$$

$$p_j := -p_j^g + p_j^l + p^{\mathrm{ev}} v_a, \quad \forall j \in \mathcal{M}^{\mathrm{cs}}, a \in \mathcal{A}^{\mathrm{cs}} \tag{R18}$$

$$\underline{p}_j^g \leq p_j^g \leq \bar{p}_j^g, \quad \underline{q}_j^g \leq q_j^g \leq \bar{q}_j^g, \quad \forall j \in \mathcal{M}^{\mathrm{g}} \tag{R19}$$

$$\underline{\nu}_j \leq \nu_j \leq \bar{\nu}_j, \quad \forall j \in \mathcal{M} \tag{R20}$$

is a Nash equilibrium of the game Ξ.

Proof. We first form the partial Lagrangian with respect to constraints (R7) and (R14) as follows:

$$\begin{aligned} \mathcal{L} = {} & F_{\mathrm{T}} + F_{\mathrm{E}} - \sum_{a \in \mathcal{A}^{\mathrm{cs}}} \lambda_j p^{\mathrm{ev}} v_a + \sum_{w \in \mathcal{W}} \mu_w \left(\sum_{r \in \mathcal{R}_w} f_{rw} - d_w \right) \\ & + \sum_{j=1}^{n} \lambda_j \left(-p_j^g + p_j^l + p^{\mathrm{ev}} v_a - P_{ij} + R_{ij} L_{ij} + \sum_{k:(j,k) \in \mathcal{B}} P_{jk} \right) \end{aligned} \tag{R21}$$

where λ_j is the Lagrange multipliers associated with (R14), and n is the number of buses in the power network.

The following optimality conditions need to hold for the route flows:

$$f_{rw}\frac{\partial \mathcal{L}}{\partial f_{rw}} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial f_{rw}} \geq 0, \quad \forall r \in \mathcal{R}_w, w \in \mathcal{W} \tag{R22}$$

where

$$\frac{\partial \mathcal{L}}{\partial f_{rw}} = \frac{\partial \mathcal{L}_{\mathrm{T}}}{\partial f_{rw}} + \frac{\partial}{\partial f_{rw}} \left(- \sum_{a \in \mathcal{A}^{\mathrm{cs}}} \lambda_j p^{\mathrm{ev}} v_a + \sum_{j=1}^{n} \lambda_j p^{\mathrm{ev}} v_a \right) \tag{R23}$$

where by virtue of (R18), charging demand $p^{\mathrm{ev}} v_a$ only appears in the power balance constraints (R14) when traversing a charging arc $a \in \mathcal{A}^{\mathrm{cs}}$. Hence, (R23) yields

$$\frac{\partial \mathcal{L}}{\partial f_{rw}} = \frac{\partial \mathcal{L}_{\mathrm{T}}}{\partial f_{rw}} \tag{R24}$$

Therefore, (R22) is equivalent to (R11) and the equivalence between problem (R12)-(R20) and (R6)-(R9) is established.

For the generators, a necessary and sufficient condition for a minimum $\mathbf{p}^g > 0$ is to satisfy (since we adopt the definitions in [R9] that the maximum of a potential function is a Nash equilibrium, its Lagrangian is the negative of $\mathcal{L}$)

$$-\frac{\partial \mathcal{L}}{\partial p_j^g} = \lambda_j - \frac{\partial G_j \left(p_j^g \right)}{\partial p_j^g} = 0 \tag{R25}$$

such that $\partial \mathcal{L} / \partial p_j^g = \partial \Lambda_j / \partial p_j^g$ holds. Consequently, Φ is a potential function for the generators [R11], and the proof is complete. $\square$

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