Statistics 110: Introduction to Probability — Section Midterm Review, Fall 2018

Practice Problems

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Don't Make Category Errors!

Characterize the following as a probability, sample space/support, event, random variable, or distribution

(a) $Binom(n = 1, p = 0.25)$. This is the same thing as $Bern(0.25)$

(b) Ω

(c) $x|y$

(d) λ, where $\lambda \in S$ and $S = \{0, 1, 2, 3, \cdots, \infty\}$

(e) S, where $\lambda \in S$ and $S = \{0, 1, 2, 3, \cdots, \infty\}$

(f) p, where p = -$\frac{1}{2}$

(g) 0.3145

(h) I_j, where $I_j \sim Bern(0.5)$

(i) All x where $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$

(j) All $P(X = x)$ where $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$

(k) $Z = z$, where $z \in \Omega$

(l) P, where $P \in (0, 1)$

(m) $\varnothing$

(n) $P(Y) = 80\%$, where Y is true when $X = x$

(o) Y, where Y is 0 when $X = x$

(p) $\frac{P(A|B)P(B)}{P(A)}$

(q) $\sum_{k=0}^{n} \binom{n}{k} (\frac{1}{4})^k (\frac{3}{4})^{n-k}$, where $n > 0$

(r) $\lim_{n\to\infty}(1 - \frac{1}{n})^n$

(s) X, where $X = \sum_{k=1}^{N} \frac{1}{k}$, and $N > 1$

(t) $X = x|N = n$, where $X = \sum_{k=1}^{N} \frac{1}{k}$, and $N > 1$

(u) $N - X = x|N = n$, where $X = \sum_{k=1}^{N} \frac{1}{k}$, and $N > 1$

(v) $N - X|N = n$, where $N \sim HGeom(w = 3, b = 5, k = 2)$ and $X \sim HGeom(w = 3, b = 5, k = 2)$

(w) $Y^2|X$ where $Y|X \sim NBin(r, p)$

(x) Everything to the right of the "|" in $A|B, C, D, \cdots$

(y) Your score on the midterm

(z) You acing the midterm!

Counting

In the game of Avalon, 7 players are dealt four “good" characters and three “bad" characters. Consider the following:

(a) How many possible arrangements of unique good and bad players are there?

(b) We now make a distinction between generic “goodies" and two special good players, Merlin and Percival. How many more possible arrangements of unique good and bad players are there?

(c) In addition to the two special good characters, we now make a distinction between generic “baddies" and two special bad players, Mordrid and Morgana. How many more possible arrangements of unique good and bad players are there?

(d) A “probabilistic" version of Avalon is played where we still have 7 players but 14 cards are dealt, with the pack containing 7 “good" characters and 7 “bad" characters, no special characters. How many possible arrangements of unique good and bad players are there? (Hint: consider the extreme case where all players are “good" characters)

(e) Assume we play the “independent" version of Avalon where 7 freshmen are dealt 7 cards, with the pack containing the four special cards (Merlin, Percival, Mordrid, and Morgana) in addition to 2 generic good and 1 generic bad cards. We now assign all the cards to all players randomly such that each card has the equal probability of being assigned to any one of the student regardless of whether that particular student has been already dealt a card. This will mean it could be the case where one student possesses all of the cards and the other 6 students possesses none of the cards. Additionally, by the pigeonhole principle, this will mean there will be at least one freshman with at least 1 card. How many possible arrangements of cards are **played among the seven players**?

Conditional Probability

In a game of Avalon, players go on missions, where goodies pass missions and baddies try to fail missions. After each mission, everybody discusses who they think are baddies, switching out a supposed baddy on the team with a supposed goody. Therefore, it could be in a baddy's best interest to pass a mission in order to stay on the team so he can fail a mission in the future.

The game starts, and you must propose a mission. So far, n missions have passed, but there is a possibility that a baddie has been on the team this whole time and will fail the next mission.

We want to find whether this next mission consisting of you (a "goodie") and the existing team consists of all "goodies." Let G be the event that this team consists of all goodies, $p = P(G)$ be the probability that this team consists of all goodies, and $q = 1 - p$. Let M_j be the event that the j-th mission passes.

(a) Assume for this part that each mission passed is conditionally independent given that this team consists of all goodies and, similarly, that each mission passed is conditionally independent given that this team consists at least one baddie. Let $a = P(M_j|G)$ and $b = P(M_j|G^c)$, where a and b don't depend on j. Find the posterior probability that the team contains all goodies given that all n of the n missions have passed.

(b) Suppose that the mission is passed on all n tests. However, it is likely that Merlin has voiced out all of the goodies who should be on the team since Merlin knows all the goodies. Therefore, all missions will pass. However, Merlin disclosing this information is not desirable since the baddies win the game if they guess correctly who Merlin is.

Let X be the event that Merlin discloses information. Assume the probability that Merlin discloses information is $P(X) = c$ and that G and X are independent. If Merlin is *not* on the team, then the missions are conditionally independent on G. Let $a_0 = P(M_j|G, X^c)$ and $b_0 = P(M_j|G^c, X^c)$. Find the posterior probability that the team consists of all goodies given that all n missions have passed.

Expected Values and Standard Deviation of Indicator Random Variables

Suppose 300 people are in Stat110, each completed a test. We make 300 copies of each person's test, grade these 300^2 copies and mix the tests, handing them out randomly to each person.

(a) What is the expected number of people who get their own test?

(b) What is the variance of the number of people who get their own test?

(c) Can we use a Poisson to approximate the probability of there being two people who receive their test back?

Inequality Practice

Let X and Y be independent and identical (i.i.d) random variables. I_j is an indicator variable.

Fill out the inequalities ($\leq, \geq, =$). Also, spot the category error.

(a)
$$E[Y|2X = x] \quad = \quad E[Y|X = x]$$

(b)
$$P(X + Y = 2) \quad \geq \quad P(X = 1)P(Y = 1)$$

(c)
$$E[X|Y] \quad ? \quad E[X|Y = 2]$$

(d)
$$E[\frac{X^2}{X^2 + Y^2}] \quad \geq \quad \frac{1}{4}$$

(e)
$$Var(X^2) \quad \leq \quad E(X^4)$$

(f)
$$E[X^2] \quad \geq \quad E[XY]$$

(g)
$$E[2X + 2] \quad \leq \quad 2E[X] + 4$$

(h)
$$E[|X|] \quad \geq \quad |E[X]|$$

(i)
$$E[I_j] + P(I_j = 0) \quad = \quad 1$$

(j)
$$E[I_j(1 - I_j)] \quad \leq \quad E[1 - I_j]$$

Min and Max

Let $X \sim \text{Bin}\left(n, \frac{1}{2}\right)$ and $Y \sim \text{Bin}\left(n+1, \frac{1}{2}\right)$ independently.

(a) Let V = $\min(X, Y)$ be the smaller of X and Y, and let $W = \max(X, Y)$. So if X crystalizes to x and Y crystalizes to y, then V crystalizes to $\min(x, y)$ and W crystalizes to $\max(x, y)$. Find $E(V) + E(W)$.

(b) Show that $E|X - Y| = E(W) - E(V)$, with notation as in (a).

(c) Compute $\text{Var}(n - X)$.

True and False

(a) If X and Y have the same CDF, they have the same expectation.

(b) If X and Y have the same expectation, they have the same CDF.

(c) If X and Y have the same CDF, they must be dependent.

(d) If X and Y have the same CDF, they must be independent.

(e) If X and Y have the same distribution, $P(X < Y)$ is at most $^1/_2$.

(f) If X and Y are independent and have the same distribution, $P(X < Y)$ is at most $^1/_2$.

(g) If X and Y share the same expected values, symmetry can be applied.

(h) It is impossible for X and Y to be dependent and share the same distribution without being identical.