Conditional Probability

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Conditional Probability

Conditional Probability - Suppose we observe event B and are interested in the probability of event A occurring given this information. Then,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Bayes' Rule - This is arguably one of the most important concepts and tools you will learn in this course.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bridging Conditional Probability and Sets

An intuitive way to visualize conditional probability is to think about the intersection of sets. In order to find the intersection of two different sets A and B, we establish one of these sets to be our sample space and find the likely occurrence of the other set within this established sample space.

$$P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)$$

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots A_n) = P(A_1|A_2 \cap A_3 \cap \cdots A_n)P(A_2|A_3 \cap \cdots A_n) \cdots P(A_{n-1}|A_n)$$

Law of Total Probability (LOTP)

A common theme in this course is that it is far easier to solve a problem by breaking it down into smaller, simpler components than tackling it head-on. LOTP is one such tools that will allow you to do so. Suppose you want to find the probability of some event B, and you can partition the sample space into disjoint events $A_1, A_2, \ldots, A_n$. Then,

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i) \quad (1)$$

$$= \sum_{i=1}^{n} P(B \cap A_i) \quad (2)$$

We often use LOTP with Bayes' rule! Specifically, the denominator of Bayes' rule, $P(B)$, is often difficult to calculate outright, so we will instead calculate it in terms of LOTP.

Extra Conditioning

Incorporating extra information C is a simple extension of Bayes' rule and LOTP:

$$
\begin{aligned}
P(A|B,C) &= \frac{P(A \cap B|C)}{P(B|C)} &(3)\\
&= \frac{P(B|A,C)P(A|C)}{P(B|C)} &(4)\\
P(B|C) &= \sum_{i=1}^{n} P(B|A_i,C)P(A_i|C) &(5)\\
&= \sum_{i=1}^{n} P(B \cap A_i|C) &(6)
\end{aligned}
$$

Disjoint vs. Independent

Disjoint, or mutually exclusive, events are events that cannot occur simultaneously. That is, observing event A precludes the possibility of also observing event B. We can state this equivalently as

$$P(A \cap B) = 0$$

Independent events are events such that observing event B yields no information about the possibility of also observing event A. That is, conditioning on observing event B, the probability of observing event A is unchanged.

$$P(A|B) = P(A)$$

We can apply this result to Bayes' rule and quickly demonstrate an alternative definition of independence:

$$P(A \cap B) = P(A)P(B)$$

Another form of independence is conditional independence. Two events A and B are said to be conditionally independent given C if

$$P(A \cap B|C) = P(A|C)P(B|C)$$

However, just as pairwise independence does not imply independence (and vice versa), conditional independence does not imply independence (and vice versa).

Practice Problems

Example 1. Birth Probabilities.
A couple tells you they plan on having two children.

(a) Let A be the event that at least one of their kids is a girl. Assuming that having a boy or a girl are equally likely. What is $P(A)$?

(b) Let B be the event that at least one of their kids is a boy. What is $P(A \cap B)$?

(c) What is $P(B|A)$?

(d) Are A and B independent events? Are A and B disjoint events?

Example 2. Taking Tests.
Fred decides to take a series of n tests, to diagnose whether he has a certain disease (any individual test is not perfectly reliable, so he hopes to reduce his uncertainty by taking multiple tests). Let D be the event that he has the disease, $p = P(D)$ be the prior probability that he has the disease, and $q = 1 - p$. Let T_j be the event that he tests positive on the jth test.

(a) Assume for this part that the tests are conditionally independent given Fred's disease status. Let $a = P(T_j|D)$ and $b = P(T_j|D^c)$, where a and b don't depend on j. Find the posterior probability that Fred has the disease given that he tests positive on all n of the n tests.

(b) Suppose that Fred tests positive on all n tests. However, some people have a certain gene that makes them *always* test positive. Let G be the event that Fred has the gene. Assume that $P(G) = \frac{1}{2}$ and that D and G are independent. If Fred does *not* have the gene, then the test results are conditionally independent given his disease status. Let $a_0 = P(T_j|D, G^c)$ and $b_0 = P(T_j|D^c, G^c)$, where a_0 and b_0 don't depend on j. Find the posterior probability that Fred has the disease, given that he tests positive on all n of the tests.

Example 3. Law and Order.
Adapted from Jimmy Lin's section notes.

(a) A three-person jury has two members who independently have some probability p of making the correct decision, and a third member who makes each decision with a coin flip. A one-person jury has probability p of making the right decision. Which jury has a higher probability of making the right decision?

(b) Breaking news! The defendant is mega-corporation USGSO, which is on trial for violating some poorly defined staffing regulations. The corporation's representative arrives at the courthouse, but because USGSO keeps changing its employment policy, the representative doesn't know if he needs to stay or not. Over the next hour, we takes n steps, right or left with equal probability, in his confusion. At the end of the hour, if he is not back at the courthouse, the judge will rule him guilty. Otherwise, the trial will proceed with the jury from part (a). What is the probability that USGSO will be convicted?

(c) The judge decides that the trial needs members from a different council to enforce the honor of the court. Shira, the main enforcer, misses trials with probability $\frac{2}{20}$ and Tim, the substitute, misses trials with probability $\frac{4}{20}$. At least one of them misses trials with probability $\frac{5}{20}$. However, because Shira and Tim are always busy writing section notes for Stat 110, their absences are not necessarily independent. Given this information, what is the probability that the trial will not occur?

Example 4. "Fun" Interview Problem 1.
Suppose the Harvard Consulting, Investment, and Tech Group™currently consists of two freshmen and some number of upperclassmen. A new student joins the group, but she forgot to indicate what year she was in! At the next club meeting, a recruiter from BainBookSachs™comes in and plucks a lucky student to join their ranks. Given that the student is a freshman, what is the probability that the student that just joined was a freshman? Suppose that freshmen and upperclassmen are

equally likely to join HCITG.

Example 5. "Fun" Interview Problem 2.
Welcome to your interview! On your table you will find a revolver with **six** chambers. I've loaded two real bullets **side-by-side**, while the remaining four chambers are empty. Let me spin the chamber and aim at you... Click. No bullet. Now, I'm going to shoot one more time, but do you want me to spin the barrel again first or just pull the trigger? (Note: This is an actual interview question that's fairly common in first-round finance interviews)