Statistics 110: Introduction to Probability — Section 3 Solutions, Fall 2018

Random Variables and their Distributions

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Practice Problems

Example 1. Calvin and Hobbes.
Calvin and Hobbes play a match consisting of a series of games, where Calvin has probability p of winning each game (independently). They play with a "win by two" rule: the first player to win two games more than his opponent wins the match. Find the probability that Calvin wins the match (in terms of p) in two different ways:

(a) by conditioning, using the Law of Total Probability

(b) by interpreting the problem as a gambler's ruin problem.

Solution
LOTP Solution: Let C be the event that Calvin wins the match. Consider the first two games. There are 4 possible outcomes, 3 of which result in Calvin not losing immediately. Therefore, we condition on each of these possibilities: Let WW represent Calvin winning both games, and WL, LW represent a win then a loss or a loss then a win respectively. Using LOTP, we get:

$$\begin{aligned}
P(C) &= P(C|WW)P(WW) + P(C|WL)P(WL) + P(C|LW)P(LW) \\
&= p^2 + 2pq \cdot P(C) \\
&\Downarrow \\
P(C) &= \frac{p^2}{1-2pq} \\
&= \boxed{\frac{p^2}{p^2+q^2}}
\end{aligned}$$

Gambler's Ruin Solution: We can interpret this problem as a Gambler's Ruin where each player starts out with \$2. Calvin wins if he gets to \$4. Therefore, we have $N = 4$ and $i = 2$, so the probability of Calvin winning is:

$$\frac{1-\left(\frac{q}{p}\right)^2}{1-\left(\frac{q}{p}\right)^4} = \frac{p^2(p^2-q^2)}{p^4-q^4} = \boxed{\frac{p^2}{p^2+q^2}}$$

which is indeed the same answer as before.

Example 2. Mixture of Binomials.
There are two coins, one with probability p_1 of Heads and the other with probability p_2 of Heads. One of the coins is randomly chosen (with equal probabilities for the two coins). It is then flipped $n \geq 2$ times. Let X be the number of times it lands Heads.

(a) Find the PMF of X.

(b) What is the distribution of X if $p_1 = p_2$?

(c) Give an intuitive explanation of why X is not Binomial for $p_1 \neq p_2$ (its distribution is called a mixture of two Binomials). You can assume that n is large for your explanation, so that the frequentist interpretation of probability can be applied.

Solution

(a) By LOTP, conditioning on which coin is chosen, we have

$$P(X = k) = \frac{1}{2}\binom{n}{k} p_1^k (1 - p_1)^{n-k} + \frac{1}{2}\binom{n}{k} p_2^k (1 - p_2)^{n-k}$$

for $k = 0, 1, \ldots, n$.

(b) For $p_1 = p_2$, the above expression reduces to the $\text{Bin}(n, p_1)$ PMF.

(c) A mixture of two Binomials is not Binomial (except in the degenerate case $p_1 = p_2$). Marginally, each toss has probability $\frac{p_1+p_2}{2}$ of landing Heads, but the tosses are not independent since earlier tosses give information about which coin was chosen, which in turn gives information about later tosses.
Let n be large, and imagine repeating the entire experiment many times (each repetition consists of choosing a random coin and flipping it n times). We would expect to see either approximately np_1 Heads about half the time, and approximately np_2 Heads about half the time. In contrast, with a $\text{Bin}(n, p)$ distribution we would expect to see approximately np Heads; no fixed choice of p can create the behavior described above.

Example 3. Symmetry.
For the following 2 exercises, think about how symmetry may be used to avoid unnecessary calculations.

(a) Suppose X and Y are i.i.d. $\text{Bin}(n, p)$. What is $P(X < Y)$?

(b) Can you construct two random variables X and Y both distributed Bin(3, $\frac{1}{2}$) such that $P(X = Y) = 0$?

Solution

(a) First, we note that $P(X < Y) = P(Y < X)$ by symmetry, and that $P(X < Y) + P(X = Y) + P(Y < X) = 1$. Hence,

$$\begin{aligned}
P(X < Y) &= \frac{1}{2}(1 - P(X = Y)) \\
&= \frac{1}{2}\left(1 - \sum_{k=0}^{n} P(X = Y | X = k) P(X = k)\right) \\
&= \frac{1}{2}\left(1 - \sum_{k=0}^{n} P(Y = k) P(X = k)\right) \\
&= \frac{1}{2}\left(1 - \sum_{k=0}^{n}\left(\binom{n}{k} p^k (1 - p)^{n-k}\right)^2\right)
\end{aligned}$$

(b) Yes, let $Y = 3 - X$. Then, there is no way that X and Y take on the same value because their sum would have to be 3.

Example 4. Counting Cards.
In the game Texas Hold'em, players combine two of their cards that are hidden to everyone else with five community cards to make the best possible five-card hand. The game is played with a standard deck of 52 cards. A flush is where all 5 cards belong to the same suit.

Suppose you are holding 2 spades in your hand, and there are 2 spades showing among the three community cards. What is the probability that you hit the flush?

Solution
Since we currently see 4 spades, there are 9 other spades that can be used. Of the 52 cards in the deck, we know what 5 of them are, so we have 47 card values that remain to be seen. To find this probability, we can use the Hyper-Geometric distribution, specifically of $\text{HGeom}(9, 38, 2)$, since there are 9 desirable cards, 38 undesirable cards, and 2 cards that are being drawn. Now, we are interested in the probability that we observe either one or two spades in the two cards we draw. Or $P(X = 1) + P(X = 2)$, where X is distributed as above.

$$\frac{\binom{9}{1}\binom{38}{1}}{\binom{47}{2}} + \frac{\binom{9}{2}}{\binom{47}{2}}$$