Statistics 110: Introduction to Probability	Section 1, Fall 2018

Probability and Counting

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Probability

Naive Definition - If all outcomes are equally likely, the probability of event A happening is:

$$P_{\text{naive}}(A) = \frac{\text{number of outcomes favorable to A}}{\text{number of outcomes}}$$

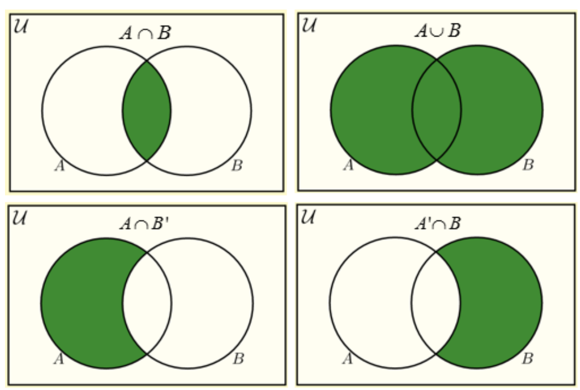
Intersection - Given two events A and B, $A \cap B$ means A *and* B.

Union - Given two events A and B, $A \cup B$ means A *or* B.

Complement - Given an event A, A^C is called A's complement, and means when "A does *not* occur", meaning everything that's not in A.

De Morgan's Laws - A useful identity that can make calculations easier by relating unions to intersections. Analogous results hold with more than two sets.

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$



Principle of Inclusion-Exclusion - For any events $A_1, \dots, A_n$,

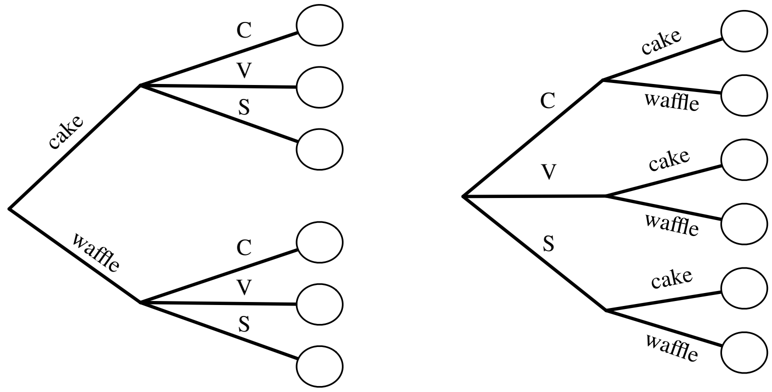
$$P(\bigcup_{i=1}^{n} A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \cdots + (-1)^{n+1} P(A_1 \cap \cdots \cap A_n)$$

In the small cases that we will usually deal with, this can be written as:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\quad + P(A \cap B \cap C). \end{aligned}$$

Counting

Multiplication Rule - If we have n decisions to make and the j-th decision has r_j outcomes, then the total number of potential outcomes is $r_1 \cdot r_2 \cdot \cdots \cdot r_{n-1} \cdot r_n$



Binomial Coefficient Formula - For $k \leq n$, we have

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem -

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Factorial - The number of ways to order n objects is given as

$$n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$$

Sampling Table - The sampling table gives the number of possible samples of size k out of a population of size n, under various assumptions about how the sample is collected.

	Order Matters	**Order Doesn't Matter**
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Practice Problems

Example 1. Story Proof Practice.
Story proofs are a fundamental and useful way that we will go about proving important results, especially later in the course. To that end, provide story proofs for each of the following results:

(a)
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

(b)
$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

(c)
$$\frac{n!}{(n-k)!k!} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k \cdot (k-1) \cdots 1}$$

Example 2. So Many Committees.
An organization with $2n$ people consists of n married couples. A committee of size k is selected, with all possibilities equally likely. Find the probability that there are exactly j married couples within the committee.

Example 3. Counting Chocolates.
Suppose you have 15 chocolate bars and 10 section students. How many ways can you distribute the chocolate bars to your section students, in each of the following scenarios?

(a) The chocolate bars are interchangeable (i.e. it doesn't matter where each particular bar goes).

(b) The chocolate bars are interchangeable, but each student must receive at least one.
Hint: First make sure each student gets a chocolate bar and then figure out what to do with the remaining 5.

(c) The chocolate bars are not interchangeable.

(d) The chocolate bars are not interchangeable, and each student must receive at least one.
Hint: Consider randomly giving the chocolate bars to students and then apply inclusion-exclusion.

Example 4. Poker Probabilities..
Suppose we have a standard 52-card deck, from which you are dealt five cards. Compute the probability of each of the following hands:

(a) A royal flush (getting 10, Jack, Queen, King, and Ace of the same suit).

(b) A flush (all of the cards are of the same suit).

(c) A straight (all five cards are in consecutive order)

(d) A three-of-a-kind (three cards show the same number, and the other two cards do not form a pair)

(e) A two-pair (two cards form a pair and another two cards form a different pair)

Example 5. Fun Interview Problem!.
Suppose the probability of at least one car passing you at an intersection over the course of twenty minutes is given by 0.9. What is the probability that at least one car passes you over the course of five minutes? Assume that time intervals of the same length have the same probability of observing at least one car.