Statistics 110: Introduction to Probability — Section 6 Solutions, Fall 2018

MGFs and Joint, Conditional, and Marginal Distributions

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Practice Problems

Example 1. Counting Cars.
Cars pass by a certain point on a road according to a Poisson process with rate λ cars/minute. Let $N_t \sim \text{Pois}(\lambda t)$ be the number of cars that pass by that point in the time interval $[0, t]$, with t measured in minutes.

(a) A certain device is able to count cars as they pass by, but it does not record the arrival times. At time 0, the counter on the device is reset to 0. At time 3 minutes, the device is observed and it is found that exactly 1 car had passed by. Given this information, find the conditional CDF of when that car arrived. Also describe in words what the result says.

(b) In the late afternoon, you are counting blue cars. Each car that passes by is blue with probability b, independently of all other cars. Find the joint PMF and marginal PMFs of the number of blue cars and number of non-blue cars that pass by the point in 10 minutes.

Solution

(a) Let T_1 be the arrival time of the first car to arrive after time 0. Unconditionally, $T_1 \sim \text{Expo}(\lambda)$. Given $N_3 = 1$, for $0 \le t \le 3$, we have

$$P(T_1 \le t | N_3 = 1) = P(N_t \ge 1 | N_3 = 1) = \frac{P(N_t \ge 1, N_3 = 1)}{P(N_3 = 1)} = \frac{P(N_t = 1, N_3 = 1)}{P(N_3 = 1)}$$

By definition of the Poisson process, the numerator is

$$P(N_t = 1, N_3 = 1) = P\Big(N_{[0,t]} = 1, N_{(t,3]} = 0\Big) = P\Big(N_{[0,t]} = 1\Big)P\Big(N_{(t,3]} = 0\Big) = e^{-\lambda t}\lambda t e^{-\lambda(3-t)} = \lambda t e^{-3\lambda}$$

and the denominator is $e^{-3\lambda}3\lambda$. Hence,

$$P(T_1 \le t | N_3 = 1) = \frac{\lambda t e^{-3\lambda}}{e^{-3\lambda}3\lambda} = \frac{t}{3}$$

for $0 \le t \le 3$ (and 0 for $t < 0$ and 1 for $t > 3$).
This says that the conditional distribution of the first arrival time, given that there was exactly one arrival in $[0, 3]$, is $\text{Unif}(0, 3)$.

(b) Let X and Y be the number of blue and non-blue cars that pass by in those 10 minutes respectively, and $N = X + Y$. Then $N \sim \text{Pois}(10\lambda)$ and $X|N \sim \text{Bin}(N, b)$. By the chicken-egg story, X and Y are independent with $X \sim \text{Pois}(10\lambda b)$, $Y \sim \text{Pois}(10\lambda(1-b))$. The joint PMF is the product of the marginal PMFs:

$$P(X = x, Y = y) = \frac{e^{-10\lambda b}(10\lambda b)^x}{x!}\frac{e^{-10\lambda(1-b)}(10\lambda(1-b))^y}{y!}$$

for all nonnegative integers x, y.

Example 2. Linear Combination of Independent Normals.
Let $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$. Use the MGF to show that for any values of $a, b \neq 0$, $Y = aX_1 + bX_2$ is also Normal. (You may use the fact that the MGF of a $\mathcal{N}(\mu, \sigma^2)$ distribution is $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.)

Solution
We know that the MGF for the sum of 2 independent random variables is the product of their individual MGFs. We also know that the MGF of a scalar c times a random variable is equal to that random variable's MGF with ct subsituted for t.

$$\begin{aligned}
M_Y(t) &= \exp\left(\mu_1(at) + \frac{1}{2}\sigma_1^2(at)^2\right) \cdot \exp\left(\mu_2(bt) + \frac{1}{2}\sigma_2^2(bt)^2\right) \\
&= \exp\left(\mu_1(at) + \frac{1}{2}\sigma_1^2(at)^2 + \mu_2(bt) + \frac{1}{2}\sigma_2^2(bt)^2\right) \\
&= \exp\left((a\mu_1 + b\mu_2)t + \left(\frac{(a\sigma_1)^2 + (b\sigma_2)^2}{2}\right)t^2\right)
\end{aligned}$$

This is the $\mathcal{N}\left(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2\right)$ MGF.

Example 3. Return of Nick and Hibbert.
One of two doctors, Dr. Hibbert and Dr. Nick, is called upon to perform a series of n surgeries. Let H be the indicator random variable for Dr. Hibbert performing the surgeries, and suppose that $E(H) = p$. Given tht Dr. Hibbert is performing the surgeries, each surgery is successful with probability a, independently. Given that Dr. Nick is performing the surgeries, each surgery is successful with probability b independently. Let X be the number of successful surgeries.

(a) Find the joint PMF of H and X.

(b) Find the marginal PMF of X.

(c) Find the conditional PMF of H given $X = k$.

Solution

(a) We find the joint PMF by recalling the definition of conditional probability.

$$P(X = x|H = h) = \frac{P(H = h, X = x)}{P(H = h)} \implies P(H = h, X = x) = P(X = x|H = h)P(H = h)$$

From here, we use the fact that H can only take on two values and define a piecewise function.

$$P(X = x|H = h)P(H = h) = \begin{cases} \binom{n}{x}a^x(1-a)^{n-x}p, \text{ if } h = 1 \\ \binom{n}{x}b^x(1-b)^{n-x}(1-p) \text{ if } h = 0 \end{cases}$$

(b) To find the marginal PMF of X, we must eliminate H from the joint PMF. To do so, we sum over all possible values of H.

$$P(X = x) = P(H = 1, X = x) + P(H = 0, X = x)$$

where we plug in our piecewise function from above.

(c) The conditional PMF is found using the definition of conditional probability.

$$P(H = h|X = x) = \frac{P(H = h, X = x)}{P(X = x)}$$

where the joint PMF is defined as above.

Example 4. Probabilities Using Joint CDFs and PDFs.

Let X and Y be continuous random variables. What is the probability that (X, Y) falls into the 2-D rectangle $[a_1, a_2] \times [b_1, b_2]$ in terms of a) the joint CDF of X and Y, $F(x, y)$, and b) the joint PDF of x and Y, $f(x, y)$?

Solution

(a) $F(a_2, b_2) - F(a_1, b_2) - F(a_2, b_1) + F(a_1, b_1)$. Draw a picture to see this.

(b) We can integrate the joint PDF over the region of interest to get the probability:

$$P((X, Y) \in [a_1, a_2] \times [b_1, b_2]) = \int_{b_1}^{b_2} \int_{a_1}^{a_2} f(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

Example 5. LogNormal Moments.
Let $Z \sim \mathcal{N}(0, 1)$ and $Y = e^Z$. Then Y has a LogNormal distribution (because its log is Normal!).

(a) Find **all** the moments of Y. That is, find $E(Y^n)$ for all n.

(b) Show the MGF of Y does not exist, by arguing that the integral $E(e^{tY})$ diverges for $t > 0$.

Solution

(a)

$$E(Y^n) = E\left(\left(e^Z\right)^n\right) = E\left(e^{nZ}\right)$$

We may be tempted to use LOTUS at this point, but it is easier to recognize that $E(e^{nZ})$ is the MGF of Z, evaluated at $t = n$. In other words, $E(e^{nZ}) = M_Z(n)$. Since $M_Z(t) = e^{t^2/2}$, we conclude that $M_Z(n) = e^{n^2/2}$.

(b)

$$E\left(e^{tY}\right) = E\left(e^{te^Z}\right) = \int_{-\infty}^{\infty} e^{te^z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{te^z - z^2/2} \mathrm{d}z$$

Suppose $t > 0$. Then as z grows larger, te^z will grow at a faster rate than $z^2/2$. Hence, $te^z - z^2/2$ will explode. Therefore, the integral diverges for all $t > 0$. Since $E(e^{tY})$ is not finite on an open interval around 0, the MGF of Y does not exist.