Statistics 110: Introduction to Probability — Section 5 Solutions, Fall 2018

Continuous Random Variables

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Practice Problems

Example 1. Uniform Power.
Let $U \sim \text{Unif}(-1, 1)$

(a) Compute $E(U), \text{Var}(U), E(U^4)$.

(b) Find the PDF and CDF of U^2. Is it also a uniform distribution?

Solution
When trying to compute the PDF of a transformation, we usually work from the CDF first. Note that the PDF of U is $f(x) = \frac{1}{2}$.

(a) $E(U) = 0$ because the distribution is symmetric about 0. We need to calculate $E(U^2)$ for the variance, so we have:

$$E(U^2) = \int_{-1}^{1} u^2 \cdot \frac{1}{2}\, \mathrm{d}u = \left[\frac{1}{6}u^3\right]_{-1}^{1} = \frac{1}{3}$$

Therefore, $\text{Var}(U) = E(U^2) - E(U)^2 = \boxed{\frac{1}{3}}$

Next, we do the same for $E(U^4)$:

$$E(U^4) = \int_{-1}^{1} u^4 \cdot \frac{1}{2}\, \mathrm{d}u = \left[\frac{1}{10}u^5\right]_{-1}^{1} = \boxed{\frac{1}{5}}$$

(b) We first find the CDF.

$$P(U^2 < k) = P(-\sqrt{k} < U < \sqrt{k}) = \frac{2\sqrt{k}}{2} = \sqrt{k}$$

, which we can easily calculate graphically. If this wasn't possible to do graphically, then we would integrate the PDF of U between $-k$ and k.

Therefore, the PDF is:

$$\frac{d}{dk}P(U^2 < k) = \boxed{\frac{1}{2\sqrt{k}}}$$

This is definitely **not** a uniform distribution. This also shows that U^k is not uniform anymore for any $k > 1$.

Example 2. Normal Squared.
Let $Z \sim N(0, 1)$ with CDF Φ. The PDF of Z^2 is the function given by:

$$g(w) = \frac{1}{\sqrt{2\pi w}} e^{-w/2}$$

with a support of $w \geq 0$.

(a) Find expressions for $E(Z^4)$ as integrals in two different ways, one based on the PDF of Z and the other based on the PDF of Z^2.

(b) Find $E(Z^2 + Z + \Phi(Z))$.

(c) Find the CDF of Z^2 in terms of Φ; do not find the PDF of g.

Solution

(a) Let $W = Z^2$, so $W^2 = Z^4$. By LOTUS,

$$E(Z^4) = \int_{-\infty}^{\infty} z^4 \varphi(z) dz = \int_{0}^{\infty} w^2 g(w) dw,$$

where $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the PDF of Z, and g is as above. (Using techniques from Chapter 6, it turns out that this reduces to a very simple answer: $E(Z^4) = 3$.)

(b) By linearity, this is $E(Z^2) + E(Z) + E(\Phi(Z))$. The second term is 0 and the first term is 1 since $E(Z) = 0, \mathrm{Var}(Z) = 1$. The third term is $1/2$ since by universality of the Uniform, $\Phi(Z) \sim \mathrm{Unif}(0,1)$. Thus, the value is $3/2$.

(c) For $w \leq 0$, the CDF of Z^2 is 0. For $w > 0$, the CDF of Z^2 is

$$P(Z^2 \leq w) = P(-\sqrt{w} \leq Z \leq \sqrt{w}) = \Phi(\sqrt{w}) - \Phi(-\sqrt{w}) = 2\Phi(\sqrt{w}) - 1.$$

Example 3. Universality of the Uniform.
Let $U \sim \mathrm{Unif}(0,1)$, and let $X = -(\log(1-U))^{1/3}$. Find the CDF and PDF of X.

Solution
We calculate $P(x \leq X)$, but use the substitution given to try and solve a Uniform distribution's CDF instead:

$$\begin{aligned}
P(X \leq x) &= P(-(\log(1-U))^{1/3} \leq x) \\
&= P(\log(1-U)^{1/3}) \geq -x) \\
&= P(\log(1-U) \geq -x^3) \\
&= P(1 - e^{-x^3} \geq U) \\
&= P(U \leq 1 - e^{-x^3}) \\
&= \boxed{(1 - e^{-x^3})}
\end{aligned}$$

which is the CDF of X.

The PDF of x is the derivative, so:

$$f(x) = \frac{\partial}{\partial x}(1 - e^{-x^3}) = 3x^2 e^{-x^3}$$

Example 4. Continuous RV manipulation.
Let X be a continuous r.v. with CDF F and PDF f.

(a) Find the conditional CDF X given that $X > a$ (where a is a constant with $P(X > a) \neq 0$).

(b) Find the conditional PDF of X given $X > a$ (this is the derivative of the conditional CDF).

(c) Check that the conditional PDF from (b) is a valid PDF, by showing directly that it is nonnegative and integrates to 1.

Solution

(a) We have $P(X \le x | X > a) = 0$ for $x \le a$. For $x > a$,

$$P(X \le x | X > a) = \frac{P(a < X \le x)}{P(X > a)} = \frac{F(x) - F(a)}{1 - F(a)}.$$

(b) The derivative of the conditional CDF is $f(x)/(1 - F(a))$ for $x > a$, and 0 otherwise.

(c) We have $f(x)/(1 - F(a)) \ge 0$ since $f(x) \ge 0$. And

$$\int_a^\infty \frac{f(x)}{1 - F(a)} dx = \frac{1}{1 - F(a)} \int_a^\infty f(x) dx = \frac{1 - F(a)}{1 - F(a)} = 1.$$