Statistics 110: Introduction to Probability — Section 3 Solutions, Fall 2018

Random Variables and their Distributions

TFs: Justin Zhu (justinzhu@college)

Practice Problems

Example 1. Calvin and Hobbes.
Calvin and Hobbes play a match consisting of a series of games, where Calvin has probability p of winning each game (independently). They play with a "win by two" rule: the first player to win two games more than his opponent wins the match. Find the probability that Calvin wins the match (in terms of p) in two different ways:

(a) by conditioning, using the Law of Total Probability

(b) by interpreting the problem as a gambler's ruin problem.

Solution
LOTP Solution: Let C be the event that Calvin wins the match. Consider the first two games. There are 4 possible outcomes, 3 of which result in Calvin not losing immediately. Therefore, we condition on each of these possibilities: Let WW represent Calvin winning both games, and WL, LW represent a win then a loss or a loss then a win respectively. Using LOTP, we get:

$$\begin{aligned}
P(C) &= P(C|WW)P(WW) + P(C|WL)P(WL) + P(C|LW)P(LW) \\
&= p^2 + 2pq \cdot P(C) \\
&\Downarrow \\
P(C) &= \frac{p^2}{1-2pq} \\
&= \boxed{\frac{p^2}{p^2+q^2}}
\end{aligned}$$

Gambler's Ruin Solution: We can interpret this problem as a Gambler's Ruin where each player starts out with \$2. Calvin wins if he gets to \$4. Therefore, we have $N = 4$ and $i = 2$, so the probability of Calvin winning is:

$$\frac{1-\left(\frac{q}{p}\right)^2}{1-\left(\frac{q}{p}\right)^4} = \frac{p^2(p^2-q^2)}{p^4-q^4} = \boxed{\frac{p^2}{p^2+q^2}}$$

which is indeed the same answer as before.

Example 2. Symmetry.
For the following 2 exercises, think about how symmetry may be used to avoid unnecessary calculations.

(a) Suppose X and Y are i.i.d. $\text{Bin}(n, p)$. What is $P(X < Y)$?

(b) Can you construct two random variables X and Y both distributed Bin(3, $\frac{1}{2}$) such that $P(X = Y) = 0$?

Solution

(a) First, we note that $P(X < Y) = P(Y < X)$ by symmetry, and that $P(X < Y) + P(X = Y) + P(Y < X) = 1$. Hence,

$$\begin{aligned} P(X < Y) &= \frac{1}{2}(1 - P(X = Y)) \\ &= \frac{1}{2}\left(1 - \sum_{k=0}^{n} P(X = Y|X = k)P(X = k)\right) \\ &= \frac{1}{2}\left(1 - \sum_{k=0}^{n} P(Y = k)P(X = k)\right) \\ &= \frac{1}{2}\left(1 - \sum_{k=0}^{n} \left(\binom{n}{k} p^k (1-p)^{n-k}\right)^2\right) \end{aligned}$$

(b) Yes, let $Y = 3 - X$. Then, there is no way that X and Y take on the same value because their sum would have to be 3.

Example 3. Counting Cards.
In the game Texas Hold'em, players combine two of their cards that are hidden to everyone else with five community cards to make the best possible five-card hand. The game is played with a standard deck of 52 cards. A flush is where all 5 cards belong to the same suit.

Suppose you are holding 2 spades in your hand, and there are 2 spades showing among the three community cards. What is the probability that you hit the flush?

Solution
Since we currently see 4 spades, there are 9 other spades that can be used. Of the 52 cards in the deck, we know what 5 of them are, so we have 47 card values that remain to be seen. To find this probability, we can use the Hyper-Geometric distribution, specifically of HGeom$(9, 38, 2)$, since there are 9 desirable cards, 38 undesirable cards, and 2 cards that are being drawn. Now, we are interested in the probability that we observe either one or two spades in the two cards we draw. Or $P(X = 1) + P(X = 2)$, where X is distributed as above.

$$\frac{\binom{9}{1}\binom{38}{1}}{\binom{47}{2}} + \frac{\binom{9}{2}}{\binom{47}{2}}$$