Conditional Expectation

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Practice Problems

Example 1. Chicken-Egg Revisited.
Recall in the Chicken-Egg problem that we have a chicken that lays eggs, where the number of eggs N follows the distribution $N \sim \text{Pois}(\lambda)$. Given that the chicken lays N eggs, the number of eggs that hatch X follows the distribution $X|N \sim \text{Bin}(N, p)$. Find $E(X)$ and $Var(X)$.

Solution
Use Adam's and Eve's Law!

$$\begin{aligned} E(X) &= E[E(X|N)] = E[Np] = pE[N] = \lambda p \\ Var(X) &= E[Var(X|N)] + Var[E(X|N)] \\ &= E[Np(1-p)] + Var[Np] \\ &= \lambda p(1-p) + p^2\lambda \\ &= \lambda p \end{aligned}$$

Example 2. Consumerism.
When customers enter a particular store, each makes a purchase with probability p, independently. Given that a customer makes a purchase, the amount spent has mean μ (in dollars) and variance σ^2. Find the mean and variance of how much a random customer spends (note that the customer may spend nothing).

Solution
Let X be the amount a random customer spends at the store, and let I be the indicator that a random customer makes a purchase. We then apply the Law of Total Expectation:

$$\begin{aligned} E[X] &= E[X|I=0]P(I=0) + E[X|I=1]P(I=1) = \mu p \\ E[X^2] &= E[X^2|I=0]P(I=0) + E[X^2|I=1]P(I=1) \\ &= E[X^2|I=1]P(I=1) \\ &= (\mu^2 + \sigma^2)p \\ Var[X] &= E[X^2] - E[X]^2 \\ &= \sigma^2 p + \mu^2 p(1-p) \end{aligned}$$

Example 3. Trapped Miners.
A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 minutes. The other two doors lead to tunnels that will return him to the mine after 5 and 7 minutes of travel each. The miner is equally likely to choose any of the doors at any time. What is the expected amount of time until he reaches safety?

Solution
Let W be the total amount of time taken until he reaches safety, and D_1, D_2, and D_3 be the events of going through doors 1, 2, and 3 respectively.

Using LoTE, we have

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\begin{aligned}
E(W) &= E(W|D_1)P(D_1) + E(W|D_2)P(D_2) + E(W|D_3)P(D_3) \\
&= 3\frac{1}{3} + (5 + E(W))\frac{1}{3} + (7 + E(W))\frac{1}{3} \\
\frac{1}{3}E(W) &= 5 \\
E(W) &= 15
\end{aligned}
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