TER
Distribution des valeurs propres du laplacien :
Loi asymptotique de Weyl

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Contents

1 Explicite Calculation for specific shapes

1.1 Rectangle Case

Consider $\Omega =]0, a[$, we have the following problem

$$\begin{cases} \frac{d^2}{dx^2}\phi(x) = -\lambda\phi(x) \\ \phi(0) = \phi(a) = 0 \end{cases} \tag{1}$$

This gives solution $\phi_k(x) = \sin(\frac{k\pi}{a}x)$, with corresponding Dirichlet eigenvalues $\lambda_k = (\frac{k\pi}{a})^2$, for $k \in \mathbb{N}^*$. The counting function:

$$N(\lambda) = \#\{k \in \mathbb{N}^* | \lambda_k \leq \lambda\} = \#\{k \in \mathbb{N}^* | (\frac{k\pi}{a})^2 \leq \lambda\} \sim \frac{a}{\pi}\sqrt{\lambda}$$

Now consider $\Omega =]0, a_1[\times]0, a_2[\times \cdots \times]0, a_n[$, by symmtricity and separation of variable we get $\phi_{j_1,j_2,\ldots,j_n} = sin(\frac{j_1\pi}{a_1}x_1)sin(\frac{j_2\pi}{a_2}x_2)\ldots sin(\frac{j_n\pi}{a_n}x_n)$, with eigenvalues $\lambda_{j_1,j_2,\ldots,j_n} = (\frac{j_1\pi}{a_1})^2 + (\frac{j_2\pi}{a_2})^2 + \cdots + (\frac{j_n\pi}{a_n})^2$, for $j_1, j_2, \ldots, j_n \in \mathbb{N}^*$. The counting function is then given by

$$N(\lambda) = \#\{(j_1, j_2, \ldots, j_n) \in \mathbb{N}^{*n} \mid (\frac{j_1\pi}{a_1})^2 + (\frac{j_2\pi}{a_2})^2 + \ldots (\frac{j_n\pi}{a_n})^2 \leq \lambda\}$$

Each lattice point corresponds to one cube, thus the number of lattice points is bounded by $1/2^n$ volume of the ellipsoid $E_\lambda = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid (\frac{\pi}{a_1\sqrt{\lambda}}x_1)^2 +$

$(\frac{\pi}{a_2\sqrt{\lambda}}x_2)^2 + \dots (\frac{\pi}{a_n\sqrt{\lambda}}x_3)^n \leq 1\}$, note ω_n the volume of the unit ball of dimention n, we have

$$N(\lambda) \leq \frac{1}{2^n} vol(E_\lambda) = \frac{\lambda^{\frac{n}{2}}}{(2\pi)^n} a_1 a_2 \dots a_n \omega_n = \frac{\lambda^{\frac{n}{2}}}{(2\pi)^n} vol(\Omega)\omega_n$$

The union of unit cubes on the other hand contains the positive cut of E_λ shrinking 1 unit on each axis, note this smaller ellipse E'_λ, we have

$$E'_\lambda = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid (\frac{x_1}{\frac{a_1\sqrt{\lambda}}{\pi} - 1})^2 + (\frac{x_2}{\frac{a_2\sqrt{\lambda}}{\pi} - 1})^2 + \dots + (\frac{x_n}{\frac{a_n\sqrt{\lambda}}{\pi} - 1})^2 \leq 1\}$$

Thus

$$N(\lambda) \geq \frac{1}{2^n} vol(E'_\lambda) = \frac{\lambda^{\frac{n}{2}}}{(2\pi)^n} a_1 a_2 \dots a_n \omega_n + o(\lambda^{\frac{n}{2}}) = \frac{\lambda^{\frac{n}{2}}}{(2\pi)^n} vol(\Omega)\omega_n + o(\lambda^{\frac{n}{2}})$$

We conclude in the rectangle case:

$$N(\lambda) \sim \frac{\lambda^{\frac{n}{2}}}{(2\pi)^n} vol(\Omega)\omega_n$$

Which fits the Weyl's law.
The we get the same result under Neumann condition by simply replacing sin in the reasoning above by cos and $\mathbb{N}^*$ to $\mathbb{N}$.

1.2 Disk Case

Here we assumpe $\Omega = \{x \in \mathbb{R}^2 \mid |x| < a\}$

$$\begin{cases} \Delta u(x) = -\lambda u(x) \\ u(x) = 0 \text{ on } \partial\Omega \end{cases} \tag{2}$$

We switch to polar coordinates:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = -\lambda u \\ u(r,\theta) = 0 \text{ when } r = a \end{cases} \tag{3}$$

We first seek a solution of the system above in the form: $u(r,\theta) = R(r)\Theta(\theta)$, so the first equation of (5) becomes:

$$\frac{r^2 R'' + rR'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta}$$

Since the two sides depends respectly on r and θ, they equal to constant. Also since we consider a disk we must assume that Θ is 2π-periodic and therefore $\Theta = e^{i\omega\theta}$, so $-\frac{\Theta''}{\Theta} = n^2$, with $n = 0, 1, 2, \dots$, we end up with

$$r^2 R''(r) + rR'(r) + (r^2\lambda - n^2)R(r) = 0$$

Note that with the same $n > 0$, we have two eigenmodes $\Theta = e^{in\theta}$ and $\Theta = e^{-in\theta}$.
With a subsitution of variable $x = \sqrt{\lambda}r$, and $T(x) = R(x/\sqrt{\lambda})$, we get

$$x^2 T''(x) + xT'(x) + (x^2 - n^2)T(x) = 0$$

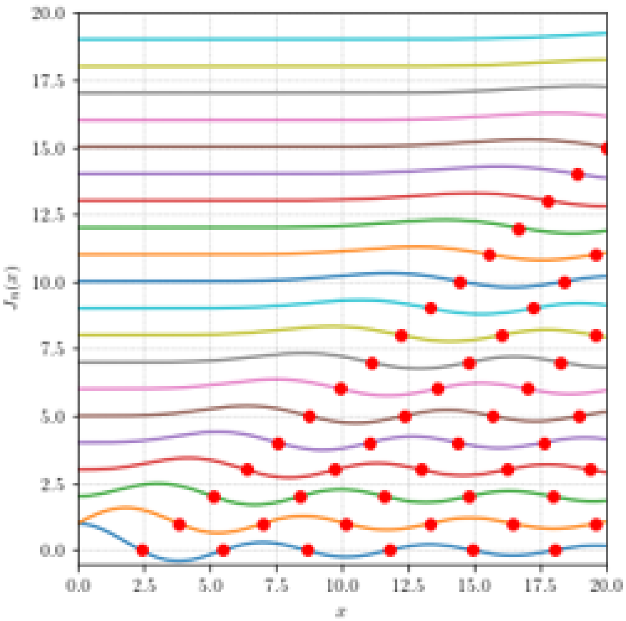

Figure 1: An illustration of our lattice point problem

This is Bessel's equation, its solution is Bessel's function $J_n(x)$.
Impossing the Dirichlet boundary condition, we need to have λ such that $J_n(a\sqrt{\lambda}) = 0$.
Impossing the Neumann condition, we need to have λ such that $J_n'(a\sqrt{\lambda}) = 0$.

Note m-th zero of Bessel function J_n as j_{nm}, we know that J_n have no common zeros given two different n, thus our problem of counting eigenvalues less then α reduces to count:

$$\#\{n \in \mathbb{N}, m \in \mathbb{N}^* | j_{nm} \leq a\sqrt{\alpha}\}$$

This is not easy task as Bessel function does not have an explicit form. In [1], the author managed to convert this problem into a counting lattice points in a cusp shaped area, here we give a rough estimation. We have from [2] an estimation of J_n as:

$$J_n(n \sec \beta) = \sqrt{2/(\pi n \tan \beta)}\{\cos(n \tan \beta - n\beta - \frac{1}{4}\pi) + O(1/n)\}, \beta \in]0, \pi/2[$$

Rewrite this, we get:

$$J_n(x) \approx \sqrt{\frac{2}{\pi\sqrt{x - n^2}}} \cos(n \tan arccos\frac{n}{x} - n \arccos\frac{n}{x} - \frac{\pi}{4}), x > n$$

The estimation only concerns $x > n$, but this does not affect the actual counting as the smallest zero of J_n is bigger then n anyway.
This function hits 0 when:

$$n \tan \arccos\frac{n}{x} - n \arccos\frac{n}{x} - \frac{1}{4}\pi = k\pi + \pi/2, k \in \mathbb{Z}$$

Note this function is continuous, greater then $-\frac{1}{\pi}$ and increasing on $x > n$, thus the possible k for $x \leq \alpha$ will be $1, 2, \ldots, \left\lfloor \frac{n}{\pi}(\tan \arccos \frac{n}{a\sqrt{\alpha}} - \arccos \frac{n}{a\sqrt{\alpha}}) \right\rfloor$. Integrate this on n, from 0 to $a\sqrt{\alpha}$, we get exactly $\frac{a^2\alpha}{8}$, note that each n corresponds to 2 eigenmodes as discussed before, and the Weyl's law counts in multiplicity, we get an estimation $\frac{a^2\alpha}{4}$, which is exactly the main term of Weyl's asymptotic for 2d-disk.
We can also notice from the last expression that the bigger the domain is, the smaller eignvalues are, and thus the bigger the counting function is. This fits the some results in the following section.

2 A proof of Weyl's law by variational caracterization (Weyl's original way)

2.1 Spectral theory for laplacian on bounded domain

Let $\Omega \subset \mathbb{R}^n$ denote a bounded domain with smooth boundary. Denote E as $H^1(\Omega)$ or $H_0^1(\Omega)$, for each $f \in L^2(\Omega)$, there exists a unique $u \in E$ such that $\langle u, v \rangle_{H^1} = \langle f, v \rangle, \forall v \in E$. And the map:

$$\begin{aligned} A : &L^2 \to E \\ &f \mapsto u \end{aligned}$$

is linear and bounded. In fact, consider linear application $F(v) = \langle f, v \rangle$ on E, F is bounded because:

$$F(v) \leq \|v\| \|f\| \leq C \|f\| \|v\|_{H^1}$$

The latter $\leq$ is because the continuous embedding $H^1(\Omega)$ to $L^2(\Omega)$. Hense by Riesz Representation Theorem on E, there exists a unique $u \in E$ such that $\langle u, v \rangle_{H^1} = F(v) = \langle f, v \rangle, \forall v \in E$. Also A is bounded because $\|u\|_{H^1}^2 = \langle u, u \rangle_{H^1} = \langle f, u \rangle \leq \|u\| \|f\| \leq C \|u\|_{H^1} \|f\|$, thus $\|u\|_{H^1} = \|A(f)\|_{H^1} \leq C \|f\|$. Now we have well defined $A : E \to L^2(\Omega)$, define:

$$\begin{aligned} B : &L^2(\Omega) \to E \to L^2(\Omega) \\ &f \mapsto a(f) \mapsto a(f) \end{aligned}$$

By composing A and the identity map from E to $L^2(\Omega)$, we have the following spectral theorem:

Theorem There exists an orthonormal basis of $L^2(\Omega)$ consisting eigenfunctions of B.

Proof:

$B : L^2(\Omega) \xrightarrow[A]{E} \xrightarrow[identity]{} L^2(\Omega)$ is compact, since A is bounded and E embeds compactly into $L^2(\Omega)$ (Rellich's Lemma).

B is also autoadjoint on $L^2(\Omega)$ since:

$$\begin{aligned}\forall f, g \in L^2, \langle B(f), g\rangle &= \overline{\langle g, B(f)\rangle} \\ &= \overline{\langle B(g), B(f))\rangle}_{H^1} \\ &= \langle B(f), B(g)\rangle_{H^1} \\ &= \langle f, B(g)\rangle\end{aligned}$$

Hence the spectral theorem of compact autoadjoint operators on Hilbert space provides an orthonormal basis (u_i) for L^2 consisting eigenvectors of B, with

$$B(u_i) = \rho_i u_i$$

for a sequence of $(\rho_i) \to 0$.

Lemma The eigenvalues (ρ_i) we have found in the last theorem are non zero

Proof:

If some $\rho_i = 0$, then $B(u_i) = 0$, so $\forall v \in E, \langle u_i, v\rangle = \langle B(u_i), v\rangle_{H^1} = 0$, using the density of E in $L^2(\Omega)$ ($C^\infty(\Omega) \subset H_0^1 \subset H^1(\Omega)$, and $C^\infty(\Omega)$ is dense in $L^2(\Omega)$), we have $\forall v \in L^2(\Omega), \langle u_i, v\rangle = 0$, that would require $u_i = 0$.

Corollary The eigenfunctions $(u_i) \in L^2$ we have found in the last theorem actually belongs to E

Proof:

Since ρ_i are non zero, we can divide by eigenvalue and get $B(u_i/\rho_i) = u_i$, that shows u_i lays in the range of B, that is E.

We have found an spectral theory of application $B : L^2(\Omega) \to L^2(\Omega)$, what does it have to do with laplacian?

- For the Dirichlet case, we have

  $$B(u_i) = \rho_i u_i$$

  The way to caracterize $B(u_i)$ is:

  $$\langle B(u_i), v\rangle_{H^1} = \rho_i \langle u_i, v\rangle_{H^1} = \langle u_i, v\rangle, \forall v \in H_0^1(\Omega)$$

  Note $\lambda_i = 1/\rho_i - 1$, we have thus:

  $$\int_\Omega \nabla u_i \cdot \nabla v = \lambda_i \int_\Omega u_i v, \forall v \in H_0^1(\Omega)$$

  λ_i are the eigenvalues of Dirichlet Laplacian, and the last equation is exactly the weak form of:

  $$-\Delta u_i = \lambda_i u_i$$

Note that we only know that $u_i \in H^1(\Omega)$ thus having 1-weakly derivative, so $-\Delta u_i$ does not necessarily make sense. However $-\Delta$ is a second order elliptic operator and the elliptic regularity implies that u_i is 2-weakly differentiable in Ω. Further more, repeatedly apply the eigenvalue equation to u_i we get u_i is actually $H^\infty(\Omega)$, then by Sobolev Embedding we can conclude that $u_i \in C^\infty(\Omega)$. Thus the eigenequation is satisfied classically. The boundary condition $u_i|_{\partial\Omega}$ is satisfied natually since $u_i \in H_0^1(\Omega)$.
Obviously we have $\lambda_i \to \infty$ as $\rho_i \to 0$. But also we have $\lambda_i > 0$ as $\lambda_i = \frac{\int_\Omega (\nabla u_i)^2}{\int_\Omega u_i^2} \geq 0$ and if $\lambda_i = 0$, then $\int_\Omega (\nabla u_i)^2 = 0$ and by the Sobolev Inequality, $\int_\Omega u_i^2 = 0$, but this is impossible since u_i has norm 1.

- For the Neumann case, we note eigenvalues μ_i. We have similar reasoning except $u_i \in H^1(\Omega)$. The boundary condition could no longer be satisfied natually, but we have:

$$\int_\Omega \nabla u_i \cdot \nabla v = \mu_i \int_\Omega u_i v, \forall v \in H^1(\Omega) \text{ instead of } H_0^1(\Omega)$$

By taking v from $H_0^1 \subset H^1$, we take advantage of the ellipcity, and conclude $-\Delta u_i = \mu_i u_i$ is satisfied classically. Taking v from H^1
H_0^1 we have (Green's formula):

$$\int_\Omega (-\Delta u_i) v + \int_{\partial\Omega} \frac{\partial u_i}{\partial n} v = \mu_i \int_\Omega u_i v, \forall v \in C^\infty(\overline{\Omega})$$

As we already have $-\Delta u_i = \mu_i u_i$, we must have:

$$\int_{\partial\Omega} \frac{\partial u_i}{\partial n} v = 0, \forall v \in C^\infty(\overline{\Omega})$$

Note that $C^\infty(\overline{\Omega})|_{\partial\Omega}$ is dense in $L^2(\partial\Omega)$, we conclude that:

$$\frac{\partial u_i}{\partial n} = 0$$

Thus for all u_i, the Neumann boundary condition is satisfied.
Also we have $\mu_i \geq 0$ by similar reasoning in the Dirichlet case, but the first Neumann eigenvalue is 0 because we can always choose a constant function as eigenfunction (which belongs to $H^1(\Omega)$ but not $H_0^1(\Omega)$) for the eignvalue 0.

2.2 Variational Caracterization

Lemma (Min-max principle for Laplacian of Dirichlet condition):
Denote $R(u) = \frac{\|\nabla u\|^2}{\|u\|^2} = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$, we have:

$$\lambda_n = \inf_{U \subset H_0^1(\Omega) \text{ is a n-dim subspace}} \sup_{u \in U} R(u)$$

Proof:

For $u \in H_0^1(\Omega)$, we have:

$$\|u\|^2 = \sum_{n=1}^{\infty} |\langle u_n, u\rangle|^2$$

let H_n denote $H_0^1(\Omega) \cap span(u_1, \dots, u_n)$, So we have

$$\begin{aligned}
\|\nabla u\|^2 &= \int_\Omega -\Delta u \cdot u \\
&= \langle -\Delta \sum_{n=1}^{\infty} \langle u, u_n\rangle u, u_n\rangle \\
&= \sum_{n=1}^{\infty} \langle -\langle u, u_n\rangle \Delta u_n, u_n\rangle \\
&= \sum_{n=1}^{\infty} \lambda_n |\langle u, u_n\rangle|^2
\end{aligned}$$

If $u \in H_{n-1}$, then $\|\nabla u\|^2 = \sum_{i=n}^{\infty} \lambda_i |\langle u, u_i\rangle|^2$, with $\lambda_i \geq \lambda_n$, thus

$$\|\nabla u\|^2 \geq \lambda_n \sum_{i=n}^{\infty} |\langle u, u_i\rangle|^2 + \lambda_n \underbrace{\sum_{i=1}^{n-1} |\langle u, u_i\rangle|^2}_{=0} = \lambda_n \|u\|^2$$

Thus we have

$$\inf_{u \in H_{n-1}} R(u) \geq \lambda_n$$

Also by choosing $u = u_n \in H_{n-1}$, we have the infinimum atteined, thus

$$\inf_{u \in H_n} R(u) = \lambda_n$$

Also we have:

$$\sup_{u \in span(u_1, \dots, u_{n-1})} R(u) = \lambda_n$$

Now we have on direction:

$$\lambda_n \geq \inf \sup_{u \in U} R(u)$$

For every n-d subspace $U \subset H_0^1(\Omega)$, we have $\exists u_\alpha \in U \cap H_{n-1}$, thus:

$$\forall U, \ \sup_{u \in U} R(u) \geq R(u_\alpha) \geq \inf_{u \in H_{n-1}} R(u) = \lambda_n$$

Thus we have the other direction (and thus the lemma):

$$\lambda_n \leq \inf_{U \subset H_0^1(\Omega)} \sup_{u \in U} R(u)$$

Similarily we have the variational caracterization of μ_i for the Neumann case as:

$$\mu_n = \inf_{U \subset H^1(\Omega) \text{ is a n-dim subspace}} \sup_{u \in U} R(u)$$

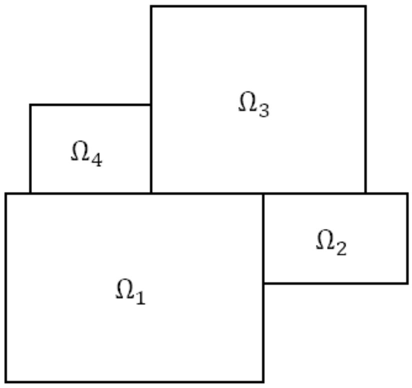

Figure 2: An example of finite union of disjoint adjacent rectangles

Lemma: $\mu_n \leq \lambda_n$

Proof:

$H_0^1 \subset H^1$, thus by last lemma, μ_n is taking infinimum value from a larger set.

Lemma (Domain monotonicity) For two bounded domain $\Omega_1 \subset \Omega_2$, $\lambda_{n,\Omega_1} \leq \lambda_{n,\Omega_2}$

Proof:

Same reasoning as previous lemma, every subspace for the case of Ω_1 is also valid in Ω_2 by extending to zero, thus the *min* in the Min-Max lemma ranges over more subspaces for Ω_2, thus $\lambda_{n,\Omega_1} \leq \lambda_{n,\Omega_2}$.

Suppose first that Ω is a finite union of disjoint adjacent rectangles, and $\Omega = \bigcup_{i \in I} \Omega_i$. Taking $\tilde{H}_0^1(\Omega) = \{u \in L^2, \forall i \in I, u|_{\Omega_i} \in H_0^1(\Omega_i)\}$, we define similarly $\tilde{H}^1(\Omega)$, We have chain of inclusions:

$$\forall i \in I, \tilde{H}_0^1(\Omega) \subset H_0^1(\Omega) \subset H^1(\Omega) \subset \tilde{H}^1(\Omega)$$

We suppose each Ω_i has Dirichlet eigenvalues $\lambda_{i,1}, \lambda_{i,2}, \dots$, we group all the eigenvalues of all the rectangles together and form a single sequence $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots$, same way we form $\tilde{\mu}_1, \tilde{\mu}_2, \dots$.
Here we have a variational caracterization for these $\tilde{\lambda}_i$s and $\tilde{\mu}_i$s. we extend basis $u_{i,1}, u_{i,2}, \dots$ corresponding to Dirichlet eigenvalues of each subdomain Ω_i with zero to the whole domain Ω, then group them together, we get a orthenormal basis $\tilde{u}_i$ for $\tilde{H}_0^1$, in a similar way we get a orthenormal basis $\tilde{v}_i$ for $\tilde{H}^1$. Apply these basis to the Min-Max lemma, we get that:

$$\tilde{\lambda}_i = \inf_{U \subset \tilde{H}_0^1(\Omega) \text{ is a n-d subspace}} \sup_{u \in U} R(u)$$

and

$$\tilde{\mu}_i = \inf_{U \subset \tilde{H}^1(\Omega) \text{ is a n-d subspace}} \sup_{u \in U} R(u)$$

Because of the chain of inclusions of $H_0^1, H^1, \tilde{H}_0^1, \tilde{H}^1$, we have:

$$\tilde{\lambda}_n \geq \lambda_n \geq \mu_n \geq \tilde{\mu}_n$$

And

$$N_{\tilde{\mu}}(\alpha) \geq N_{\mu}(\alpha) \geq N_{\lambda}(\alpha) \geq N_{\tilde{\lambda}}(\alpha)$$

Denote the counting function of Ω_i as $N_{\lambda,i}(\alpha)$, we have by construction of $\tilde{\lambda}_i$:

$$N_{\tilde{\lambda}}(\alpha) = \sum_{i \in I} N_{\lambda,i}(\alpha)$$

We have already proofed in the rectangle case that:

$$N_{\lambda,i}(\alpha) \sim \frac{\alpha^{\frac{n}{2}}}{(2\pi)^n} vol(\Omega_i)\omega_n$$

Thus:

$$N_{\tilde{\lambda}}(\alpha) \sim \sum_{i \in I} N_{\lambda,i}(\alpha) = \frac{\alpha^{\frac{n}{2}}}{(2\pi)^n}(\sum_{i \in I} vol(\Omega_i))\omega_n = \frac{\alpha^{\frac{n}{2}}}{(2\pi)^n} vol(\Omega)\omega_n$$

With the same reasoning, we also get for the Neumann eigenvalue:

$$N_{\tilde{\mu}}(\alpha) \sim \sum_{i \in I} N_{\mu,i}(\alpha) = \frac{\alpha^{\frac{n}{2}}}{(2\pi)^n}(\sum_{i \in I} vol(\Omega_i))\omega_n = \frac{\alpha^{\frac{n}{2}}}{(2\pi)^n} vol(\Omega)\omega_n$$

Thus we conclude:

$$N_{\lambda}(\alpha) \sim \frac{\alpha^{\frac{n}{2}}}{(2\pi)^n} vol(\Omega)\omega_n$$

New let $\Omega \subset \mathbb{R}^n$ be any bounded domain. Given any $\epsilon > 0$, there exists Ω_1 and Ω_2 finite unions of rectangles such that $\Omega_1 \subset \Omega \subset \Omega_2$ and $vol(\Omega_2$
$\Omega_1) < \epsilon$. Domain monotonicity implies that $\lambda_n(\Omega_2) \leq \lambda_n(\Omega) \leq \lambda_n(\Omega_1)$, and hence $N_{\lambda,1}(\lambda) \leq N_{\lambda}(\lambda) \leq N_{\lambda,2}(\lambda)$. Consider the following:

$$\frac{\omega_n}{(2\pi)^n} vol(\Omega_1)\alpha^{n/2} + o(1) \leq N_{\lambda}(\alpha) \leq \frac{\omega_n}{(2\pi)^n} vol(\Omega_2)\alpha^{n/2} + o(1).$$

Thus

$$\frac{\omega_n}{(2\pi)^n}(vol(\Omega) - \epsilon)\alpha^{n/2} + o(1) \leq N_{\lambda}(\alpha) \leq \frac{\omega_n}{(2\pi)^n}(vol(\Omega) + \epsilon)\alpha^{n/2} + o(1).$$

With $\epsilon \to 0$, we conclude:

$$N_{\lambda}(\alpha) \sim \frac{\omega_n}{(2\pi)^n} vol(\Omega)\alpha^{n/2}$$

References

[1] Yves Colin De Verdière. *On the remainder in the Weyl formula for the Euclidean disk*. 2011. arXiv: `1105.2233 [math-ph]`.

[2] URL: `http://people.math.sfu.ca/~cbm/aands/page_366.htm`.