Assignmet11

November 2020

1

We have the Hamiltonian for the spherical pendulum:

$$H = \frac{1}{2a^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2(\theta)}\right) - mga\sin(\theta) \tag{1}$$

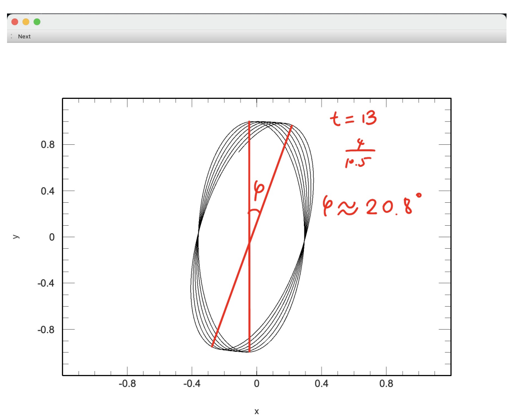
Then we can get the Hamiltonian equations:

$$\frac{\partial H}{\partial p_\theta} = \frac{\partial \theta}{\partial t}, \quad \frac{\partial H}{\partial p_\phi} = \frac{\partial \phi}{\partial t}, \quad \frac{\partial H}{\partial \theta} = -\frac{\partial p_\theta}{\partial t}, \quad \frac{\partial H}{\partial \phi} = -\frac{\partial p_\phi}{\partial t}$$

In this question, we set $m = a = g = 1$, together with the initial conditions:

$$\theta(0) = \theta_0 = 0.3, \quad \phi(0) = 0, \quad p_\theta(0) = 0, \quad p_\phi(0) = l = 1$$

By solving this OEDs, we get:



From the diagram we can read the precession rate is around 1.6° per unit time.

2

2.1

If $\xi = 1/2$, we have:

$$\frac{dx}{d\tau} = \eta \tag{2}$$
$$\frac{d\eta}{d\tau} = -\eta - x \tag{3}$$

We can rewrite the equation as:

$$\frac{du}{d\tau} = \frac{-1 + \sqrt{3}i}{2} u \tag{4}$$
$$\frac{dv}{d\tau} = \frac{-1 - \sqrt{3}i}{2} v \tag{5}$$

where $u = [(-1 - \sqrt{3}i)/2] + \eta, v = [(-1 + \sqrt{3}i)/2]x + \eta$. And for the simplicity, we set the initial conditions as: $x(0) = 0, \eta(0) = 1$. Therefore, we get the analytic solution:

$$x = e^{(-\tau/2)} \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}\tau\right)$$
$$\eta = e^{(-\tau/2)} \cos\left(\frac{\sqrt{3}}{2}\tau\right)$$

2.2

The deviation between the numerical solution and the true solution at point $\tau = 10$ is showed in following diagram.

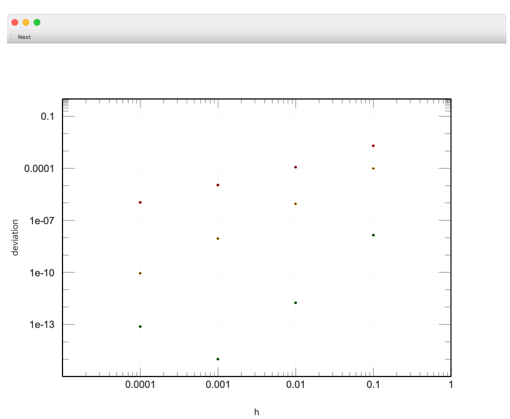


Figure 1: The deviation for Euler(red), Midpoints(orange), Trapezoid(blue, overlap with midpoints) and CRK(green).

2.3 Adaptive step size control

The deviation between the numerical solution and the true solution at point $\tau = 10$ is showed in following diagram.

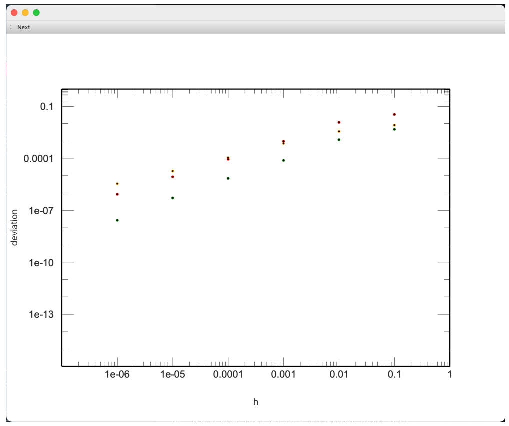


Figure 2: The deviation for Euler(red), Midpoints(orange), Trapezoid(blue, overlap with midpoints) and CRK(green).

3

3.1

We have the Lagrangian for the double pendulum:

$$L = \frac{m_1 + m_2}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \cos(\theta_1) + m_2 g l_2 \cos(\theta_2) \tag{6}$$

And the canonical momentum are:

$$p_{\theta_1} = \frac{\partial L}{\dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$p_{\theta_2} = \frac{\partial L}{\dot{\theta}_2} = m_2 l_1^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

3.2

Therefore we get the Hamiltonian:

$$H = \frac{m_1 + m_2}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2) \tag{7}$$

$$= \frac{m_2 l_2^2 p_{\theta_1}^2 + (m_1 + m_2) l_1^2 p_{\theta_2}^2 - 2 m_2 l_1 l_2 p_{\theta_1}^2 p_{\theta_2}^2 \cos(\theta_1 - \theta_2)}{2 m_2 l_1^2 l_2^2 (m_1 + m_2 \sin^2(\theta_1 - \theta_2))} - (m1 + m2) g l_1 \cos(\theta_1) - m2 g l_2 \cos(\theta_2) \tag{8}$$

3.3

For the simplicity, we set $g = l_1 = l_2 = m_1 = m_2 = 1$.

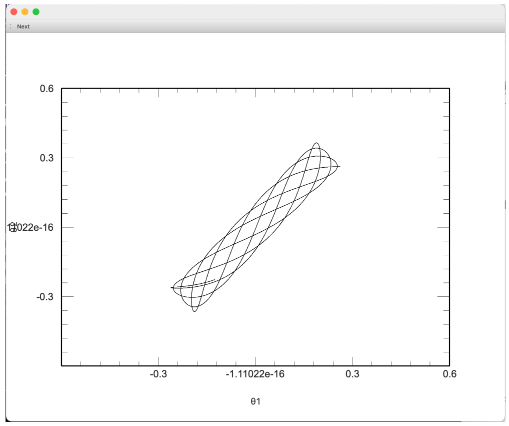


Figure 3: $\theta_1(0) = \pi/12, \theta_2(0) = \pi/12$

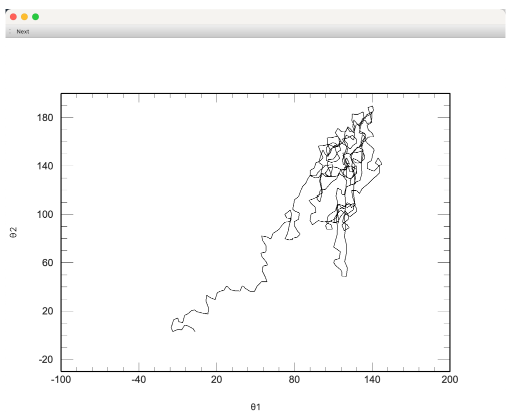


Figure 4: $\theta_1(0) = \pi, \theta_2(0) = \pi$

4

4.1

$$\begin{cases} dSdt = bN - (\lambda + d)S + R \\ \frac{dE}{dt} = \lambda S - (d + a)E \\ \frac{dI}{dt} = aE - (d + \gamma)I \\ \frac{dR}{dt} = \gamma I(+d)R \end{cases} \tag{9}$$

4.2

By summing the four equations above, we get $dN/dt = bN - dN$. If $b = d$, we have $dN/dt = 0$, the population is static.

4.3

This model assume the disease mortality are the same for all four categories.

4.4

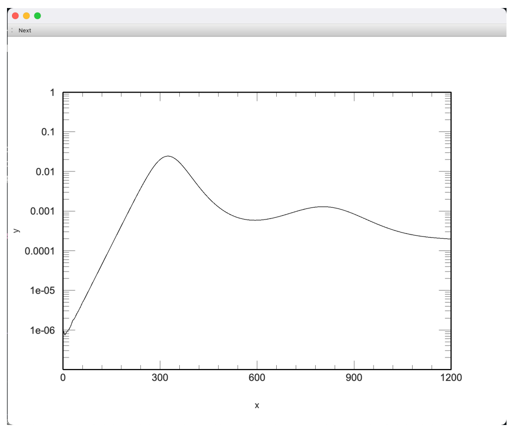


Figure 5: The infected population ratio