Condensed Matter Physics Lecture 2

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Abstract

introduction to the general technique of Bosonization of 1-D systems

Contents

1 Electrons, Spin on Lattices

Hamiltonian of electron in 1D Hubbard model

$$H_{hub} = -t \sum_{j\sigma} \left(c_{j\sigma}^{\dagger} c_{j+1\sigma} + \text{h.c.} \right) + U \sum_{j} c_{j\uparrow}^{\dagger} c_{j\uparrow} c_{j\downarrow}^{\dagger} c_{j\downarrow} \tag{1}$$

where $\left\{ \hat{c}_{i,\sigma}, \hat{c}_{j,\sigma'}^{\dagger} \right\} = \delta_{i,j} \delta_{\sigma,\sigma'}$

Hamiltonian of XXZ model

$$H_{XXZ} = J \sum_{j} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right) \tag{2}$$

the anisotropy parameter in special case $\Delta = 1$ give Heisenberg model

Both models can be solved exactly in one dimension by means of the celebrated Bethe Ansatz

1.1 non-interacting limit $(U = 0)$

the Hamiltonian can be easily diagonalized by Fourier transformation

$$c_{k\sigma}^{\dagger} = \sum_{j=1}^{L} \frac{e^{ikj}}{\sqrt{L}} c_{j\sigma}^{\dagger} \Leftrightarrow c_{j\sigma}^{\dagger} = \sum_{k \in BZ} \frac{e^{-ikj}}{\sqrt{L}} c_{k\sigma}^{\dagger} \tag{3}$$

the fourier transformation of non-interacting Hamiltonian

$$\begin{aligned}
H_0 &= -t \sum_{j\sigma} \left(c_{j\sigma}^{\dagger} c_{j+1\sigma} + \text{h.c.} \right) && (4)\\
&= -t \sum_{j\sigma} \left[\sum_{k \in BZ} \frac{e^{-ikj}}{\sqrt{L}} c_{k\sigma}^{\dagger} \sum_{p \in BZ} \frac{e^{ip(j+1)}}{\sqrt{L}} c_{p\sigma} + \text{h.c.} \right] && (5)\\
&= -\frac{t}{L} \sum_{k,p \in BZ} \sum_{j\sigma} e^{ij(p-k)} e^{ip} c_{k\sigma}^{\dagger} c_{p\sigma} + \text{h.c.} && (6)\\
&= -\mathrm{t} \sum_{\mathrm{k} \in \mathrm{BZ}} \mathrm{e}^{\mathrm{ik}} \mathrm{c}_{\mathrm{k}\sigma}^{\dagger} \mathrm{c}_{\mathrm{k}\sigma} + \text{h.c.} && (7)\\
&= -2t \sum_{k \in BZ} \cos(k) c_{k\sigma}^{\dagger} c_{k\sigma} && (8)
\end{aligned}$$

where we use $\frac{1}{L} \sum_j e^{i\left(k-k'\right)} j = \delta_{k,k'}$

In real space, a $N \times N$ matrix

$$H = \begin{pmatrix} \cdots & \vec{c} & \cdots \end{pmatrix} \begin{pmatrix} 0 & -t & 0 & 0 & \cdots & 0 & -t \\ -t & 0 & -t & 0 & \cdots & 0 & 0 \\ 0 & -t & 0 & -t & \cdots & 0 & 0 \\ 0 & 0 & -t & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ -t & 0 & 0 & 0 & \cdots & -t & 0 \end{pmatrix} \begin{pmatrix} \vdots \\ \vec{c} \\ \vdots \end{pmatrix} \tag{9}$$

In Momentum space,a 1×1 matrix

$$H = -2t \sum_{k \in BZ} c^{\dagger}_{k\sigma} \cos(k) c_{k\sigma} \tag{10}$$

Translational invariant, k is a good quantum number, this is a good basis to solve the problem

1.2 Ground state

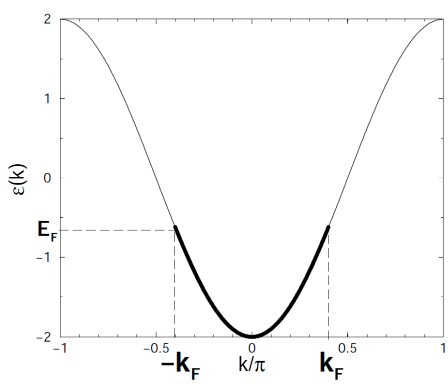

Figure 1: The ground state for N electrons corresponds to filling up all the states. The highest occupied level is the Fermi level, its energy the Fermi energy E_F and its wave-vector the Fermi wave-vector k_F

Ground state

$$|G\rangle = \prod_{k \in BZ} c^{\dagger}_{k\sigma} |0\rangle \tag{11}$$

in Fock space, if we written in terms of wavefunction $\langle x_i|c_k^{\dagger}|0\rangle = \psi_k\left(x_i\right)$
Slater determinant manybody function

$$\langle \vec{x}|G\rangle = \begin{vmatrix} \psi_{k1}\left(\vec{r}_1\right) & \psi_{k1}\left(\vec{r}_2\right) & \cdots & \psi_{k1}\left(\vec{r}_N\right) \\ \psi_{k2}\left(\vec{r}_1\right) & \psi_{k2}\left(\vec{r}_2\right) & \cdots & \psi_{k2}\left(\vec{r}_N\right) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{N}\left(\vec{r}_1\right) & \psi_{kN}\left(\vec{r}_2\right) & \cdots & \psi_{kN}\left(\vec{r}_{kN}\right) \end{vmatrix} \tag{12}$$

2 Linearized spectrum

We are interest in the spectrum near the Fermi momentum and Fermi energy, which can be Linearized.

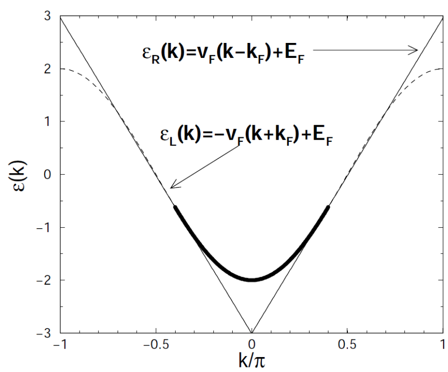

Figure 2: Linearization of the spectrum near the Fermi momentum and Fermi energy

$$H_0 = \int_{-L/2}^{L/2} dx \psi^{\dagger}(x) \left(-i\partial_x\right) \psi(x) + h.c \tag{13}$$

we are still thinking about a box $\left[-\frac{L}{2}, \frac{L}{2}\right]$, then $k = \frac{2\pi}{L} n$ where $n \in \mathbb{Z}$.
the Lattice spacing in the limit of $(a \to 0)$, such that the Brillouin zone $Bz \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right] \to [-\infty, \infty]$
here the operator

$$\begin{aligned} \psi(x) &\equiv \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{+\infty} e^{ikx} c_k \\ \psi^{\dagger}(x) &\equiv \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{+\infty} e^{-ikx} c_k^{\dagger} \end{aligned} \tag{14}$$

the inverse transformation.

$$c_k = \frac{1}{\sqrt{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx e^{-ikx} \psi(x) \tag{15}$$

one should notes that the Ground state has finite number of particles

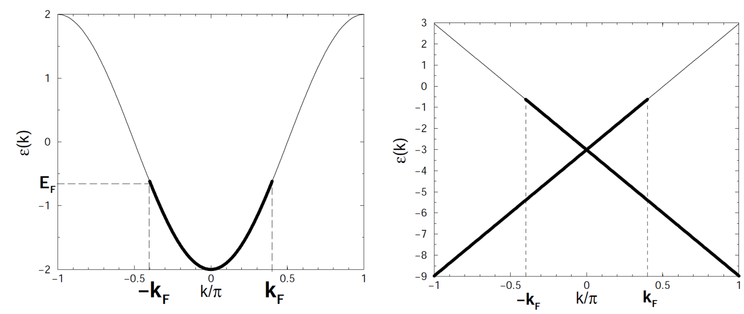

Figure 3: Linearization of the spectrum near the Fermi momentum and Fermi energy

Since Vaccum $|0\rangle_0$ has infinite number particle. ,To define the number particle, we consider the difference between the a state $|N\rangle$ and $|0\rangle_0$.
Here the number operator define as

$$\hat{N} = \sum_k \left[c_k^\dagger c_k - \left\langle c_k^\dagger c_k \right\rangle_0 \right] \tag{16}$$

where the operation of normal-ordering a string of creation and annihilation operators

$$c_k^\dagger c_k - \left\langle c_k^\dagger c_k \right\rangle_0 =: c_k^\dagger c_k : \tag{17}$$

obvious any operator with normal ordering

$$\langle : ABCD \ldots : \rangle_0 = 0 \tag{18}$$

Let us suppose

$$\begin{aligned} c_k|0\rangle_0 &= 0, \quad k > 0 \\ c_k^\dagger|0\rangle_0 &= 0, \quad k \leq 0 \end{aligned} \tag{19}$$

Then we can define N-particle ground state

$$\begin{aligned} c_N^\dagger c_{N-1}^\dagger \cdots c_1^\dagger |0\rangle_0 &\equiv |N\rangle_0 (N > 0) \\ c_{N+1} c_{N+2} \cdots c_{-1} c_0 |0\rangle_0 &\equiv |N\rangle_0 (N < 0) \end{aligned} \tag{20}$$

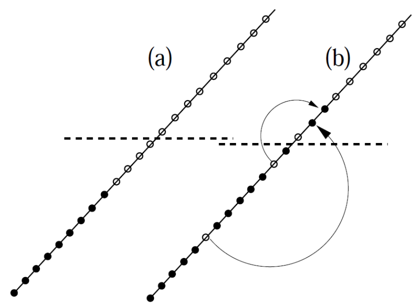

Figure 4: (a)The four highest energy particles of $|0\rangle_0$ have been removed. (b)Two particle-hole excitations on the ground state

2.1 Hilbert space

The Hilbert space is spanned by the Fock space

$$\mathcal{H} = \oplus_{n\in\mathbb{Z}}\mathcal{H}_n = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \tag{21}$$

where $\mathcal{H}_n$ is n-particle Hilbert space.
we can define a operator, which is constructed from all of particle-hole excitations.
This operator characterizes the density fluctuation and should behave like a boson.

2.2 Density operators

define an important operators, the density operators,

$$\rho(q) \equiv \sum_k c^\dagger_{k+q} c_k \tag{22}$$

where $\rho(-q) = \rho^\dagger(q)$

Now we check the commutation relationship

$$[\rho(p),\rho(q)] = \sum_{kk'} \left[c^\dagger_{k'+p}c_{k'}, c^\dagger_{k+q}c_k\right] \tag{23}$$

$$= \sum_{kk'} \left\{c^\dagger_{k'+p}\left[c_{k'}, c^\dagger_{k+q}c_k\right] + \left[c^\dagger_{k'+p}, c^\dagger_{k+q}c_k\right]c_{k'}\right\} \tag{24}$$

$$= \sum_{kk'} \left\{c^\dagger_{k'+p}\delta_{k',k+q}c_k - c^\dagger_{k+q}c_{k'}\delta_{k'+p,k}\right\} \tag{25}$$

$$= \sum_{k} \left[c^\dagger_{k+p+q}c_k - c^\dagger_{k+q}c_{k-p}\right] \tag{26}$$

if $p \neq -q$ then $[\rho(p),\rho(q)] = \rho(p+q) - \rho(p+q) = 0$
if $p = -q$

$$[\rho(p),\rho(-p)] = \sum_k \left[: c^\dagger_k c_k : + \left\langle c^\dagger_k c_k\right\rangle_0 - : c^\dagger_{k-p}c_{k-p} : - \left\langle c^\dagger_{k-p}c_{k-p}\right\rangle_0\right] \tag{27}$$

make the shift $k-p \to k$ within the normal ordering sign because it does not introduce infinite quantities.

$$\hat{N} = \sum_k : c^\dagger_k c_k =: c^\dagger_{k-p}c_{k-p} \tag{28}$$

The two terms cancel

$$[\rho(p),\rho(-p)] = \sum_k \left[\left\langle c^\dagger_k c_k\right\rangle_0 - \left\langle c^\dagger_{k-p}c_{k-p}\right\rangle_0\right] = -\frac{L}{2\pi}p \tag{29}$$

where $p = \Delta k = \frac{2\pi}{L}$

*comment: if we shift the summation of k' by $k'-q$ at before, we end up with different cofficient of commutator

$$[\rho(p),\rho(-p)] = -\frac{L}{\pi}p \tag{30}$$

This means we use different regularization method, once we choose one, we fix it to avoid subtlety.

In summary, we can normalize the commutator

$$[\rho(p),\rho(q)] = -\frac{Lp}{2\pi}\delta_{p,-q} \tag{31}$$

and define the boson operators

$$\begin{aligned} b_q &= \sqrt{\frac{2\pi}{Lq}}\rho(-q)(q>0) \\ b_q^\dagger &= \sqrt{\frac{2\pi}{Lq}}\rho(q) \quad (q>0) \end{aligned} \tag{32}$$

we have

$$[b_q, b_{q'}] = \left[b_q^\dagger, b_{q'}^\dagger\right] = 0 \quad \left[b_q, b_{q'}^\dagger\right] = \delta_{q,q'} \tag{33}$$

and the Density operators

$$\rho(q) = \begin{cases} \sqrt{\frac{Lq}{2\pi}} b_q^\dagger & q>0 \\ \sqrt{\frac{L|q|}{2\pi}} b_{-q} & q<0 \end{cases} \tag{34}$$

Bilinear Fermionic operators is

$$\begin{aligned} \psi^\dagger(x)\psi(x) &= \frac{1}{L}\sum_{kp} e^{i(k-p)x} c_p^\dagger c_k \\ &= \frac{1}{L}\sum_{kq} e^{-iqx} c_{k+q}^\dagger c_k \\ &= \frac{1}{L}\sum_{q>0} \left[e^{-iqx}\rho(q) + e^{iqx}\rho(-q)\right] \\ &= \frac{1}{\sqrt{2\pi L}}\sum_{q>0}\sqrt{q}\left[e^{iqx} b_q + e^{-iqx} b_q^\dagger\right] \end{aligned} \tag{35}$$

2 fermions $\rightarrow$ 1 Boson

4 fermions $\rightarrow$ 2 Boson

interacting fermionic system $\rightarrow$ 1 Quadratic form of Bosonic field

obvious $\left[b_q, \hat{N}\right] = \left[b_{q,}^\dagger, \hat{N}\right] = 0 \quad b_q|N\rangle_0 = 0, \quad \forall q, N$

Physically, this means that particle-hole excitations preserve total particle number.

and $|N\rangle_0$ means that the N-particle ground state without particle-hole excitations

3 Fermion in terms of Bosons

3.1 Klein Factors

The Hilbert space as we know

$$\mathcal{H} = \oplus_{n\in\mathbb{Z}}\mathcal{H}_n \tag{36}$$

where $|n\rangle_0 \in \mathcal{H}_n$ and

$$G[\{b_q^\dagger\}]|N\rangle_0 \in \mathcal{H}_n \tag{37}$$

where $G[\{b_q^\dagger\}]$ is function of $b_q^\dagger$.
To connect the sub-Hilbert space with different particle number, we need "fermionic operator"(Klein Factors)

define Klein factors

$$\begin{aligned} F^\dagger|N\rangle_0 = & \quad |N+1\rangle_0 \quad F|N\rangle_0 = |N-1\rangle_0 \\ \left[F, b_q^\dagger\right] = & \quad [F, b_q] = \left[F^\dagger, b_q^\dagger\right] = \left[F^\dagger, b_q\right] = 0 \end{aligned} \tag{38}$$

the shape of particle-hole distribution $G\left[\{b_q^\dagger\}\right]|N\rangle_0$ are the same for different sector $|N\rangle$, $|N+1\rangle$
since

$$\begin{aligned} F^\dagger|N\rangle = F^\dagger f\left[\{b_q^\dagger\}\right]|N\rangle_0 = f\left[\{b_q^\dagger\}\right]F^\dagger|N\rangle_0 = f\left[\{b_q^\dagger\}\right]|N+1\rangle_0 \\ F|N\rangle = Ff\left[\{b_q^\dagger\}\right]|N\rangle_0 = f\left[\{b_q^\dagger\}\right]F|N\rangle_0 = f\left[\{b_q^\dagger\}\right]|N-1\rangle_0 \end{aligned} \tag{39}$$

we also have

$$[F, \hat{N}] = F \quad \left[F^\dagger, \hat{N}\right] = -F^\dagger \tag{40}$$

thus we can construct the entire Hilbert space by $F, F^\dagger, b_q^\dagger$ and $|N\rangle$

3.2 fermion in terms of bosons

Next important step is written the original ferimion $\psi(x)$ in terms of bosons,$\hat{N}$ and Klein factors.
from the commutators

$$\psi(x)|N\rangle_0 \quad \propto \quad \exp\left[\sum_{q>0}\alpha_q(x)b_q^\dagger\right]|N-1\rangle_0 \tag{41}$$

$$\psi(x)|N\rangle_0 \quad = \quad \Lambda(x)\exp\left[\sum_{q>0}\alpha_q(x)b_q^\dagger\right]F|N\rangle_0 \tag{42}$$

on the N-particle ground state

$$[b_q, \psi(x)]\,|N\rangle_0 = (b_q\psi(x) - \psi(x)b_q)|N\rangle_0 = b_q\psi(x)|N\rangle_0 \tag{43}$$

$$= -\sqrt{\frac{2\pi}{Lq}}e^{-iqx}\psi(x)|N\rangle_0 \equiv \alpha_q(x)\psi(x)|N\rangle_0 \tag{44}$$

where $b_q|N\rangle_0 = 0$,thus $\psi(x)|N\rangle_0$ is an eigenstate of b_q with corresponding eigenvalue $\alpha_q(x) = -\sqrt{\frac{2\pi}{Lq}}e^{-iqx}$

$$b_q|\alpha\rangle_0 = \alpha|\alpha\rangle_0] \tag{45}$$

this kind of states are called coherent states. and the state without fixed number of particles

$$|\alpha\rangle \propto \exp\left[\alpha b^\dagger - \alpha^* b\right]|0\rangle \tag{46}$$

here we have

$$\psi(x)|N\rangle = \Lambda(x)\exp\left[\sum_{q>0}\alpha_q(x)b_q^\dagger - \alpha_q^*(x)b_q\right]|N-1\rangle \tag{47}$$

The normalization $\Lambda(x)$ is determined by

$$_0\langle N-1|\psi(x)|N\rangle_0 = \Lambda(x)_0\langle N|\exp\left[\sum_{q>0}\alpha_q(x)b_q^\dagger - \alpha_q^*(x)b_q\right]|N-1\rangle_0 \tag{48}$$

$$_0\langle N-1|\frac{1}{\sqrt{L}}\sum_{k=-\infty}^{+\infty}e^{ikx}c_k|N\rangle_0 = \Lambda(x)_0\langle N||N\rangle_0 \tag{49}$$

$$\frac{1}{\sqrt{L}}e^{i\frac{2\pi Nx}{L}} = \Lambda(x) \tag{50}$$

one can read off the fermionic operator

$$\psi(x) = \frac{F}{\sqrt{L}}e^{i\frac{2\pi\hat{N}x}{L}}\exp\left[\sum_{q>0}\alpha_q(x)b_q^\dagger - \alpha_q^*(x)b_q\right] \tag{51}$$

This is one of the most important results of bosonization. the expression of the fermionic annihilation operator in terms of the bosons, F and $\hat{N}$.

and $\exp\left[\sum_{q>0}\alpha_q(x)b_q^\dagger - \alpha_q^*(x)b_q\right]$ is displacement operator

3.3 Phase operator

let us define the Phase operator

$$\phi(x) = \frac{i}{\sqrt{L}} \sum_{q>0} \frac{1}{\sqrt{q}} e^{-\alpha q/2} \left[e^{iqx} b_q - e^{-iqx} b_q^\dagger \right] \tag{52}$$

since the phase and density are canonical conjugate operators, we have

$$[\phi(x), \rho(y)] = i\delta(x-y) \tag{53}$$

$$\rho(x) = \psi^\dagger(x)\psi(x) = \frac{1}{\sqrt{2\pi L}} \sum_{q>0} \sqrt{q} e^{-\alpha q/2} \left[e^{iqx} b_q + e^{-iqx} b_q^\dagger \right] \tag{54}$$

Proof:

$$\begin{aligned} [\phi(x), \rho(y)] &= i\frac{1}{L} \sum_{q,k} \sqrt{\frac{k}{q}} e^{iqx} e^{-iky} && (55) \\ &= i\frac{1}{L} \sum_{q} e^{iq(x-y)} = i\delta(x-y) && (56) \end{aligned}$$

where the first step we use $\left[b_q, b_k^\dagger\right] = \delta_{q,k}$. end proof.

since

$$\begin{aligned} \phi(x) &= i \sum_q \left[\alpha_q(x) b_q^\dagger - \alpha_q^*(x) b_q \right] && (57) \\ \alpha_q(x) &= -\sqrt{\frac{2\pi}{Lq}} e^{-iqx} && (58) \end{aligned}$$

the fermionic operator is

$$\psi(x) = \frac{F}{\sqrt{2\pi\alpha}} e^{i\frac{2\pi \hat{N} x}{L}} \exp[-i\phi(x)] \tag{59}$$

Here we write down the Fermion in terms of bosonic field $\phi(x)$

4 Bosonization Linearized Hamiltonian

4.1 non-interacting part

Linearized Hamiltonian only for the right branch

$$H_0 = v_F \int_{-L/2}^{L/2} dx \psi^\dagger(x) \left(-i\partial_x\right) \psi(x) \tag{60}$$

before write down the Hamiltonian in terms of $\phi(x)$, let's look at some subtleties

4.1.1 propagator

the propagator of the linearized Hamiltonian

$$H_0 = v_F \int dp \psi^\dagger(p) p \psi(p) \tag{61}$$

with energy $E = v_F p$

we need to use a 'regulator': this is a way to introduce a cut off, know there is a lattice underline (UV cut off)

$$\Psi(x,t) = \int \frac{dp}{2\pi} \Psi(p) e^{ipx} e^{-\frac{1}{2}\alpha|p|} e^{-iv_F pt} \tag{62}$$

we will take $\alpha \to 0$ at the end.
then, the propagator

$$\begin{aligned}
\langle \psi(x,t)\psi^\dagger(0)\rangle &= \int \frac{dp}{2\pi} e^{-\frac{1}{2}\alpha|p|} \int \frac{dq}{2\pi} e^{-\frac{1}{2}\alpha|q|} e^{ip(x-v_F t)} \langle \psi(p)\psi^\dagger(q)\rangle && (63)\\
&= \int \frac{dp}{2\pi} e^{ip(x-v_F t)} e^{-\alpha|p|} && (64)\\
&= \frac{1}{2\pi} \frac{1}{\alpha - i\left(x - v_F t\right)} && (65)\\
&= \frac{i}{2\pi} \frac{1}{(x - v_F t) + i\alpha} && (66)
\end{aligned}$$

where we used $\psi(p)\psi^\dagger(q)\rangle = 2\pi\delta(p-q)$ and $p > 0$ only one branch.
if we don't introduction α then $x = 0$ will be singularity

4.1.2 point splitting

Ambiguity for putting operators at the same point

$$\psi^\dagger(x+a)\psi(x) = e^{-i\frac{2\pi}{L}Na} e^{-i\phi(x+a)} e^{-i\phi(x)} \tag{67}$$

use $e^A e^B = e^{A+B} e^{[A,B]/2}$

$$
\begin{aligned}
[\phi(x+a), \phi(x)] &\sim \int \frac{dp}{2\pi} \int \frac{dq}{2\pi} e^{-\frac{1}{2}\alpha|q| - \frac{1}{2}\alpha|p|} \frac{1}{\sqrt{pq}} [e^{-iqa} b_q^\dagger - e^{iqa} b_q, b_p^\dagger - b_p] && (68) \\
&= \int \frac{dp}{2\pi} \frac{e^{-\alpha p}}{|p|} (e^{ipa} - e^{-ipa}) && (69) \\
&= \frac{1}{2\pi} \ln(\frac{\alpha}{\alpha - ia}) && (70)
\end{aligned}
$$

where we used $e^{-ipa} \to 1$ when a is small
in the limit $a \to 0$ $\alpha \to 0$

$$
\begin{aligned}
\psi^\dagger(x+a)\psi(x) &= \frac{1}{2\pi\alpha}(1 - i\frac{2\pi}{L}Na)(\frac{\alpha}{\alpha - ia})(i\partial_x\phi(x)a) && (71) \\
&= \frac{i}{2\pi a} + \frac{\hat{N}}{L} - \frac{1}{\sqrt{2\pi}}\partial_x\phi && (72)
\end{aligned}
$$

the first term divergent from infinite
if we normal order it

$$
:\psi^\dagger(x+a)\psi(x):= \frac{\hat{N}}{L} - \frac{1}{\sqrt{2\pi}}\partial_x\phi \qquad (73)
$$

we remove the fermi sea contribution

Question:what is $:\psi^\dagger(x+a)(-i\partial_x)\psi(x):=?$
one should use the same 'point splitting' procedure to write down $:\psi^\dagger(x)(\partial_x\psi(x))$ in terms of $\phi(x)$.
here we can also use another approach

$$
\begin{aligned}
:\psi^\dagger(x)(\partial_x\psi(x)): &= \lim_{x\to\varepsilon} g\frac{1}{2\pi}\psi^\dagger(x)(\frac{\psi(x+\varepsilon) - \psi(x-\varepsilon)}{2\varepsilon}) && (74) \\
&= \frac{1}{4\pi}(\partial_x\psi)^2 && (75)
\end{aligned}
$$

Here we drop the total derivative since it represents the boundary terms $\int_{B_1}^{B_2} dx\partial_x\phi = \partial_{B_2} - \partial_{B_1}$

we have the bosonized Hamiltonian

$$
\begin{aligned}
H_0 &= v_F \int dx :\psi^\dagger(x)\,(-i\partial_x)\,\psi(x): && (76) \\
&= \frac{1}{4\pi} v_F \int dx\,(\partial_x\phi)^2 && (77)
\end{aligned}
$$

4.1.3 two branches

Now, consider Hamiltonian

$$
\begin{aligned}
H_0 &= v_F \int dx \left[\psi_R^\dagger(x)\left(-i\partial_x\right)\psi_R(x) + \psi_L^\dagger(x)\left(i\partial_x\right)\psi_L(x)\right] \\
&= \frac{v_F}{4\pi}\int dx\left(\partial_x\phi_R\right)^2 + \left(\partial_x\phi_L\right)^2
\end{aligned} \tag{78}
$$

The Hilbert space is spanned by left and right Hilbert space

$$
\mathcal{H} = \mathcal{H}_R \oplus \mathcal{H}_L \quad \mathcal{H}_\alpha = \oplus_{n\in\mathbb{Z}}\mathcal{H}_{\alpha,n} \tag{79}
$$

where $\alpha = L, R$.

the Klein factor F_α where also$\alpha = L, R$

$$
F_L|N_R, N_L\rangle_0 = |N_R, N_{L-1}\rangle \tag{80}
$$

$$
F_R|N_R, N_L\rangle_0 = |N_{R-1}, N_L\rangle \tag{81}
$$

4.1.4 Sin-Gordon Model

We can change variables

$$
\begin{aligned}
\theta &= \frac{1}{\sqrt{2}}\left[\phi_L + \phi_R\right] \\
\phi &= \frac{1}{\sqrt{2}}\left[\phi_R - \phi_L\right]
\end{aligned} \tag{82}
$$

Hamiltonian

$$
H_0 = \frac{v_F}{4\pi}\int dx\left(\partial_x\theta\right)^2 + \left(\partial_x\phi\right)^2 \tag{83}
$$

for convince, we rescale the field $\theta \to \sqrt{2\pi}\theta$,$\phi \to \sqrt{2\pi}\phi$

$$
H_0 = \frac{v_F}{2}\int dx\left(\partial_x\theta\right)^2 + \left(\partial_x\phi\right)^2 \tag{84}
$$

recall

$$
\phi_R(x) = -i\sum_q \frac{1}{\sqrt{2\pi L q}}\left[e^{-iqx}b_q^\dagger - e^{iqx}b_q\right] \tag{85}
$$

$$
\partial_x\phi_R(x) = -\frac{1}{\sqrt{2\pi L}}\sum_q \sqrt{q}\left[e^{-iqx}b_q^\dagger + e^{iqx}b_q\right] = -\rho_R(x) \tag{86}
$$

the commutation relation

$$
[\phi_R(x), -\rho_R(y)] = [\phi_R(x), \partial_y\phi_R(y)] = -i\delta(x-y) \tag{87}
$$

$$
[\phi_L(x), \rho_L(y)] = [\phi_L(x), \partial_y\phi_L(y)] = i\delta(x-y) \tag{88}
$$

in summary $[\phi(x), \partial_y \phi(y)] = i\delta(x-y)$

$\Pi(x) = \partial_x \phi(x)$ is the conjugate momentum of $\phi(x)$ Hamiltonian can be written as

$$H_0 = \frac{v_F}{2} \int dx[(\Pi(x))^2 + (\partial_x \phi)^2] + V\cos(\alpha\phi) \tag{89}$$

the term$V\cos(\alpha\phi)$ is interaction. the Hamiltonian is sine-Gordon model

4.2 interacting part

Now we consider the interacting case

$$H = H_0 + H_{int} \tag{90}$$

where the interacting term

$$H_{int} = \int_{-L/2}^{L/2} dx \left[\frac{g_4}{2} \left(\psi_R^\dagger \psi_R \psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L \psi_L^\dagger \psi_L \right) + g_2 \left(\psi_R^\dagger \psi_R \psi_L^\dagger \psi_L \right) \right] \tag{91}$$

where $\psi_R^\dagger \psi_R \psi_R^\dagger \psi_R$ and $\psi_L^\dagger \psi_L \psi_L^\dagger \psi_L$is forward scattering
the $\psi_R^\dagger \psi_R \psi_L^\dagger \psi_L$ is backward scattering.
Umklapp terms,$\psi_R^\dagger \psi_L \psi_R^\dagger \psi_R$, $\psi_R^\dagger \psi_L \psi_R^\dagger \psi_L$ average to zero.therefore, we focus on others.

4.2.1 Luttinger Model

only consider g_2, g_4term

$$H_0 = \frac{v_F}{2} \int dx \left(\partial_x \theta\right)^2 + \left(\partial_x \phi\right)^2 \tag{92}$$

and $\psi_\alpha^+ \psi_\alpha = -\frac{1}{\sqrt{2\pi}} \left(\partial_x \phi_\alpha\right)$ where $\alpha = L, R$

$$\begin{aligned} \phi_R &= \frac{1}{\sqrt{2}}(\theta - \phi) \\ \phi_L &= \frac{1}{\sqrt{2}}(\theta + \phi) \end{aligned} \tag{93}$$

then $\psi_{R/L}^\dagger \psi_{R/L} = \frac{1}{2\sqrt{\pi}} \left(\partial_x \theta \mp \partial_x \phi\right)$

forward scattering

$$\psi_R^\dagger \psi_R \psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L \psi_L^\dagger \psi_L = \frac{1}{2\pi} \int dx[\left(\partial_x \theta\right)^2 + \left(\partial_x \phi\right)^2] \tag{94}$$

backward scattering

$$\psi_R^\dagger \psi_R \psi_L^\dagger \psi_L = \frac{1}{4\pi} \int dx [(\partial_x \theta)^2 - (\partial_x \phi)^2] \tag{95}$$

Total Hamiltonian

$$\begin{aligned} H &= H_0 + H_{int} && (96) \\ &= \frac{v}{2} \int dx \left[K \left(\partial_x \theta\right)^2 + \frac{1}{K} \left(\partial_x \phi\right)^2 \right] && (97) \end{aligned}$$

where $v = \sqrt{\left(1+\overline{g}_4\right)^2 - \overline{g}_2^2}$ $\quad K = \sqrt{\frac{1+\overline{g}_4-\overline{g}_2}{1+\overline{g}_4+\overline{g}_2}}$
and $\overline{g}_4 = \frac{g_4}{2\pi}$ $\quad \overline{g}_2 = \frac{g_2}{2\pi}$

4.2.2 interpretation

(1)Since we can write $\Pi(x) = \partial_x \theta(x)$ as the conjugate variable of θ. the interacting fermion Hamiltonian can be expressed in terms of free Boson theory with normalizad v and stiffness K

(2)self Dual $\phi \leftrightarrow \theta$ at $K \leftrightarrow \frac{1}{K}$
then K is also called the Luttinger parameter

(3)backward scattering g_2 change K
$g < 1, \quad K > 1$ corresponds to repulsive interactions
$g > 1, \quad K < 1$ corresponds to attractive interactions

(4)As we learn from the free Boson theory
zero sound mode (phonon) is oscillated point

(5)Beyond linearization:
add term $\left(\partial_x \phi\right)^3$, breaks particle hole symmetry. interacting of sound modes.