Linear Bench-marking on Gkeyll Simulation of Ion Acoustic Instability with Vlasov-Poisson Solver

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1 Linear Theory

Starting from Vlasov-Poisson Equation,

$$\begin{gathered}\frac{\partial f_\alpha}{\partial t}+v\cdot\nabla f_\alpha-\frac{q_\alpha}{m_\alpha}(\nabla\varphi)\cdot\frac{\partial f_\alpha}{\partial v}=0\\ -\nabla^2\varphi=4\pi\sum q_\alpha\int \mathrm{d}^3 v f_\alpha\end{gathered}\tag{1}$$

where α stands for different species. Follow the standard routine of linear theory, one will arrive at the dielectric function

$$\epsilon(p,k)=1-\sum_\alpha\frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{i}{n_\alpha}\int \mathrm{d}^3\boldsymbol{v}\frac{1}{p+ik\cdot\boldsymbol{v}}\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}},p=-i\omega+\gamma\tag{2}$$

Given k, one can always choose the x axis to be along k,

$$\epsilon(p,k)=1-\sum_\alpha\frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{i}{n_\alpha}\int \mathrm{d}v_x\frac{F_\alpha'(v_x)}{v_x-ip/k}\tag{3}$$

where $F_\alpha(v_x)$ is the 1D distribution function. When $F_\alpha(v_x)$ is Maxwellian, define u and ζ_α

$$u=v_x/v_{th\alpha},\ \zeta_\alpha=ip/kv_{th\alpha},\ F_{\alpha0}(v_x)=n_\alpha\left(\frac{1}{\pi v_{th\alpha}^2}\right)^{1/2}e^{-v_x^2/v_{th\alpha}^2}\tag{4}$$

Then

$$\begin{gathered}\frac{1}{n_\alpha}\int \mathrm{d}v_x\frac{F_\alpha'(v_x)}{v_x-ip/k}=-\frac{2}{\sqrt{\pi}v_{\mathrm{th}\alpha}^2}\int \mathrm{d}u\frac{ue^{-u^2}}{u-\zeta_\alpha}=-\frac{2}{v_{\mathrm{th}\alpha}^2}\left[1+\zeta_\alpha Z\left(\zeta_\alpha\right)\right]\\ Z(\zeta)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}du\frac{e^{-u^2}}{u-\zeta}\end{gathered}\tag{5}$$

Finally, the dielectric function can be written as

$$\epsilon(p,k)=1+\sum_\alpha\frac{1+\zeta_\alpha Z\left(\zeta_\alpha\right)}{k^2\lambda_{\mathrm{D}\alpha}^2}\tag{6}$$

where $\lambda_{D\alpha}=v_{T\alpha}/\omega_{p\alpha}$ is the Debye length of certain species. Note that if the Maxwellian distribution has a mean flow, this amounts to a shift by some mean velocity u_α and all one needs to do to adjust the above results is to shift the argument of Z accordingly $\zeta_\alpha\to\zeta_\alpha-u_\alpha/v_{th\alpha}$. Theoretical value of the frequency and growth rate can be obtained by solving

$$\epsilon(p,k)=0\tag{7}$$

2 Linear Benchmarking

The unchanged physical parameters are listed in Table 1.

Table 1: Physical parameters

v_{Te}	u_e	v_{Ti}	u_i	T_e/T_i	m_i/m_e	c_s
0.02	/	0.000566	0.0	50	25	0.004

The linear benchmarking comprises two parts. The first part is to vary electron drift velocity while keeping the wave number being constant. And the second part is to vary wave number while keeping drift velocity constant. The detailed simulation parameters can be obtained from the input files.

All the simulations have been run to the saturation of linear stage. The frequencies are calculated by doing Fourier transformation on the electric field (in x direction) at the center of the box. The growth rates are calculated by doing linear fit of the x component of "fieldEnergy.bp" in linear stage.

2.1 1x1v

Table 2: keep $k = 10 * 2\pi$, vary electron drift velocity

$\boldsymbol{u_e}$	γ	γ_{theory}	ω	ω_{theory}
0.005	0.0036	0.0038	0.167	0.170
0.0075	0.0078	0.0080	0.167	0.171
0.01	0.00119	0.0122	0.170	0.172
0.0125	0.00161	0.0162	0.173	0.173
0.015	0.00199	0.0202	0.182	0.176

Table 3: keep $u_e = 0.01$, vary wave number

$\boldsymbol{k/2\pi}$	γ	γ_{theory}	ω	ω_{theory}
10	0.0119	0.0122	0.170	0.172
12	0.0095	0.0097	0.183	0.187
15	0.0053	0.0059	0.208	0.204
18	0.0002	0.0003	0.234	0.224

2.2 2x2v

In this part, wave number is still set to be along with x axis. The numbers of grids in real space and velocity space has been reduced to make the simulation less expensive.

Table 4: keep $k = 10 * 2\pi$, vary electron drift velocity

$\boldsymbol{u_e}$	γ	γ_{theory}	ω	ω_{theory}
0.005	0.0037	0.0038	0.172	0.170
0.01	0.00117	0.0122	0.170	0.172
0.015	0.00192	0.0202	0.179	0.176

Table 5: keep $u_e = 0.01$, vary wave number

$\boldsymbol{k/2\pi}$	γ	γ_{theory}	ω	ω_{theory}
10	0.0117	0.0122	0.172	0.172
12	0.0095	0.0097	0.183	0.187
15	0.0058	0.0059	0.209	0.204