2x2v Vlasov simulations of ion-acoustic turbulence using the Gkeyll code

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I. INTRODUCTION

The purpose of these notes is to describe the simulation set up and compile all of the calculations that are necessary for setting up a simulation and justifying the numerical choices. We work in SI units because it is easiest to move from these units to code units.

II. DESCRIPTION OF THE SIMULATION

We simulate a uniform unmagnetized plasma with initially cold Maxwellian ions and hot shifted Maxwellian electrons. We use Cartesian coordinates with two spatial dimensions and two velocity space dimensions. The initial current is ion-acoustic stable, and is increasing linearly in time on a time scale $\tau_{\rm d}$ that is slow compared to an ion-acoustic frequency but fast compared to a collision frequency. The final value of the current density is $J_{\rm final} = env_{T_{\rm e}}$. All modes are excited at each time step with a stochastic noise source to model spontaneous emission.

Model Equations

$$
\begin{aligned}
\frac{\partial F^\sigma}{\partial t} + \boldsymbol{v} \cdot \frac{\partial F^\sigma}{\partial \boldsymbol{x}} + \frac{e_\sigma}{m_\sigma}\left(E_z \hat{\boldsymbol{z}} + \delta \boldsymbol{E}\right) \cdot \frac{\partial F^\sigma}{\partial \boldsymbol{v}} =& \sum_{\sigma'} \nu_{\sigma/\sigma'} v_{T_{\sigma/\sigma'}}^2 \frac{\partial}{\partial \boldsymbol{v}} \cdot \left(\frac{\partial F^\sigma}{\partial \boldsymbol{v}} + \frac{\boldsymbol{v} - \boldsymbol{u}_{\sigma/\sigma'}}{v_{T_{\sigma/\sigma'}}^2} F^\sigma \right) \\
&+ \left(\frac{\partial F^\sigma}{\partial t} \right)^{\text{ext}}, && (1) \\
\frac{\partial \delta \boldsymbol{B}}{\partial t} =& \underbrace{-\nabla \times \delta \boldsymbol{E}}_{=0 \text{ we hope}}, && (2) \\
\underbrace{(\nabla \times \boldsymbol{B})^{\text{ext}}(T)}_{-\mu_0 \boldsymbol{J}^{\text{ext}}} =& \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial T}, && (3) \\
-\mu_0 \delta \boldsymbol{J}^{\text{ext}} =& \mu_0 \delta \boldsymbol{J}[\delta \boldsymbol{E}] + \mu_0 \epsilon_0 \frac{\partial \delta \boldsymbol{E}}{\partial t}, && (4) \\
\nabla \cdot \delta \boldsymbol{E}(t=0) =& \frac{1}{\epsilon_0} \sum_{\sigma'} \int \mathrm{d}^3 v' \, e_{\sigma'} F^{\sigma'}(t=0). && (5)
\end{aligned}
$$

- We have explicitly separated the oscillating $(\delta \boldsymbol{E}, \delta \boldsymbol{J})(\boldsymbol{x}, t)$ and spatially uniform $(\boldsymbol{E}, \boldsymbol{J})$ parts, for clarity. Also $T = t$, but we have used a new variable to emphasize the slow time dependence.

- $\nu_{\sigma/\sigma'}$ is a parameter that we choose that corresponds roughly to a mean time for a particle of species σ to scatter by 90 degrees due to collisions with species σ'. Both $v_{T_{\sigma/\sigma'}}$ and $u_{\sigma/\sigma'}$ are parameters that are constructed to make the model collision operator conserve momentum and energy (cf. `https://gkeyll.readthedocs.io/en/latest/dev/collisionmodels.html`).

- As long as $\boldsymbol{J}^{\text{ext}}$ is curl-free, then we should not excite an electromagnetic mode.

- In the larger context which we are trying to model, there is a reconnecting magnetic field $B_y(x)$ that is associated with $\boldsymbol{J}^{\text{ext}}$, but we do not include that field in the kinetic equations. Thus, we are assuming that we are sufficiently close to $x = 0$ that the reconnecting field has no important effect. There is also potentially a guide field. We could model that, but we need to understand the simpler problem first.

- The collisions are necessary for numerical regularization. This is an energy sink. We hope that the collisions are small enough that this is an insignificant energy loss channel.

- It is a phenomenological fact that Gkeyll does not supply sufficient numerical noise to source instabilities, so we need to add a term $\delta \boldsymbol{J}^{\text{ext}}$ that supplies the necessary perturbation for any instability to grow. We hope that the energy added via $\delta \boldsymbol{J}^{\text{ext}}$ is insignificant. In fact, it should really be of the same order as the collisional energy loss rate.
  - We really should be adding a source term in the ion kinetic equation to maintain quasineutrality with the stochastic wave source $\delta \boldsymbol{J}$.

Boundary conditions: 2x2v

- **No spatial variation in x:** (forbid waves propagating across the layer): $\partial_x = 0$
  - This *models* (as opposed to *approximates*) the effect of a layer which is so narrow that waves propagate out of the current layer before they can e-fold. In this circumstance, there is no amplification of the wave energy in this region of k space, and thus no quasilinear relaxation of the distribution function along v_x. Does BB agree with this?
    - A more realistic setup would be to have a box that is very coarse in x (i.e. the grid spacing Δx is large) so that k_x is forced to be small. The main computational cost would be including v_x. Perhaps it could be a coarse grid in v_x as well?
    - A second idealized setup that would be numerically tractable would be to use cylindrical coordinates in both space and velocity space and assume symmetry in ϕ and v_ϕ. Gkeyll is not currently set up to do this.
- **No v_x dependence:** Assume $F^\sigma(y, z, v_x, v_y, v_z, t) = F^\sigma(y, z, v_y, v_z, t)F^\sigma(v_x)$. Is this right?? Or are we simply able to integrate over v_x??
- **Spatial boundaries:** Periodic in y and z:

$$\begin{pmatrix} \boldsymbol{E} \\ F^{\mathrm{i}} \\ F^{\mathrm{e}} \end{pmatrix} (y + nL_y, z + mL_z, v_y, v_z) = \begin{pmatrix} \boldsymbol{E} \\ F^{\mathrm{i}} \\ F^{\mathrm{e}} \end{pmatrix} (x, y, z, v_y, v_z) \tag{6}$$

- **velocity space boundaries:** Specular reflection of particles at boundaries in velocity space (Gkeyll implementation). This prevents particle loss (which would violate quasineutrality). The velocity space boundaries are set at $v_{\alpha,\mathrm{max}} = 6 \times v_T$
- **distribution functions at $t = 0$:** Ions at rest, electrons flowing along v_z

$$\hat{F}^{\mathrm{e}}(\boldsymbol{v}, t = 0) = \frac{1}{(2\pi T_{\mathrm{e},0}/m_{\mathrm{e}})^{2/2}} \exp\left[-\left(\frac{\frac{1}{2}m_{\mathrm{e}}|\boldsymbol{v} - u_{\mathrm{e},z0}|^2}{T_{\mathrm{e},0}}\right)\right], \tag{7}$$

$$\hat{F}^{\mathrm{i}}(\boldsymbol{v}, t = 0) = \frac{1}{(2\pi T_{\mathrm{i},0}/m_{\mathrm{i}})^{2/2}} \exp\left[-\left(\frac{\frac{1}{2}m_{\mathrm{i}}|\boldsymbol{v}|^2}{T_{\mathrm{i},0}}\right)\right] \tag{8}$$

Source/Drive

DC

There is a "DC" external current $\boldsymbol{J}^{\text{ext}}$ in Ampère's law (that models the effect of magnetic flux being driven together by the inflows) $\delta\boldsymbol{J}^{\text{ext}}$

$$\boldsymbol{J}^{\text{ext}} = J_0\,(1 + t/\tau_{\text{d}})\,\hat{\boldsymbol{z}} \tag{9}$$

AC

as well as a stochastic wave source current

$$\delta\boldsymbol{J}^{\text{ext}} = \sum_{\alpha}\delta(t - t_\alpha)\frac{1}{\epsilon_0}\sum_{n_y, n_z \in \mathcal{N}}|\delta\boldsymbol{E}_{\boldsymbol{n}}(t_\alpha)|\hat{\boldsymbol{n}}\sin(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x} + \phi(t_\alpha)) \tag{10}$$

that is associated with a stochastic source term in the ion-kinetic equation:

$$\left(\frac{\partial\delta F^{\text{i,ext}}}{\partial t}\right)_{\text{noise injection}} = \sum_{\alpha}\delta(t - t_\alpha)\hat{F}^{\text{i}}(\boldsymbol{v})\sum_{n_y, n_z \in \mathcal{N}}\chi_0(n_y, n_z)\cos(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x} + \phi_{\boldsymbol{n}}(t_\alpha)) \tag{11}$$

I wonder if we should only have an AC source term in Ampere's law, and not have a corresponding term in the kinetic equation. On the other hand, if there is no source term in the kinetic equation, I have no idea how one would get the spontaneous emission term that shows up in the kinetic equation in the quasilinear theory

Physical Parameters

$$
\begin{aligned}
\frac{m_\mathrm{e}}{m_\mathrm{i}} &= \frac{1}{25}, && (12)\\
\frac{T_{\mathrm{e},0}}{T_{\mathrm{i},0}} &= 100, && (13)\\
|u_{\mathrm{e},0} - \underbrace{u_{\mathrm{i},0}}_{=0.0}| &= 2.0\sqrt{\frac{T_{\mathrm{e},0}}{m_\mathrm{i}}} = 0.4\, v_{T_\mathrm{e}}, && (14)\\
|u_{\mathrm{e},f} - u_{\mathrm{i},f}| &= v_{T_\mathrm{e}} && (15)\\
\omega_\mathrm{pe}\tau_\mathrm{d} &= 0.005, && (16)\\
\nu_\mathrm{e/e} &= 0.001\,\omega_\mathrm{pe} && (17)\\
\nu_\mathrm{e/i} &= 0.001\,\omega_\mathrm{pe} && (18)\\
\nu_\mathrm{i/e} &= 0.001\frac{m_\mathrm{e}}{m_\mathrm{i}}\omega_\mathrm{pe} && (19)\\
\nu_\mathrm{i/i} &= 0.001\,\sqrt{\frac{m_\mathrm{e}}{m_\mathrm{i}}}\left(\frac{T_\mathrm{e}}{T_\mathrm{i}}\right)^{3/2}\omega_\mathrm{pe} && (20)
\end{aligned}
$$

- From Eq. (17), and

$$
\nu_\mathrm{e/e} = \frac{ne^4 \log(\Lambda)}{12\pi^{3/2}\epsilon_0^2 m_\mathrm{e}^{1/2} T_\mathrm{e}^{3/2}} = \frac{\log(\Lambda)}{12\pi^{3/2}} \underbrace{\frac{1}{n}\left(\frac{n^{1/2}e}{\epsilon_0^{1/2} T_\mathrm{e}^{1/2}}\right)^3}_{=N_\mathrm{D}^{-1}} \underbrace{\left(\frac{n^{1/2}e}{\epsilon_0 m_\mathrm{e}^{1/2}}\right)}_{=\omega_\mathrm{pe}} \tag{21}
$$

  we have $N_\mathrm{D} \approx 10^{-3}$, assuming $\log(\Lambda) \approx 10$ - 50.

- See Eq (43) of `http://www.euro-fusionscipub.org/wp-content/uploads/2014/11/EFDR07001.pdf` for mass ratio and temperature ratio collisionality dependence. This is consistent with the default collisionality computed by Gkeyll.

Numerical Parameters

These are parameters that do not have physical meaning, but do have meaning in the numerical scheme we are using.

$$(v_{\sigma,\mathrm{min}}, v_{\sigma,\mathrm{max}}) = (-6.0\, v_{T_\sigma}, 6.0\, v_{T_\sigma}), \tag{22}$$

$$(N_v, p_{\mathrm{order}}) = (256, 2) \implies \Delta v_\sigma = \frac{12 v_{T_\sigma}}{256 \cdot 3} = v_{T_\sigma}/64, \tag{23}$$

$$v_{T_\mathrm{e}}/c = 0.02 \implies \frac{\lambda_{\mathrm{De}}}{d_\mathrm{e}} = 0.02 \tag{24}$$

III. NOISE

A. Initial conditions

In the simulation, we anticipate that the ion-acoustic instability will be near marginal stability for a range of k's throughout the development of the current. We know that the instability needs to be seeded by a source term in the Vlasov-Maxwell system in order to grow, so we must provide a stochastic source term that models the effect of ion-acoustic plasmon creation at random times whose interarrival time is short compared to the the ion-acoustic growth/damping time.

- In the quasilinear derivation, one averages over an intermediate time that is long compared to the real period, but short compared to the damping rate. This time average is equivalent to some sort of statistical average, where the stochastic source can be replaced by its average.

1. *Collisionless kinetic eigenfunctions*

As a preliminary exercise, let us compute the source term that corresponds to the initial condition

$$\delta F^\sigma(t=t_0,\boldsymbol{x},\boldsymbol{v}) = \sum_{n_y,n_z\in\mathcal{N}} \frac{-\frac{e_\sigma}{m_\sigma}|\delta\boldsymbol{E}_{\boldsymbol{n}}|\hat{\boldsymbol{k}}_{\boldsymbol{n}}\cdot\frac{\partial F^\sigma}{\partial \boldsymbol{v}}}{(\omega_{\boldsymbol{k}}-\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{v})^2+\gamma_{\boldsymbol{k}}^2}\times \tag{25}$$

$$[(\omega_{\boldsymbol{k}}-\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{v})\cos(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}})+\gamma_{\boldsymbol{k}}\sin(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}})]$$

$$\delta\boldsymbol{E}(t=t_0,\boldsymbol{x},\boldsymbol{v}) = \sum_{n_y,n_z\in\mathcal{N}} |\delta\boldsymbol{E}_{\boldsymbol{n}}|\hat{\boldsymbol{k}}_{\boldsymbol{n}}\sin(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}}) \tag{26}$$

This is the form of the late time behavior of the linear initial value problem for weakly damped Fourier modes, and is valid for $\gamma \ll \omega$. In the language of quasilinear theory, we have the ion-acoustic plasmon born at $t=t_0$. The electrostatic field that is induced must satisfy

$$\nabla\cdot\delta\boldsymbol{E} = \frac{1}{\epsilon_0}\sum_\sigma e_\sigma \delta n_\sigma \tag{27}$$

Here,

$$\mathcal{N} = \{(n_x,n_y) : n_y=0, n_z>0\}\cup\{(n_x,n_y) : n_y>0\} \tag{28}$$

It is straightforward to see, geometrically, that

$$\mathcal{N}\cup(-\mathcal{N}) = (n_{y,\min},\ldots,n_{y,\max})\times(n_{z,\min},\ldots,n_{z,\max})-(0,0)$$

In words: taking all of the wave numbers in $\mathcal{N}$ along with the wave numbers obtained switching the sign for each of these values gives the set of all waves that fit in the box. We split it this way because the amplitude and phase for each wave must satisfy reality conditions

$$|\delta\boldsymbol{E}_{\boldsymbol{n}}| = |\delta\boldsymbol{E}_{-\boldsymbol{n}}| \tag{29}$$

$$\phi_{\boldsymbol{n}} = -\phi_{-\boldsymbol{n}} \tag{30}$$

1. If we use this precise kinetic form of the eigenfunction, we must take equal care in choosing $\omega_{\boldsymbol{k}}$ to satisfy the precise dispersion relation obtained by using these eigenfunctions. (This dispersion relation is obtained by substituting (25) and (26) into (27))

2. The form (25) is only approximate when there are collisional corrections. These collisional corrections can be important near the resonant velocity for weakly damped modes.

2. Collisionless fluid eigenfunctions

ions

$$\delta F^{\mathrm{i}}(t=t_0,\boldsymbol{x},\boldsymbol{v}) = \sum_{n_y,n_z\in\mathcal{N}} \frac{-\frac{e_{\mathrm{i}}}{m_{\mathrm{i}}}|\delta\boldsymbol{E}_{\boldsymbol{n}}|\hat{\boldsymbol{k}}_{\boldsymbol{n}}\cdot\left(-\frac{\boldsymbol{v}}{v_{T_{\mathrm{i}}}^2}F_{\mathrm{M}}^{\mathrm{i}}\right)}{(\omega_{\boldsymbol{k}}-\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{v})^2+\gamma_{\boldsymbol{k}}^2}\times \tag{31}$$

$$[(\omega_{\boldsymbol{k}}-\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{v})\cos(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}})+\gamma_{\boldsymbol{k}}\sin(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}})]$$

$$\approx \frac{1}{e_{\mathrm{i}}}\sum_{n_y,n_z\in\mathcal{N}}\frac{e_{\mathrm{i}}^2 n_0}{\epsilon_0 m_{\mathrm{i}}|\boldsymbol{k}_{\boldsymbol{n}}|^2 v_{T_{\mathrm{i}}}^2}\left(\epsilon_0|\boldsymbol{k}_{\boldsymbol{n}}||\delta\boldsymbol{E}_{\boldsymbol{n}}|\right)\underbrace{\frac{\boldsymbol{k}\cdot\boldsymbol{v}/\omega_{\boldsymbol{k}}}{(1-\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{v}/\omega_{\boldsymbol{k}})}}_{\approx\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}+\left(\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right)^2}\hat{F}_{\mathrm{M}}^{\mathrm{i}}\cos(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}}) \tag{32}$$

$$\approx \frac{1}{e_{\mathrm{i}}}\sum_{n_y,n_z\in\mathcal{N}}\frac{\omega_{\mathrm{pi}}^2}{\epsilon_0|\boldsymbol{k}_{\boldsymbol{n}}|^2 v_{T_{\mathrm{i}}}^2}\left(\epsilon_0|\boldsymbol{k}_{\boldsymbol{n}}||\delta\boldsymbol{E}_{\boldsymbol{n}}|\right)\left[\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega_{\boldsymbol{k}}}+\left(\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega_{\boldsymbol{k}}}\right)^2\right]\hat{F}_{\mathrm{M}}^{\mathrm{i}}\cos(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}}) \tag{33}$$

It looks to me like we need to include both terms in square brackets in Eq. (33). The first is the approximate current response, and contributes in Ampere's law, while the second is the leading nonzero charge response, and is needed to satisfy the Poisson constraint at $t=t_0$.

electrons

$$\delta F^{\mathrm{e}}(t=t_0,\boldsymbol{x},\boldsymbol{v}) = \sum_{n_y,n_z\in\mathcal{N}} \frac{-\frac{e_{\mathrm{e}}}{m_{\mathrm{e}}}|\delta\boldsymbol{E}_{\boldsymbol{n}}|\hat{\boldsymbol{k}}_{\boldsymbol{n}}\cdot\left(-\frac{\boldsymbol{v}-\boldsymbol{V}_{\mathrm{e}}}{v_{T_{\mathrm{e}}}^2}F_{\mathrm{M}}^{\mathrm{e}}\right)}{(\omega_{\boldsymbol{k}}-\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{v})^2+\gamma_{\boldsymbol{k}}^2}\times \tag{34}$$

$$[(\omega_{\boldsymbol{k}}-\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{v})\cos(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}})+\gamma_{\boldsymbol{k}}\sin(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}})]$$

$$\approx \frac{1}{e_{\mathrm{e}}}\sum_{n_y,n_z\in\mathcal{N}}\frac{e_{\mathrm{e}}^2 n_0}{\epsilon_0 m_{\mathrm{e}}|\boldsymbol{k}_{\boldsymbol{n}}|^2 v_{T_{\mathrm{e}}}^2}\left(\epsilon_0|\boldsymbol{k}_{\boldsymbol{n}}||\delta\boldsymbol{E}_{\boldsymbol{n}}|\right)\underbrace{\frac{-1}{1-\frac{\omega_{\boldsymbol{k}}-\boldsymbol{k}\cdot\boldsymbol{V}_{\mathrm{e}}}{\boldsymbol{k}_{\boldsymbol{n}}\cdot(\boldsymbol{v}-\boldsymbol{V}_{\mathrm{e}})}}}_{\approx-1}\hat{F}_{\mathrm{M}}^{\mathrm{e}}\cos(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}}) \tag{35}$$

$$\approx \frac{1}{e_{\mathrm{e}}}\sum_{n_y,n_z\in\mathcal{N}}\frac{\omega_{\mathrm{pe}}^2}{|\boldsymbol{k}_{\boldsymbol{n}}|^2 v_{T_{\mathrm{e}}}^2}\left(\epsilon_0|\boldsymbol{k}_{\boldsymbol{n}}||\delta\boldsymbol{E}_{\boldsymbol{n}}|\right)(-1)\,\hat{F}_{\mathrm{M}}^{\mathrm{e}}\cos(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}}) \tag{36}$$

displacement current

$$\delta\boldsymbol{E}(t=t_0,\boldsymbol{x},\boldsymbol{v}) = \sum_{n_y,n_z\in\mathcal{N}}|\delta\boldsymbol{E}_{\boldsymbol{n}}|\hat{\boldsymbol{k}}_{\boldsymbol{n}}\sin(\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}}) \tag{37}$$

Dispersion relation

Using Eqs. (33), (36), and (37), Poisson's equation gives the much simpler textbook dispersion equation

$$1 - \frac{\omega_{\rm pi}^2}{\omega^2} + \frac{1}{k^2\lambda_{\rm De}^2} = 0 \implies \omega_{\boldsymbol{k}} = \frac{\omega_{\rm pi}}{\sqrt{1 + 1/(k^2\lambda_{\rm De}^2)}} \tag{38}$$

Again, while the fluid eigenfunctions are approximations of the $t \to \infty$ asymptotic behavior of the IVP for a Fourier mode (with no collisions), the dispersion relation obtained from them is exact, so Poisson's equation is exactly satisfied.

fluid approximation as Hermite truncation

Note: The ion eigenfunction has the form

$$\delta F^{\rm i} \sim \left[{\rm He}_0(\boldsymbol{k}\cdot\boldsymbol{v}/\omega) + {\rm He}_1(\boldsymbol{k}\cdot\boldsymbol{v}/\omega) + {\rm He}_2(\boldsymbol{k}\cdot\boldsymbol{v}/\omega)\right] F_0 \tag{39}$$

and the electron fluid eigenfunction has the approximate form

$$\delta F^{\rm e} \sim \left[{\rm He}_0(\boldsymbol{k}\cdot\boldsymbol{v}/\omega)\right] F_0 \tag{40}$$

These polynomial truncations are from the large and small argument expansions of

$$\frac{\xi}{1-\xi} \approx \begin{cases} \xi + \xi^2, & \xi \ll 1 \\ -1 & \xi \gg 1 \end{cases} \tag{41}$$

with $\xi \sim k v_{T_\sigma}/\omega$

B. Initial conditions as a source

If we add the source terms

$$\left(\frac{\partial F^\sigma}{\partial t}\right)_{\text{initial fluct.}} = \delta(t-t_0)\left[F_0^\sigma(\boldsymbol{v}) + \delta F^\sigma(t=t_0,\boldsymbol{x},\boldsymbol{v})\right] \tag{42}$$

$$-\mu_0\delta\boldsymbol{J}_{\text{initial fluct.}} = \mu_0\epsilon_0\delta\boldsymbol{E}(t=t_0,\boldsymbol{x})\delta(t-t_0) \tag{43}$$

then this is equivalent to the initial conditions

$$F^\sigma(t=t_0) = F_0^\sigma(\boldsymbol{v}) + \delta F^\sigma(t=t_0,\boldsymbol{x},\boldsymbol{v}) \tag{44}$$

$$\boldsymbol{E}(t=t_0) = \delta\boldsymbol{E}(t=t_0,\boldsymbol{x}) \tag{45}$$

A solution and its problem (part I: a single source)

The phase space functions

$$\delta\boldsymbol{E}(\boldsymbol{x},t;\boldsymbol{n},|\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|,\phi_{\boldsymbol{n},\alpha(\boldsymbol{n})},t_{\alpha(\boldsymbol{n})}) = \left(\frac{|\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|}{2\mathrm{i}}\hat{\boldsymbol{k}}_{\boldsymbol{n}}\mathrm{e}^{-\mathrm{i}(\omega_{\boldsymbol{k}}+\mathrm{i}\gamma_{\boldsymbol{k}})(t-t_{\alpha(\boldsymbol{n})})}\mathrm{e}^{\mathrm{i}\phi_{\alpha(\boldsymbol{n})}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}} + \text{c.c.}\right)\theta(t-t_{\alpha(\boldsymbol{n})}) \tag{46}$$

$$\delta f^\sigma(\boldsymbol{x},\boldsymbol{v},t;\dots) = \left(\frac{-\frac{e_\sigma}{m_\sigma}\frac{|\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|}{2}\hat{\boldsymbol{k}}_{\boldsymbol{n}}\cdot\frac{\partial F_\sigma}{\partial\boldsymbol{v}}}{\omega_{\boldsymbol{k}}+\mathrm{i}\gamma_{\boldsymbol{k}}-\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{v}}\mathrm{e}^{-\mathrm{i}(\omega_{\boldsymbol{k}}+\mathrm{i}\gamma_{\boldsymbol{k}})(t-t_{\alpha(\boldsymbol{n})})}\mathrm{e}^{\mathrm{i}\phi_{\alpha(\boldsymbol{n})}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}} + \text{c.c.}\right)\theta(t-t_{\alpha(\boldsymbol{n})}), \tag{47}$$

with $\omega_{\boldsymbol{k}} + \mathrm{i}\gamma_{\boldsymbol{k}}$ determined by the exact dispersion equation, are an exact *solution* to the *problem*

$$\frac{\partial\delta f^\sigma}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f^\sigma}{\partial\boldsymbol{x}} + \frac{e_\sigma}{m_\sigma}\delta\boldsymbol{E}\cdot\frac{\partial F_\sigma}{\partial\boldsymbol{v}} = \mathcal{S}^\sigma, \tag{48}$$

$$\epsilon_0\frac{\partial\delta\boldsymbol{E}}{\partial t} + \sum_{\sigma'}\int \mathrm{d}^3v' e_{\sigma'}\boldsymbol{v}'\delta f^{\sigma'} = -\delta\boldsymbol{J}^{\text{ext}} \tag{49}$$

with

$$\mathcal{S}^\sigma(\boldsymbol{x},\boldsymbol{v},t;\boldsymbol{n},|\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|,\phi_{\boldsymbol{n},\alpha(\boldsymbol{n})},t_{\alpha(\boldsymbol{n})}) = \left(\frac{-\frac{e_\sigma}{m_\sigma}\frac{|\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|}{2}\hat{\boldsymbol{k}}_{\boldsymbol{n}}\cdot\frac{\partial F_\sigma}{\partial\boldsymbol{v}}}{\omega_{\boldsymbol{k}}+\mathrm{i}\gamma_{\boldsymbol{k}}-\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{v}}\mathrm{e}^{\mathrm{i}\phi_{\alpha(\boldsymbol{n})}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}} + \text{c.c.}\right)\delta(t-t_{\alpha(\boldsymbol{n})}) \tag{50}$$

$$-\delta\boldsymbol{J}^{\text{ext}}(\boldsymbol{x},t;\boldsymbol{n},|\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|,\phi_{\boldsymbol{n},\alpha(\boldsymbol{n})},t_{\alpha(\boldsymbol{n})}) = \left(\frac{|\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|}{2\mathrm{i}}\hat{\boldsymbol{k}}_{\boldsymbol{n}}\mathrm{e}^{\mathrm{i}\phi_{\alpha(\boldsymbol{n})}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}} + \text{c.c.}\right)\delta(t-t_{\alpha(\boldsymbol{n})}) \tag{51}$$

and $\delta f^\sigma(t < t_{\alpha(\boldsymbol{n})}) = \delta\boldsymbol{E}(t < t_{\alpha(\boldsymbol{n})}) = 0$. (This solution also satisfies the Poisson equation, which is a constraint equation that we don't write here.)

A solution and its problem (part II: a superposition of sources)

Eqs (48) and (49) with

$$\mathcal{S}^\sigma(\boldsymbol{x},\boldsymbol{v},t) = \sum_{\boldsymbol{n}\in\mathcal{N}} \sum_{\alpha(\boldsymbol{n})=0}^{\infty} \mathcal{S}^\sigma(\boldsymbol{x},\boldsymbol{v},t;\boldsymbol{n},|\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|,\phi_{\boldsymbol{n},\alpha(\boldsymbol{n})},t_{\alpha(\boldsymbol{n})}) \tag{52}$$

$$-\delta\boldsymbol{J}^{\mathrm{ext}}(\boldsymbol{x},t) = \sum_{\boldsymbol{n}\in\mathcal{N}} \sum_{\alpha(\boldsymbol{n})=0}^{\infty} -\delta\boldsymbol{J}^{\mathrm{ext}}(\boldsymbol{x},t;\boldsymbol{n},|\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|,\phi_{\boldsymbol{n},\alpha(\boldsymbol{n})},t_{\alpha(\boldsymbol{n})}) \tag{53}$$

and $\delta f^\sigma(t\to-\infty) = \delta\boldsymbol{E}(t\to-\infty) = 0$ has the (sum of IVP solutions) solution

$$\delta\boldsymbol{E}(\boldsymbol{x},t) = \sum_{\boldsymbol{n}\in\mathcal{N}} \sum_{\alpha(\boldsymbol{n})=0}^{\infty} \delta\boldsymbol{E}(\boldsymbol{x},t;\boldsymbol{n},|\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|,\phi_{\boldsymbol{n},\alpha(\boldsymbol{n})},t_{\alpha(\boldsymbol{n})}) \tag{54}$$

$$\delta f^\sigma(\boldsymbol{x},\boldsymbol{v},t) = \sum_{\boldsymbol{n}\in\mathcal{N}} \sum_{\alpha(\boldsymbol{n})=0}^{\infty} \delta f^\sigma(\boldsymbol{x},\boldsymbol{v},t;\boldsymbol{n},|\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|,\phi_{\boldsymbol{n},\alpha(\boldsymbol{n})},t_{\alpha(\boldsymbol{n})}) \tag{55}$$

We can express each individual linear IVP solution analytically, in terms of a Fourier integral, but it is better to write out it's time-asymptotic form:

$$\delta\boldsymbol{E}(\boldsymbol{x},t;\boldsymbol{n},|\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|,\phi_{\boldsymbol{n},\alpha(\boldsymbol{n})},t_{\alpha(\boldsymbol{n})}) = |\delta\boldsymbol{E}_{\boldsymbol{n},\alpha(\boldsymbol{n})}|\hat{\boldsymbol{k}}\,\mathrm{e}^{\gamma_{\boldsymbol{k}}(t-t_{\alpha(\boldsymbol{n})})}\sin\left[\boldsymbol{k}_{\boldsymbol{n}}\cdot\boldsymbol{x}-\omega^{\mathrm{ia}}_{\boldsymbol{k}}(t-t_{\alpha(\boldsymbol{n})})+\phi_{\boldsymbol{n},\alpha(\boldsymbol{n})}\right] \tag{56}$$

Stochastic phases and interarrival times

If we assume that $\phi_{\boldsymbol{n},\alpha(\boldsymbol{n})}$ is an instantiation of a random variable $\Phi_{\boldsymbol{n},\alpha(\boldsymbol{n})} \sim \mathrm{Uniform}(0,\pi)$, and $t_{\alpha(\boldsymbol{n})} = t_{\alpha(\boldsymbol{n})-1} + \Delta T_{\alpha(\boldsymbol{n})}$ with the interarrival times distributed exponentially: $\Delta T_{\alpha(\boldsymbol{n})} \sim \mathrm{Expo}(\nu_{\mathbf{n}})$. Note that we are essentially building in the random-phase approximation here, as a part of a statistical model of the higher order corrections to Vlasov.

In terms of the random interarrival times and phases, the spontaneous emission rate is

$$\frac{\partial}{\partial t}\left(\epsilon_0\frac{\partial|\delta\boldsymbol{E}|^2}{2}\right)_{\text{sp. emission}} = -\delta\boldsymbol{E}\cdot\delta\boldsymbol{J}^{\mathrm{ext}} \tag{57}$$

C. Spontaneous emission as a source

As our simulation proceeds, the system becomes unstable to a progressively broader range of wave numbers. At the same time, there is Landau damping of any initially stable waves that are excited at $t = t_0$, and this phase mixed structure in velocity space is dissipated via collisions prior to reaching the grid scale. Thus we need continual excitation of all waves (i.e. we need spontaneous emission).

1. *Microscopic (i.e. stochastic) spontaneous emission*

Physically, each particle randomly Cerenkov radiates ion-sound waves at a rate whose time average corresponds to the continuum quasilinear spontaneous emission rate. If time is treated as a continuous variable, then the inter-arrival times $\{\Delta t(\alpha(\boldsymbol{n}))\}_{\alpha=1,\ldots,\infty}$ are iid exponential random variables:

$$\Delta t(\alpha(\boldsymbol{n})) \sim \mathrm{Expo}(\lambda(\boldsymbol{n})) \tag{58}$$

If time is treated as a discrete variable, then the inter-arrival times, in number of time steps, are geometric random variables. That is, each particle flips a coin with probability p of getting 'H' at each time step. Every time the coin lands 'H', the particle emits an ion-acoustic plasmon.

The amplitude of the electric field associated with 1 ion-acoustic plasmon in state $\boldsymbol{n}$ per unit volume is

$$|\delta \mathbb{E}_{\boldsymbol{n}}| = \sqrt{\frac{2\hbar\omega_{\boldsymbol{k}}}{\frac{\partial(\epsilon_0 \varepsilon_{\boldsymbol{k}} \omega_{\boldsymbol{k}})}{\partial \omega_{\boldsymbol{k}}}}} \tag{59}$$

The total energy of ion-acoustic plasmons injected per unit volume from spontaneous emission is

$$\left\langle S_{\boldsymbol{n}}^{\mathrm{micro}} \right\rangle \sim \lambda^{\mathrm{micro}}(\boldsymbol{n}) \hbar\omega_{\boldsymbol{k}} \tag{60}$$

The rate $\lambda^{\mathrm{micro}}(\boldsymbol{n})$ is determined by setting this rate equal to the classical rate, which is roughly

$$S_{\boldsymbol{n}}^{\mathrm{cl}} \sim 2\gamma_{\mathrm{s}} T_{\mathrm{e}} \tag{61}$$

2. *Continuum-averaged stochastic source in the WKB limit*

The theoretical expression for the source terms are

$$\left(\frac{\partial F^\sigma}{\partial t}\right)_{\text{sp. emission}} = \frac{1}{m_\sigma}\frac{\partial}{\partial \boldsymbol{v}} \cdot \sum_{b'} \int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \boldsymbol{k}' \hat{w}_\sigma^{b'} \delta(\omega_{\boldsymbol{k}'}^{b'} - \boldsymbol{k}' \cdot \boldsymbol{v}) F^\sigma(\boldsymbol{v}) \tag{62}$$

$$\left(\frac{\partial W_{\boldsymbol{k}}^b}{\partial t}\right)_{\text{sp. emission}} = \sum_{\sigma'} \int \mathrm{d}^3 v' \hat{w}_{\sigma'}^b \delta(\omega_{\boldsymbol{k}}^b - \boldsymbol{k} \cdot \boldsymbol{v}') \omega_{\boldsymbol{k}}^b F^{\sigma'}(\boldsymbol{v}') \tag{63}$$

with

$$\hat{w}_\sigma^b = 2\frac{e_\sigma^2/\epsilon_0}{k^2}\frac{1}{\frac{\partial \varepsilon}{\partial \omega_{\boldsymbol{k}}^b}}\pi \tag{64}$$

Maxwellian electrons and ions

With Maxwellian ions at rest and shifted Maxwellian electrons, the general expression (63) gives (cf. SubcriticalResistivityEnhancement.pdf) the quasilinear wave equation

$$\frac{\partial W_{\boldsymbol{k}}}{\partial t} = -2\gamma_{\mathrm{s}}\left[\left((1-\boldsymbol{k}\cdot\boldsymbol{u}_{\mathrm{e}}/\omega_{\boldsymbol{k}})W_{\boldsymbol{k}}-T_{\mathrm{e}}\right)+\delta_{\mathrm{M}}\left(W_{\boldsymbol{k}}-T_{\mathrm{i}}\right)\right] \tag{65}$$

with

$$\gamma_{\mathrm{e}\boldsymbol{k}} = \gamma_{\mathrm{s}}\left(1-\boldsymbol{k}\cdot\boldsymbol{u}_{\mathrm{e}}/\omega_{\boldsymbol{k}}\right)\underbrace{\exp\left(-\frac{(\omega_{\boldsymbol{k}}/|k|-\boldsymbol{k}\cdot\boldsymbol{u}_{\mathrm{e}}/|k|)^2}{2v_{T_{\mathrm{e}}}^2}\right)}_{\approx 1} \tag{66}$$

$$\gamma_{\mathrm{i}\boldsymbol{k}} = \gamma_{\mathrm{s}}\delta_{\mathrm{M}}, \tag{67}$$

$$\gamma_{\mathrm{s}} = \pi\frac{1}{\sqrt{2\pi}}\omega_{\boldsymbol{k}}\frac{1}{k^2\lambda_{\mathrm{De}}^2}\frac{1}{\partial\varepsilon'/\partial\omega}\frac{1}{kv_{T_{\mathrm{e}}}}, \tag{68}$$

$$= \sqrt{\frac{\pi}{2}}\frac{\omega_{\boldsymbol{k}}}{\omega_{\mathrm{pe}}}\frac{1}{k^3\lambda_{\mathrm{De}}^3}\frac{1}{\partial\varepsilon'/\partial\omega}, \tag{69}$$

$$\delta_{\mathrm{M}} = \sqrt{\frac{m_{\mathrm{i}}}{m_{\mathrm{e}}}}\frac{T_{\mathrm{e}}^{3/2}}{T_{\mathrm{i}}^{3/2}}\exp\left(-\frac{(\omega_{\boldsymbol{k}}/|k|)^2}{2v_{T_{\mathrm{i}}}^2}\right) \tag{70}$$

In particular

$$\left(\frac{\partial W_{\boldsymbol{k}}}{\partial t}\right)_{\mathrm{sp/e}} = 2\gamma_{\mathrm{s}}T_{\mathrm{e}} \tag{71}$$

$$\left(\frac{\partial W_{\boldsymbol{k}}}{\partial t}\right)_{\mathrm{sp/i}} = 2\gamma_{\mathrm{s}}\delta_{\mathrm{M}}T_{\mathrm{i}} \tag{72}$$

If we use the fluid dispersion relation (38) to approximate $\partial\varepsilon'/\partial\omega$ and ω' in Eq. (68), we obtain

$$\gamma_{\mathrm{s}} \approx \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\frac{\frac{1}{(k\lambda_{\mathrm{De}})^3}}{\left(1+\frac{1}{(k\lambda_{\mathrm{De}})^2}\right)^2}\omega_{\mathrm{pe}}, \tag{73}$$

$$\left(\frac{\partial W_{\boldsymbol{k}}}{\partial t}\right)_{\mathrm{sp/e}} = \sqrt{\frac{\pi}{2}}\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\frac{\frac{1}{(k\lambda_{\mathrm{De}})^3}}{\left(1+\frac{1}{(k\lambda_{\mathrm{De}})^2}\right)^2}\omega_{\mathrm{pe}}T_{\mathrm{e}} \tag{74}$$

Ion acoustic spectrum for a slowly-increasing current

$$W_{\boldsymbol{k}} = \frac{T_{\mathrm{e}}+\delta_{\mathrm{M}}T_{\mathrm{i}}}{(1+\delta_{\mathrm{M}})-\boldsymbol{k}\cdot\boldsymbol{u}_{\mathrm{e}}/\omega_{\boldsymbol{k}}} \tag{75}$$

This expression requires $\partial_t W \ll (\partial_t W)_{\mathrm{sp/e}}$. This will be satisfied provided

$$(1-(u/u_{\mathrm{crit}}))^2 \ll \sqrt{m_{\mathrm{e}}/m_{\mathrm{i}}}\tau_u \omega_{\mathrm{pi}} \tag{76}$$

(Note: this is the criterion for the Baalrud result to hold)

(Note: This criterion will be satisfied until the system gets quite close to marginality since the current increases so slowly compared to ω_{pi})

(Note: this criterion assumes that the only reason γ changes is because of the current. That is, the change in γ due to the current increase is assumed to be large compared to the change in γ due to quasilinear relaxation of the distribution functions)

Ion acoustic spectrum for a rapidly-increasing current

In the opposite extreme, we can consider a case where the current increases so rapidly that the electrons and ions remain Maxwellian (so, in particular, the expression for γ holds) for a much longer time. Here the solution is simply the integral of Eq. (III C 2)

1. Note that the spontaneous emission rate depends on the value of the distribution function (not the slope), so even when quasilinear relaxation changes the shape of F^σ, as long as the area under the curve is not dramatically altered, the Maxwellian expressions for spontaneous emission should be a good approximation.

For 2x2v and a discrete box, we have

$$\left(\frac{\partial F^\sigma}{\partial t}\right)_{\text{sp. emission}} = \frac{1}{m_\sigma}\frac{\partial}{\partial \boldsymbol{v}} \cdot \sum_{b'} \sum_{\boldsymbol{n}\in\mathcal{N}} \frac{2^2}{L_x L_y} \boldsymbol{k}' \hat{w}_\sigma^{b'} \delta(\omega_{\boldsymbol{k}'}^{b'} - \boldsymbol{k}' \cdot \boldsymbol{v}) F^\sigma(\boldsymbol{v}) \tag{77}$$

3. Model stochastic source

For now, we simply inject noise at every time step t_α with a stochastic amplitude and phase:

$$\left(\frac{\partial F^\sigma}{\partial t}\right)_{\text{sp. fluct.}} = \sum_\alpha \delta(t-t_\alpha) \sum_{n_y,n_z\in\mathcal{N}} \chi_0(n_y,n_z)\cos(\boldsymbol{k_n}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}}(t_\alpha)) \tag{78}$$

$$\mu_0\delta\boldsymbol{J}_{\text{sp. fluct.}} = -\mu_0\epsilon_0 \sum_\alpha \delta(t-t_\alpha) \sum_{n_y,n_z\in\mathcal{N}} |\delta\boldsymbol{E_n}(t_\alpha)|\hat{\boldsymbol{n}}\sin(\boldsymbol{k_n}\cdot\boldsymbol{x}+\phi_{\boldsymbol{n}}(t_\alpha)) \tag{79}$$

- What distribution do we sample $|\delta\boldsymbol{E_n}(t_\alpha)|$ from? Answer: If we assume that our numerical time step is always large compared to the mean time between radiation of ion-acoustic plasmons (for every mode), then the number of plasmons in mode $\boldsymbol{n}$ that were born in the numerical time step Δt is going to be Poisson distributed with mean $m_{\boldsymbol{n}} = \lambda_{\boldsymbol{n}}\Delta t$. (If this is sufficiently large, we can approximate a Poisson random variable with a Normal distribution with the same mean and variance.)

$$|\delta\boldsymbol{E_n}| \sim \text{Pois}(\lambda_{\boldsymbol{n}}\Delta t) \tag{80}$$

$$\phi_{\boldsymbol{n}}(t_\alpha) \sim \text{Uniform}(0,2\pi) \tag{81}$$

  Note: this is really only useful for knowing the shape of the distribution for $|\Delta\boldsymbol{E_n}|$

  – We want to make sure we are injecting a reasonable amount of wave energy; large enough to reasonably approximate the spontaneous emission rate in the wave energy equation but small enough that the momentum and energy source in the ion kinetic equation is insignificant. The rate of wave energy injection is roughly

$$\frac{\partial(\text{field energy per unit volume})}{\partial t} \sim \frac{|\delta\boldsymbol{E_n}|^2}{\Delta t} \sim \frac{T_\mathrm{e}}{L_xL_yL_z}\underbrace{\gamma}_{\sim\sqrt{\frac{m_\mathrm{e}}{m_\mathrm{i}}}\omega_{\mathrm{pi}}} \tag{82}$$

  where Δt is the time step; γ is the characteristic damping rate of ion-sound waves; and $|\delta\boldsymbol{E_n}|$ is a characteristic (e.g. sample average) scale for the fluctuating field amplitude. I get the line in red by assuming that $|\delta E_{\boldsymbol{n}}|^2 \sim T_\mathrm{e}/\mathcal{V}$ in equilibrium, and also that in equilibrium the emission rate balances the damping rate. (Another way to say this is that I am choosing the model spontaneous emission rate – or rather the size of the wave source at each time step – so that in equilibrium, the fluctuation levels are correct.)

- We only perturb the ions. This is simply because it is an effective way to excite predominantly the ion-acoustic branch.

D. Estimating the trapping width

$$\delta \boldsymbol{E}_{\boldsymbol{k}} = -\mathrm{i}\boldsymbol{k}\delta\varphi_{\boldsymbol{k}} \tag{83}$$

The effective potential energy for a particle moving at speed $\omega_{\boldsymbol{k}}/k + \delta v$ is

$$U(x) = -U_0 \cos(kx) \tag{84}$$

We estimate the frequency of oscillation of a trapped particle oscillating about $x = 0$ using

$$U(x) \approx -U_0 \frac{k^2 x^2}{2} \tag{85}$$

(Yes, the wave number k is the effective spring constant, which is also typically denoted by k) so that the oscillation frequency ω_{B} is

$$\omega_{\mathrm{B}} = \sqrt{\frac{U_0 k^2}{m}} = \sqrt{\frac{e\delta\varphi_{\boldsymbol{k}} k^2}{m}} = \sqrt{\frac{e\delta E_{\boldsymbol{k}} k}{m}} \tag{86}$$

We estimate that a single mode will saturate if it reaches an amplitude where the bounce frequency is equal to the linear growth rate:

$$\gamma_{\boldsymbol{k}} \sim \omega_{\mathrm{B,max}} \sim \sqrt{\frac{e\delta E_{\boldsymbol{k},\mathrm{max}} k}{m}} \implies \delta E_{\boldsymbol{k},\mathrm{max}} \sim \frac{m\gamma_{\boldsymbol{k}}^2}{ek} \tag{87}$$

Finally, the trapping width of the saturated wave is given by

$$m\delta v^2 \sim U_{0,\mathrm{max}} \sim e\varphi_{\boldsymbol{k},\mathrm{max}} \sim \frac{m\gamma_{\boldsymbol{k}}^2}{k^2} \implies \delta v \sim \frac{\gamma_{\boldsymbol{k}}}{k} \tag{88}$$

Note that this does not depend on mass.

E. Estimating the distance between phase velocities

$$\Delta(\omega/k) = \frac{\omega_{\boldsymbol{k}+\delta\boldsymbol{k}}}{\boldsymbol{k}+\delta\boldsymbol{k}} - \frac{\omega_{\boldsymbol{k}}}{\boldsymbol{k}} \sim \frac{\delta\boldsymbol{k}}{k}\frac{\omega_{\boldsymbol{k}}}{\boldsymbol{k}} \sim \frac{2\pi}{kL}\frac{\omega}{k} \implies \delta v_\phi \sim v_\phi \frac{\lambda}{L} \tag{89}$$

F. Requiring the trapping width to be larger than the distance between phase velocities

$$\delta v_{\boldsymbol{k}} \gtrsim \Delta(\omega/k) \implies \frac{\gamma_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}} \gtrsim \frac{2\pi}{kL} \tag{90}$$

Note that $\gamma_{\boldsymbol{k}}\tau_{\mathrm{CS}} > 1$, so a sufficient condition for (90) is $L/\lambda \gtrsim \omega_{\boldsymbol{k}}\tau_{\mathrm{CS}}$

Setting numerical collisionality

In early simulations, it was observed that solving the Vlasov equations lead to the creation of structure at the grid scale in velocity space. This problem was solved by adding collisions. The Gkeyll code has a Lenard-Bernstein collision operator implemented. It is possible to estimate the level of collisions necessary to dissipate energy at the grid scale in velocity. This estimate is obtained by considering what level of collisions would dissipate energy at the velocity grid scale in one time step. We can estimate this via the formula

$$\nu \sim \alpha \frac{\Delta v^2}{\Delta t}, \tag{91}$$

where α is usually between 3 and 5 depending on the specific algorithm. In a standard setup of our problem, for the electrons we have

$$\Delta v_{\mathrm{e}} = \frac{v_{\mathrm{e,max}} - v_{\mathrm{e,min}}}{\text{num effective grid points}} = \frac{6.0\, v_{T_{\mathrm{e}}} - (-6.0\, v_{T_{\mathrm{e}}})}{3 \cdot 256} = 0.02{\times}1.56{\times}10^{-2}\, c = 3.125{\times}10^{-4}\, c, \tag{92}$$

and thus we obtain, given a typical $\omega_{\mathrm{pe}}\Delta t = 1.41 \times 10^{-3}$, $\nu_e \sim 6.9 \times 10^{-5}\ \alpha$. For the ions, we have instead

$$\Delta v_i = 256/(3 * 256) * v_{thi} = 256/(3 * 256) * 0.0141 * 0.02\ c = 9.47 \times 10^{-5}\ c, \tag{93}$$

and thus $\nu_i \sim 6.36 \times 10^{-6}\ \alpha$. Picking an intermediate value of $\alpha = 4$, we arrive at a floor for our collisionality of $\nu_e \sim 2.8 \times 10^{-4}$ and $\nu_i \sim 2.54 \times 10^{-5}$. Note that for our mass ratio of 25, $\nu_e/\nu_i = 5$ and thus we only have an effective floor on ν_e.

G. Exciting box-scale Langmuir waves

To see how this works, consider the $k = 0$ component of Ohm's law, frozen ions (for simplicity), together with (3):

$$m_\mathrm{e} n \frac{\partial u_{\mathrm{e},z}}{\partial T} = e_\mathrm{e} n E_z(T) + \underbrace{R(T)}_{-\nu_\mathrm{eff} m_\mathrm{e} n u_{\mathrm{e},z}} \tag{94}$$

$$-J^\mathrm{ext}(T) = J_{\mathrm{e},z} + \epsilon_0 \frac{\partial E_z}{\partial T} \tag{95}$$

Using $J_{\mathrm{e},z} = e_\mathrm{e} n u_{\mathrm{e},z}$, we have

$$\frac{\partial J_{\mathrm{e},z}}{\partial T} + \nu_\mathrm{eff} J_z = \omega_\mathrm{pe}^2 \epsilon_0 E_z \tag{96}$$

$$-J^\mathrm{ext}(T) = J_{\mathrm{e},z} + \epsilon_0 \frac{\partial E_z}{\partial T}, \tag{97}$$

or finally

$$-J^\mathrm{ext}(T) = J_{\mathrm{e},z} + \underbrace{\frac{1}{\omega_\mathrm{pe}^2} \frac{\partial \nu_\mathrm{eff} J_{\mathrm{e},z}}{\partial T}}_{\text{e/i friction}} + \underbrace{\frac{1}{\omega_\mathrm{pe}^2} \frac{\partial^2 J_{\mathrm{e},z}}{\partial T^2}}_{\text{electron inertia}}, \tag{98}$$

Note:

1. both electron inertia and the friction are going to be small terms in Eq. (98), as long as the time scale is large compared to the Langmuir period of oscillation.

2. When the electron-ion friction dominates electron inertia in Ohm's law, we can drop the last term on the right side of Eq. (98).

3. If we do not include an antenna current or a ghost current, we will set up box-scale plasma waves and T will *not* be a slow time scale.