

# Notes of GTC Formula in Detailed (GTC User Manual, 1/3)

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## Abstract

Here, I try to give detailed derivations of the electromagnetic (EM) formula for the Gyrokinetic Toroidal Code (GTC<sup>1</sup>) ([Lin1998]), which can be seen as the combination and with extension of the previous tutorials and the papers [Holod2009] and [Deng2012]. Part of this document can be also found in [Wang2011] notes and [Deng2011] PhD thesis, however, with typos fixed<sup>2</sup> here. To check step by step, one can understand the assumptions used and also may find possible weaknesses/problems in GTC.

This document can be also seen as part of the GTC user manual. The total manual should include the formula (this document, 1/3), the implement in the code (2/3) and the application (how to use, data analysis tool, examples, 3/3).

One who is only interested in the final equations can find the summary and normalization in Sec. 8.

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<sup>1</sup> GTC Website: <http://phoenix.ps.uci.edu/GTC/>

<sup>2</sup> If you still find mistakes/typos in this document or possible problems in GTC, please tell me, thanks.

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## 1 Introduction

Main advantage of gyrokinetic approach is that the gyroaveraging can reduce the dimensionality of particle dynamics from 6 to 5.

To implement the GTC code, we should consider the following problems firstly:

- 1) Gyrokinetic equation
- 2) Field equations
- 3) (Delta-f formula)
- 4) (Fluid-kinetic electron model)
- 5) Magnetic coordinate
- 6) PIC method for coding
- 7) Parallel computation

In this document, I just discuss the first five problems. The main difficulty in understanding the gyrokinetic theory is the changes between different coordinates --- particle coordinate (x), guiding-center coordinate (GC) and gyro-center coordinate (GY) --- for not only the distribution function but also the field variables. Keeping this in mind, then, I think, the following should be easier<sup>3</sup>.

## 2 Gyrokinetic Theory

Firstly, I list the kinetic and field equations for full kinetic approach. Then, provide some preliminary knowledge of

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<sup>3</sup> Due to the complexity, it's commonly for beginner to take months or even longer to understand the gyrokinetic approach.

gyrokinetic theory, such as the gyrokinetic ordering, guiding center transformation, guiding center motion equations.

## 2.1 Full kinetic equation and Maxwell field equations

In full kinetic approach, we use kinetic equation

$$\frac{d}{dt} f(\mathbf{x}, \mathbf{v}, t) = \left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial}{\partial \mathbf{v}} - C_\alpha \right] f = 0, \quad (0.1)$$

and, Maxwell equations (Gaussian units)

$$\nabla \cdot \mathbf{E} = -4\pi\rho, \quad (0.2)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (0.3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (0.4)$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{J}. \quad (0.5)$$

For gyrokinetic approach, the basic equations are variants of the above equations.

## 2.2 Gyrokinetic ordering

Ordering:

- a) Small amplitude:  $\delta f / F_0 \sim e\delta\phi / T \sim \delta B / B_0 \sim \delta \ll 1$ .
- b) Low frequency:  $\omega / \Omega_i \sim \delta$ .
- c) Anisotropic:  $k_{||} / k_{\perp} \sim \rho_i / L_0 \sim \delta$ ,  $k_{\perp} \rho_i \sim 1$ . With,  $L_0 = \nabla \ln B$ .

For detailed discussion of the gyrokinetic assumptions, one can refer to [Brizard2007]. I comment here, the same ordering  $\delta$  used here for all small variables is just for convenient, and the third assumption  $k_{\perp} \rho_i \sim 1$  is also not a must.

## 2.3 Guiding center transformation

First key step is coordinate transformation, from particle coordinate to guiding center coordinate; second step is average over gyro-angle  $\vartheta$ .

$$(\mathbf{x}, \mathbf{v}) \xrightarrow{\text{Step1: To GC coordinate}} (\mathbf{X}_{GC}, v_{||}, \mu, \vartheta) \xrightarrow{\text{Step2: Gyro-averaging to GY coordinate}} (\mathbf{X}_{GY}, v_{||}, \mu) \quad (0.6)$$

In the first step of (0.6), kinetic equation (0.1) changes to,

$$\begin{aligned} 0 = & \frac{\partial F}{\partial t} + \left( v_{||} + \frac{q}{m} \frac{\mathbf{E} \times \mathbf{b}}{\Omega} \right) \cdot \frac{\partial F}{\partial \mathbf{X}} - \Omega \frac{\partial F}{\partial \vartheta} + \nu \cdot \left( \frac{\partial \mathbf{p}}{\partial \mathbf{X}} \cdot \frac{\partial F}{\partial \mathbf{X}} + \frac{\partial \mu}{\partial \mathbf{X}} \cdot \frac{\partial F}{\partial \mu} + \frac{\partial v_{||}}{\partial \mathbf{X}} \cdot \frac{\partial F}{\partial v_{||}} + \frac{\partial \vartheta}{\partial \mathbf{X}} \cdot \frac{\partial F}{\partial \vartheta} \right) \\ & + \frac{q}{m} \mathbf{E} \cdot \left( \frac{\mathbf{v}_{\perp}}{B} \frac{\partial F}{\partial \mu} + \mathbf{b} \frac{\partial F}{\partial v_{||}} + \frac{\mathbf{b} \times \mathbf{v}_{\perp}}{v_{||}^2} \frac{\partial F}{\partial \vartheta} \right). \end{aligned} \quad (0.7)$$

with the collision term dropped. The second step will give Eq. (0.14). Here, we see that Eq. (0.1) is simple in expressions, while Eqs. (0.7) and (0.14) are very complicate: to reduce the dimensions, then sacrifice the simplicity in expressions. We do not have better choices yet.

## 2.4 Guiding center motion in the inhomogeneous magnetic field

The equations are

$$\dot{\mathbf{X}} = v_{\parallel} \mathbf{b}_0 + \mathbf{v}_d, \quad (0.8)$$

$$\dot{v}_{\parallel} = -\frac{\mu}{m} \mathbf{b}_0 \cdot \nabla B_0, \quad (0.9)$$

$$\dot{\mu} = 0, \quad (0.10)$$

where, magnetic drift velocity

$$\mathbf{v}_d = \mathbf{v}_c + \mathbf{v}_g, \quad (0.11)$$

magnetic curvature drift

$$\mathbf{v}_c = \frac{v_{\parallel}^2}{\Omega} \nabla \times \mathbf{b}_0, \quad \left( \text{with } \Omega \equiv \frac{B_0 Z}{mc} \right) \quad (0.12)$$

magnetic gradient drift

$$\mathbf{v}_g = \frac{\mu}{m\Omega} \mathbf{b}_0 \times \nabla B_0. \quad (0.13)$$

We should note here, for guiding center (GC) motion, the field variables are all with no perturbations. While, we will see below, in gyrocenter (GY), this will be more complicated, because of that all field variables are changed with time caused by perturbations.

## 2.5 Gyrocenter equations of motion and gyrokinetic equation

Gyrokinetic equation of plasmas in the inhomogeneous magnetic field, using the gyrocenter position  $\mathbf{X}$ , magnetic moment  $\mu$ , and parallel velocity  $v_{\parallel}$  as a set of independent variables in the five dimensional phase space,

$$\frac{d}{dt} f_{\alpha}(\mathbf{X}, \mu, v_{\parallel}, t) = \left[ \frac{\partial}{\partial t} + \dot{\mathbf{X}} \cdot \nabla + \dot{v}_{\parallel} \frac{\partial}{\partial v_{\parallel}} - C_{\alpha} \right] f_{\alpha} = 0, \quad (0.14)$$

Here  $\nabla$  is in GY coordinate

Where,

$$\dot{\mathbf{X}} = v_{\parallel} \frac{\langle \mathbf{B} \rangle_c}{B_0} + \langle \mathbf{v}_E \rangle_c + \mathbf{v}_d, \quad (0.15)$$

$$\dot{v}_{\parallel} = -\frac{1}{m_{\alpha}} \frac{\langle \mathbf{B}^* \rangle_c}{B_0} \cdot \left( \mu \nabla B_0 + Z_{\alpha} \nabla \langle \phi \rangle_c \right) - \frac{Z_{\alpha}}{m_{\alpha} c} \frac{\partial \langle A_{\parallel} \rangle_c}{\partial t}, \quad (0.16)$$

and, magnetic moment as an adiabatic invariable,  $\mu$  term is dropped,

$$\dot{\mu} = 0. \quad (0.17)$$

Here, index  $\alpha = e, i, f$ , stands for the particle species (core electrons, core ions and fast ions),  $Z_{\alpha}$  is the particle charge, and  $m_{\alpha}$  is the particle mass.  $\mathbf{B}_0 \equiv B_0 \mathbf{b}_0$  is the equilibrium magnetic field,  $\langle \mathbf{B} \rangle_c \equiv \mathbf{B}_0 + \langle \delta \mathbf{B} \rangle_c$ , and

$$\langle \mathbf{B}^* \rangle_c = \mathbf{B}_0^* + \langle \delta \mathbf{B} \rangle_c = \mathbf{B}_0 + \frac{B_0 v_{\parallel}}{\Omega_{\alpha}} \nabla \times \mathbf{b}_0 + \langle \delta \mathbf{B} \rangle_c. \quad (0.18)$$

$E \times B$  drift velocity

$$\langle \mathbf{v}_E \rangle_c = \frac{c \mathbf{b}_0 \times \nabla \langle \phi \rangle_c}{B_0}. \quad (0.19)$$

Magnetic drift velocity

$$\mathbf{v}_d = \mathbf{v}_c + \mathbf{v}_g, \quad (0.20)$$

where the magnetic curvature drift is

$$\mathbf{v}_c = \frac{v_{\parallel}^2}{\Omega_{\alpha}} \nabla \times \mathbf{b}_0, \quad (0.21)$$

and the grad-B drift is

$$\mathbf{v}_g = \frac{\mu}{m_{\alpha} \Omega_{\alpha}} \mathbf{b}_0 \times \nabla B_0. \quad (0.22)$$

Eqs. (0.14)-(0.22) are the standard gyrokinetic theory results (see the review paper [Brizard2007]), then I won't give too many further explanations here.

In the early version of GTC (e.g., [Holod2009]), the approximation of zero equilibrium is made,

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B}_0 = 0. \quad (0.23)$$

So, Eq. (0.21) can be rewritten to

$$\mathbf{v}_c = \frac{v_{\parallel}^2}{\Omega_{\alpha}} \nabla \times \frac{\mathbf{B}_0}{B_0} = \frac{v_{\parallel}^2}{\Omega_{\alpha}} \left[ \underbrace{\frac{\nabla \times \mathbf{B}_0}{B_0}}_{\text{dropped}} - \frac{\nabla B_0}{B_0^2} \times \mathbf{B}_0 \right] \approx \frac{v_{\parallel}^2}{\Omega_{\alpha}} \mathbf{b}_0 \times \frac{\nabla B_0}{B_0}, \quad (0.24)$$

Then, Eq. (0.20)

$$\mathbf{v}_d \approx \frac{m_{\alpha} v_{\parallel}^2 + \mu B_0}{m_{\alpha} \Omega_{\alpha}} \mathbf{b}_0 \times \frac{\nabla B_0}{B_0}. \quad (0.25)$$

However, in [Deng2012],  $\nabla \times \mathbf{B}_0$  terms are kept. I keep the general form in this document.

The collision operator  $C_{\alpha}$  in Eq. (0.14) has been implemented in GTC, but we will omit it in the following sections.

In (0.14), there is no gyro-angle variable  $\vartheta$ , which is removed out by gyro-phase average using the operator,

$$\langle a \rangle_c = \bar{a} = \int \frac{d\vartheta_c}{2\pi} d\mathbf{x} \delta(\mathbf{X} + \mathbf{p} - \mathbf{x}) a. \quad (0.26)$$

All the field variables should be gyro-phase averaged in Eqs. (0.14)-(0.16). This is also called the first gyro-phase average, which is from particle coordinate to gyro-center coordinate (forward,  $\mathbf{x} \rightarrow \mathbf{X}$ ). The second gyro-phase average will be met in field equation (e.g., Eq. (0.34)), which is from gyro-center to particle coordinate (backward,  $\mathbf{X} \rightarrow \mathbf{x}$ ). For simplifying the notation, operator (0.26) has not been written out in [Holod2009] and also in most of the following sections.

## 3 The Field Equations

### 3.1 Which equations we used

In **linear** gyrokinetic theory, the field variables  $\delta\phi$ ,  $\delta\psi$  and  $\delta B_{\parallel}$  are commonly used for convenience,  $\delta\phi$  is the perturbed electrostatic potential,  $\delta\psi = \delta A_{\parallel} \cdot \omega / ck_{\parallel}$  is a quantity related to the parallel component of the perturbed magnetic vector potential, and  $\delta B_{\parallel}$  is the parallel component of the perturbed magnetic field. And the field equations are quasi-neutrality condition and (parallel and perpendicular) Ampere's law. Three variables and three equations, then the system is closed.

In low-beta ( $\beta \equiv 8\pi T n / B_0^2 \ll 1$ ), we can exclude the compressional component of the magnetic field perturbation by assuming  $\delta B_{\parallel} = 0$ . In MHD limit,  $\delta E_{\parallel} = 0$ , i.e.,  $\delta\phi_{\parallel} \equiv \delta\phi - \delta\psi = 0$ . Then the remaining variable is  $\delta\psi$  (or,  $\delta A_{\parallel}$ ), which will give the shear Alfvén wave. This is also why (shear) Alfvén wave or Alfvén eigenmode (AE) is important.

To understand the above descriptions, a very straightforward way is to see the linear dispersion relation 3-by-3 matrix, which will show these very clearly (See e.g., [Xie2011]).

In GTC, we just consider low-beta, then we use  $\delta B_{\parallel} = 0$  assumption, thus

$$\delta \mathbf{B} = \delta \mathbf{B}_{\perp} = \nabla \times \lambda \mathbf{B}_0, \quad (0.27)$$

with  $\lambda = A_{\parallel} / B_0$ .

Only two field variables are remained, and then only two equations are needed: quasi-neutrality condition (reduced to Poisson's equation<sup>4</sup>) and parallel Ampere's law.

### 3.2 Poisson equation

As described in the above sub-section, the electrostatic potential can be found using gyrokinetic Poisson's equation.

The original Poisson equation,

$$\nabla^2 \phi(\mathbf{x}) = \int F d\mathbf{v} = -4\pi \sum_{\alpha} n_{\alpha}(\mathbf{x}) Z_{\alpha}. \quad (0.28)$$

The density of species described by gyrokinetic equation is

$$n_{\alpha} = \bar{n}_{\alpha} + n_{0\alpha} \frac{Z_{\alpha}}{T_{\alpha}} (\tilde{\phi} - \phi), \quad (0.29)$$

where,

$$\bar{n}_{\alpha} = \langle n_{\alpha} \rangle_c. \quad (0.30)$$

Eq. (0.28) becomes

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<sup>4</sup> A strange thing! In full kinetic, Poisson equation is just used in ES (electrostatic) simulation, and we won't use it directly in EM case. However, Eq. (0.31) tells us this is feasible in gyrokinetic simulation.

$$\underbrace{\nabla^2 \phi(\mathbf{x})}_{\approx 0, \text{ quasineutral}} + \underbrace{4\pi \sum_{\alpha} (\tilde{\phi}_{\alpha} - \phi) \frac{n_{0\alpha} Z_{\alpha}^2}{T_{\alpha}}}_{\text{Polarization density response}} = -4\pi \sum_{\alpha} \bar{n}_{\alpha} Z_{\alpha}. \quad (0.31)$$

For electrons,  $k_{\perp} \rho_e \rightarrow 0$ ,

$$\tilde{\phi}_e - \phi \approx 0 \quad (0.32)$$

Assuming a single dominant ion species,

$$\frac{4\pi Z_i^2 n_i}{T_i} (\phi - \tilde{\phi}) = 4\pi (Z_i n_i - e n_e), \quad (0.33)$$

Or, with fast ions,

$$\frac{Z_i^2 n_i}{T_{\perp i}} (\phi - \tilde{\phi}_i) + \frac{Z_f^2 n_f}{T_{\perp f}} (\phi - \tilde{\phi}_f) = \sum_{\alpha=i,e,f} Z_{\alpha} \delta n_{\alpha}. \quad (0.34)$$

### 3.3 Parallel Ampere's law

The vector potential satisfies the gyrokinetic parallel Ampere's law,

$$\nabla_{\perp}^2 A_{\parallel} = \frac{4\pi}{c} (e n_e u_{\parallel e} - Z_i n_i u_{\parallel i}), \quad (0.35)$$

Or, with fast ions,

$$\frac{c}{4\pi} \nabla_{\perp}^2 A_{\parallel} = - \sum_{\alpha=i,e,f} \delta J_{\parallel \alpha}. \quad (0.36)$$

The second gyro-phase averaged potential  $\tilde{\phi}$ , the gyro-center density  $n_{\alpha}$  and parallel velocity  $u_{\parallel \alpha}$  or current  $\delta J_{\parallel \alpha}$  in the particle coordinate are defined as the fluid moments of the corresponding distribution functions,

$$\tilde{\phi}_{\alpha}(\mathbf{x}, t) = \frac{1}{n_{\alpha}} \int_{\mathbf{X} \rightarrow \mathbf{x}} d\mathbf{v} f_{\alpha}(\mathbf{X}, \mu, v_{\parallel}, t) \langle \phi \rangle_c(\mathbf{X}, t), \quad (0.37)$$

$$n_{\alpha}(\mathbf{x}, t) = n_{0\alpha}(\mathbf{x}, t) + \delta n_{\alpha}(\mathbf{x}, t) = \int_{\mathbf{X} \rightarrow \mathbf{x}} d\mathbf{v} (f_{0\alpha}(\mathbf{X}, \mu, v_{\parallel}, t) + \delta f_{\alpha}(\mathbf{X}, \mu, v_{\parallel}, t)), \quad (0.38)$$

$$\begin{aligned} n_{\alpha}(\mathbf{x}, t) u_{\parallel \alpha}(\mathbf{x}, t) &= \int_{\mathbf{X} \rightarrow \mathbf{x}} d\mathbf{v} Z_{\alpha} v_{\parallel} (f_{0\alpha}(\mathbf{X}, \mu, v_{\parallel}, t) + \delta f_{\alpha}(\mathbf{X}, \mu, v_{\parallel}, t)) \\ &= J_{\parallel 0\alpha}(\mathbf{x}, t) + \delta J_{\parallel \alpha}(\mathbf{x}, t), \end{aligned} \quad (0.39)$$

Here, the integral symbol represents the integral over the gyro-center velocity space and the transformation between the gyro-center and particle coordinates:

$$\int_{\mathbf{X} \rightarrow \mathbf{x}} d\mathbf{v} \equiv \int \frac{2\pi B_0}{m} d\mu dv_{\parallel} \int \frac{d\vartheta_c}{2\pi} d\mathbf{X} \delta(\mathbf{X} + \mathbf{p} - \mathbf{x}). \quad (0.40)$$

From coordinates transform Jacobian

The electrostatic  $\phi$  and vector potential  $A_{\parallel}$  in Eqs. (0.15)-(0.16) are **being gyroaveraged** for ions or taken at the gyrocenter position for electrons. Because, for electrons, the position and the gyro-center are not distinguished ( $\mathbf{x}_e \approx \mathbf{X}_e$ )

for their small gyro-radii ( $k_{\perp} \rho_e \ll 1$ ) in the drift-kinetic limit, then, their density and current are simply:

$$\delta n_e = \int_{GC} d\mathbf{v} \delta f_e, \quad (0.41)$$

$$\delta J_{\parallel e} = -e \int_{GC} d\mathbf{v} v_{\parallel} \delta f_e, \quad (0.42)$$

where,

$$\int_{GC} d\mathbf{v} \equiv \int \frac{2\pi B_0}{m} d\mu dv_{\parallel}. \quad (0.43)$$

Note: The field equations are in particle coordinate ( $\mathbf{x}$ ), while the gyrokinetic equation is in gyrocenter coordinate ( $\mathbf{X}$ ).

So far, Eqs. (0.14)-(0.43) can be readily implemented in a non-perturbative (full-f) simulation.

However, for numerical reasons, we need go further. Due to large ion-electron mass ratio, ion and electron motions have very distinct temporal and spatial scales. Therefore, the gyrokinetic equation is solved using different strategies for ions and electrons in GTC. The ions are solved using direct particle-in-cell (PIC) method and the electrons are solved using a fluid-kinetic hybrid model. In GTC, we also include delta-f ( $\delta f$ ) simulation. Then, we need also formulate them.

## 4 $\delta f$ Formula for Ions

### 4.1 General form

The distribution function can be decomposed into equilibrium and perturbed parts  $f_{\alpha} = f_{0\alpha} + \delta f_{\alpha}$ , with the equilibrium part satisfying the gyrokinetic equation,

$$\frac{\partial f_{0\alpha}}{\partial t} + \left( v_{\parallel} \mathbf{B}_0 + \mathbf{v}_d \right) \cdot \nabla f_{0\alpha} - \frac{\mu}{m_{\alpha}} \frac{\mathbf{B}_0^*}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}} f_{0\alpha} - C_{\alpha} f_{0\alpha} = 0. \quad (0.44)$$

Considering (0.44) and expanding Eq. (0.14), gives,

$$0 = \underbrace{\left[ \frac{\partial}{\partial t} + \left( v_{\parallel} \frac{\mathbf{B}_0}{B_0} + \mathbf{v}_d \right) \cdot \nabla - \frac{1}{m_{\alpha}} \frac{\mathbf{B}_0^*}{B_0} \cdot \mu \nabla B_0 \frac{\partial}{\partial v_{\parallel}} \right] f_{0\alpha}}_{=0, \text{ equilibrium part}} + \left( v_{\parallel} \frac{\delta \mathbf{B}}{B_0} + \mathbf{v}_E \right) \cdot \nabla f_{0\alpha} - \left[ \frac{\delta \mathbf{B}}{B_0} \cdot \mu \nabla B_0 + Z_{\alpha} \frac{\mathbf{B}^*}{B_0} \cdot \nabla \phi + \frac{Z_{\alpha}}{c} \frac{\partial A_{\parallel}}{\partial t} \right] \frac{1}{m_{\alpha}} \frac{\partial}{\partial v_{\parallel}} f_{0\alpha} + \frac{d\delta f_{\alpha}}{dt}. \quad (0.45)$$

Defining the particle weight as  $w_{\alpha} \equiv \delta f_{\alpha} / f_{\alpha}$ , we can get,

$$\begin{aligned} \frac{dw_{\alpha}}{dt} &= \frac{d}{dt} \left( \frac{\delta f_{\alpha}}{f_{\alpha}} \right) = \frac{1}{f_{\alpha}} \frac{d\delta f_{\alpha}}{dt} - \frac{\delta f_{\alpha}}{f_{\alpha}^2} \frac{df_{\alpha}}{dt} = \frac{1}{f_{\alpha}} \frac{d\delta f_{\alpha}}{dt} \\ &= \left( 1 - \frac{\delta f_{\alpha}}{f_{\alpha}} \right) \frac{1}{f_{0\alpha}} \frac{d\delta f_{\alpha}}{dt} = -(1 - w_{\alpha}) \frac{1}{f_{0\alpha}} \frac{df_{0\alpha}}{dt}. \end{aligned} \quad (0.46)$$

Using Eq. (0.45), we get (Eq. (8) in [Holod2009])

$$\frac{dw_\alpha}{dt} = (1 - w_\alpha) \left[ -\left( v_\parallel \frac{\delta \mathbf{B}}{B_0} + \mathbf{v}_E \right) \cdot \frac{\nabla f_{0\alpha}}{f_{0\alpha}} + \left( \mu \frac{\delta \mathbf{B}}{B_0} \cdot \nabla B_0 + Z_\alpha \frac{\mathbf{B}^*}{B_0} \cdot \nabla \phi + \frac{Z_\alpha}{c} \frac{\partial A_\parallel}{\partial t} \right) \frac{1}{m_\alpha} \frac{1}{f_{0\alpha}} \frac{\partial f_{0\alpha}}{\partial v_\parallel} \right]. \quad (0.47)$$

The dynamic equation (0.47) (for the perturbative simulation) or Eq. (0.14) (for full-f simulation) together with the field equations (0.34) and (0.35) form the basic closed system of equations for the nonlinear gyrokinetic simulations.

## 4.2 Drift Maxwellian distribution

Core (background) ions and electrons are always close to Maxwellian distribution, while fast ions are always modeled in slowing-down distribution (derived from the collision theory). Here, for simplicity, we assume the equilibrium distribution for all three species is isotropic drift/shifted Maxwellian:

$$f_{0\alpha} = \frac{n_{0\alpha}}{(2\pi v_{th0,\alpha}^2)^{3/2}} \exp \left[ -\frac{(v_\parallel - u_{||0\alpha})^2 + 2\mu B_0 / m_\alpha}{2v_{th0,\alpha}^2} \right], \quad \alpha = i, e, f, \quad (0.48)$$

where  $u_{||0\alpha}$  is the parallel equilibrium flow velocity, and  $v_{th0,\alpha} = \sqrt{T_{0\alpha} / m_\alpha}$  is the thermal velocity.

Substituting (0.48) to (0.47), we can easily get,

$$\frac{dw_\alpha}{dt} = (1 - w_\alpha) \left\{ -\left( v_\parallel \frac{\delta \mathbf{B}}{B_0} + \mathbf{v}_E \right) \cdot \boldsymbol{\kappa} + \frac{u_{||0\alpha} \mu}{T_{0\alpha}} \frac{\delta \mathbf{B}}{B_0} \cdot \nabla B_0 - \frac{Z_\alpha}{T_{0\alpha}} (v_\parallel - u_{||0\alpha}) \left( \mathbf{b}_0 \cdot \nabla \phi + \frac{1}{c} \frac{\partial A_\parallel}{\partial t} \right) \right. \\ \left. - \frac{Z_\alpha}{T_{0\alpha}} \left[ \mathbf{v}_g + \left( 1 - \frac{u_{||0\alpha}}{v_\parallel} \right) \mathbf{v}_c \right] \cdot \nabla \phi - \underbrace{\frac{Z_\alpha}{T_{0\alpha}} \left[ (v_\parallel - u_{||0\alpha}) \frac{\delta \mathbf{B}}{B_0} \right] \cdot \nabla \phi}_{\text{dropped in [Holod2009]&[Deng2012]?}} \right\}, \quad (0.49)$$

where

$$\boldsymbol{\kappa} \equiv \frac{\nabla f_{0\alpha}}{f_{0\alpha}} (n_{0\alpha}, T_{0\alpha}, u_{||0\alpha}) = \frac{\nabla n_{0\alpha}}{n_{0\alpha}} + \left[ \frac{m_\alpha (v_\parallel - u_{||0\alpha})^2 + 2\mu B_0}{2T_{0\alpha}} - \frac{3}{2} \right] \frac{\nabla T_{0\alpha}}{T_{0\alpha}} + \frac{m_\alpha (v_\parallel - u_{||0\alpha}) \nabla u_{||0\alpha}}{T_{0\alpha}}, \quad (0.50)$$

represents the background density, temperature, and parallel flow gradients. And the magnetic field gradients

$$\boldsymbol{\kappa}_{B_0} \equiv \frac{\nabla f_{0\alpha}}{f_{0\alpha}} (B_0) = \frac{\mu \nabla B_0}{T_{0\alpha}}, \quad (0.51)$$

has been included to  $\mathbf{v}_g$  in Eq. (0.49), i.e.,

$$\mathbf{v}_E \cdot \boldsymbol{\kappa}_{B_0} = \left( \frac{c \mathbf{b}_0 \times \nabla \phi}{B_0} \right) \cdot \left( \frac{\mu \nabla B_0}{T_{0\alpha}} \right) = -\frac{c}{B_0} \frac{\mu}{T_{0\alpha}} \mathbf{b}_0 \times \nabla B_0 \cdot \nabla \phi = -\frac{Z_\alpha}{T_{0\alpha}} \left( \frac{\mu}{m_\alpha \Omega_\alpha} \mathbf{b}_0 \times \nabla B_0 \right) \cdot \nabla \phi = \frac{Z_\alpha}{T_{0\alpha}} \mathbf{v}_g \cdot \nabla \phi. \quad (0.52)$$

Note,

$$\frac{\nabla f_{0\alpha}}{f_{0\alpha}} = \boldsymbol{\kappa} + \boldsymbol{\kappa}_{B_0}. \quad (0.53)$$

The parallel velocity shift  $u_{||0}$  is taken as local equilibrium parallel flow velocity, which is calculated using the radial force balance equation and neoclassical collisionality (see the references in [Holod2009]),

$$u_{||} = -\frac{cgT_{0i}}{Z_i B_0} \left[ \frac{Z_i}{T_{0i}} \frac{\partial \phi_0}{\partial \psi} + (1-k) \frac{\partial \ln T_{0i}}{\partial \psi} + \frac{\partial \ln n_{0i}}{\partial \psi} \right]. \quad (0.54)$$

The corresponding neoclassical poloidal flow is

$$u_\theta = k \frac{cgT_{0i}B_{0\theta}}{Z_i B_0^2} \frac{\partial \ln T_{0i}}{\partial \psi}. \quad (0.55)$$

Here  $k$  is the poloidal rotation factor ( $k \approx 1.17$  in the banana regime), and  $\phi_0$  is the equilibrium electrostatic potential.

## 5 Fluid-kinetic Hybrid Electron Model

### 5.1 Electron continuity equation

Integrating (0.14) in the drift-kinetic limit term by term over the guiding center velocity space gives (first-order moment):

$$\int_{GC} d\mathbf{v} \partial_t (f_0 + \delta f) = \partial_t \delta n, \quad (0.56)$$

and,

$$\begin{aligned} \int_{GC} d\mathbf{v} \dot{\mathbf{X}} \cdot \nabla f_0 &= \int_{GC} d\mathbf{v} \left[ v_{\parallel} \frac{\mathbf{B}_0 + \delta \mathbf{B}}{B_0} + \mathbf{v}_E + \frac{v_{\parallel}^2}{\Omega_\alpha} \nabla \times \mathbf{b}_0 + \frac{\mu}{m_\alpha \Omega_\alpha} \mathbf{b}_0 \times \nabla B_0 \right] \cdot \nabla f_0 \\ &= (\mathbf{B}_0 + \delta \mathbf{B}) \cdot \nabla \left( \frac{n_0 u_{\parallel 0}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_0}{B_0} \right) + \frac{c \nabla \times \mathbf{b}_0}{Z} \cdot \nabla \left( \frac{P_{\parallel 0}}{B_0} \right) + \frac{c \mathbf{b}_0 \times \nabla B_0}{Z} \cdot \nabla \left( \frac{P_{\perp 0}}{B_0^2} \right), \end{aligned} \quad (0.57)$$

$$\begin{aligned} \int_{GC} d\mathbf{v} \dot{\mathbf{X}} \cdot \nabla \delta f &= \int_{GC} d\mathbf{v} \left[ v_{\parallel} \frac{\mathbf{B}_0 + \delta \mathbf{B}}{B_0} + \mathbf{v}_E + \frac{v_{\parallel}^2}{\Omega_\alpha} \nabla \times \mathbf{b}_0 + \frac{\mu}{m_\alpha \Omega_\alpha} \mathbf{b}_0 \times \nabla B_0 \right] \cdot \nabla \delta f \\ &= (\mathbf{B}_0 + \underbrace{\delta \mathbf{B}}_{\text{nonlinear}}) \cdot \nabla \left( \frac{n_0 \delta u_{\parallel}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{\delta n}{B_0} \right) + \underbrace{\frac{c \nabla \times \mathbf{b}_0}{Z} \cdot \nabla \left( \frac{\delta P_{\parallel}}{B_0} \right)}_{\text{nonlinear}} + \frac{c \mathbf{b}_0 \times \nabla B_0}{Z} \cdot \nabla \left( \frac{\delta P_{\perp}}{B_0^2} \right), \end{aligned} \quad (0.58)$$

and,

$$\begin{aligned} \int_{GC} d\mathbf{v} \dot{v}_{\parallel} \partial_{v_{\parallel}} f_0 &= \frac{2\pi B_0}{m} \int d\mu \int dv_{\parallel} \dot{v}_{\parallel} \partial_{v_{\parallel}} f_0 = \frac{2\pi B_0}{m} \int d\mu \left[ \underbrace{\dot{v}_{\parallel} f_0 \Big|_{v_{\parallel}=-\infty}^{v_{\parallel}=\infty}}_{=0} - \int dv_{\parallel} f_0 \right. \\ &\quad \left. - \frac{\nabla \times \mathbf{b}_0}{m_\alpha \Omega_\alpha} \cdot (\mu \nabla B_0 + Z_\alpha \nabla \langle \phi \rangle_c) \right] \\ &= \frac{c \nabla \times \mathbf{b}_0}{Z B_0} \cdot \left( P_{\perp 0} \frac{\nabla B_0}{B_0} + Z_\alpha n_0 \nabla \phi \right), \end{aligned} \quad (0.59)$$

$$\int_{GC} d\mathbf{v} \dot{v}_{\parallel} \partial_{v_{\parallel}} \delta f = \frac{c \nabla \times \mathbf{b}_0}{Z B_0} \cdot \left( \delta P_{\perp} \frac{\nabla B_0}{B_0} + \underbrace{Z_\alpha \delta n \nabla \phi}_{\text{nonlinear}} \right), \quad (0.60)$$

where,

$$n_0 = \int_{GC} d\mathbf{v} f_0, \quad (0.61)$$

$$\delta n = \int_{GC} d\mathbf{v} \delta f, \quad (0.62)$$

$$u_{\parallel 0} = \frac{1}{n_0} \int_{GC} d\mathbf{v} v_{\parallel} f_0, \quad (0.63)$$

$$\delta u_{\parallel} = \frac{1}{n_0} \int_{GC} d\mathbf{v} v_{\parallel} \delta f, \quad (0.64)$$

$$P_{\parallel 0} = \int_{GC} d\mathbf{v} m v_{\parallel}^2 f_0, \quad (0.65)$$

$$\delta P_{\parallel} = \int_{GC} d\mathbf{v} m v_{\parallel}^2 \delta f, \quad (0.66)$$

$$P_{\perp 0} = \int_{GC} d\mathbf{v} \mu B_0 f_0, \quad (0.67)$$

$$\delta P_{\perp} = \int_{GC} d\mathbf{v} \mu B_0 \delta f. \quad (0.68)$$

To the definition of  $\delta u_{\parallel}$  in Eq. (0.64) (also, Eq. (0.79)), one may argue that,

$$\frac{1}{n_0} \int_{GC} d\mathbf{v} v_{\parallel} \delta f = \frac{1}{n_0} \delta J_{\parallel e} = \frac{1}{n_0} \delta (n u_{\parallel e}) = \frac{n_0 \delta u_{\parallel e} + u_{\parallel 0e} \delta n}{n_0} = \delta u_{\parallel e} + \frac{\delta n}{n_0} u_{\parallel 0e}. \quad (0.69)$$

Eq. (0.69) has an external  $\delta n$  term than Eq. (0.64). For simplify, we use Eq. (0.64), which will not influent the bellowing discussions, but just should keep in mind that the definition of  $\delta u_{\parallel e}$  is different from traditional case.

To get Eqs. (0.57)-(0.58), one should note the integral with gradient operator,

$$\int_{GC} \nabla(\dots) d\mathbf{v} = \int \frac{2\pi B_0}{m} \nabla(\dots) d\mu dv_{\parallel} = B_0 \int \nabla \left[ \frac{2\pi B_0}{m} \frac{(\dots)}{B_0} \right] d\mu dv_{\parallel} = B_0 \nabla \int \frac{(\dots)}{B_0} d\mathbf{v}. \quad (0.70)$$

Collecting the above integrals, we get an equilibrium equation and a perturbed equation. The equilibrium equation is

$$\mathbf{B}_0 \cdot \nabla \left( \frac{n_0 u_{\parallel 0}}{B_0} \right) + \frac{c \nabla \times \mathbf{b}_0}{Z} \cdot \nabla \left( \frac{P_{\parallel 0}}{B_0} \right) + \frac{c \mathbf{b}_0 \times \nabla B_0}{Z} \cdot \nabla \left( \frac{P_{\perp 0}}{B_0^2} \right) + \frac{c \nabla \times \mathbf{b}_0 \cdot \nabla B_0}{Z B_0^2} P_{\perp 0} = 0, \quad (0.71)$$

and the perturbed continuity equation is

$$\begin{aligned} 0 &= \partial_t \delta n + \delta \mathbf{B} \cdot \nabla \left( \frac{n_0 u_{\parallel 0}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_0}{B_0} \right) + \mathbf{B}_0 \cdot \nabla \left( \frac{n_0 \delta u_{\parallel}}{B_0} \right) + \frac{c \nabla \times \mathbf{b}_0}{Z} \cdot \nabla \left( \frac{\delta P_{\parallel}}{B_0} \right) + \frac{c \mathbf{b}_0 \times \nabla B_0}{Z} \cdot \nabla \left( \frac{\delta P_{\perp}}{B_0^2} \right) \\ &\quad + \frac{c \nabla \times \mathbf{b}_0 \cdot \nabla B_0}{Z B_0^2} \delta P_{\perp} + \frac{c \nabla \times \mathbf{b}_0 \cdot n_0 \nabla \phi}{B_0} + \underbrace{\delta \mathbf{B} \cdot \nabla \left( \frac{n_0 \delta u_{\parallel}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{\delta n}{B_0} \right) + \frac{c \delta n}{B_0} \nabla \times \mathbf{b}_0 \cdot \nabla \phi}_{\text{nonlinear}} \\ &= \partial_t \delta n + \delta \mathbf{B} \cdot \nabla \left( \frac{n_0 u_{\parallel 0}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_0}{B_0} \right) + \mathbf{B}_0 \cdot \nabla \left( \frac{n_0 \delta u_{\parallel}}{B_0} \right) \\ &\quad - n_0 (\delta \mathbf{v}_* + \mathbf{v}_E) \cdot \frac{\nabla B_0}{B_0} + \frac{c}{Z} \frac{\nabla \times \mathbf{B}_0}{B_0} \cdot \frac{\nabla \delta P_{\parallel}}{B_0} + \frac{c \nabla \times \mathbf{B}_0}{Z B_0^3} \cdot \nabla B_0 (\delta P_{\perp} - \delta P_{\parallel}) + \frac{c n_0}{B_0} \frac{\nabla \times \mathbf{B}_0}{B_0} \cdot \nabla \phi \\ &\quad + \delta \mathbf{B} \cdot \nabla \left( \frac{n_0 \delta u_{\parallel}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{\delta n}{B_0} \right) + \frac{c \delta n}{B_0^2} \mathbf{b}_0 \times \nabla B_0 \cdot \nabla \phi + \frac{c \delta n}{B_0^2} \nabla \times \mathbf{b}_0 \cdot \nabla \phi. \end{aligned} \quad (0.72)$$

where

$$\delta \mathbf{v}_* = \frac{c}{n_0 Z B_0^2} \mathbf{b}_0 \times (\nabla \delta P_{\perp} + \nabla \delta P_{\parallel}). \quad (0.73)$$

is the perturbed diamagnetic drift velocity.

Noting that Eq. (0.19) and,

$$\nabla \times \mathbf{b}_0 = \nabla \times \frac{\mathbf{B}_0}{B_0} = \frac{1}{B_0} \nabla \times \mathbf{B}_0 + \frac{\mathbf{B}_0 \times \nabla B_0}{B_0^2}, \quad (\nabla \times (f\mathbf{A}) = f \nabla \times \mathbf{A} + \nabla f \times \mathbf{A}), \quad (0.74)$$

are used, and many,

$$\mathbf{B}_0 \times \nabla B_0 \cdot \nabla B_0, \quad (\mathbf{A} \times \mathbf{B} \cdot \mathbf{B} = 0), \quad (0.75)$$

terms are vanished.

For electrons ( $Z_e = -e$ ), (0.72) rewrites to ([Deng2012] Eq. (28)),

$$0 = \partial_t \delta n_e + \delta \mathbf{B} \cdot \nabla \left( \frac{n_{0e} \delta u_{||e}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_{0e}}{B_0} \right) + \mathbf{B}_0 \cdot \nabla \left( \frac{n_{0e} \delta u_{||e}}{B_0} \right) \\ - \frac{c \nabla \times \mathbf{b}_0}{e} \cdot \nabla \left( \frac{\delta P_{||e}}{B_0} \right) - \frac{c \mathbf{b}_0 \times \nabla B_0}{e} \cdot \nabla \left( \frac{\delta P_{||e}}{B_0^2} \right) - \frac{c \nabla \times \mathbf{b}_0 \cdot \nabla B_0}{e B_0^2} \delta P_{||e} + \frac{c \nabla \times \mathbf{b}_0}{B_0} \cdot n_{0e} \nabla \phi \\ + \underbrace{\delta \mathbf{B} \cdot \nabla \left( \frac{n_{0e} \delta u_{||e}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{\delta n_e}{B_0} \right) + \frac{c \delta n_e}{B_0} \mathbf{b}_0 \times \nabla B_0 \cdot \nabla \phi + \frac{c \delta n_e}{B_0^2} \nabla \times \mathbf{B}_0 \cdot \nabla \phi}_{\text{nonlinear}}, \quad (0.76)$$

Or,

$$0 = \partial_t \delta n_e + \mathbf{B}_0 \cdot \nabla \left( \frac{n_{0e} \delta u_{||e}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_{0e}}{B_0} \right) - n_{0e} (\delta \mathbf{v}_{*e} + \mathbf{v}_E) \cdot \frac{\nabla B_0}{B_0} \\ + \underbrace{\delta \mathbf{B} \cdot \nabla \left( \frac{n_{0e} \delta u_{||e}}{B_0} \right) + \frac{c \nabla \times \mathbf{B}_0}{B_0^2} \cdot \left[ - \frac{\nabla \delta P_{||e}}{e} - \frac{(\delta P_{||e} - \delta P_{||e}) \nabla B_0}{e B_0} + n_{0e} \nabla \phi \right]}_{\text{parallel equilibrium flow and finite } \nabla \times \mathbf{B}_0 \text{ terms}} \\ + \underbrace{\delta \mathbf{B} \cdot \nabla \left( \frac{n_{0e} \delta u_{||e}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{\delta n_e}{B_0} \right) + \frac{c \delta n_e}{B_0^2} \mathbf{b}_0 \times \nabla B_0 \cdot \nabla \phi + \frac{c \delta n_e}{B_0^2} \nabla \times \mathbf{B}_0 \cdot \nabla \phi}_{\text{nonlinear terms}}. \quad (0.77)$$

Eq. (0.76), which is exactly the same equation as (0.77), will be used when reduced to MHD limit.

## 5.2 Electron adiabatic response

### 5.2.1 Electron parallel motion

The perturbed parallel flow  $\delta u_{||e}$  is calculated by inverting the Ampere's law Eq. (0.36),

$$n_{0e} \delta u_{||e} = \frac{c}{4\pi} \nabla_{\perp}^2 \delta A_{||} + \sum_{\alpha=i,f} \frac{Z_{\alpha}}{e} n_{0\alpha} \delta u_{||\alpha}, \quad (0.78)$$

where,

$$\delta u_{||\alpha} = \frac{1}{n_{0\alpha}} \int_{\mathbf{x} \rightarrow \mathbf{x}} d\mathbf{v} v_{||} \delta f_{\alpha}, \quad \alpha = i, f. \quad (0.79)$$

An effective potential and an inductive potential are defined to represent  $\delta E_{\parallel}$  and  $\partial_t \delta A_{\parallel}$  (See also Sec. 3.1),

$$\delta E_{\parallel} \equiv -\mathbf{b}_0 \cdot \nabla \delta \phi_{\text{eff}}, \quad (0.80)$$

$$\partial_t \delta A_{\parallel} \equiv c \mathbf{b}_0 \cdot \nabla \delta \phi_{\text{ind}}, \quad (0.81)$$

Noting that the non-zonal parallel electric field is (also: linear tearing mode is removed)

$$\delta E_{\parallel} = -\mathbf{b}_0 \cdot \nabla \delta \phi - \frac{1}{c} \partial_t \delta A_{\parallel}, \quad (0.82)$$

we have

$$\delta \phi_{\text{ind}} = \delta \phi_{\text{eff}} - \delta \phi. \quad (0.83)$$

Please note the different definitions in [Holod2009] and here (also, [Deng2011]),  $\delta \phi_{\text{eff, ind}}^{[\text{here}]} = \phi_{\text{eff, ind}}^{[\text{Holod2009}]}$ , which is OK, since we choose equilibrium electrostatic potential be zero.

In GTC, an optional antenna is implemented through an external potential perturbation in  $\delta \phi_{\text{ind}}$ . When the antenna is turned on,  $\delta \phi_{\text{ind}}$  becomes,

$$\delta \phi_{\text{ind}} = \delta \phi_{\text{eff}} - \delta \phi + \delta \phi_{\text{ant}}, \quad (0.84)$$

where  $\delta \phi_{\text{ant}}$  is the external potential perturbation from the antenna.

### 5.2.2 Lowest order solution (adiabatic response)

The perturbed pressures  $\delta P_{\perp e}$  and  $\delta P_{\parallel e}$  in (0.77) and the effective potential  $\delta \phi_{\text{eff}}$  are calculated from solving the electron perturbed distribution  $\delta f_e$  order by order based on the ordering of  $\omega / k_{\parallel} v_{\parallel}$ :

$$\delta f_e = \delta f_e^{(0)} + \delta h_e^{(1)} + \delta h_e^{(2)} + \dots, \quad (0.85)$$

$$\delta P_{\perp e} = \delta P_{\perp e}^{(0)} + \delta P_{\perp e}^{(1)} + \delta P_{\perp e}^{(2)} + \dots, \quad (0.86)$$

$$\delta P_{\parallel e} = \delta P_{\parallel e}^{(0)} + \delta P_{\parallel e}^{(1)} + \delta P_{\parallel e}^{(2)} + \dots, \quad (0.87)$$

$$\delta \phi_{\text{eff}} = \delta \phi_{\text{eff}}^{(0)} + \delta \phi_{\text{eff}}^{(1)} + \delta \phi_{\text{eff}}^{(2)} + \dots. \quad (0.88)$$

The lowest order is the adiabatic response. The kinetic effects are treated in higher order terms. Now, let's begin to solve the lowest order.

Rewriting (0.45),

$$\begin{aligned} 0 = & \left( v_{\parallel} \frac{\partial \mathbf{B}}{B_0} \cdot \nabla - \frac{\mu}{m_e} \frac{\partial \mathbf{B}}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}} \right) f_{0e} + \frac{e}{m_e} \mathbf{b}_0 \cdot \nabla \delta \phi_{\text{eff}} \frac{\partial}{\partial v_{\parallel}} f_{0e} \\ & + \frac{d \delta f_e}{dt} + \left[ \mathbf{v}_E \cdot \nabla + \frac{e}{m_e} \left( \frac{\mathbf{v}_c}{v_{\parallel}} + \frac{\partial \mathbf{B}}{B_0} \right) \cdot \nabla \phi \frac{\partial}{\partial v_{\parallel}} \right] f_{0e}, \end{aligned} \quad (0.89)$$

where,

$$\begin{aligned} \frac{d\delta f_e}{dt} = & v_{\parallel} \mathbf{b}_0 \cdot \nabla \delta f_e + \left[ \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla - \frac{\mu}{m_e} \frac{\mathbf{B}_0^*}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}} \right] \delta f_e \\ & + \left[ \left( v_{\parallel} \frac{\partial \mathbf{B}}{\partial B_0} + \mathbf{v}_E \right) \cdot \nabla - \frac{\mu}{m_e} \frac{\partial \mathbf{B}}{\partial B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}} + \frac{e}{m_e} \frac{\mathbf{B}_0^* + \delta \mathbf{B}}{B_0} \cdot \nabla \phi \frac{\partial}{\partial v_{\parallel}} + \frac{e}{m_e c} \frac{\partial A_{\parallel}}{\partial t} \frac{\partial}{\partial v_{\parallel}} \right] \delta f_e. \end{aligned} \quad (0.90)$$

Keeping only the leading order (colored terms) in (0.89) and (0.90), and still assuming drift Maxwellian distribution,

$$\begin{aligned} & \left( v_{\parallel} \frac{\partial \mathbf{B}}{\partial B_0} \cdot \nabla - \frac{\mu}{m_e} \frac{\partial \mathbf{B}}{\partial B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}} \right) f_{0e} = \left[ v_{\parallel} \frac{\partial \mathbf{B}}{\partial B_0} \cdot \nabla - \frac{\mu}{m_e} \frac{\partial \mathbf{B}}{\partial B_0} \cdot \nabla B_0 \frac{m_e (v_{\parallel} - u_{\parallel 0e})}{-T_e} \right] f_{0e} \\ & = \left[ v_{\parallel} \frac{\partial \mathbf{B}}{\partial B_0} \cdot \left( \nabla + \frac{\mu}{T_e} \cdot \nabla B_0 \right) - \frac{u_{\parallel 0e} \mu}{T_e} \frac{\partial \mathbf{B}}{\partial B_0} \cdot \nabla B_0 \right] f_{0e} = \left[ v_{\parallel} \frac{\partial \mathbf{B}}{\partial B_0} \cdot \left( \nabla + \nabla \mu \Big|_{v_{\perp}} \frac{\partial}{\partial \mu} \right) - \frac{u_{\parallel 0e} \mu}{T_e} \frac{\partial \mathbf{B}}{\partial \mu} \cdot \nabla B_0 \right] f_{0e} \quad (0.91) \\ & = v_{\parallel} \frac{\partial \mathbf{B}}{\partial B_0} \cdot \nabla f_{0e} \Big|_{v_{\perp}} - \frac{u_{\parallel 0e} \mu}{T_e} \frac{\partial \mathbf{B}}{\partial B_0} \cdot \nabla B_0 f_{0e}, \end{aligned}$$

$$\frac{e}{m_e} \mathbf{b}_0 \cdot \nabla \delta \phi_{\text{eff}} \frac{\partial}{\partial v_{\parallel}} f_{0e} = -(v_{\parallel} - u_{\parallel 0e}) \frac{f_{0e} e}{T_e} \mathbf{b}_0 \cdot \nabla \delta \phi_{\text{eff}}. \quad (0.92)$$

The notation  $\nabla f \Big|_{v_{\perp}}$  means derivative taken at  $v_{\perp} = \text{const}$  instead of  $\mu = \text{const}$ .

The leading order of Eq. (0.89) for  $\omega/k_{\parallel}v_{\parallel}$ , (Eq. (13) in [Holod2009])

$$v_{\parallel} \mathbf{b}_0 \cdot \nabla \delta f_e^{(0)} = -v_{\parallel} \frac{\partial \mathbf{B}}{\partial B_0} \cdot \nabla f_{0e} \Big|_{v_{\perp}} + v_{\parallel} \frac{f_{0e} e}{T_e} \mathbf{b}_0 \cdot \nabla \delta \phi_{\text{eff}}. \quad (0.93)$$

Note, here, the parallel flow terms in Eqs. (0.91) and (0.92) are added into higher order (**assumption, weak parallel flow**,  $u_{\parallel 0e} \ll v_{th,e}$ ).

Clebsch representation for toroidal magnetic field is

$$\mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B} = \nabla(\psi_0 + \delta\psi) \times \nabla(\alpha_0 + \delta\alpha), \quad (0.94)$$

where  $\psi$  is the poloidal flux label, and  $\alpha = q(\psi)\theta - \zeta$  is the magnetic field line label with  $\theta$  and  $\zeta$ , respectively, being the poloidal and toroidal angles in magnetic coordinates. So, the equilibrium  $\mathbf{B}_0$  and the perturbed  $\delta \mathbf{B}$  are

$$\mathbf{B}_0 = \nabla \psi_0 \times \nabla(\alpha_0 + \delta\alpha), \quad (0.95)$$

$$\delta \mathbf{B} = \nabla \psi_0 \times \nabla \delta \alpha + \nabla \delta \psi \times \nabla \alpha_0 + \underbrace{\nabla \delta \psi \times \nabla \delta \alpha}_{\text{dropped}}. \quad (0.96)$$

Plugging (0.96) into Eq. (0.93),

$$\begin{aligned}
\frac{\delta \mathbf{B}}{B_0} \cdot \nabla f_{0e} \Big|_{v_\perp} &= \frac{\nabla \psi_0 \times \nabla \delta \alpha + \nabla \delta \psi \times \nabla \alpha_0}{B_0} \cdot \nabla f_{0e} \Big|_{v_\perp} = \frac{\nabla \psi_0 \times \nabla \delta \alpha + \nabla \delta \psi \times \nabla \alpha_0}{B_0} \cdot \left( \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \nabla \psi_0 + \frac{\partial f_{0e}}{\partial \alpha_0} \Big|_{v_\perp} \nabla \alpha_0 \right) \\
&= \frac{\nabla \delta \psi \times \nabla \alpha_0}{B_0} \cdot \nabla \psi_0 \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} + \frac{\nabla \psi_0 \times \nabla \delta \alpha}{B_0} \cdot \nabla \alpha_0 \frac{\partial f_{0e}}{\partial \alpha_0} \Big|_{v_\perp} \\
&= -\frac{\nabla \psi_0 \times \nabla \alpha_0}{B_0} \cdot \nabla \left( \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta \psi + \frac{\partial f_{0e}}{\partial \alpha_0} \Big|_{v_\perp} \delta \alpha \right) = -\mathbf{b}_0 \cdot \nabla \left( \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta \psi + \frac{\partial f_{0e}}{\partial \alpha_0} \Big|_{v_\perp} \delta \alpha \right).
\end{aligned} \tag{0.97}$$

Eq. (0.93) can be rewritten as

$$\begin{aligned}
v_{\parallel} \mathbf{b}_0 \cdot \nabla \delta f_e^{(0)} &= v_{\parallel} \mathbf{b}_0 \cdot \nabla \left( \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta \psi + \frac{\partial f_{0e}}{\partial \alpha_0} \Big|_{v_\perp} \delta \alpha \right) + v_{\parallel} \frac{f_{0e} e}{T_e} \mathbf{b}_0 \cdot \nabla \delta \phi_{\text{eff}} \\
&= v_{\parallel} \mathbf{b}_0 \cdot \nabla \left( \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta \psi + \frac{\partial f_{0e}}{\partial \alpha_0} \Big|_{v_\perp} \delta \alpha + \frac{f_{0e} e}{T_e} \delta \phi_{\text{eff}} \right) - \underbrace{v_{\parallel} \delta \phi_{\text{eff}} \mathbf{b}_0 \cdot \nabla \left( \frac{f_{0e} e}{T_e} \right)}_{\text{dropped}}
\end{aligned} \tag{0.98}$$

In Eq. (0.98), we have used the assumption,

$$\mathbf{b}_0 \cdot \nabla f_{0e} = \mathbf{b}_0 \cdot \nabla T_e = 0. \tag{0.99}$$

Obviously, when  $k_{\parallel} v_{\parallel} \neq 0$  (i.e., the zonal part cannot be hold), Eq. (0.98) has a solution ([Holod2009] Eq. (14))

$$\delta f_e^{(0)} = \frac{ef_{0e}}{T_e} \delta \phi_{\text{eff}} + \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta \psi + \frac{\partial f_{0e}}{\partial \alpha_0} \Big|_{v_\perp} \delta \alpha. \tag{0.100}$$

Since the electron response is adiabatic only for non-zonal ( $k_{\parallel} \neq 0$ ) modes, Eq. (0.93) excludes zonal components (with  $k_{\parallel} = 0$ ) of the perturbed potentials. The zonal components will be discussed in the following section.

To find  $\delta \phi_{\text{eff}}$  in the lowest order, we integrate Eq. (0.100) over the velocity space,

$$\delta n_e^{(0)} = \frac{en_{0e}}{T_e} \delta \phi_{\text{eff}} + \frac{\partial n_{0e}}{\partial \psi_0} \delta \psi + \frac{\partial n_{0e}}{\partial \alpha_0} \delta \alpha, \tag{0.101}$$

i.e. ([Holod2009] Eq. (16)),

$$\frac{e \delta \phi_{\text{eff}}^{(0)}}{T_e} = \frac{\delta n_e}{n_{0e}} - \frac{\delta \psi}{n_{0e}} \frac{\partial n_{0e}}{\partial \psi_0} - \frac{\delta \alpha}{n_{0e}} \frac{\partial n_{0e}}{\partial \alpha_0}. \tag{0.102}$$

The magnetic field line perturbations  $\delta \psi$  and  $\delta \alpha$  can be found using Eqs. (0.27) and (0.96), and with the assumption (0.23) (i.e.,  $\nabla \times \mathbf{B}_0 = 0$ ). Eq. (0.27) gives,

$$\begin{aligned}
\delta \mathbf{B} &= \nabla \times \lambda(\psi_0, \alpha_0) \mathbf{B}_0 = -\mathbf{B}_0 \times \nabla \lambda + \underbrace{\lambda \nabla \times \mathbf{B}_0}_{\text{dropped}} \\
&= -\mathbf{B}_0 \times \left( \frac{\partial \lambda}{\partial \psi_0} \nabla \psi_0 + \frac{\partial \lambda}{\partial \alpha_0} \nabla \alpha_0 \right) = \nabla \psi_0 \times \left( \frac{\partial \lambda}{\partial \psi_0} \mathbf{B}_0 \right) - \left( \frac{\partial \lambda}{\partial \alpha_0} \mathbf{B}_0 \right) \times \nabla \alpha_0.
\end{aligned} \tag{0.103}$$

Comparing Eq. (0.103) with Eq. (0.96), we get

$$\nabla \psi_0 \times \left( \frac{\partial \lambda}{\partial \psi_0} \mathbf{B}_0 \right) - \left( \frac{\partial \lambda}{\partial \alpha_0} \mathbf{B}_0 \right) \times \nabla \alpha_0 = \nabla \psi_0 \times \nabla \delta \alpha + \nabla \delta \psi \times \nabla \alpha_0. \quad (0.104)$$

From Eq. (0.104), we get,

$$\begin{aligned} \frac{\partial \lambda}{\partial \psi_0} \mathbf{B}_0 &= \nabla \delta \alpha, \\ \frac{\partial \lambda}{\partial \alpha_0} \mathbf{B}_0 &= -\nabla \delta \psi. \end{aligned} \quad (0.105)$$

Using  $\mathbf{B}_0$  dot Eq. (0.105) gives ([Holod2009] Eq. (17)),

$$\begin{aligned} \frac{\partial \lambda}{\partial \psi_0} &= \frac{\mathbf{B}_0 \cdot \nabla \delta \alpha}{B_0^2}, \\ \frac{\partial \lambda}{\partial \alpha_0} &= -\frac{\mathbf{B}_0 \cdot \nabla \delta \psi}{B_0^2}. \end{aligned} \quad (0.106)$$

Combining Eq. (0.106) and Eq. (0.81), we can easily get ([Holod2009] Eq. (18)),

$$\begin{aligned} \frac{\partial \delta \alpha}{\partial t} &= c \frac{\partial \delta \phi_{\text{ind}}}{\partial \psi_0}, \\ \frac{\partial \delta \psi}{\partial t} &= -c \frac{\partial \delta \phi_{\text{ind}}}{\partial \alpha_0}. \end{aligned} \quad (0.107)$$

The pressure terms in Eq. (0.77) are correspondingly decomposed into two parts: through the separation of the electron distribution function into a zero-order adiabatic response  $\delta f^{(0)}$  and a higher order non-adiabatic response  $\delta h$ . Using the solution Eq. (0.100), we can write ([Holod2009] Eqs. (19)-(20))

$$\begin{aligned} \delta P_{\perp e}^{(0)} &= \int d\mathbf{v} \mu B_0 \delta f_e^{(0)} = n_{0e} e \delta \phi_{\text{eff}}^{(0)} + \frac{\partial(n_{0e} T_e)}{\partial \psi_0} \delta \psi + \frac{\partial(n_{0e} T_e)}{\partial \alpha_0} \delta \alpha, \\ \delta P_{\parallel e}^{(0)} &= \int d\mathbf{v} m_e v_{\parallel}^2 \delta f_e^{(0)} = n_{0e} e \delta \phi_{\text{eff}}^{(0)} + \frac{\partial(n_{0e} T_e)}{\partial \psi_0} \delta \psi + \frac{\partial(n_{0e} T_e)}{\partial \alpha_0} \delta \alpha. \end{aligned} \quad (0.108)$$

### 5.2.3 Check the orderings in Eq. (0.89)

We need to verify that the black terms in Eq. (0.89) are of higher order.

(...)

## 5.3 Kinetic equation for electron non-adiabatic response

### 5.3.1 $\delta f$ formulation

Higher order non-adiabatic response  $\delta h_e$

$$f_e = f_{0e} + \delta f_e = f_{0e} + \delta f_e^{(0)} + \delta h_e. \quad (0.109)$$

Operator  $L$  can also be separated into an equilibrium and perturbed parts,

$$L = L_0 + \delta L, \quad (0.110)$$

with,

$$L_0 = \frac{\partial}{\partial t} + \left( v_{\parallel} \mathbf{b}_0 + \mathbf{v}_d \right) \cdot \nabla - \frac{\mu}{m_e} \frac{\mathbf{B}_0^*}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}}, \quad (0.111)$$

$$\delta L = \left( v_{\parallel} \frac{\partial \mathbf{B}}{\partial B_0} + \mathbf{v}_E \right) \cdot \nabla - \left[ \frac{\mu}{m_e} \frac{\partial \mathbf{B}}{\partial B_0} \cdot \nabla B_0 - \frac{e}{m_e} \frac{1}{B_0} \left( \mathbf{B}_0^* + \underbrace{\delta \mathbf{B}}_{\text{dropped}} \right) \cdot \nabla \phi - \frac{e}{m_e c} \frac{\partial A_{\parallel}}{\partial t} \right] \frac{\partial}{\partial v_{\parallel}} \quad (0.112)$$

Since  $L f_e = 0$  and  $L_0 f_{0e} = 0$ , we should have

$$L \delta h_e = -L_0 f_{0e} - \delta L f_{0e} - L_0 \delta f_e^{(0)} - \underbrace{\delta L \delta f_e^{(0)}}_{\text{dropped}}. \quad (0.113)$$

Rewriting Eqs. (0.89) and (0.90),

$$\begin{aligned} 0 &= \nu_{\parallel} \mathbf{b}_0 \cdot \nabla \delta f_e + \left( \nu_{\parallel} \frac{\delta \mathbf{B}}{B_0} \cdot \nabla - \frac{\mu}{m_e} \frac{\delta \mathbf{B}}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}} \right) f_{0e} + \frac{e}{m_e} \mathbf{b}_0 \cdot \nabla \phi_{\text{eff}} \frac{\partial}{\partial v_{\parallel}} f_{0e} \\ &\quad + \left[ \mathbf{v}_E \cdot \nabla + \frac{e}{m_e} \left( \frac{\mathbf{v}_c}{v_{\parallel}} + \frac{\delta \mathbf{B}}{B_0} \right) \cdot \nabla \phi \frac{\partial}{\partial v_{\parallel}} \right] f_{0e} + \left[ \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla - \frac{\mu}{m_e} \frac{\mathbf{B}_0^*}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}} \right] \delta f_e \\ &\quad + \left[ \left( \nu_{\parallel} \frac{\delta \mathbf{B}}{B_0} + \mathbf{v}_E \right) \cdot \nabla - \frac{\mu}{m_e} \frac{\delta \mathbf{B}}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}} + \frac{e}{m_e} \frac{\mathbf{B}_0^* + \delta \mathbf{B}}{B_0} \cdot \nabla \phi \frac{\partial}{\partial v_{\parallel}} + \frac{e}{m_e c} \frac{\partial A_{\parallel}}{\partial t} \frac{\partial}{\partial v_{\parallel}} \right] \delta f_e. \end{aligned} \quad (0.114)$$

Considering Eq.(0.109), substituting Eq. (0.114) into Eq. (0.113), and ignoring higher orders,

$$\begin{aligned} L \delta h_e &= - \underbrace{\left[ \nu_{\parallel} \mathbf{b}_0 \cdot \nabla \delta f_e^{(0)} + \left( \nu_{\parallel} \frac{\delta \mathbf{B}}{B_0} \cdot \nabla - \frac{\mu}{m_e} \frac{\delta \mathbf{B}}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}} \right) f_{0e} + \frac{e}{m_e} \mathbf{b}_0 \cdot \nabla \phi_{\text{eff}} \frac{\partial}{\partial v_{\parallel}} f_{0e} \right]}_{=0} \\ &\quad - \left[ \mathbf{v}_E \cdot \nabla + \frac{e}{m_e} \left( \frac{\mathbf{v}_c}{v_{\parallel}} + \frac{\delta \mathbf{B}}{B_0} \right) \cdot \nabla \phi \frac{\partial}{\partial v_{\parallel}} \right] f_{0e} - \left[ \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla - \frac{\mu}{m_e} \frac{\mathbf{B}_0^*}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}} \right] \delta f_e^{(0)} \\ &\quad - \underbrace{\left[ \left( \nu_{\parallel} \frac{\delta \mathbf{B}}{B_0} + \mathbf{v}_E \right) \cdot \nabla - \frac{\mu}{m_e} \frac{\delta \mathbf{B}}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{\parallel}} + \frac{e}{m_e} \frac{\mathbf{B}_0^* + \delta \mathbf{B}}{B_0} \cdot \nabla \phi \frac{\partial}{\partial v_{\parallel}} + \frac{e}{m_e c} \frac{\partial A_{\parallel}}{\partial t} \frac{\partial}{\partial v_{\parallel}} \right] \delta f_e^{(0)}}_{\text{dropped}}. \end{aligned} \quad (0.115)$$

The choice of  $\delta f_e^{(0)}$  in Sec. 5.2.2 (Eq. (0.93)) makes the first line in Eq. (0.115) be,

$$-\left[ \frac{u_{\parallel 0} f_{0e}}{T_e} e \mathbf{b}_0 \cdot \nabla \phi_{\text{eff}} - \frac{u_{\parallel 0} f_{0e}}{T_e} \mu \frac{\delta \mathbf{B}}{B_0} \cdot \nabla B_0 \right]. \quad (0.116)$$

The  $\delta f_e^{(0)}$  part in Eq. (0.115),

$$\begin{aligned} & \left[ \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla - \frac{\mu}{m_e} \frac{\mathbf{B}_0^*}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{||}} \right] \delta f_e^{(0)} = f_{0e} \left[ L_0 - v_{||} \mathbf{b}_0 \cdot \nabla \right] \frac{\delta f_e^{(0)}}{f_{0e}} - \frac{\delta f_e^{(0)}}{f_{0e}} \left[ L_0 f_{0e} - \underbrace{v_{||} \mathbf{b}_0 \cdot \nabla f_{0e}}_{\text{dropped}} \right] \\ &= f_{0e} \left[ \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla - \underbrace{\frac{\mu}{m_e} \frac{\mathbf{B}_0^*}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{||}}}_{\text{for simplify, then dropped}} \right] \frac{\delta f_e^{(0)}}{f_{0e}} \approx f_{0e} \left[ \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla \right] \frac{\delta f_e^{(0)}}{f_{0e}}. \end{aligned} \quad (0.117)$$

The  $f_{0e}$  part in Eq. (0.115), with drift Maxwellian distribution,

$$\begin{aligned} & \left[ \mathbf{v}_E \cdot \nabla + \frac{e}{m_e} \frac{\mathbf{v}_c}{v_{||}} \cdot \nabla \phi \frac{\partial}{\partial v_{||}} \right] f_{0e} = \left[ \mathbf{v}_E \cdot \nabla + \frac{e}{m_e} \frac{\mathbf{v}_c}{v_{||}} \cdot \nabla \phi \frac{-m_e(v_{||} - u_{||0})}{T_e} \right] f_{0e} \\ &= \left[ \mathbf{v}_E \cdot \nabla - \frac{e}{T_e} \left( 1 - \frac{u_{||0}}{v_{||}} \right) \mathbf{v}_c \cdot \nabla \phi \right] f_{0e}. \end{aligned} \quad (0.118)$$

Or, for without parallel flow,

$$\begin{aligned} & \left[ \mathbf{v}_E \cdot \nabla + \frac{e}{m_e} \frac{\mathbf{v}_c}{v_{||}} \cdot \nabla \phi \frac{\partial}{\partial v_{||}} \right] f_{0e} = \left[ \mathbf{v}_E \cdot \nabla + \frac{e}{m_e} \frac{\mathbf{v}_c}{v_{||}} \cdot \nabla \phi \frac{-m_e v_{||}}{T_e} \right] f_{0e} = \left[ \mathbf{v}_E \cdot \nabla - \frac{e}{T_e} (\mathbf{v}_d - \mathbf{v}_g) \cdot \nabla \phi \right] f_{0e} \\ &= \left[ \mathbf{v}_E \cdot \nabla + \frac{e}{T_e} \left( \frac{\mu}{m_e \Omega_e} \mathbf{b}_0 \times \nabla B_0 \right) \cdot \nabla \phi - \frac{e}{T_e} \mathbf{v}_d \cdot \nabla \phi \right] f_{0e} \\ &= \left[ \mathbf{v}_E \cdot \nabla + \frac{c \mathbf{b}_0 \times \nabla \phi}{B_0} \cdot \left( -\mu \frac{\nabla B_0}{B_0} \right) \left( -\frac{B_0}{T_e} \right) - \frac{e}{T_e} \mathbf{v}_d \cdot \nabla \phi \right] f_{0e} = \left[ \mathbf{v}_E \cdot \left( \nabla + \nabla \mu \Big|_{v_{||}} \frac{\partial}{\partial \mu} \right) - \frac{e}{T_e} \mathbf{v}_d \cdot \nabla \phi \right] f_{0e} \\ &= \left[ \underbrace{\mathbf{v}_E \cdot \nabla \ln(f_{0e}) \Big|_{v_{||}}}_{\text{??}} - \frac{e}{T_e} \mathbf{v}_d \cdot \nabla \phi \right] f_{0e}. \end{aligned} \quad (0.119)$$

So, Eq. (0.115) simplifies to

$$\begin{aligned} L\delta h_e &= \mathbf{v}_E \cdot \nabla f_{0e} - \frac{ef_{0e}}{T_e} \left( 1 - \frac{u_{||0}}{v_{||}} \right) \mathbf{v}_c \cdot \nabla \phi + f_{0e} \left[ \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla \right] \frac{\delta f_e^{(0)}}{f_{0e}} \\ &\quad - \underbrace{\frac{u_{||0} f_{0e}}{T_e} e \mathbf{b}_0 \cdot \nabla \phi_{\text{eff}} + \frac{u_{||0} f_{0e}}{T_e} \mu \frac{\delta \mathbf{B}}{B_0} \cdot \nabla B_0}_{\text{parallel flow terms}} - \underbrace{f_{0e} \left[ \frac{\mu}{m_e} \frac{\mathbf{B}_0^*}{B_0} \cdot \nabla B_0 \frac{\partial}{\partial v_{||}} \right] \frac{\delta f_e^{(0)}}{f_{0e}}}_{\text{dropped}} \end{aligned} \quad (0.120)$$

If without parallel flow,

$$L\delta h_e = f_{0e} \left[ - \left( \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla \right) \frac{\delta f_e^{(0)}}{f_{0e}} - \mathbf{v}_E \cdot \nabla \ln(f_{0e}) \Big|_{v_{||}} + \frac{e}{T_e} \mathbf{v}_d \cdot \nabla \phi \right]. \quad (0.121)$$

Adding back the zonal terms, Eq. (0.120) is

$$L\delta h_e = f_{0e} \left\{ \begin{aligned} & \mathbf{v}_E \cdot \mathbf{k} - \frac{e}{T_e} \left[ \mathbf{v}_g + \left( 1 - \frac{u_{||0}}{v_{||}} \right) \mathbf{v}_c \right] \cdot \nabla \phi + \left[ \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla \right] \frac{\delta f_e^{(0)}}{f_{0e}} \\ & - \underbrace{\frac{c \mathbf{b}_0 \times \nabla \phi_{00}}{B_0} \cdot \nabla \frac{\delta f_e^{(0)}}{f_{0e}} + \frac{e(v_{||} - u_{||0})}{c T_e} \frac{\partial A_{||00}}{\partial t}}_{\text{zonal terms}} - \underbrace{\frac{u_{||0}}{T_e} e \mathbf{b}_0 \cdot \nabla \phi_{\text{eff}} + \frac{u_{||0}}{T_e} \mu \frac{\delta \mathbf{B}}{B_0} \cdot \nabla B_0}_{\text{parallel flow terms}} \end{aligned} \right\} \quad (0.122)$$

Eq. (0.121) is,

$$L\delta h_e = f_{0e} \left[ -\left( \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla \right) \frac{\delta f_e^{(0)}}{f_{0e}} - \mathbf{v}_E \cdot \nabla \ln(f_{0e}) \Big|_{v_\perp} + \frac{e}{T_e} \mathbf{v}_d \cdot \nabla \phi - \underbrace{\frac{c\mathbf{b}_0 \times \nabla \phi_{00}}{B_0} \cdot \nabla \frac{\delta f_e^{(0)}}{f_{0e}}}_{\text{zonal terms}} + \frac{ev_{||}}{cT_e} \frac{\partial A_{||00}}{\partial t} \right]. \quad (0.123)$$

Define  $w_e = \delta h_e / f_e$  and assume  $\delta f_e^{(0)} / f_{0e} \ll 1$ . Then, the  $\delta f$  formulation for electrons is ([Holod2009] Eq. (22)),

$$\begin{aligned} \frac{dw_e}{dt} &= \frac{1}{f_e} L\delta h_e = \frac{f_{0e}}{f_e} L\delta h_e = \left( 1 - \frac{\delta f_e^{(0)}}{f_e} - w_e \right) L\delta h_e \\ &\approx \left( 1 - \frac{\delta f_e^{(0)}}{f_{0e}} - w_e \right) \left[ -\left( \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla \right) \frac{\delta f_e^{(0)}}{f_{0e}} - \mathbf{v}_E \cdot \nabla \ln(f_{0e}) \Big|_{v_\perp} + \frac{e}{T_e} \mathbf{v}_d \cdot \nabla \phi - \underbrace{\frac{c\mathbf{b}_0 \times \nabla \phi_{00}}{B_0} \cdot \nabla \frac{\delta f_e^{(0)}}{f_{0e}}}_{\text{zonal terms}} + \frac{eu_{||}}{cT_e} \frac{\partial A_{||00}}{\partial t} \right]. \end{aligned} \quad (0.124)$$

While, we can also define  $w_e = \delta h_e / f_{0e}$ , then,

$$\begin{aligned} \frac{dw_e}{dt} &= L \left( \frac{\delta h_e}{f_{0e}} \right) = \frac{1}{f_{0e}} L\delta h_e - \frac{\delta h_e}{f_{0e}^2} Lf_{0e} = \frac{1}{f_{0e}} L\delta h_e + \frac{\delta h_e}{f_{0e}^2} (L\delta h_e + L\delta f_e^{(0)}) \\ &= \left[ 1 + \frac{\delta h_e}{f_{0e}} \left( 1 + \frac{L\delta f_e^{(0)}}{L\delta h_e} \right) \right] \frac{1}{f_{0e}} L\delta h_e \approx \left[ 1 + \frac{\delta h_e}{f_{0e}} \left( 1 + \frac{\delta f_e^{(0)}}{\delta h_e} \right) \right] \frac{1}{f_{0e}} L\delta h_e = \left[ 1 + \frac{\delta f_e^{(0)}}{f_{0e}} + w_e \right] \frac{1}{f_{0e}} L\delta h_e \\ &= \left( 1 + \frac{\delta f_e^{(0)}}{f_{0e}} + w_e \right) \left[ -\left( \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla \right) \frac{\delta f_e^{(0)}}{f_{0e}} - \mathbf{v}_E \cdot \nabla \ln(f_{0e}) \Big|_{v_\perp} + \frac{e}{T_e} \mathbf{v}_d \cdot \nabla \phi - \underbrace{\frac{c\mathbf{b}_0 \times \nabla \phi_{00}}{B_0} \cdot \nabla \frac{\delta f_e^{(0)}}{f_{0e}}}_{\text{zonal terms}} + \frac{eu_{||}}{cT_e} \frac{\partial A_{||00}}{\partial t} \right]. \end{aligned} \quad (0.125)$$

Where, using Eq. (0.100)

$$\frac{\delta f_e^{(0)}}{f_{0e}} = \frac{e}{T_e} \delta \phi_{\text{eff}} + \frac{\partial \ln f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta \psi + \frac{\partial \ln f_{0e}}{\partial \alpha_0} \Big|_{v_\perp} \delta \alpha, \quad (0.126)$$

And using Eq. (0.107),

$$\frac{\partial}{\partial t} \frac{\delta f_e^{(0)}}{f_{0e}} = \frac{e}{T_e} \frac{\partial \delta \phi_{\text{eff}}^{(0)}}{\partial t} + c \frac{\partial \ln f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \frac{\partial \delta \phi_{\text{ind}}}{\partial \alpha_0} + c \frac{\partial \ln f_{0e}}{\partial \alpha_0} \Big|_{v_\perp} \frac{\partial \delta \phi_{\text{ind}}}{\partial \psi_0}. \quad (0.127)$$

From Eq.(0.122), with parallel flow, Eqs. (0.124) and (0.125) change to,

$$\frac{dw_e}{dt} \approx \left( 1 - \frac{\delta f_e^{(0)}}{f_{0e}} - w_e \right) \left\{ \begin{aligned} &\mathbf{v}_E \cdot \mathbf{K} - \frac{e}{T_e} \left[ \mathbf{v}_g + \left( 1 - \frac{u_{||0}}{v_{||}} \right) \mathbf{v}_c \right] \cdot \nabla \phi + \left[ \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla \right] \frac{\delta f_e^{(0)}}{f_{0e}} \\ &- \underbrace{\frac{c\mathbf{b}_0 \times \nabla \phi_{00}}{B_0} \cdot \nabla \frac{\delta f_e^{(0)}}{f_{0e}}}_{\text{zonal terms}} + \frac{e(v_{||} - u_{||0})}{cT_e} \frac{\partial A_{||00}}{\partial t} - \underbrace{\frac{u_{||0}}{T_e} e \mathbf{b}_0 \cdot \nabla \phi_{\text{eff}} + \frac{u_{||0}}{T_e} \mu \frac{\partial \mathbf{B}}{B_0} \cdot \nabla B_0}_{\text{parallel flow terms}} \end{aligned} \right\}, \quad (0.128)$$

And

$$\frac{dw_e}{dt} \approx \left( 1 + \frac{\delta f_e^{(0)}}{f_{0e}} + w_e \right) \left\{ \begin{aligned} & \mathbf{v}_E \cdot \mathbf{k} - \frac{e}{T_e} \left[ \mathbf{v}_g + \left( 1 - \frac{u_{||0}}{v_{||}} \right) \mathbf{v}_c \right] \cdot \nabla \phi + \left[ \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla \right] \frac{\delta f_e^{(0)}}{f_{0e}} \\ & - \underbrace{\frac{c \mathbf{b}_0 \times \nabla \phi_{00}}{B_0} \cdot \nabla \frac{\delta f_e^{(0)}}{f_{0e}}}_{\text{zonal terms}} + \frac{e(v_{||} - u_{||0})}{c T_e} \frac{\partial A_{||00}}{\partial t} - \underbrace{\frac{u_{||0}}{T_e} e \mathbf{b}_0 \cdot \nabla \phi_{\text{eff}} + \frac{u_{||0}}{T_e} \mu \frac{\delta \mathbf{B}}{B_0} \cdot \nabla B_0}_{\text{parallel flow terms}} \end{aligned} \right\}. \quad (0.129)$$

In order to find the first order correction  $\delta\phi_{\text{eff}}^{(1)}$ , we integrate the perturbed distribution function using Eq. (0.100),

$$\frac{e \delta\phi_{\text{eff}}^{(1)}}{T_e} = - \frac{\delta n_e^{(1)}}{n_{0e}}, \quad (0.130)$$

where,

$$\delta n_e^{(1)} = \int d\mathbf{v} \delta h_e. \quad (0.131)$$

Equations (0.124)-(0.131) form a complete system for the first order correction of the nonadiabatic response. This procedure can be repeated to achieve higher accuracy. It was found that the second order accuracy is needed to recover the trapped electron response.

### 5.3.2 The equilibrium current terms

All the  $\nabla \times \mathbf{B}_0$  terms are retained in our formula. Also implemented is the parallel Ampere's law

$$\frac{c}{4\pi} \mathbf{b}_0 \cdot \nabla \times \mathbf{B}_0 = \sum_{\alpha=i,f} Z_\alpha n_{0\alpha} u_{||0\alpha} - e n_{0e} u_{||0e}, \quad (0.132)$$

being enforced on the equilibrium flows of all species for given  $\mathbf{B}_0$  from equilibrium solver such as EFIT.

Also from Eq. (0.132), the electron equilibrium flow is expressed as

$$u_{||0e} = \sum_{\alpha=i,f} \frac{Z_\alpha n_{0\alpha}}{e n_{0e}} u_{||0\alpha} - \frac{c}{4\pi e n_{0e}} \mathbf{b}_0 \cdot \nabla \times \mathbf{B}_0. \quad (0.133)$$

Substituting Eq. (0.133) into Eq. (0.77),

$$\begin{aligned} 0 = & \partial_t \delta n_e + \mathbf{B}_0 \cdot \nabla \left( \frac{n_{0e} \delta u_{||e}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_{0e}}{B_0} \right) - n_{0e} (\delta \mathbf{v}_{*e} + \mathbf{v}_E) \cdot \frac{\nabla B_0}{B_0} \\ & + \delta \mathbf{B} \cdot \nabla \left( \sum_{\alpha=i,f} \frac{Z_\alpha n_{0\alpha}}{e B_0} u_{||0\alpha} - \frac{c}{4\pi e B_0} \mathbf{b}_0 \cdot \nabla \times \mathbf{B}_0 \right) + \frac{c \nabla \times \mathbf{B}_0}{B_0^2} \cdot \left[ - \underbrace{\frac{\nabla \delta P_{||e}}{e}}_{(\text{I})} - \underbrace{\frac{(\delta P_{\perp e} - \delta P_{||e}) \nabla B_0}{e B_0}}_{(\text{II})} + n_{0e} \nabla \phi \right] \\ & + \delta \mathbf{B} \cdot \nabla \left( \frac{n_{0e} \delta u_{||e}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{\delta n_e}{B_0} \right) + \frac{c \delta n_e}{B_0^2} \mathbf{b}_0 \times \nabla B_0 \cdot \nabla \phi + \frac{c \delta n_e}{B_0^2} \nabla \times \mathbf{B}_0 \cdot \nabla \phi. \end{aligned} \quad (0.134)$$

Term (II) compared with term (I) is of order  $1/(k_{\perp} R_0) \ll 1$ , so it can be dropped.

### 5.3.3 Explanations to the neglected terms

(...)

## 6 Zonal Fields

Zonal flow is nonlinearly self-generated large scale  $\mathbf{E} \times \mathbf{B}$  flow with electrostatic potential having  $k_{\parallel} = 0$ .

As mentioned in Sec. 5.2.2, the adiabatic solution of the drift-kinetic equation (0.100) has no zonal components in it, since electron adiabatic only for the non-zonal components. The lowest order zonal response can be found from,

$$\frac{\partial \langle \delta f_e^{(0)} \rangle}{\partial t} = f_{0e} \frac{u_{\parallel e} e}{c T_e} \frac{\partial \langle A_{\parallel} \rangle}{\partial t}, \quad (0.135)$$

giving

$$\langle \delta f_e^{(0)} \rangle = f_{0e} \frac{u_{\parallel e} e}{c T_e} \langle A_{\parallel} \rangle, \quad (0.136)$$

where the flux-surface averaging is defined as

$$\langle A \rangle \equiv \frac{\int J d\theta d\zeta A}{\int J d\theta d\zeta}. \quad (0.137)$$

Since, zonal component is  $m=0$  and  $n=0$  part, with  $m, n$  the poloidal and toroidal mode number. Then, an equivalent definition is  $\langle A \rangle \equiv A_{00}$ .

Thus, the Ampere's law for zonal field can be

$$\langle \nabla_{\perp}^2 A_{\parallel} \rangle - \frac{1}{\delta_e^2} \langle A_{\parallel} \rangle = \frac{4\pi n_0}{c} \left( e \langle \delta u_{\parallel e}^{(1)} \rangle - Z_i \langle \delta u_{\parallel i} \rangle \right), \quad (0.138)$$

where  $n_0 \delta u_{\parallel e}^{(1)} = \int d\mathbf{v} v_{\parallel} \delta h_e$  and  $\delta_e = c / \omega_{pe}$  is the electron collisionless skin depth.

As we can see from Eq. (0.138), zonal currents on scale length larger than  $\delta_e$  are strongly reduced by the electron shielding, thus  $\nabla_{\perp}^2$  in Eq. (0.138) can be neglected for the ion scale turbulence since typically  $\rho_i > \delta_e$ . Note that there is no such screening for  $k_{\parallel} \neq 0$  component of  $A_{\parallel}$  since the electron response is dominantly adiabatic. Thus, the solution for the zonal field is ([Holod2009], Eq. (42))

$$\langle A_{\parallel} \rangle = \frac{4\pi n_0 c}{\omega_{pe}^2} \left( Z_i \langle \delta u_{\parallel i} \rangle - e \langle \delta u_{\parallel e}^{(1)} \rangle \right). \quad (0.139)$$

Or, with fast ions,

$$A_{\parallel 00} = \frac{4\pi c e}{\omega_{pe}^2} \left( \sum_{\alpha=i,f} \frac{Z_{\alpha} n_{0\alpha}}{e n_{0e}} \delta u_{\parallel \alpha 00} - \delta u_{\parallel e 00}^{(1)} \right). \quad (0.140)$$

The zonal component of the electrostatic potential can be found from the gyrokinetic Poisson equation, with using the Pade approximation,

$$\frac{n_i Z_i^2}{m_i \Omega_i^2} \nabla_{\perp}^2 \langle \phi \rangle = \underbrace{(1 - \rho_i^2 \nabla_{\perp}^2)}_{\text{from Pade approximation}} (e \langle \delta n_e \rangle - Z_i \langle \delta n_i \rangle). \quad (0.141)$$

Or, with fast ions,

$$\frac{c^2 (m_i n_{0i} + m_f n_{0f})}{B_0^2} \nabla_{\perp}^2 \phi_{00} = -(1 - \rho_i^2 \nabla_{\perp}^2) \sum_{\alpha=i,e,f} Z_{\alpha} \delta n_{\alpha 00}. \quad (0.142)$$

Here, Pade approximation (only used the lowest order) is from the Bessel function  $J_0(k_{\perp} \rho_i)$  related term, which comes from gyro-average.

$$\frac{1}{J_0^2(k_{\perp} \rho_i)} \approx \begin{cases} \frac{1}{1 - (k_{\perp} \rho_i)^2}, & k_{\perp} \rho_i \ll 1 \\ \frac{1}{\frac{2}{\pi k_{\perp} \rho_i} \cos^2 \left( k_{\perp} \rho_i - \frac{\pi}{4} \right)}, & k_{\perp} \rho_i \gg 1 \end{cases} \quad (0.143)$$

$$\approx 1 + (k_{\perp} \rho_i)^2 = 1 - \rho_i^2 \nabla_{\perp}^2.$$

In the long wavelength limit ( $k_{\perp} \rho_{i,f} \ll 1$ ) and high aspect ratio ( $R_0 / a \gg 1$ ) limit, Eq. (0.142) reduces to,

$$\frac{c^2 (m_i n_{0i} + m_f n_{0f})}{B_0^2} \nabla_{\perp}^2 \phi_{00} = - \sum_{\alpha=i,e,f} Z_{\alpha} \delta n_{\alpha 00}. \quad (0.144)$$

Alternatively, the general solution of the gyrokinetic Poisson equation (0.33) can be averaged over flux surface to obtain the zonal electrostatic potential.

## 7 Formulation in Magnetic Coordinates

For tokamak simulation, it is convenient to use the toroidal magnetic coordinate system  $(\psi, \theta, \zeta)$ , where  $\psi$  is a poloidal magnetic flux function,  $\theta$  is a poloidal angle, and  $\zeta$  is a toroidal angle. Straight field line condition

$$\frac{\mathbf{B}_0 \cdot \nabla \zeta}{\mathbf{B}_0 \cdot \nabla \theta} = q(\psi). \quad (0.145)$$

The covariant representation of the magnetic field in this system is

$$\mathbf{B}_0 = \delta \nabla \psi + I \nabla \theta + g \nabla \zeta, \quad (0.146)$$

the contravariant representation is

$$\mathbf{B}_0 = q \nabla \psi \times \nabla \theta - \nabla \psi \times \nabla \zeta, \quad (0.147)$$

and the Jacobian is

$$J^{-1} = \nabla \psi \cdot \nabla \theta \times \nabla \zeta = \frac{B_0^2}{gq + I}. \quad (0.148)$$

The radial component  $\delta$  is small and usually neglected. The curl of the field line then reads

$$\nabla \times \mathbf{B}_0 = g' \nabla \psi \times \nabla \zeta + (I' - \partial_\theta \delta) \nabla \psi \times \nabla \theta, \quad (0.149)$$

where the prime symbol ('') denotes the derivative with respect to  $\psi$ . The parallel component writes

$$\mathbf{b}_0 \cdot \nabla B_0 = B_0 \frac{g(I' - \partial_\theta \delta) - Ig'}{gq + I}. \quad (0.150)$$

The particle motion equations in the magnetic coordinate are,

$$\dot{\zeta} = \frac{v_{\parallel} B_0 (q + \rho_c I' + I \partial_\psi \lambda)}{D} - c \frac{I}{D} \left[ \frac{1}{Z_\alpha} \frac{\partial \epsilon}{\partial B_0} \frac{\partial B_0}{\partial \psi} + \frac{\partial \phi}{\partial \psi} \right], \quad (0.151)$$

$$\dot{\theta} = \frac{v_{\parallel} B_0 (1 - \rho_c g' - g \partial_\psi \lambda)}{D} + c \frac{g}{D} \left[ \frac{1}{Z_\alpha} \frac{\partial \epsilon}{\partial B_0} \frac{\partial B_0}{\partial \psi} + \frac{\partial \phi}{\partial \psi} \right], \quad (0.152)$$

$$\dot{\psi} = \frac{c}{Z_\alpha} \frac{\partial \epsilon}{\partial B_0} \left( \frac{I}{D} \frac{\partial B_0}{\partial \zeta} - \frac{g}{D} \frac{\partial B_0}{\partial \theta} \right) + \frac{cI}{D} \frac{\partial \phi}{\partial \zeta} - \frac{cg}{D} \frac{\partial \phi}{\partial \theta} + v_{\parallel} B_0 \left( \frac{g}{D} \frac{\partial \lambda}{\partial \theta} - \frac{I}{D} \frac{\partial \lambda}{\partial \zeta} \right), \quad (0.153)$$

$$\begin{aligned} \dot{\rho}_{\parallel} = & -c \frac{(1 - \rho_c g' - g \partial_\psi \lambda)}{D} \left[ \frac{1}{Z_\alpha} \frac{\partial \epsilon}{\partial B_0} \frac{\partial B_0}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \right] - c \frac{(q + \rho_c I' + I \partial_\psi \lambda)}{D} \left[ \frac{1}{Z_\alpha} \frac{\partial \epsilon}{\partial B_0} \frac{\partial B_0}{\partial \zeta} + \frac{\partial \phi}{\partial \zeta} \right] \\ & + c \frac{(I \partial_\zeta \lambda - g \partial_\theta \lambda)}{D} \left[ \frac{1}{Z_\alpha} \frac{\partial \epsilon}{\partial B_0} \frac{\partial B_0}{\partial \psi} + \frac{\partial \phi}{\partial \psi} \right] - \frac{\partial \lambda}{\partial t}, \end{aligned} \quad (0.154)$$

where,

$$D = gq + I + \rho_c (gI' - Ig'). \quad (0.155)$$

The radial derivatives of poloidal and toroidal currents are

$$I' = \frac{dI}{d\psi}, \quad g' = \frac{dg}{d\psi}. \quad (0.156)$$

The modified parallel canonical momentum is

$$\rho_c = \rho_{\parallel} + \lambda, \quad (0.157)$$

with

$$\rho_{\parallel} = \frac{v_{\parallel}}{\Omega_\alpha} \equiv \frac{m_\alpha c}{Z_\alpha B_0} v_{\parallel}. \quad (0.158)$$

For short notation we use,

$$\frac{\partial \epsilon}{\partial B_0} = \mu + \frac{Z_\alpha^2}{m_\alpha c^2} \rho_{\parallel}^2 B_0. \quad (0.159)$$

Keeping terms up to first order in the perturbations, Eq. (0.49) can be written as,

$$\begin{aligned} \frac{dw_\alpha}{dt} &= (1-w_\alpha) \left\{ -\left( v_\parallel \frac{\delta \mathbf{B}}{B_0} + \mathbf{v}_E \right) \cdot \mathbf{k} + \frac{Z_\alpha}{T_{0\alpha}} (v_\parallel - u_{\parallel 0\alpha}) E_\parallel - \frac{Z_\alpha}{T_{0\alpha}} \left[ \mathbf{v}_g + \left( 1 - \frac{u_{\parallel 0\alpha}}{v_\parallel} \right) \mathbf{v}_c \right] \cdot \nabla \phi \right\} \\ &= (1-w_\alpha) \left\{ -\frac{c}{B_0} \mathbf{b}_0 \times \nabla (\phi - v_\parallel \lambda B_0) \cdot \frac{\nabla f_{0\alpha}}{f_{0\alpha}} \Big|_{v_\perp} + \frac{Z_\alpha}{T_{0\alpha}} (v_\parallel - u_{\parallel 0\alpha}) E_\parallel - \frac{Z_\alpha}{T_{0\alpha}} \left[ \mathbf{v}_g + \left( 1 - \frac{u_{\parallel 0\alpha}}{v_\parallel} \right) \mathbf{v}_c \right] \cdot \nabla \phi \right\}, \end{aligned} \quad (0.160)$$

Assuming only radial dependence of the background density and temperature, the first term in the square brackets of Eq. (0.160) becomes,

$$\frac{c}{B_0} \mathbf{b}_0 \times \nabla (\phi - v_\parallel \lambda B_0) \cdot \frac{\nabla f_{0\alpha}}{f_{0\alpha}} \Big|_{v_\perp} = \frac{c}{B_0^2 J} \left[ I \frac{\partial}{\partial \zeta} - g \frac{\partial}{\partial \theta} \right] (\phi - v_\parallel \lambda B_0) \cdot \frac{1}{f_{0\alpha}} \frac{\partial f_{0\alpha}}{\partial \psi} \Big|_{v_\perp}. \quad (0.161)$$

The scalar product is

$$\begin{aligned} \left[ \mathbf{v}_g + \left( 1 - \frac{u_{\parallel 0\alpha}}{v_\parallel} \right) \mathbf{v}_c \right] \cdot \nabla \phi &= \left[ \frac{\mu}{m_\alpha \Omega_\alpha} \mathbf{b}_0 \times \nabla B_0 + \left( 1 - \frac{u_{\parallel 0\alpha}}{v_\parallel} \right) \frac{v_\parallel^2}{\Omega_\alpha} \left( \frac{1}{B_0} \underbrace{\nabla \times \mathbf{B}_0}_{\text{dropped}} - \frac{\nabla B_0}{B_0^2} \times \mathbf{B}_0 \right) \right] \cdot \nabla \phi \\ &= \frac{c}{Z_\alpha B_0^2} \left[ \mu + \left( 1 - \frac{u_{\parallel 0\alpha}}{v_\parallel} \right) \frac{m_\alpha v_\parallel^2}{B_0} \right] \mathbf{B}_0 \times \nabla B_0 \cdot \nabla \phi \\ &= \frac{c}{Z_\alpha B_0^2} \left[ \mu + \left( 1 - \frac{u_{\parallel 0\alpha}}{v_\parallel} \right) \frac{m_\alpha v_\parallel^2}{B_0} \right] \frac{1}{J} \left( g \frac{\partial B_0}{\partial \psi} \frac{\partial \phi}{\partial \theta} - I \frac{\partial B_0}{\partial \psi} \frac{\partial \phi}{\partial \zeta} - g \frac{\partial B_0}{\partial \theta} \frac{\partial \phi}{\partial \psi} \right), \end{aligned} \quad (0.162)$$

assuming  $\nabla \times \mathbf{B}_0 = 0$  and axisymmetry of the magnetic field.

The non-zonal component of parallel electric field is

$$\delta E_\parallel = -b_0 \cdot \nabla \phi_{\text{eff}} = -\frac{1}{B_0} \frac{1}{J} \left( \frac{\partial \phi_{\text{eff}}}{\partial \theta} + q \frac{\partial \phi_{\text{eff}}}{\partial \zeta} \right), \quad (0.163)$$

and the zonal component is

$$\langle E_\parallel \rangle = -\frac{1}{c} \frac{\partial \langle A_\parallel \rangle}{\partial t}. \quad (0.164)$$

The high-order electron drift-kinetic equation (0.124) reads,

$$\begin{aligned} \frac{dw_e}{dt} &= \left( 1 - \frac{\delta f_e^{(0)}}{f_{0e}} - w_e \right) \left[ -\left( \frac{\partial}{\partial t} + \mathbf{v}_d \cdot \nabla \right) \frac{\delta f_e^{(0)}}{f_{0e}} - \mathbf{v}_E \cdot \nabla \ln(f_{0e}) \Big|_{v_\perp} + \frac{e}{T_e} \mathbf{v}_d \cdot \nabla \phi \right. \\ &\quad \left. - \frac{\mathbf{c} \mathbf{b}_0 \times \nabla \phi_{00}}{B_0} \cdot \nabla \frac{\delta f_e^{(0)}}{f_{0e}} + \frac{e u_\parallel}{c T_e} \frac{\partial A_{\parallel 00}}{\partial t} \right] \\ &= \left( 1 - \frac{\delta f_e^{(0)}}{f_{0e}} - w_e \right) \left[ -\mathbf{v}_E \cdot \frac{\nabla f_{0e}}{f_{0e}} \Big|_{v_\perp} - \frac{\partial}{\partial t} \frac{\delta f_e^{(0)}}{f_{0e}} - \mathbf{v}_d \cdot \nabla \frac{\delta f_e^{(0)}}{f_{0e}} + \frac{e}{T_e} \mathbf{v}_d \cdot \nabla \phi \right. \\ &\quad \left. - \frac{\mathbf{c} \mathbf{b}_0 \times \nabla \phi_{00}}{B_0} \cdot \nabla \frac{\delta f_e^{(0)}}{f_{0e}} + \frac{e u_\parallel}{c T_e} \frac{\partial A_{\parallel 00}}{\partial t} \right] \\ &= . \end{aligned} \quad (0.165)$$

with the adiabatic solution

$$\frac{\delta f_e^{(0)}}{f_{0e}} = \frac{e}{T_e} \delta\phi_{\text{eff}}^{(0)} + \frac{1}{f_{0e}} \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta\psi, \quad (0.166)$$

and background inhomogeneity drive

$$\mathbf{v}_E \cdot \frac{\nabla f_{0e}}{f_{0e}} \Big|_{v_\perp} = \frac{c}{B_0^2 J} \left[ I \frac{\partial \phi}{\partial \zeta} - g \frac{\partial \phi}{\partial \theta} \right] \frac{1}{f_{0e}} \frac{\partial f_{0e}}{\partial \psi} \Big|_{v_\perp}. \quad (0.167)$$

Using Eq. (0.107), the time derivative becomes,

$$\frac{\partial}{\partial t} \frac{\delta f_e^{(0)}}{f_{0e}} = \frac{e}{T_e} \frac{\partial \delta\phi_{\text{eff}}^{(0)}}{\partial t} - \frac{c}{f_{0e}} \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \frac{1}{q} \frac{\partial \delta\phi_{\text{ind}}^{(0)}}{\partial \theta}. \quad (0.168)$$

The inductive potential and field-line perturbation convected by magnetic drift is

$$\begin{aligned} & \mathbf{v}_d \cdot \nabla \left( \frac{e}{T_e} \delta\phi_{\text{ind}}^{(0)} + \frac{1}{f_{0e}} \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta\psi \right) \\ &= \frac{c}{e B_0^2} \left[ \mu + \left( 1 - \frac{u_{||0e}}{v_{||}} \right) \frac{m_e v_{||}^2}{B_0} \right] \frac{1}{J} \left( I \frac{\partial B_0}{\partial \psi} \frac{\partial \phi}{\partial \zeta} + g \frac{\partial B_0}{\partial \theta} \frac{\partial \phi}{\partial \psi} - g \frac{\partial B_0}{\partial \psi} \frac{\partial \phi}{\partial \theta} \right) \left( \frac{e}{T_e} \delta\phi_{\text{ind}}^{(0)} + \frac{1}{f_{0e}} \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta\psi \right). \end{aligned} \quad (0.169)$$

Finally, the zonal flow convection terms are

$$\left[ \mathbf{v}_g + \left( 1 - \frac{u_{||0\alpha}}{v_{||}} \right) \mathbf{v}_c \right] \cdot \nabla \langle \phi \rangle = \frac{c}{e B_0^2} \left[ \mu + \left( 1 - \frac{u_{||0\alpha}}{v_{||}} \right) \frac{m_\alpha v_{||}^2}{B_0} \right] \frac{1}{J} g \frac{\partial B_0}{\partial \theta} \frac{\partial \langle \phi \rangle}{\partial \psi}, \quad (0.170)$$

and,

$$\mathbf{b}_0 \times \nabla \frac{\delta f_e^{(0)}}{f_{0e}} \cdot \nabla \phi_{00} = \frac{\partial \phi_{00}}{\partial \psi} \frac{1}{B_0 J} \left( I \frac{\partial}{\partial \zeta} - g \frac{\partial}{\partial \theta} \right) \times \left( \frac{e}{T_e} \delta\phi_{\text{eff}}^{(0)} + \frac{1}{f_{0e}} \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta\psi \right). \quad (0.171)$$

## 8 Summary of the Equations and Normalization

Here is the summary of the above equations we used in GTC and the basic units and normalization.

### 8.1 Units and normalization

Basic units:

- Time: inverse on-axis cyclotron frequency of proton  $\Omega_p^{-1} = m_p c / e B_a$ ,
- Length: tokamak major radius  $R_0$ ,
- Charge: proton charge  $e$ , and
- Mass: proton mass  $m_p$ .

Dimensionless number:  $\beta_e = 8\pi n_a T_e / B_0^2$ .

Normalizations ( $B_a$  is the on-axis equilibrium magnetic field and  $n_a$  is the on-axis equilibrium electron density):

Magnetic field

$$\hat{B} = B / B_a.$$

Density

$$\hat{n} = n / n_a.$$

Scalar potential

$$\hat{\phi} = \frac{e}{m_p R_0^2 \Omega_p^2} \phi.$$

Vector potential

$$\hat{A} = \frac{A}{B_a R_0}.$$

Current

$$\hat{I} = \frac{I}{B_a R_0}.$$

Poloidal flux function (same for  $\alpha$ )

$$\hat{\psi} = \frac{\psi}{B_a R_0^2}.$$

Velocity

$$\hat{v} = \frac{v}{R_0 \Omega_p}.$$

Temperature (energy)

$$\hat{T} = \frac{T}{m_p R_0^2 \Omega_p^2}.$$

Pressure

$$\hat{P} = \frac{P}{n_a m_p R_0^2 \Omega_p^2}.$$

## 8.2 Normalized ion equations

Normalized Eq. (0.160)

$$\frac{dw_\alpha}{dt} = (1 - w_\alpha) \left\{ -\frac{c}{B_0} \mathbf{b}_0 \times \nabla (\phi - v_{||} \lambda B_0) \cdot \frac{\nabla f_{0\alpha}}{f_{0\alpha}} \Big|_{v_\perp} + \frac{Z_\alpha}{T_{0\alpha}} (v_{||} - u_{||0\alpha}) E_{||} - \frac{Z_\alpha}{T_{0\alpha}} \left[ \mathbf{v}_g + \left( 1 - \frac{u_{||0\alpha}}{v_{||}} \right) \mathbf{v}_c \right] \cdot \nabla \phi \right\}, \quad (0.172)$$

where

$$\mathbf{b}_0 \times \nabla (\phi - v_{||} \lambda B_0) \cdot \frac{\nabla f_{0\alpha}}{f_{0\alpha}} \Big|_{v_\perp} = \frac{1}{B_0 J} \left[ I \frac{\partial}{\partial \zeta} - g \frac{\partial}{\partial \theta} \right] (\phi - v_{||} \lambda B_0) \cdot \frac{1}{f_{0\alpha}} \frac{\partial f_{0\alpha}}{\partial \psi} \Big|_{v_\perp}. \quad (0.173)$$

$$E_{||} = -\frac{1}{B_0} \frac{1}{J} \left( \frac{\partial \phi_{\text{eff}}}{\partial \theta} + q \frac{\partial \phi_{\text{eff}}}{\partial \zeta} \right) - \frac{1}{c} \frac{\partial \langle A_{||} \rangle}{\partial t}, \quad (0.174)$$

$$\left[ \mathbf{v}_g + \left( 1 - \frac{u_{||0\alpha}}{v_{||}} \right) \mathbf{v}_c \right] \cdot \nabla \phi = \frac{c}{Z_\alpha B_0^2} \left[ \mu + \left( 1 - \frac{u_{||0\alpha}}{v_{||}} \right) \frac{m_\alpha v_{||}^2}{B_0} \right] \frac{1}{J} \left( g \frac{\partial B_0}{\partial \psi} \frac{\partial \phi}{\partial \theta} - I \frac{\partial B_0}{\partial \psi} \frac{\partial \phi}{\partial \zeta} - g \frac{\partial B_0}{\partial \theta} \frac{\partial \phi}{\partial \psi} \right), \quad (0.175)$$

### 8.3 Normalized equations of motion

Equations of motion (0.151)-(0.154),

$$\dot{\zeta} = \frac{v_{||} B_0 (q + \rho_c I' + I \partial_\psi \lambda)}{D} - \frac{I}{D} \left[ \frac{1}{Z_\alpha} \frac{\partial \epsilon}{\partial B_0} \frac{\partial B_0}{\partial \psi} + \frac{\partial \phi}{\partial \psi} \right], \quad (0.176)$$

$$\dot{\theta} = \frac{v_{||} B_0 (1 - \rho_c g' - g \partial_\psi \lambda)}{D} + \frac{g}{D} \left[ \frac{1}{Z_\alpha} \frac{\partial \epsilon}{\partial B_0} \frac{\partial B_0}{\partial \psi} + \frac{\partial \phi}{\partial \psi} \right], \quad (0.177)$$

$$\dot{\psi} = \frac{1}{Z_\alpha} \frac{\partial \epsilon}{\partial B_0} \left( \frac{I}{D} \frac{\partial B_0}{\partial \zeta} - \frac{g}{D} \frac{\partial B_0}{\partial \theta} \right) + \frac{I}{D} \frac{\partial \phi}{\partial \zeta} - \frac{g}{D} \frac{\partial \phi}{\partial \theta} + v_{||} B_0 \left( \frac{g}{D} \frac{\partial \lambda}{\partial \theta} - \frac{I}{D} \frac{\partial \lambda}{\partial \zeta} \right), \quad (0.178)$$

$$\begin{aligned} \dot{\rho}_{||} = & -\frac{(1 - \rho_c g' - g \partial_\psi \lambda)}{D} \left[ \frac{1}{Z_\alpha} \frac{\partial \epsilon}{\partial B_0} \frac{\partial B_0}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \right] - \frac{(q + \rho_c I' + I \partial_\psi \lambda)}{D} \left[ \frac{1}{Z_\alpha} \frac{\partial \epsilon}{\partial B_0} \frac{\partial B_0}{\partial \zeta} + \frac{\partial \phi}{\partial \zeta} \right] \\ & + \frac{(I \partial_\zeta \lambda - g \partial_\theta \lambda)}{D} \left[ \frac{1}{Z_\alpha} \frac{\partial \epsilon}{\partial B_0} \frac{\partial B_0}{\partial \psi} + \frac{\partial \phi}{\partial \psi} \right] - \frac{\partial \lambda}{\partial t}, \end{aligned} \quad (0.179)$$

where,

$$D = gq + I + \rho_c (gI' - Ig'), \quad (0.180)$$

$$I' = \frac{dI}{d\psi}, \quad g' = \frac{dg}{d\psi}, \quad (0.181)$$

$$\rho_c = \rho_{||} + \lambda, \quad (0.182)$$

$$\rho_{||} = \frac{v_{||}}{\Omega_\alpha} \equiv \frac{m_\alpha}{Z_\alpha B_0} v_{||}, \quad (0.183)$$

$$\frac{\partial \epsilon}{\partial B_0} = \mu + \frac{Z_\alpha^2}{m_\alpha} \rho_{||}^2 B_0. \quad (0.184)$$

### 8.4 Normalized electron equations

Dimensionless continuity equation (0.77)

$$\begin{aligned}
0 = & \underbrace{\partial_t \delta n_e + B_0 \mathbf{b}_0 \cdot \nabla \left( \frac{n_{0e} \delta u_{||e}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_{0e}}{B_0} \right) - n_{0e} (\delta \mathbf{v}_{*e} + \mathbf{v}_E) \cdot \frac{\nabla B_0}{B_0}}_{[\text{Holod2009}] \text{ Eq. (10) terms}} \\
& + \underbrace{\delta \mathbf{B} \cdot \nabla \left( \frac{n_{0e} u_{||0e}}{B_0} \right) + \frac{\nabla \times \mathbf{B}_0}{B_0^2} \cdot \left[ -\frac{\nabla \delta P_{||e}}{e} - \frac{(\delta P_{\perp e} - \delta P_{||e}) \nabla B_0}{e B_0} + n_{0e} \nabla \phi \right]}_{\text{parallel equilibrium flow and finite } \nabla \times \mathbf{B}_0 \text{ terms}} \\
& + \underbrace{\delta \mathbf{B} \cdot \nabla \left( \frac{n_{0e} \delta u_{||e}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{\delta n_e}{B_0} \right) + \frac{\delta n_e}{B_0^2} \mathbf{b}_0 \times \nabla B_0 \cdot \nabla \phi + \frac{\delta n_e}{B_0^2} \nabla \times \mathbf{B}_0 \cdot \nabla \phi}_{\text{nonlinear terms}}
\end{aligned} \tag{0.185}$$

where,

$$\mathbf{v}_E = \frac{\mathbf{b}_0 \times \nabla \phi}{B_0}, \tag{0.186}$$

$$\delta \mathbf{v}_* = \frac{1}{n_0 B_0^2} \mathbf{b}_0 \times (\nabla \delta P_{\perp} + \nabla \delta P_{||}), \tag{0.187}$$

$$\delta P_{\perp e}^{(0)} = n_{0e} \delta \phi_{\text{eff}}^{(0)} + \frac{\partial (n_{0e} T_e)}{\partial \psi_0} \delta \psi, \tag{0.188}$$

$$\delta P_{||e}^{(0)} = n_{0e} \delta \phi_{\text{eff}}^{(0)} + \frac{\partial (n_{0e} T_e)}{\partial \psi_0} \delta \psi. \tag{0.189}$$

Inverse Ampere's law Eq. (0.78)

$$\delta u_{||e} = \frac{2T_{ea}}{\beta_{ea}} \frac{1}{n_{0e}} \nabla_{\perp}^2 \delta A_{||} + \sum_{\alpha=i,f} \frac{Z_{\alpha} n_{0\alpha}}{n_{0e}} \delta u_{||\alpha}, \tag{0.190}$$

where,  $\beta_{ea}$  is the on-axis value of  $\beta_e$  and  $T_{ea}$  is the normalized on-axis electron temperature.

High order electron drift-kinetic equation

$$\begin{aligned}
\frac{dw_e}{dt} = & \left( 1 - \frac{\delta f_e^{(0)}}{f_{0e}} - w_e \right) \left[ \begin{aligned} & -\mathbf{v}_E \cdot \frac{\nabla f_{0e}}{f_{0e}} \Big|_{v_{\perp}} - \frac{\partial}{\partial t} \frac{\delta f_e^{(0)}}{f_{0e}} - \mathbf{v}_d \cdot \nabla \frac{\delta f_e^{(0)}}{f_{0e}} + \frac{e}{T_e} \mathbf{v}_d \cdot \nabla \phi \\ & - \frac{c \mathbf{b}_0 \times \nabla \phi_{00}}{B_0} \cdot \nabla \frac{\delta f_e^{(0)}}{f_{0e}} + \frac{eu_{||}}{cT_e} \frac{\partial A_{||00}}{\partial t} \end{aligned} \right] \\
= .
\end{aligned} \tag{0.191}$$

with

$$\frac{\delta f_e^{(0)}}{f_{0e}} = \frac{1}{T_e} \delta \phi_{\text{eff}}^{(0)} + \frac{1}{f_{0e}} \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_{\perp}} \delta \psi, \tag{0.192}$$

$$\mathbf{v}_E \cdot \frac{\nabla f_{0e}}{f_{0e}} \Big|_{v_{\perp}} = \frac{1}{B_0^2 J} \left[ I \frac{\partial \phi}{\partial \zeta} - g \frac{\partial \phi}{\partial \theta} \right] \frac{1}{f_{0e}} \frac{\partial f_{0e}}{\partial \psi} \Big|_{v_{\perp}}, \tag{0.193}$$

$$\frac{\partial}{\partial t} \frac{\delta f_e^{(0)}}{f_{0e}} = \frac{1}{T_e} \frac{\partial \delta \phi_{\text{eff}}^{(0)}}{\partial t} - \frac{1}{f_{0e}} \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \frac{1}{q} \frac{\partial \delta \phi_{\text{ind}}^{(0)}}{\partial \theta}, \quad (0.194)$$

$$\left[ \mathbf{v}_g + \left( 1 - \frac{u_{||0\alpha}}{v_{||}} \right) \mathbf{v}_c \right] \cdot \nabla \langle \phi \rangle = \frac{1}{B_0^2 J} \left[ \mu + \left( 1 - \frac{u_{||0\alpha}}{v_{||}} \right) \frac{m_\alpha v_{||}^2}{B_0} \right] g \frac{\partial B_0}{\partial \theta} \frac{\partial \langle \phi \rangle}{\partial \psi}, \quad (0.195)$$

$$\mathbf{b}_0 \times \nabla \frac{\delta f_e^{(0)}}{f_{0e}} \cdot \nabla \phi_{00} = \frac{\partial \phi_{00}}{\partial \psi} \frac{1}{B_0 J} \left( I \frac{\partial}{\partial \zeta} - g \frac{\partial}{\partial \theta} \right) \times \left( \frac{\delta \phi_{\text{eff}}^{(0)}}{T_e} + \frac{1}{f_{0e}} \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta \psi \right), \quad (0.196)$$

$$\begin{aligned} & \mathbf{v}_d \cdot \nabla \left( \frac{e}{T_e} \delta \phi_{\text{ind}}^{(0)} + \frac{1}{f_{0e}} \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta \psi \right) \\ &= \frac{1}{B_0^2 J} \left[ \mu + \left( 1 - \frac{u_{||0e}}{v_{||}} \right) \frac{m_e v_{||}^2}{B_0} \right] \left( I \frac{\partial B_0}{\partial \psi} \frac{\partial \phi}{\partial \zeta} + g \frac{\partial B_0}{\partial \theta} \frac{\partial \phi}{\partial \psi} - g \frac{\partial B_0}{\partial \psi} \frac{\partial \phi}{\partial \theta} \right) \left( \frac{1}{T_e} \delta \phi_{\text{ind}}^{(0)} + \frac{1}{f_{0e}} \frac{\partial f_{0e}}{\partial \psi_0} \Big|_{v_\perp} \delta \psi \right). \end{aligned} \quad (0.197)$$

## 8.5 Normalized field equations

Inductive potential Eq. (0.83)

$$\delta \phi_{\text{ind}} = \delta \phi_{\text{eff}} - \delta \phi. \quad (0.198)$$

Evolution equation for the vector potential (0.81)

$$\partial_t \delta A_{||} = \mathbf{b}_0 \cdot \nabla \delta \phi_{\text{ind}}. \quad (0.199)$$

Perturbed magnetic field label evolution (0.107)

$$\frac{\partial \delta \psi}{\partial t} = - \frac{\partial \delta \phi_{\text{ind}}}{\partial \alpha_0}. \quad (0.200)$$

Adiabatic response equation (0.102)

$$\frac{\delta \phi_{\text{eff}}^{(0)}}{T_e} = \frac{\delta n_e}{n_{0e}} - \frac{\delta \psi}{n_{0e}} \frac{\partial n_{0e}}{\partial \psi_0}. \quad (0.201)$$

Gyrokinetic Poisson equation (0.33)

$$\frac{Z_i^2 n_i}{T_i} (\phi - \tilde{\phi}) = Z_i n_i - e n_e. \quad (0.202)$$

Poisson equation for zonal component (0.142)

$$\frac{(m_i n_{0i} + m_f n_{0f})}{B_0^2} \nabla_\perp^2 \phi_{00} = - (1 - \rho_i^2 \nabla_\perp^2) \sum_{\alpha=i,e,f} Z_\alpha \delta n_{\alpha 00}, \quad (0.203)$$

where  $\rho_i^2 = m_i T_i / (Z_i^2 B_0^2)$

Zonal component of vector potential (0.140)

$$A_{||00} = \frac{4\pi ce}{\omega_{pe}^2} \left( \sum_{\alpha=i,f} \frac{Z_\alpha n_{0\alpha}}{n_{0e}} \delta u_{||\alpha 00} - \delta u_{||e 00}^{(1)} \right). \quad (0.204)$$

## 9 Reduction to Ideal MHD Limit

Here, we prove that in the linear and long wavelength limit of  $k_\perp \rho_{i,f} \ll 1$ , the above gyrokinetic formulation reduces to the ideal MHD limit eigenmode equation.

### 9.1 Linear ideal MHD theory with equilibrium current

Linearized Momentum equation

$$(n_{0i}m_i + n_{0f}m_f)(\partial_t + \mathbf{v}_0 \cdot \nabla) \delta \mathbf{v} + \nabla \cdot \delta \mathbb{P} = \frac{1}{c} (\delta \mathbf{J}_\perp \times \mathbf{B}_0 + \mathbf{J}_0 \times \delta \mathbf{B}), \quad (0.205)$$

where the leading order of the equilibrium velocity is the ion diamagnetic velocity

$$\mathbf{v}_0 = \mathbf{v}_* = \frac{n_{0i}m_i \mathbf{v}_{*i} + n_{0f}m_f \mathbf{v}_{*f}}{n_{0i}m_i + n_{0f}m_f}, \quad (0.206)$$

with  $\mathbf{v}_{*\alpha}$  defined as

$$\mathbf{v}_{*\alpha} \equiv \frac{c \mathbf{b}_0 \times \nabla P_{0\alpha}}{Z_\alpha B_0 n_{0\alpha}}. \quad (0.207)$$

The leading order of the perturbed velocity is the  $\mathbf{E} \times \mathbf{B}$  drift

$$\delta \mathbf{v}_\perp = \frac{c \mathbf{b}_0 \times \nabla \delta \phi}{B_0}. \quad (0.208)$$

In Eq. (0.205), the  $\delta \mathbf{v} \cdot \nabla \mathbf{v}_0$  term is dropped due to being higher order compared with  $\mathbf{v}_0 \cdot \nabla \delta \mathbf{v}$ .

In the following derivation, linear normal mode substitution  $\partial_t \rightarrow -i\omega$ ,  $\mathbf{b}_0 \cdot \nabla \rightarrow ik_{||}$  will be applied. By taking

$\mathbf{B}_0 \times$  of Eq. (0.205),

$$\frac{1}{c} \mathbf{B}_0 \times (\delta \mathbf{J}_\perp \times \mathbf{B}_0 + \mathbf{J}_0 \times \delta \mathbf{B}) = (n_{0i}m_i + n_{0f}m_f) \mathbf{B}_0 \times (\partial_t + \mathbf{v}_* \cdot \nabla) \frac{c \mathbf{b}_0 \times \nabla \delta \phi}{B_0} + \mathbf{B}_0 \times \nabla \cdot \delta \mathbb{P}, \quad (0.209)$$

i.e.,

$$\begin{aligned}
\delta \mathbf{J}_\perp &= \frac{c(n_{0i}m_i + n_{0f}m_f)}{-B_0^2} \mathbf{B}_0 \times (\partial_t + \mathbf{v}_* \cdot \nabla) \frac{c\mathbf{b}_0 \times \nabla \delta\phi}{B_0} - \frac{\mathbf{B}_0 \times \mathbf{J}_0 \times \delta \mathbf{B}}{-B_0^2} + \frac{c\mathbf{B}_0}{-B_0^2} \times \nabla \cdot \delta \mathbb{P} \\
&= \frac{i(\omega - \omega_{*P})c^2}{4\pi v_A^2} \mathbf{B}_0 \times \frac{\mathbf{b}_0 \times \nabla \delta\phi}{B_0} + \frac{\mathbf{B}_0 \times \mathbf{J}_0 \times \delta \mathbf{B}}{B_0^2} - \frac{c\mathbf{B}_0}{B_0^2} \times \nabla \cdot \delta \mathbb{P} \\
&= \frac{i(\omega - \omega_{*P})c^2}{4\pi v_A^2} \nabla_\perp \delta\phi + \frac{\mathbf{b}_0}{B_0} \times J_{||0} \mathbf{b}_0 \times \delta \mathbf{B}_\perp - \frac{c\mathbf{b}_0}{B_0} \times \nabla \cdot \delta \mathbb{P}
\end{aligned} \tag{0.210}$$

where,

$$\mathbf{B}_0 \times \delta \mathbf{J}_\perp \times \mathbf{B}_0 = \underbrace{(\mathbf{B}_0 \cdot \delta \mathbf{J}_\perp) \mathbf{B}_0}_{=0} - (\mathbf{B}_0 \cdot \mathbf{B}_0) \delta \mathbf{J}_\perp = -B_0^2 \delta \mathbf{J}_\perp, \tag{0.211}$$

is used. And, with,

$$v_A^2 = \frac{cB_0^2}{4\pi(n_{0i}m_i + n_{0f}m_f)}, \tag{0.212}$$

$$\omega_{*P} = -i\mathbf{v}_* \cdot \nabla. \tag{0.213}$$

In MHD ( $\delta \mathbf{E}_{||} = 0$ ) and low-beta ( $\delta \mathbf{B}_{||} = 0$ ) limit, we have

$$\partial_t \delta A_{||} = -c\mathbf{b}_0 \cdot \nabla \delta\phi, \quad (\text{i.e., } \delta A_{||} = ck_{||} \delta\phi / \omega) \tag{0.214}$$

$$\delta \mathbf{B}_\perp = \nabla \times (\delta A_{||} \mathbf{b}_0) = \underbrace{\delta A_{||} \nabla \times \mathbf{b}_0}_{\text{dropped}} + \nabla \delta A_{||} \times \mathbf{b}_0 \approx \nabla \delta A_{||} \times \mathbf{b}_0 = \nabla (ck_{||} \delta\phi / \omega) \times \mathbf{b}_0. \tag{0.215}$$

By plugging the parallel Ampere's law

$$\delta J_{||} = \frac{c}{4\pi} \mathbf{b}_0 \cdot \nabla \times \delta \mathbf{B}_\perp = \frac{c}{4\pi} \mathbf{b}_0 \cdot \nabla \times (\nabla (ck_{||} \delta\phi / \omega) \times \mathbf{b}_0) = \frac{c^2}{4\pi \omega} \nabla_\perp^2 (k_{||} \delta\phi), \tag{0.216}$$

$$J_{||0} = \frac{c}{4\pi} \mathbf{b}_0 \cdot \nabla \times \mathbf{B}_0, \tag{0.217}$$

and Eq. (0.210) into quasi-neutrally and charge conservation equation, we obtain

$$\begin{aligned}
0 &= \nabla \cdot \delta \mathbf{J} = \nabla \cdot \delta \mathbf{J}_\perp + \nabla \cdot \delta \mathbf{J}_\parallel \\
&= \nabla \cdot \left( \frac{i(\omega - \omega_{*P})c^2}{4\pi} \frac{1}{v_A^2} \nabla_\perp \delta\phi \right) + \nabla \cdot \left( \frac{\mathbf{b}_0}{B_0} \times J_{\parallel 0} \mathbf{b}_0 \times \delta \mathbf{B}_\perp \right) - \nabla \cdot \left( \frac{c\mathbf{b}_0}{B_0} \times \nabla \cdot \delta \mathbb{P} \right) + \nabla \cdot \left( \frac{\delta J_\parallel}{B_0} \mathbf{B}_0 \right) \\
&= \frac{i(\omega - \omega_{*P})c^2}{4\pi} \nabla \cdot \left( \frac{1}{v_A^2} \nabla_\perp \delta\phi \right) + \nabla \cdot \left( \frac{J_{\parallel 0}}{B_0} \delta \mathbf{B}_\perp \right) - c \nabla \cdot \left( \frac{\mathbf{b}_0}{B_0} \times \nabla \cdot \delta \mathbb{P} \right) + \nabla \cdot \left( \frac{\delta J_\parallel}{B_0} \mathbf{B}_0 \right) \\
&= \frac{i(\omega - \omega_{*P})c^2}{4\pi} \nabla \cdot \left( \frac{1}{v_A^2} \nabla_\perp \delta\phi \right) - c \nabla \cdot \left( \frac{\mathbf{b}_0}{B_0} \times \nabla \cdot \delta \mathbb{P} \right) + \delta \mathbf{B}_\perp \cdot \nabla \left( \frac{J_{\parallel 0}}{B_0} \right) + \mathbf{B}_0 \cdot \nabla \left( \frac{\delta J_\parallel}{B_0} \right) \\
&\quad + \underbrace{\frac{J_{\parallel 0}}{B_0} \nabla \cdot (\delta \mathbf{B}_\perp) + \frac{\delta J_\parallel}{B_0} \nabla \cdot \mathbf{B}_0}_{\text{dropped}} \\
&= \frac{i(\omega - \omega_{*P})c^2}{4\pi} \nabla \cdot \left( \frac{1}{v_A^2} \nabla_\perp \delta\phi \right) - c \nabla \cdot \left( \frac{\mathbf{b}_0}{B_0} \times \nabla \cdot \delta \mathbb{P} \right) \\
&\quad + \nabla (ck_\parallel \delta\phi / \omega) \times \mathbf{b}_0 \cdot \nabla \left( \frac{c}{4\pi} \frac{\mathbf{b}_0 \cdot \nabla \times \mathbf{B}_0}{B_0} \right) + \mathbf{B}_0 \cdot \nabla \left( \frac{\frac{c^2}{4\pi\omega} \nabla_\perp^2 (k_\parallel \delta\phi)}{B_0} \right), \tag{0.218}
\end{aligned}$$

i.e.,

$$\begin{aligned}
0 &= \frac{i(\omega - \omega_{*P})c^2}{4\pi} \nabla \cdot \left( \frac{1}{v_A^2} \nabla_\perp \delta\phi \right) - c \nabla \cdot \left( \frac{\mathbf{b}_0}{B_0} \times \nabla \cdot \delta \mathbb{P} \right) \\
&\quad + \frac{c^2}{4\pi\omega} \nabla (k_\parallel \delta\phi) \times \mathbf{b}_0 \cdot \nabla \left( \frac{\mathbf{b}_0 \cdot \nabla \times \mathbf{B}_0}{B_0} \right) + \frac{c^2}{4\pi\omega} \mathbf{B}_0 \cdot \nabla \left( \frac{\nabla_\perp^2 (k_\parallel \delta\phi)}{B_0} \right). \tag{0.219}
\end{aligned}$$

In the  $L_{v_A} \gg L_{\delta\phi}$  limit,

$$\nabla \cdot \left( \frac{1}{v_A^2} \nabla_\perp \delta\phi \right) \approx \frac{1}{v_A^2} \nabla_\perp^2 \delta\phi. \tag{0.220}$$

Rewriting Eq. (0.219) to get the ideal MHD equation with equilibrium current

$$\begin{aligned}
&\omega(\omega - \omega_{*P}) \nabla \cdot \left( \frac{1}{v_A^2} \nabla_\perp \delta\phi \right) - i \mathbf{B}_0 \cdot \nabla \left( \frac{\nabla_\perp^2 (k_\parallel \delta\phi)}{B_0} \right) \\
&- i \nabla (k_\parallel \delta\phi) \times \mathbf{b}_0 \cdot \nabla \left( \frac{\mathbf{b}_0 \cdot \nabla \times \mathbf{B}_0}{B_0} \right) + i \frac{4\pi\omega}{c} \nabla \cdot \left( \frac{\mathbf{b}_0}{B_0} \times \nabla \cdot \delta \mathbb{P} \right) = 0. \tag{0.221}
\end{aligned}$$

## 9.2 Reduction of gyrokinetic Poisson equation and Ampere's law

In the definition of Eq. (0.37), there is a gyro-averaging on  $\delta\phi$ , which gives rise to a Bessel function operator

$$J_0(-i\rho \nabla_\perp),$$

$$\begin{aligned}
\langle \delta\phi(\mathbf{x}) \rangle_c(\mathbf{X}) &= \int \frac{d\vartheta_c}{2\pi} d\mathbf{x} \delta(\mathbf{X} + \mathbf{p} - \mathbf{x}) \delta\phi(\mathbf{x}) = \int \frac{d\vartheta_c}{2\pi} d\mathbf{x} \delta(\mathbf{X} + \mathbf{p} - \mathbf{x}) \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= \int \frac{d\vartheta_c}{2\pi} d\mathbf{x} \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{X} + \mathbf{p})} = \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{X}} \int \frac{d\vartheta_c}{2\pi} d\mathbf{x} e^{i\mathbf{k}\cdot\mathbf{p}} \\
&= \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{X}} \underbrace{\oint \frac{d\vartheta_c}{2\pi} d\mathbf{x} e^{ik_{\perp}\rho \cos \vartheta_c}}_{J_0(k_{\perp}\rho)} = J_0(-i\rho\nabla_{\perp}) \delta\phi(\mathbf{X}).
\end{aligned} \tag{0.222}$$

The integral  $\int_{\mathbf{X} \rightarrow \mathbf{x}} d\mathbf{v}$  in Eq. (0.37) has two parts as can be seen from Eq. (0.40). The first part, which is over the gyro-center velocity space, is the same as  $\int_{GC} d\mathbf{v}$  defined in Eq. (0.43). The second part of the integral, which is the transformation between the gyro-center coordinates and the particle coordinates, gives rise to another Bessel function operator  $J_0(-i\rho\nabla_{\perp})$ ,

$$\begin{aligned}
\delta\tilde{\phi} &= \frac{1}{n_0} \int \frac{2\pi B_0}{m} d\mu dv_{\parallel} \int \frac{d\vartheta_c}{2\pi} d\mathbf{X} \delta(\mathbf{X} + \mathbf{p} - \mathbf{x}) f_0 \langle \phi \rangle_c \\
&= \frac{1}{n_0} \int \frac{2\pi B_0}{m} d\mu dv_{\parallel} \int \frac{d\vartheta_c}{2\pi} d\mathbf{X} \delta(\mathbf{X} + \mathbf{p} - \mathbf{x}) f_0 \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{X}} J_0(k_{\perp}\rho) \\
&= \frac{1}{n_0} \int \frac{2\pi B_0}{m} d\mu dv_{\parallel} \int \frac{d\vartheta_c}{2\pi} d\mathbf{X} f_0 \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{p})} J_0(k_{\perp}\rho) \\
&= \frac{1}{n_0} \int \frac{2\pi B_0}{m} f_0 d\mu dv_{\parallel} \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \underbrace{\oint \frac{d\vartheta_c}{2\pi} d\mathbf{x} e^{-ik_{\perp}\rho \cos \vartheta_c}}_{J_0(k_{\perp}\rho)} J_0(k_{\perp}\rho) \\
&= \frac{1}{n_0} \int \frac{2\pi B_0}{m} f_0 d\mu dv_{\parallel} \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \underbrace{\oint \frac{d\vartheta_c}{2\pi} d\mathbf{x} e^{-ik_{\perp}\rho \cos \vartheta_c}}_{J_0(k_{\perp}\rho)} J_0(k_{\perp}\rho) \\
&= \frac{1}{n_0} \int \frac{2\pi B_0}{m} f_0 d\mu dv_{\parallel} J_0^2(-i\rho\nabla_{\perp}) \delta\phi(\mathbf{x}).
\end{aligned} \tag{0.223}$$

Here typical gyrokinetic scaling  $k_{\perp}L_{n0} \gg 1$  is used, so the effect of coordinate transformation on  $f_0$  is ignored. In GTC, the  $J_0(-i\rho\nabla_{\perp})$  is implemented accurately in the charge scattering from each particle's gyro-center to its gyro-orbit when collecting charges and current from the particles (N-points average method).

In the long wavelength limit,

$$J_0^2(-i\rho_{\alpha}\nabla_{\perp}) = 1 + \rho_{\alpha}^2 \nabla_{\perp}^2 + O(k_{\perp}^4 \rho_{\alpha}^4) \approx 1 + \frac{\mu m_{\alpha}^2 c^2}{Z_{\alpha}^2 B_0} \nabla_{\perp}^2. \tag{0.224}$$

Then  $\delta\tilde{\phi}_{\alpha}$  becomes,

$$\delta\tilde{\phi}_{\alpha} \approx \delta\phi + \frac{m_{\alpha} T_{\perp\alpha} c^2}{Z_{\alpha}^2 B_0^2} \nabla_{\perp}^2 \delta\phi \tag{0.225}$$

Eq. (0.34) reduces to

$$\sum_{\alpha=i,e,f} Z_\alpha \delta n_\alpha = \frac{Z_i^2 n_i}{T_{\perp i}} (\phi - \tilde{\phi}_i) + \frac{Z_f^2 n_f}{T_{\perp f}} (\phi - \tilde{\phi}_f) \approx -\frac{(n_{0i} m_i + n_{0f} m_f) c^2}{B_0^2} \nabla_\perp^2 \delta \phi = -\frac{c^2}{4\pi v_A^2} \nabla_\perp^2 \delta \phi. \quad (0.226)$$

Combining Eqs. (0.226), (0.36) and (0.214),

$$\begin{aligned} -\frac{4\pi\omega^2}{c^2} \sum_{\alpha=i,e,f} Z_\alpha \delta n_\alpha &= \frac{\omega^2}{v_A^2} \nabla_\perp^2 \delta \phi. \\ \delta A_\parallel &= ck_\parallel \delta \phi / \omega \\ i\omega \frac{4\pi}{c^2} \sum_{\alpha=i,f} \delta J_{\parallel\alpha} &= -i\omega \frac{4\pi}{c^2} \left( \frac{c}{4\pi} \nabla_\perp^2 A_\parallel \right). \end{aligned} \quad (0.227)$$

### 9.3 Reduction of ion gyrokinetic equation

To obtain an equation describing  $\delta n_\alpha$  and  $\delta J_{\parallel\alpha}$  for both ion species ( $\alpha = i, f$ ), we operate  $\int_{\mathbf{x} \rightarrow \mathbf{x}} d\mathbf{v}$  on the gyrokinetic Eq. (0.14). Similar,

$$\int \frac{d\mathcal{G}_c}{2\pi} d\mathbf{X} \delta(\mathbf{X} + \mathbf{p} - \mathbf{x}) \langle \delta \mathbf{B} \rangle_c = J_0^2 (-i\rho_\alpha \nabla_\perp) \delta \mathbf{B} \approx \left( 1 + \frac{\mu m_\alpha^2 c^2}{Z_\alpha^2 B_0} \nabla_\perp^2 \right) \delta \mathbf{B}. \quad (0.228)$$

$$\int \frac{d\mathcal{G}_c}{2\pi} d\mathbf{X} \delta(\mathbf{X} + \mathbf{p} - \mathbf{x}) \langle \delta A_\parallel \rangle_c = J_0^2 (-i\rho_\alpha \nabla_\perp) \delta A_\parallel \approx \left( 1 + \frac{\mu m_\alpha^2 c^2}{Z_\alpha^2 B_0} \nabla_\perp^2 \right) \delta A_\parallel. \quad (0.229)$$

We integrate the gyrokinetic equation in the linear limit,

$$\begin{aligned} 0 &= \int_{\mathbf{x} \rightarrow \mathbf{x}} d\mathbf{v} \left( \frac{\partial}{\partial t} + \dot{\mathbf{X}} \cdot \nabla + \dot{v}_\parallel \frac{\partial}{\partial v_\parallel} \right) (f_{0\alpha} + \delta f_\alpha) \\ &= \mathbf{B}_0 \cdot \nabla \left( \frac{n_{0\alpha} u_{\parallel 0\alpha}}{B_0} \right) + \frac{c \nabla \times \mathbf{b}_0}{Z_\alpha} \cdot \nabla \left( \frac{P_{\parallel 0\alpha}}{B_0} \right) + \frac{c \mathbf{b}_0 \times \nabla B_0}{Z_\alpha} \cdot \nabla \left( \frac{P_{\perp 0\alpha}}{B_0^2} \right) + \frac{c \nabla \times \mathbf{b}_0 \cdot \nabla B_0}{Z_\alpha B_0^2} P_{\perp 0\alpha} \\ &\quad + \partial_t \delta n_\alpha + \delta \mathbf{B} \cdot \nabla \left( \frac{n_{0\alpha} u_{\parallel 0\alpha}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_{0\alpha}}{B_0} \right) + \\ &\quad \mathbf{B}_0 \cdot \nabla \left( \frac{n_0 \delta u_{\parallel\alpha}}{B_0} \right) + \frac{c \nabla \times \mathbf{b}_0}{Z_\alpha} \cdot \nabla \left( \frac{\delta P_{\parallel\alpha}}{B_0} \right) + \frac{c \mathbf{b}_0 \times \nabla B_0}{Z_\alpha} \cdot \nabla \left( \frac{\delta P_{\perp\alpha}}{B_0^2} \right) + \frac{c \nabla \times \mathbf{b}_0 \cdot \nabla B_0}{Z_\alpha B_0^2} \delta P_{\perp\alpha} + \frac{c \nabla \times \mathbf{b}_0}{B_0} \cdot n_{0\alpha} \nabla \phi \\ &\quad + \frac{m_\alpha c^2}{Z_\alpha^2 B_0} (\nabla_\perp^2 \delta \mathbf{B}) \cdot \nabla \left( \frac{q_{\perp\alpha}}{B_0^2} \right) - \frac{m_\alpha c^3 \mathbf{b}_0 \times \nabla P_{\perp 0\alpha}}{Z_\alpha^2 B_0^2} \cdot \nabla \left( \frac{\nabla_\perp^2 \delta \phi}{B_0^2} \right) \\ &\quad + \frac{m_\alpha c^3 P_{\perp 0\alpha} (3\mathbf{b}_0 \times \nabla B_0 + \nabla \times \mathbf{B}_0)}{Z_\alpha^2 B_0^2} \cdot \nabla \left( \frac{\nabla_\perp^2 \delta \phi}{B_0} \right) \\ &\quad + \underbrace{\delta \mathbf{B} \cdot \nabla \left( \frac{n_0 \delta u_{\parallel\alpha}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{\delta n_\alpha}{B_0} \right) + \frac{c \delta n_\alpha}{B_0} \nabla \times \mathbf{b}_0 \cdot \nabla \phi}_{\text{nonlinear, dropped}}. \end{aligned} \quad (0.230)$$

where,

$$q_{\perp 0\alpha} \equiv \int_{\mathbf{X} \rightarrow \mathbf{x}} d\mathbf{v} \mu B_0 v_{||} f_{0\alpha}. \quad (0.231)$$

Eq. (0.230) can be separated into the equilibrium continuity equation,

$$0 = \mathbf{B}_0 \cdot \nabla \left( \frac{n_{0\alpha} u_{||0\alpha}}{B_0} \right) + \frac{c \nabla \times \mathbf{b}_0}{Z_\alpha} \cdot \nabla \left( \frac{P_{||0\alpha}}{B_0} \right) + \frac{c \mathbf{b}_0 \times \nabla B_0}{Z_\alpha} \cdot \nabla \left( \frac{P_{\perp 0\alpha}}{B_0^2} \right) + \frac{c \nabla \times \mathbf{b}_0 \cdot \nabla B_0}{Z_\alpha B_0^2} P_{\perp 0\alpha}, \quad (0.232)$$

and the linear continuity equation,

$$\begin{aligned} 0 = & \partial_t \delta n_\alpha + \delta \mathbf{B} \cdot \nabla \left( \frac{n_{0\alpha} u_{||0\alpha}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_{0\alpha}}{B_0} \right) + \mathbf{B}_0 \cdot \nabla \left( \frac{n_0 \delta u_{||\alpha}}{B_0} \right) + \frac{c \nabla \times \mathbf{b}_0}{Z_\alpha} \cdot \nabla \left( \frac{\delta P_{||\alpha}}{B_0} \right) \\ & + \frac{c \mathbf{b}_0 \times \nabla B_0}{Z_\alpha} \cdot \nabla \left( \frac{\delta P_{\perp\alpha}}{B_0^2} \right) + \frac{c \nabla \times \mathbf{b}_0 \cdot \nabla B_0}{Z_\alpha B_0^2} \delta P_{\perp\alpha} + \frac{c \nabla \times \mathbf{b}_0}{B_0} \cdot n_{0\alpha} \nabla \phi \\ & + \underbrace{\frac{m_\alpha c^2}{Z_\alpha^2 B_0} (\nabla_\perp^2 \delta \mathbf{B}) \cdot \nabla \left( \frac{q_{\perp\alpha}}{B_0^2} \right)}_{\text{(i)}} - \underbrace{\frac{m_\alpha c^3 \mathbf{b}_0 \times \nabla P_{\perp 0\alpha}}{Z_\alpha^2 B_0^2} \cdot \nabla \left( \frac{\nabla_\perp^2 \delta \phi}{B_0} \right)}_{\text{(ii)}} \\ & + \underbrace{\frac{m_\alpha c^3 P_{\perp 0\alpha} (3 \mathbf{b}_0 \times \nabla B_0 + \nabla \times \mathbf{B}_0)}{Z_\alpha^2 B_0^2} \cdot \nabla \left( \frac{\nabla_\perp^2 \delta \phi}{B_0} \right)}_{\text{(iii)}}. \end{aligned} \quad (0.233)$$

Eqs. (0.232) and (0.233) are the same as those of the electron Eqs. (0.71) and (0.72) in the linear limit except for the last three terms in Eq. (0.233), which comes from the FLR effects. In the  $k_\perp L_{B_0} \sim k_\perp R_0 \gg 1$  limit, the term {ii} becomes,

$$\{ii\} \approx \frac{m_\alpha c^2 n_{0\alpha}}{Z_\alpha B_0^2} \frac{c \mathbf{b}_0 \times \nabla P_{\perp 0\alpha}}{Z_\alpha B_0 n_{0\alpha}} \cdot \nabla \nabla_\perp^2 \delta \phi = \frac{m_\alpha c^2 n_{0\alpha}}{Z_\alpha B_0^2} \mathbf{v}_{*\alpha} \cdot \nabla \nabla_\perp^2 \delta \phi. \quad (0.234)$$

For the thermal ion species, this term is responsible for producing the kinetic ballooning mode. We compare the ordering of this term with the other two FLR terms.

$$O\left(\frac{\{iii\}}{\{ii\}}\right) \sim \frac{L_{P_{\perp 0\alpha}}}{L_{B_0}}, \quad (0.235)$$

$$O\left(\frac{\{i\}}{\{ii\}}\right) \sim \frac{k_{||} u_{||0\alpha}}{\omega} \left( 1 + \frac{L_{P_{\perp 0\alpha}}}{L_{u_{||0\alpha}}} - 2 \frac{L_{P_{0\alpha}}}{L_{B_0}} \right). \quad (0.236)$$

For typical tokamak scaling,  $L_{B_0} \sim R_0$ ,  $L_{P_{\perp 0\alpha}} \sim a$ , and  $u_{||0i} \ll v_A \sim \omega/k_{||}$ , so  $L_{P_{\perp 0\alpha}} \ll L_{B_0}$ ,  $L_{P_{\perp 0\alpha}} \ll L_{u_{||0\alpha}}$ , and

$k_{||} u_{||0i} \ll \omega$ , the terms {i} and {iii} are not important and can be dropped. But, for fast ions,  $k_{||} u_{||0f} \ll \omega$  may not be satisfied. Keeping term {ii} as the only FLR effect, the linear ion continuity equation becomes

$$\begin{aligned} Z_\alpha \partial_t \delta n_\alpha + \mathbf{B}_0 \cdot \nabla \left( \frac{Z_\alpha n_{0\alpha} \delta u_{||\alpha}}{B_0} \right) &= -i\omega Z_\alpha \delta n_\alpha + \nabla \cdot \delta \mathbf{J}_{||\alpha} \\ &\approx -\delta \mathbf{B} \cdot \nabla \left( \frac{J_{||0\alpha}}{B_0} \right) - B_0 \mathbf{v}_E \cdot \nabla \left( \frac{Z_\alpha n_{0\alpha}}{B_0} \right) + \frac{m_\alpha c^2 n_{0\alpha}}{B_0^2} \mathbf{v}_{*\alpha} \cdot \nabla \nabla_\perp^2 \delta \phi - c \nabla \times \mathbf{b}_0 \cdot \nabla \left( \frac{\delta P_{||\alpha}}{B_0} \right) \\ &- c \mathbf{b}_0 \times \nabla B_0 \cdot \nabla \left( \frac{\delta P_{\perp\alpha}}{B_0^2} \right) - \frac{c \nabla \times \mathbf{b}_0 \cdot \nabla B_0}{B_0^2} \delta P_{\perp\alpha} - \frac{c \nabla \times \mathbf{b}_0}{B_0} \cdot Z_\alpha n_{0\alpha} \nabla \delta \phi. \end{aligned} \quad (0.237)$$

## 9.4 Reduction to ideal MHD equation

The electron continuity equation (0.76) in the linear limit is

$$\begin{aligned} -e\partial_t\delta n_e - \mathbf{B}_0 \cdot \nabla \left( \frac{en_{0e}\delta u_{||e}}{B_0} \right) &= i\omega e\delta n_e + \nabla \cdot \left( \frac{\delta \mathbf{J}_{||e}}{B_0} \right) \\ &= -\delta \mathbf{B} \cdot \nabla \left( \frac{J_{||0e}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_{0e}}{B_0} \right) \\ &\quad - c \nabla \times \mathbf{b}_0 \cdot \nabla \left( \frac{\delta P_{||e}}{B_0} \right) - c \mathbf{b}_0 \times \nabla B_0 \cdot \nabla \left( \frac{\delta P_{\perp e}}{B_0^2} \right) - \frac{c \nabla \times \mathbf{b}_0 \cdot \nabla B_0}{B_0^2} \delta P_{\perp e} + \frac{c \nabla \times \mathbf{b}_0}{B_0} \cdot en_{0e} \nabla \delta \phi. \end{aligned} \quad (0.238)$$

Plugging Eqs. (0.238) and (0.237) into (0.227), and considering quasi-neutrality  $\sum_{\alpha} Z_{\alpha} n_{\alpha 0} = 0$  and Ampere's law for equilibrium Eq. (0.132), we get

$$\begin{aligned} 0 &= \frac{\omega(\omega - \omega_{*P})}{v_A^2} \nabla_{\perp}^2 \delta \phi + i \mathbf{B}_0 \cdot \nabla \left[ \frac{\nabla_{\perp}^2(k_{||} \delta \phi)}{B_0} \right] - i \nabla(k_{||} \delta \phi) \times \mathbf{b}_0 \cdot \nabla \left( \frac{\mathbf{b}_0 \cdot \nabla \times \mathbf{B}_0}{B_0} \right) \\ &\quad - i \omega \frac{4\pi}{c} \left[ \nabla \times \mathbf{b}_0 \cdot \nabla \left( \frac{\delta P_{||}}{B_0} \right) + \mathbf{b}_0 \times \nabla B_0 \cdot \nabla \left( \frac{\delta P_{\perp}}{B_0^2} \right) + \frac{\nabla \times \mathbf{b}_0 \cdot \nabla B_0}{B_0^2} \delta P_{\perp} \right], \end{aligned} \quad (0.239)$$

where  $\delta P_{||} = \sum_{\alpha} \delta P_{||\alpha}$ ,  $\delta P_{\perp} = \sum_{\alpha} \delta P_{\perp\alpha}$  are the total parallel and perpendicular pressures, respectively.

Now, the first three terms of Eq. (0.239) match those of the ideal MHD eigenmode equation (0.221). The last term of Eq. (0.239), i.e., the pressure term, looks different than the corresponding term of Eq. (0.221). In Sec. 9.6, we show that the difference is negligible. Then the gyrokinetic formulation reduces to the ideal MHD theory in the linear and long wavelength limit. Note that this derivation does not rely on specific particle distribution functions.

Note that the third term in Eq. (0.239) is the current driving term. Most previous gyrokinetic simulations drop this current driving term. Retaining this term in this formulation gives the capability to simulate current-driven modes such as the kink mode. Eq. (0.239) shows that gyrokinetic simulation can be used to study kinetic MHD modes including interchange modes, kink modes, and shear Alfvén waves excited by energetic particles, where kinetic effects are important.

Due to resistive effect is excluded in this formulation, tearing mode<sup>5</sup> cannot be handled.

## 9.5 Estimation of some magnetic field parameters in tokamak

This part will be used in Sec. 9.6.

Noting that the safety factor  $q \approx r B_{\zeta} / (R_0 B_{\theta}) = \delta B_{\zeta} / B_{\theta}$ , where  $\delta = r/R_0$ , the equilibrium field reads

$$\mathbf{B}_0 = B_{\theta} \hat{\theta} + B_{\zeta} \hat{\zeta} = B_{\zeta} \left( \frac{\delta}{q} \hat{\theta} + \hat{\zeta} \right). \quad (0.240)$$

where  $B_{\theta}$  and  $B_{\zeta}$  are the poloidal and toroidal components, respectively, while  $\hat{\theta}$  and  $\hat{\zeta}$  are the unit vector in the poloidal and toroidal directions, respectively. The toroidal vacuum field write,

<sup>5</sup> How about (kinetic) collisionless tearing mode?

$$B_\theta = \frac{B_a R_0}{R} = \frac{B_a}{1 + \delta \cos \theta}. \quad (0.241)$$

This can be used to estimate the parallel component of  $\nabla \times \mathbf{B}_0$

$$(\nabla \times \mathbf{B}_0)_\parallel \approx \frac{\hat{\zeta}}{r} \left[ \partial_r \left( r \frac{\delta}{q} B_\zeta \right) \right] \approx \frac{\hat{\zeta}}{r} \frac{B_0}{q R_0} (2 - s), \quad (0.242)$$

where,

$$s = \frac{r}{q} \frac{dq}{dr} \quad (0.243)$$

is the magnetic shear. It is also straightforward to estimate these quantities

$$\nabla B_0 \approx -\frac{B_a R_0}{R^2} \hat{R} \approx -\frac{B_a R_0}{R^2} (\hat{r} \cos \theta - \hat{\theta} \sin \theta), \quad (0.244)$$

$$\mathbf{b}_0 \times \nabla B_0 \approx \frac{B_a R_0}{R^2} \left( -\hat{r} \sin \theta - \hat{\theta} \cos \theta + \hat{\zeta} \frac{\delta}{q} \cos \theta \right). \quad (0.245)$$

## 9.6 Negligible pressure mismatch terms between gyrokinetic and MHD theory

For comparison, we write down the pressure terms (with the  $-i\omega 4\pi/c$  coefficients removed) from the two different approaches of MHD and gyrokinetic

$$PT_{MHD} = \nabla \cdot \left( \frac{\mathbf{b}_0}{B_0} \times \nabla \cdot \delta \mathbb{P} \right), \quad (0.246)$$

$$\begin{aligned} PT_{GK} &= \nabla \times \mathbf{b}_0 \cdot \nabla \left( \frac{\delta P_\parallel}{B_0} \right) + \mathbf{b}_0 \times \nabla B_0 \cdot \nabla \left( \frac{\delta P_\perp}{B_0^2} \right) + \frac{\nabla \times \mathbf{b}_0 \cdot \nabla B_0}{B_0^2} \delta P_\perp \\ &= \frac{\mathbf{b}_0 \times \nabla B_0}{B_0^2} \cdot \nabla (\delta P_\perp + \delta P_\parallel) + \frac{\nabla \times \mathbf{B}_0}{B_0^2} \cdot \nabla \delta P_\parallel + \frac{\nabla \times \mathbf{B}_0 \cdot \nabla B_0}{B_0^3} (\delta P_\perp - \delta P_\parallel). \end{aligned} \quad (0.247)$$

Assuming  $\delta \mathbb{P}$  is diagonal, which can be justified in the long wavelength limit, i.e., FLR effects ignored

$$\delta \mathbb{P} = \delta P_\parallel \mathbf{b}_0 \mathbf{b}_0 + \delta P_\perp (\mathbb{I} - \mathbf{b}_0 \mathbf{b}_0) = \delta P_\perp \mathbb{I} + (\delta P_\parallel - \delta P_\perp) \mathbf{b}_0 \mathbf{b}_0, \quad (0.248)$$

we have

$$\begin{aligned} \nabla \cdot \left( \frac{\mathbf{b}_0}{B_0} \times \nabla \cdot \delta \mathbb{P} \right) &= \nabla \cdot \left\{ \frac{\mathbf{b}_0}{B_0} \times \nabla \cdot [\delta P_{\perp} \mathbb{I} + (\delta P_{\parallel} - \delta P_{\perp}) \mathbf{b}_0 \mathbf{b}_0] \right\} \\ &= \underbrace{\nabla \cdot \left( \frac{\mathbf{b}_0}{B_0} \times \nabla \cdot \delta P_{\perp} \right)}_{\{1\}} + \underbrace{\nabla \cdot \left\{ \frac{\mathbf{b}_0}{B_0} \times \nabla \cdot [(\delta P_{\parallel} - \delta P_{\perp}) \mathbf{b}_0 \mathbf{b}_0] \right\}}_{\{2\}}, \end{aligned} \quad (0.249)$$

$$\begin{aligned} \{1\} &= \nabla \cdot \left( \frac{\mathbf{b}_0}{B_0} \times \nabla \cdot \delta P_{\perp} \right) = \nabla \times \left( \frac{\mathbf{b}_0}{B_0} \right) \cdot \nabla \delta P_{\perp} - \underbrace{\frac{\mathbf{b}_0}{B_0} (\nabla \times \nabla \delta P_{\perp})}_{0} \\ &= \left( \frac{\nabla \times \mathbf{b}_0}{B_0} + \frac{\mathbf{b}_0 \times \nabla B_0}{B_0^2} \right) \cdot \nabla \delta P_{\perp} = \left( \frac{\nabla \times \mathbf{B}_0 + 2\mathbf{b}_0 \times \nabla B_0}{B_0^2} \right) \cdot \nabla \delta P_{\perp}, \end{aligned} \quad (0.250)$$

$$\begin{aligned} \{2\} &= \nabla \cdot \left\{ \underbrace{\frac{\mathbf{b}_0}{B_0} \times \mathbf{b}_0}_{0} \nabla \cdot [(\delta P_{\parallel} - \delta P_{\perp}) \mathbf{b}_0] + \frac{\mathbf{b}_0}{B_0} \times [(\delta P_{\parallel} - \delta P_{\perp}) \mathbf{b}_0 \cdot \nabla \mathbf{b}_0] \right\} \\ &= \nabla \cdot \left\{ \frac{\mathbf{b}_0}{B_0} \times (\delta P_{\parallel} - \delta P_{\perp}) (\nabla \times \mathbf{b}_0) \times \mathbf{b}_0 \right\} \quad (0 = \nabla(\mathbf{b}_0 \cdot \mathbf{b}_0) = 2[\mathbf{b}_0 \cdot \nabla \mathbf{b}_0 - (\nabla \times \mathbf{b}_0) \times \mathbf{b}_0] \text{ used}) \\ &= \nabla \cdot \left\{ \frac{(\delta P_{\parallel} - \delta P_{\perp})}{B_0} (\nabla \times \mathbf{b}_0)_{\perp} \right\} \\ &= (\nabla \times \mathbf{b}_0)_{\perp} \cdot \nabla \left\{ \frac{(\delta P_{\parallel} - \delta P_{\perp})}{B_0} \right\} + \frac{(\delta P_{\parallel} - \delta P_{\perp})}{B_0} \nabla \cdot [(\nabla \times \mathbf{b}_0)_{\perp}] \\ &= \left[ \frac{(\nabla \times \mathbf{B}_0)_{\perp}}{B_0} + \frac{\mathbf{b}_0 \times \nabla B_0}{B_0^2} \right] \cdot \nabla \left( \frac{\delta P_{\parallel} - \delta P_{\perp}}{B_0} \right) + \frac{(\delta P_{\parallel} - \delta P_{\perp})}{B_0} \nabla \cdot \left[ \frac{(\nabla \times \mathbf{B}_0)_{\perp}}{B_0} + \frac{\mathbf{b}_0 \times \nabla B_0}{B_0^2} \right] \\ &= \left[ \frac{(\nabla \times \mathbf{B}_0)_{\perp}}{B_0} + \frac{\mathbf{b}_0 \times \nabla B_0}{B_0^2} \right] \cdot \nabla \left( \frac{\delta P_{\parallel} - \delta P_{\perp}}{B_0} \right) + \frac{(\delta P_{\parallel} - \delta P_{\perp})}{B_0} \left\{ \nabla \cdot \left[ \frac{(\nabla \times \mathbf{B}_0)_{\perp}}{B_0} \right] - \frac{\nabla \times \mathbf{B}_0 \cdot \nabla B_0}{B_0^2} \right\}. \end{aligned} \quad (0.251)$$

Therefore,

$$\begin{aligned} PT_{MHD} = \{1\} + \{2\} &= \left( \frac{\nabla \times \mathbf{B}_0 + 2\mathbf{b}_0 \times \nabla B_0}{B_0^2} \right) \cdot \nabla \delta P_{\perp} + \frac{\mathbf{b}_0 \times \nabla B_0}{B_0^2} \cdot \nabla \delta P_{\parallel} \\ &\quad + \frac{(\nabla \times \mathbf{B}_0)_{\perp}}{B_0} \cdot \nabla \left( \frac{\delta P_{\parallel} - \delta P_{\perp}}{B_0} \right) + \frac{(\delta P_{\parallel} - \delta P_{\perp})}{B_0} \left\{ \nabla \cdot \left[ \frac{(\nabla \times \mathbf{B}_0)_{\perp}}{B_0} \right] - \frac{\nabla \times \mathbf{B}_0 \cdot \nabla B_0}{B_0^2} \right\}. \end{aligned} \quad (0.252)$$

We calculate the mismatch between  $PT_{MHD}$  and  $PT_{GK}$

$$\begin{aligned}
PT_{MHD} - PT_{GK} &= \nabla \cdot \left( \frac{\mathbf{b}_0 \times \nabla B_0}{B_0} \times \nabla \cdot \delta \mathbb{P} \right) - \left[ \underbrace{\frac{\mathbf{b}_0 \times \nabla B_0}{B_0^2} \cdot \nabla (\delta P_\perp + \delta P_\parallel)}_{\{3\}} + \underbrace{\frac{\nabla \times \mathbf{B}_0}{B_0^2} \cdot \nabla \delta P_\parallel}_{\{4\}} + \underbrace{\frac{\nabla \times \mathbf{B}_0 \cdot \nabla B_0}{B_0^3} (\delta P_\perp - \delta P_\parallel)}_{\{5\}} \right] \\
&= \frac{\nabla \times \mathbf{B}_0 \cdot \nabla \delta P_\perp}{B_0^2} - \frac{(\nabla \times \mathbf{B}_0)_\parallel}{B_0} \cdot \nabla \left( \frac{\delta P_\parallel}{B_0} \right) - \frac{(\nabla \times \mathbf{B}_0)_\perp}{B_0} \cdot \nabla \left( \frac{\delta P_\perp}{B_0} \right) + \frac{(\delta P_\parallel - \delta P_\perp)}{B_0} \nabla \cdot \left[ \frac{(\nabla \times \mathbf{B}_0)_\perp}{B_0} \right] \\
&= \frac{(\nabla \times \mathbf{B}_0)_\parallel \cdot \nabla (\delta P_\perp - \nabla \delta P_\parallel)}{B_0^2} + \frac{(\delta P_\parallel - \delta P_\perp)}{B_0} \nabla \cdot \left[ \frac{(\nabla \times \mathbf{B}_0)_\perp}{B_0} \right] \\
&\quad + 2\delta P_\parallel \frac{(\nabla \times \mathbf{B}_0)_\parallel}{B_0} \cdot \left[ \frac{\nabla B_0}{B_0^2} \right] + 2\delta P_\perp \frac{(\nabla \times \mathbf{B}_0)_\perp}{B_0} \cdot \left[ \frac{\nabla B_0}{B_0^2} \right] \\
&= \underbrace{\frac{(\nabla \times \mathbf{B}_0)_\parallel \cdot \nabla (\delta P_\perp - \nabla \delta P_\parallel)}{B_0^2}}_{\{6\}} + \underbrace{\frac{(\delta P_\parallel - \delta P_\perp)}{B_0} \nabla \cdot \left[ \frac{(\nabla \times \mathbf{B}_0)_\perp}{B_0} \right]}_{\{7\}} \\
&\quad + \underbrace{2(\delta P_\perp - \delta P_\parallel) \frac{(\nabla \times \mathbf{B}_0)_\perp}{B_0} \cdot \left[ \frac{\nabla B_0}{B_0^2} \right]}_{?}, \tag{0.253}
\end{aligned}$$

It can be immediately seen that if  $\delta P_\perp = \delta P_\parallel$ , e.g., in the isotropic or adiabatic limit, the mismatch vanishes. In the case of  $\delta P_\perp \neq \delta P_\parallel$ , assuming  $O(\delta P_\perp) \sim O(\delta P_\parallel) \sim O(\delta P_\parallel \pm \delta P_\perp)$ , the mismatch is shown to be small compared to the pressure term as follows.

Here we use the scaling of  $O(k_\parallel/k_\perp) \sim O[1/(k_\perp R_0)] \sim O(a/R_0) \sim O(\delta)$ ,  $O(\beta R_0/L_{P_0}) \sim 1$  and  $O[(2-s)/q] \sim 1$ . We first estimate the order of the terms of {3}, {4} and {5} to find out the leading order of the pressure term. Using the force balance

$$\nabla P_0 = \frac{1}{c} \mathbf{J}_0 \times \mathbf{B}_0 = \frac{1}{4\pi} (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0, \tag{0.254}$$

the perpendicular component of  $\nabla \times \mathbf{B}_0$  can be estimated

$$(\nabla \times \mathbf{B}_0)_\perp = \frac{4\pi}{B_0} \mathbf{b}_0 \times \nabla P_0 \sim \frac{\beta}{2L_{P_0}}. \tag{0.255}$$

Noticing  $\nabla B_0$  is in the perpendicular direction and its magnitude is estimated  $B_0/R_0$ ,

$$O(\{3\}) \sim \frac{k_\perp}{R_0} \frac{\delta P_{\perp\parallel}}{B_0}, \tag{0.256}$$

$$O(\{4\}) \sim \left( \frac{\beta k_\perp}{2L_{P_0}} + \frac{2-s}{q} \frac{k_\parallel}{R_0} \right) \frac{\delta P_\parallel}{B_0}, \tag{0.257}$$

$$O(\{5\}) \sim \frac{\beta}{2L_{P_0} R_0} \frac{\delta P_{\perp\parallel}}{B_0}. \tag{0.258}$$

Then,

$$O\left(\frac{\{4\}}{\{3\}}\right) \sim \left( \frac{\beta R_0}{2L_{P_0}} + \frac{2-s}{q} \frac{k_{\parallel}}{k_{\perp}} \right) \sim O(1), \quad (0.259)$$

$$O\left(\frac{\{5\}}{\{3\}}\right) \sim \frac{\beta}{2L_{P_0} k_{\perp}} \sim O(\delta). \quad (0.260)$$

The term  $\{3\}$  and  $\{4\}$  are the leading terms. Next, we only need to compare the mismatch with one of the leading order terms  $\{3\}$ . From Eq. (0.255), we have

$$\nabla \cdot [(\nabla \times \mathbf{B}_0)_{\perp}] = 4\pi \nabla \cdot \left[ \frac{\mathbf{b}_0}{B_0} \times \nabla P_0 \right] = \frac{4\pi}{B_0^2} [\nabla \times \mathbf{B}_0 + 2\mathbf{b}_0 \times \nabla B_0] \cdot \nabla P_0 \sim \frac{\beta B_0}{L_{P_0} R_0}, \quad (0.261)$$

where it is assumed that  $\nabla P_0$  is perpendicular to  $\mathbf{B}_0$ . Then we get

$$O(\{6\}) \sim \frac{2-s}{qB_0R_0} k_{\parallel} \delta P_{\perp,\parallel}, \quad (0.262)$$

$$O(\{7\}) \sim O(\{8\}) \sim \frac{\beta}{L_{P_0} R_0} \frac{\delta P_{\perp,\parallel}}{B_0}. \quad (0.263)$$

Then,

$$O\left(\frac{\{6\}}{\{3\}}\right) \sim \frac{2-s}{q} \frac{k_{\parallel}}{k_{\perp}} \sim O(\delta), \quad (0.264)$$

$$O\left(\frac{\{7\}}{\{3\}}\right) \sim O\left(\frac{\{8\}}{\{3\}}\right) \sim \frac{\beta}{k_{\perp} L_{P_0}} \sim O(\delta). \quad (0.265)$$

Hence, the mismatch is not important even in the presence of anisotropic perturbed pressure. Therefore, the gyrokinetic model reduces to the ideal MHD model in the long wavelength and linear limit.

## 10 Magnetic field model with finite $\nabla \times \mathbf{B}_0$

In this section, we keep the normalized qualities. All qualities in this section are equilibrium quantities, so the equilibrium subscript 0 for the magnetic field is omitted. The equilibrium geometry in GTC can either be taken from EFIT as numerical accurate representation, or specified as the analytic approximate model. A simple analytic equilibrium in GTC for the magnetic field model is

$$\begin{aligned} B &= 1 - \varepsilon \cos \theta + O(\varepsilon^2), \\ \delta &= 0 + O(\varepsilon), \\ I &= 0 + O(\varepsilon^2), \\ g &= 1 + O(\varepsilon^2), \\ \theta &= \theta_0 + O(\varepsilon), \\ \zeta &= \zeta_0 + O(\varepsilon), \end{aligned} \quad (0.266)$$

where  $\varepsilon = r/R_0$  is the normalized radial coordinate,  $\theta_0$  and  $\zeta_0$  are the geometric poloidal and toroidal angles, and  $\theta$  and  $\zeta$  are the corresponding magnetic coordinates. Such a field model makes all the derivatives of  $I$  and  $g$  in Eq. (0.146) zero, and thus leading to zero equilibrium current. Here we extent this simple field model to a higher-order one to recover the equilibrium current.

We assume concentric circular magnetic surfaces in this analytic equilibrium model. In the following derivation, the prime symbol ('') denotes derivative with respect to  $\psi$

$$\cdot' = \frac{d}{d\psi} = \frac{d\varepsilon}{d\psi} \frac{d}{d\varepsilon} = \frac{g}{\varepsilon} \frac{d}{d\varepsilon}. \quad (0.267)$$

In the large-aspect-ratio limit  $\varepsilon \ll 1$ , we expand the field related quantities with respect to  $\varepsilon$

$$\begin{aligned} B &= 1 - \varepsilon \cos \theta_0 + \varepsilon^2 B_2 + \varepsilon^3 B_3 + \dots, \\ \delta &= 0 + O(\varepsilon), \\ I &= \varepsilon^2 I_2 + \varepsilon^3 I_3 + \dots, \\ g &= 1 + \varepsilon^2 g_2 + \varepsilon^3 g_3 + \dots, \\ \theta &= \theta_0 + \varepsilon \theta_1 + \varepsilon \theta_2 + \dots, \\ \zeta &= \zeta_0 + \varepsilon \zeta_1 + \varepsilon \zeta_2 + \dots, \end{aligned} \quad (0.268)$$

where  $g_i$  and  $I_i$  ( $i = 2, 3, \dots$ ) are functions of the safety factor  $q$ ; and  $B_i$ ,  $\theta_i$  and  $\zeta_i$  ( $i = 1, 2, \dots$ ) are periodic function of  $\theta_0$ .

We want the field model to satisfy these conditions:

- The Jacobian satisfies  $J^{-1} = \nabla\psi \cdot \nabla\theta \times \nabla\zeta = B^2/(gq+I)$  so that  $I$  is a function of only  $\psi$  (equivalently  $\varepsilon$ , because of concentric circular flux surfaces).
- The radial component of the field is zero because of concentric circular flux surfaces:

$$B_\varepsilon = \varepsilon \delta / q + I \partial_\varepsilon \theta + g \partial_\varepsilon \zeta = 0.$$

- The field magnitude expression is consistent with the covariant representation:  $B = |\partial\nabla\psi + I\nabla\theta + g\nabla\zeta|$ .
- The field line is straight in the  $(\theta, \zeta)$  space, so Eq. (0.145) needs to be satisfied with  $q$  being independent of  $\theta$  and  $\zeta$ .

The condition for Jacobian writes

$$J^{-1} = \nabla\psi \cdot \nabla\theta \times \nabla\zeta = \frac{d\psi}{d\varepsilon} \frac{\partial_{\theta_0} \theta}{\varepsilon} \frac{1}{R} = \frac{\varepsilon}{q} \frac{\partial_{\theta_0} \theta}{\varepsilon} \frac{1}{1 + \varepsilon \cos \theta_0} = \frac{B^2}{gq+I}. \quad (0.269)$$

which is equivalent as

$$(gq+I) \partial_{\theta_0} \theta = q(1 + \varepsilon \cos \theta_0) B^2. \quad (0.270)$$

Plugging (0.268) into (0.270) gives

(...)

The final results up to  $O(\varepsilon)$  order is

$$\begin{aligned}
 B &= 1 - \varepsilon \cos \theta_0 + O(\varepsilon^2), \\
 \delta &= \varepsilon \sin \theta_0 + O(\varepsilon^2) = \varepsilon \sin \theta + O(\varepsilon^2), \\
 I &= \frac{\varepsilon^2}{q} + O(\varepsilon^4), \\
 I' &= 2 - s + O(\varepsilon^2), \\
 g &= 1 + O(\varepsilon^2), \\
 g' &= O(\varepsilon^0), \\
 \theta &= \theta_0 - \varepsilon \sin \theta_0 + O(\varepsilon^2), \\
 \theta &= \theta + \varepsilon \sin \theta + O(\varepsilon^2), \\
 \zeta &= \zeta_0 + O(\varepsilon^4),
 \end{aligned} \tag{0.271}$$

And,

$$\nabla \times \mathbf{B} = O(\varepsilon^0) \nabla \psi \times \nabla \zeta + [(2-s) - \varepsilon \cos \theta + O(\varepsilon^2)] \nabla \psi \times \nabla \theta, \tag{0.272}$$

$$\mathbf{b} \cdot \nabla B = \frac{J^{-1}}{B} [(2-s) - \varepsilon \cos \theta_0 + O(\varepsilon)]. \tag{0.273}$$

Although it is straightforward to solve the equations up to the  $O(\varepsilon^2)$  order, such a model would not be very useful because other effects come into play at the order  $O(\varepsilon^2)$  or even lower, such as the Shafranov shift and the finite pressure gradient effect. All of those effects are retained in the numerical equilibrium using EFIT solution. The field model of order  $O(\varepsilon)$ , i.e., (0.271), is good enough to recover the parallel current and is therefore implemented. This first-order set of equilibrium is similar to an earlier GTC electrostatic simulation using analytic equilibrium.

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