C247 HW1

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1. (a) $Q^TQ = I$
   i. $Q^T(Q^T)^T = Q^TQ^{-1T} = I$, $Q^{-1}(Q^{-1})^T = Q^{-1}Q = I$, so they're both orthogonal.
   ii. $|\lambda^2|x^Tx = (Qx)^TQx = x^TQ^TQx = x^Tx$, $|\lambda| = 1$
   iii. $QQ^T = I$, $|Q| = |Q^T| = +1, -1$
   iv. assume a vector x, Qx is the transformation of x. $||Qx|| = ||x||$ since $|Q|^2 = 1$

   (b) i. The SVD of $A = U\sum V^T$, eigenvectors of AA^T are the left singular vectors of A (columns of U) and eigenvectors of A^TA are the right singular vectors of A (columns of V).
   ii. Singular values of AA^T and A^TA are the same as the squares of eigenvalues of A.

   (c) i. False
   ii. False
   iii. True
   iv. False
   v. True

2. (a) i. $1 - p_B$
   ii. p_Ap_B
   iii. $(1-p_A)^{n-1}(1-p_B)^{n-1}(p_A + p_B - p_Ap_B)$
   iv. $(1-p_B)^{n-1}p_A$
   v. $(1-p_A)^{n-1}(1-p_B)^{n-1}p_Ap_B$

   (b) i. An isolated faculty member is one who is seated between two faculty members from different departments. Since there are 6 faculty members from each department, we can calculate the probability that a given faculty member is isolated and then multiply by the total number of faculty members.
   For a specific faculty member, the probability that the person to their left is from a different department is $\frac{12}{17}$ (since there are 12 faculty members not from their department out of 17 others). Similarly, the probability that the person to their right is also from a different department (and different from the person to the left) is $\frac{11}{16}$.
   So, the probability that a given faculty member is isolated is $\frac{12}{17} \times \frac{11}{16}$. Since there are 18 faculty members in total, the expected number of isolated faculties $E(X) = 8.74$.
   ii. A semi-happy faculty member is one who is seated next to exactly one faculty member from the same department. The calculation is similar to the isolated case. but here we have two scenarios: one faculty member from the same department and

one from a different department, on either side.
The probability for this scenario can be calculated as follows: First, pick one person from the same department ($\frac{5}{17}$) and then one person from a different department ($\frac{12}{16}$). Since there are two ways this can happen (same department member can be either on the left or the right), we multiply this probability by 2. $E(Y) = 7.94$

iii. A joyous faculty member is surrounded by faculty members from the same department. The probability for this is the probability that both adjacent members are from the same department.
This probability is $\frac{5}{17} \times \frac{4}{16}$ for a given faculty member. $E(Z) = 1.32$

(c) Bayes' Theorem: $P(A|B) = \dfrac{(B|A) \times P(A)}{P(B)}$
Let A be 'the man has a dangerous type of lung cancer.'
Let B be 'the man has a positive test.'
$P(A) = 0.0005, P(B|A) = 0.9, P(B|\hat{A}) = 0.01$
$P(B) = P(B|A) \times P(A) + P(B|\hat{A}) \times P(\hat{A})$

$P(A|B) = 4.31\%$

ii. $P(A|\hat{B}) = 0.0051\%$

(d) $E(Ax + b) = AE(x) = b$

(e) $cov(Ax + b) = Acov(x)A'$

3. (a) $\nabla_x x^T Ay = Ay$
   (b) $\nabla_y x^T Ay = (x^T A)^T$
   (c) $\nabla_A x^T Ay = xy^T$
   (d) $x^T(A + A^T) + b$
   (e) B^T
   (f) $B^T + B + (AB + BA)^T$
   (g) $2(A + \lambda B)$

4. Assume $Y \in R^{n\times n}, X \in R^{n\times n}, M \in R^{n\times n}$
$argmin_W f \iff argmin_W \frac{1}{2}\sum_{i=1}^{n} ||y^{(i)} - Wx^{(i)}||^2 \iff argmin_W \frac{1}{2} tr((Y - WX)^T(Y - WX)) \iff argmin_W \frac{1}{2} tr(Y^T - Y^T WX - X^T W^T Y + X^T W^T WX)$
$\nabla_W = \frac{1}{2}(-2X^T Y + 2WX^T X) = 0$
$W = (X^T X)^{-1} X^T Y$

5. Assume $Y \in R^N, X \in R^{N\times M}, \theta \in R^M$, where each is $y^{(i)}, \hat{x}^{(i)}$ arrange by columns.
$argmin_\theta f \iff argmin_\theta \frac{1}{2}\sum_{i=1}^{N}(y^{(i)} - \theta^T \hat{x}^{(i)})^2 + \frac{\lambda}{2}||\theta||_2^2 \iff argmin_\theta \frac{1}{2}(Y - X\theta)^T(Y - X\theta) + \frac{\lambda}{2}\theta^T\theta$
Get derivative: $\nabla_\theta f = (X^T X + \lambda I)\theta - X^T Y$
If $(X^T X + \lambda I)$ is invertible, $\theta * = (X^T X + \lambda I)^{-1} X^T Y$