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Virginia Tech ECE Qualifying Exam: EM Waves

11/09/16

*Note: At the end of this document in Appendix H, I have included a list of textbook title acronyms and references cited throughout this exam.

Question #1: Submarine Communicating with an Overhead Surveillance Aircraft

Parameters:

Given Parameter	Value	Unit
Airplane Altitude	10	km
Airplane Power Transmitted	200	kW
Operating Frequency	20	kHz
Submarine Receiver Sensitivity	1	μV/m
Airplane Transmit Antenna Gain	0	dBi

Major Assumptions:

- Assuming the water is seawater with the following material properites: ε_r = 81, μ_r = 0.999991, Conductivity σ = 4 siemens/m [Obtained material properties from HFSS material library and FoAE]
- Assuming the impedance at the submarine receiver is 50 Ohms in order to convert the Rx sensitivity from an E-field strength (V/m) to a power level (W) that can be added to the RF communications link budget
- Assuming the Tx and Rx antennas are polarization matched
- Assuming that the submarine has a perfectly isotropic Rx VLF antenna to receive the signal
- Assuming the isotropic radiator has perfect efficiency (efficiency factor = 1)

Answers:

a. Find maximum depth at which the submarine can receive the signal. Assuming airplane is directly overhead.

   P_t = 200 kW = 10*$\log_{10}$(200,000 W) = 53 dBW

   R_{air} = Height From Airplane to Ocean Surface = 10 km

   Gt = 1 = 0 dBi

f = 20 kHz (VLF); λ = 14,990 meters = 14.99 km

Calculate Effective Tx Antenna Aperture: $G_t = D_0 = \frac{4\pi A_{eff}}{\lambda^2}$ = 1 = $\frac{4\pi A_{eff}}{(14990\text{ m})^2}$ -> This yields an absurdly large unrealistic value so I will use the fact that this is an isotropic radiator use the surface area of the sphere to translate power transmitted to field strength at the surface.

Surface Area of the Radiation Sphere at the Seawater Surface From the Isotropic Radiating Transmit Antenna: $A_{SA} = 4\pi R^2$ = $4\pi(10{,}000\ m)^2 = 1.257 * 10^9\ m^2$

Radiated Power Density at Seawater Surface (on the air side):

Power Density = P_D= P_t/A_{SA} = (200,000 W)/($1.257 * 10^9\ m^2$) = 1.591*10^{-4} W/m^2

In Linear Form: $\frac{P_r}{P_t} = G_t G_r (\frac{\lambda}{4\pi R})^2 L_{air-seawater\ mismatch} L_{seawater\ attenuaion}$

Permittivity of Seawater: $\varepsilon = \varepsilon_r \varepsilon_0 - j\frac{\sigma}{\omega}$ = (81)*(8.854e-12 F/m) – j*(4 siemens/m)/(2 π *20000 Hz)

ε = 7.17e-10 – j*3.18e-05 F/m

From Table 7-1 in FoAE (pg. 366):

Table 7-1: Expressions for α, β, η_c, u_p, and λ for various types of media.

	Any Medium	Lossless Medium ($\sigma = 0$)	Low-loss Medium ($\varepsilon''/\varepsilon' \ll 1$)	Good Conductor ($\varepsilon''/\varepsilon' \gg 1$)	Units
$\alpha =$	$\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}-1\right]\right]^{1/2}$	0	$\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\pi f \mu \sigma}$	(Np/m)
$\beta =$	$\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}+1\right]\right]^{1/2}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$	$\sqrt{\pi f \mu \sigma}$	(rad/m)
$\eta_c =$	$\sqrt{\frac{\mu}{\varepsilon'}}\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$(1+j)\frac{\alpha}{\sigma}$	(Ω)
$u_p =$	ω/β	$1/\sqrt{\mu\varepsilon}$	$1/\sqrt{\mu\varepsilon}$	$\sqrt{4\pi f/\mu\sigma}$	(m/s)
$\lambda =$	$2\pi/\beta = u_p/f$	u_p/f	u_p/f	u_p/f	(m)

Notes: $\varepsilon' = \varepsilon$; $\varepsilon'' = \sigma/\omega$; in free space, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$; in practice, a material is considered a low-loss medium if $\varepsilon''/\varepsilon' = \sigma/\omega\varepsilon < 0.01$ and a good conducting medium if $\varepsilon''/\varepsilon' > 100$.

$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0} = \frac{4\ S/m}{(2*\pi*20000\ Hz)(81)(8.854e-12\frac{F}{m})} = 4.44e4 \gg 1$-> Good Conductor

$$\alpha = \sqrt{\pi f \mu \sigma} = \sqrt{\pi(20000\ Hz)(4\pi * 10^{-7}\ H/m)(4\ S/m)} = \text{0.562 Np/m}$$

$$\beta = \alpha = \text{0.562 rad/m}$$

$$\eta_{seawater} = (1+j)\frac{\alpha}{\sigma} = (1+j)\frac{0.562 \text{ Np/m}}{4\ S/m} = (1+j)*0.1405\ \Omega \text{ (Ohms)}$$

Phase Velocity is Seawater: $v_p = \frac{\omega}{\beta} = \sqrt{\frac{4\pi f}{\mu\sigma}} = \sqrt{\frac{4\pi(20{,}000\ Hz)}{(0.999991*4\pi*10^{-7}\ H/m)(4\ S/m)}}$ = 2.236e5 m/s

Wavelength in Seawater: $\lambda = \frac{2\pi}{\beta} = \frac{v_p}{f} = \frac{2.236\text{e}5 \text{ m/s}}{20{,}000\ Hz} = 1581.2m$

$$\eta_{air} = 377\ \Omega$$

$$\Gamma = \frac{\eta_{seawater} - \eta_{air}}{\eta_{seawater} + \eta_{air}} = \frac{(1+j)*0.1405 - 377}{(1+j)*0.1405 + 377} = -0.9993 + j*0.0007$$

Reflected Power @ Air-Seawater Interface: $|\Gamma|^2 = 0.9985$ = -0.0065 dB

Transmitted Power @ Air-Seawater Interface: T = 1 - $|\Gamma|^2$ = 1 - 0.9985 = 0.0015 = -28.24 dB

*Note: This means that there is a 28.2 dB loss in signal across the air-seawater interface. This is quite significant.

Transmitted E-Field: $T_E = 1 + \Gamma$ = 1 + (-0.9993 + j*0.0007) = 7.0e-04 + j*7.0e-04 = 9.90e-04e^{j45^o}

Calculate Transmitted E-Field at the Air-Seawater Interface on the Air Side:

$P_D = 1.591 * 10 - 4\ W/m^2 = \frac{E^2}{\eta_{air}} = \frac{E^2}{377\Omega}$; E = 0.245 V/m

Exponential Decay of E-Field Strength as Signal Propagates Through Lossy Seawater:

$$|E^t| = \left|T_E E_0^i e^{-\alpha d}\right|$$

Can use this equation to calculate depth (d) at which $|E^t| = 1$ µV/m = 10^{-6} V/m

$|E^t| = 10^{-6}$ V/m = $\left|(9.90 * 10^{-4})(0.245 \text{ V/m})e^{-(0.562 \text{ Np/m})(d)}\right|$ -> d = 9.77m

b. What happens to the results when the airplane is no longer directly overhead? How deep can the submarine be if the line-of-sight from the submarine to the airplane is inclined at 60° to the zenith?

General diagram of problem from Ulaby FoAE Section 8-1 (This problem has a Tx airplane in the sky transmitting to the submarine rather than a ship, however, the general concept and method of calculation is similar):

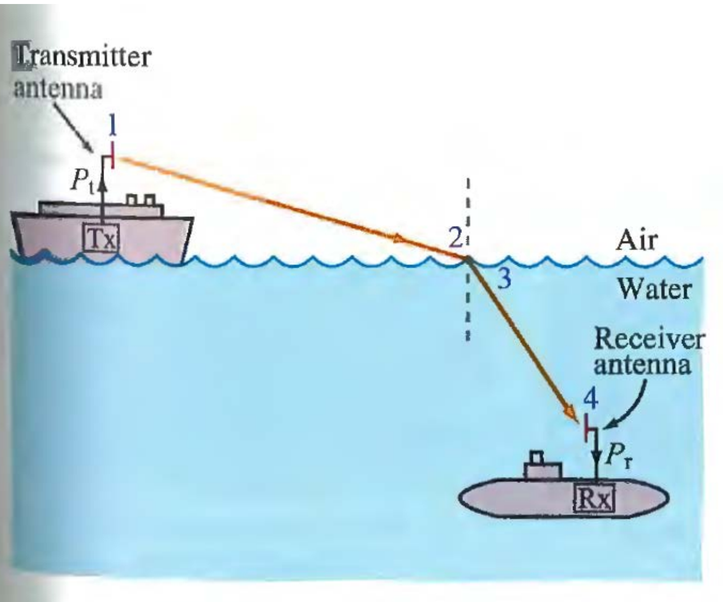


Figure 8-1: Signal path between a shipboard transmitter (Tx) and a submarine receiver (Rx).

Use Snell's law to find angle of refraction when airplane is no longer directly overhead:

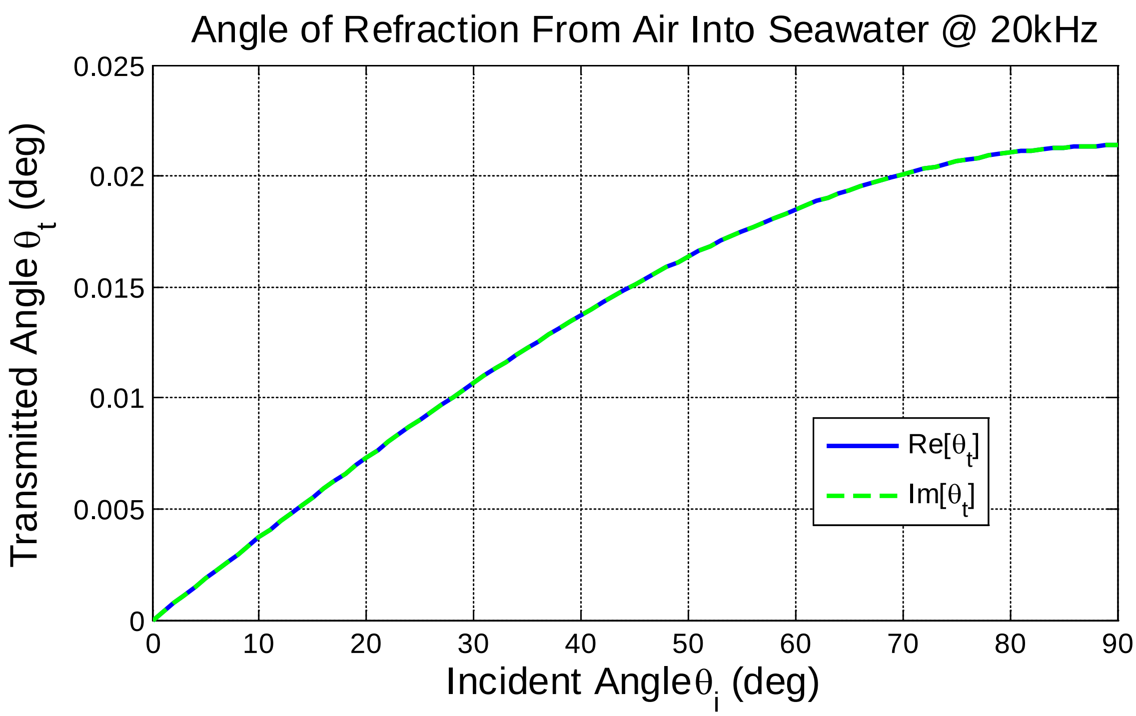
$$\sin\theta_t = \sqrt{\frac{\mu_{air}\varepsilon_{air}}{\mu_{seawater}\varepsilon_{seawater}}}\, sin\theta_i$$

$$\theta_t = \sin^{-1}\left(\sqrt{\frac{\mu_{air}\varepsilon_{air}}{\mu_{seawater}\varepsilon_{seawater}}}\, sin\theta_i\right)$$

$$\theta_t = \sin^{-1}\left(\sqrt{\frac{(4\pi*10^{-7}\ H/m)(8.854*10^{-12}\ \frac{F}{m})}{(0.999991)(4\pi*10^{-7}\ H/m)(7.17*10^{-10} - j*3.18*10^{-5}\ \text{F/m})}}\,\sin 60^o\right) = 0.0185 + j*0.0185$$

degrees

*Note: Refraction angle is complex due to complex permittivity of the lossy conductor medium (seawater)



*Note: Transmitted angle into seawater barely scans off even when the incident angle approaches grazing due to the large and complex permittivity value of seawater

At incident angles off of zenith, the reflection and transmission coefficients are polarization dependent.

Table 8-2 From FoAE pg. 391:

Property	Normal Incidence $\theta_i = \theta_t = 0$	Perpendicular Polarization	Parallel Polarization
Reflection coefficient	$\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$	$\Gamma_\perp = \dfrac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$	$\Gamma_\parallel = \dfrac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$
Transmission coefficient	$\tau = \dfrac{2\eta_2}{\eta_2 + \eta_1}$	$\tau_\perp = \dfrac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$	$\tau_\parallel = \dfrac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$
Relation of Γ to τ	$\tau = 1 + \Gamma$	$\tau_\perp = 1 + \Gamma_\perp$	$\tau_\parallel = (1 + \Gamma_\parallel) \dfrac{\cos\theta_i}{\cos\theta_t}$
Reflectivity	$R = \|\Gamma\|^2$	$R_\perp = \|\Gamma_\perp\|^2$	$R_\parallel = \|\Gamma_\parallel\|^2$
Transmissivity	$T = \|\tau\|^2 \left(\dfrac{\eta_1}{\eta_2}\right)$	$T_\perp = \|\tau_\perp\|^2 \dfrac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i}$	$T_\parallel = \|\tau_\parallel\|^2 \dfrac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i}$
Relation of R to T	$T = 1 - R$	$T_\perp = 1 - R_\perp$	$T_\parallel = 1 - R_\parallel$

Notes: (1) $\sin\theta_t = \sqrt{\mu_1\varepsilon_1/\mu_2\varepsilon_2}\,\sin\theta_i$; (2) $\eta_1 = \sqrt{\mu_1/\varepsilon_1}$; (3) $\eta_2 = \sqrt{\mu_2/\varepsilon_2}$; (4) for nonmagnetic media, $\eta_2/\eta_1 = n_1/n_2$.

Perpendicular Polarization:

At 60 degrees:

Reflection coef: $\Gamma_{\perp} = \frac{\eta_{sea} cos\theta_i - \eta_{air} cos\theta_t}{\eta_{sea} cos\theta_i + \eta_{air} cos\theta_t}$ = $\frac{((1+j)*0.1405\ \Omega)\cos(60\ deg) - (377\Omega)\cos(0.0185\ +\ j*0.0185\ \mathrm{deg})}{((1+j)*0.1405\ \Omega)\cos(60\ deg) + (377\Omega)\cos(0.0185\ +\ j*0.0185\ \mathrm{deg})}$

$\Gamma_{\perp}$= -0.9996 + 0.0004i

Transmission coef: $\tau_{\perp} = 1 + \Gamma_{\perp} = \frac{2\eta_{sea} cos\theta_i}{\eta_{sea} cos\theta_i + \eta_{air} cos\theta_t}$ = 3.7268e-04 + j*3.7254e-04

Reflectivity: $R_{\perp} = |\Gamma_{\perp}|^2$ = 0.9993

Transmissivity: $T_{\perp} = 1 - R_{\perp} = 1 - |\Gamma_{\perp}|^2$ = 7.4508e-04 = -31.3 dB

Parallel Polarization:

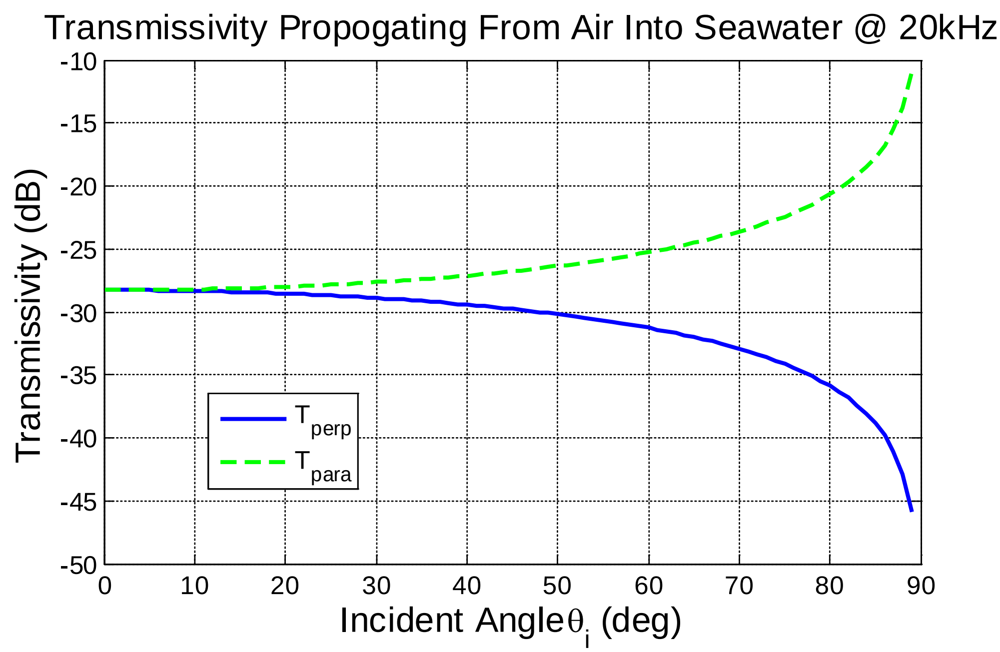
$\Gamma_{||} = \frac{\eta_{sea} cos\theta_t - \eta_{air} cos\theta_i}{\eta_{sea} cos\theta_t + \eta_{air} cos\theta_i}$ = -0.9985 + j*0.0015

Transmission coef: $\tau_{||} = (1 + \Gamma_{||}) \frac{cos\theta_i}{cos\theta_t} = \frac{2\eta_{sea} cos\theta_i}{\eta_{sea} cos\theta_t + \eta_{air} cos\theta_i}$ = 0.5007 + j*0.0007

Reflectivity: $R_{||} = \left|\Gamma_{||}\right|^2$= 0.9970

Transmissivity: $T_{||} = 1 - R_{||} = 1 - \left|\Gamma_{||}\right|^2$= 0.0030 = -25.3 dB

*Note: At 20 kHz, there is a 6dB difference in the level of power transmitted from air into seawater at an incident angle of 60 degrees off of the zenith between perpendicular and parallel polarization. This is very significant and will affect both the depth that the submarine can detect the signal as well as the data rate that can be transmitted due to the 6dB increase in SNR for parallel polarization versus perpendicular polarization.



Assuming that airplane decreases in altitude to keep the distance from the Tx antenna to the point where the signal hits the water constant at range = 10km.

Use this data to calculate the max depth of communication with the submarine as the line-of-sight of the aircraft scans off of the zenith.

$$|E^t| = \left|\tau E_0^i e^{-\alpha d}\right|$$

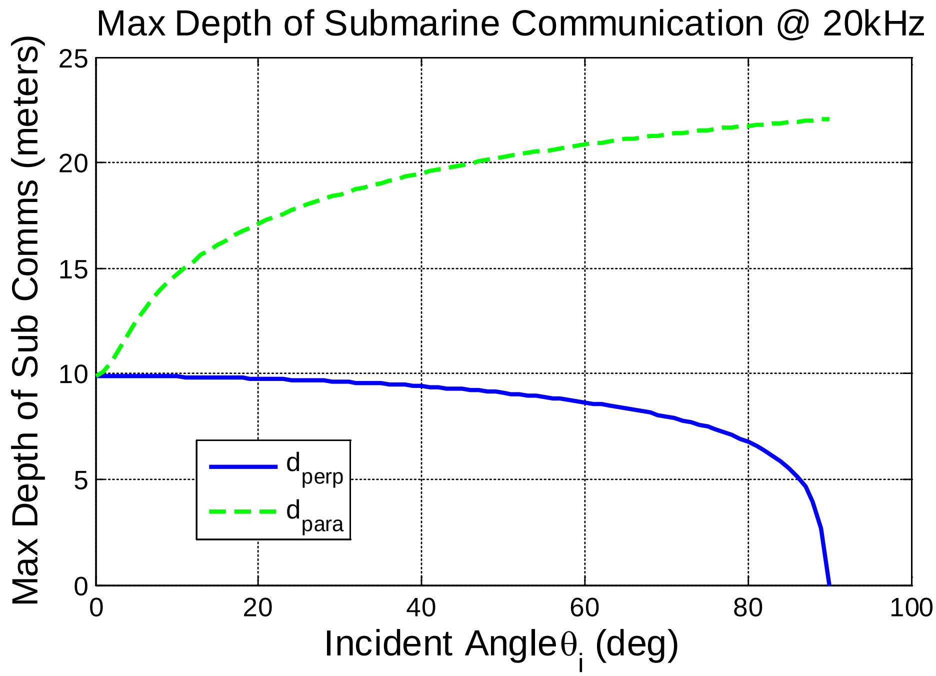
$$E^t = |\tau| E_0^i e^{-\alpha d} \text{ -> } \ln \frac{E_t}{|\tau| E_0^i} = -\alpha d$$

Max Depth of Comms to Submarine:

$$d = -\frac{1}{\alpha} \ln \frac{E_t}{|\tau| E_0^i}$$

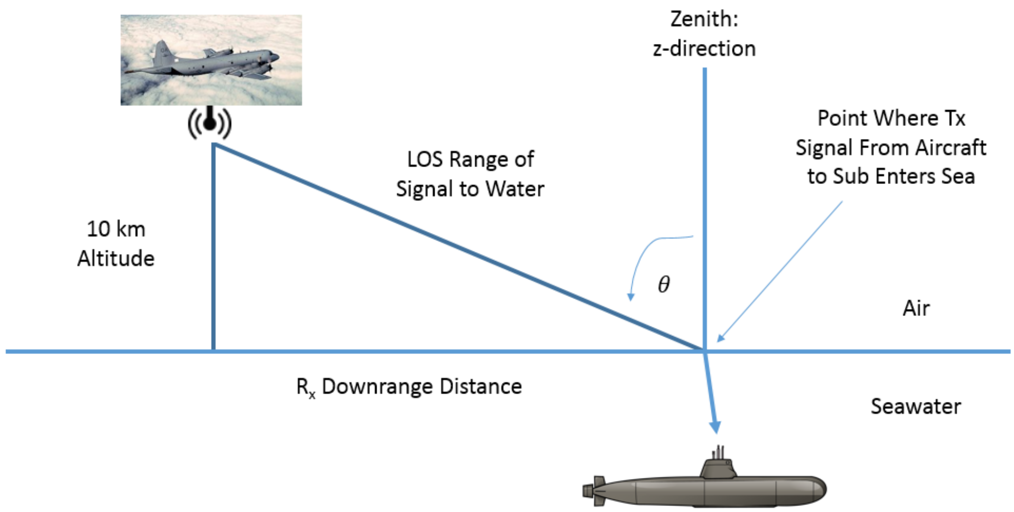
$$d_{\perp} = -\frac{1}{\alpha} \ln \frac{E_t}{|\tau_{\perp}| E_0^i} = -\frac{1}{(0.562 \text{ Np/m})} \ln \frac{10^{-6} \text{ V/m}}{|3.7268e{-}04 + j3.7254e{-}04|(0.245 \text{ V/m})} = 8.65 \text{ meters}$$

$$d_{||} = -\frac{1}{\alpha} \ln \frac{E_t}{|\tau_{||}| E_0^i} = -\frac{1}{(0.562 \text{ Np/m})} \ln \frac{10^{-6} \text{ V/m}}{|0.5007 + j0.0007|(0.245 \text{ V/m})} = 20.85 \text{ meters}$$



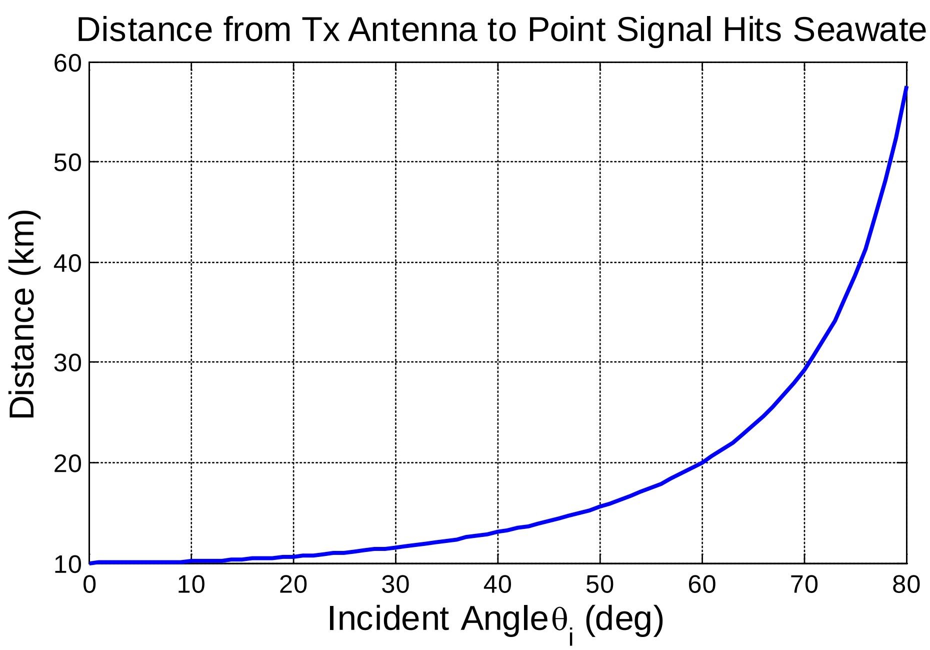
These results mean that if the incident E-field transmitted from the surveillance aircraft is parallel to the sea surface, the transmissivity and this the maximum depth of communication to the submarine increases significantly as the line-of-site angle between the aircraft. The opposite is true for perpendicular polarization. However, this scenario is not quite accurate because the line-of-sight (LOS) distance is assumed to be a constant 10km, which would mean that the airplane would need to dive to a lower altitude towards the water to increase the scan angle. We should consider the more accurate scenario, where the aircraft maintains a constant altitude of 10km and flies away from the submarine to increase the angle off of the zenith and consequently, increasing the range to the target.

Assuming that the airplane maintains a constant altitude of 10km, which is more realistic.



Need to determine from the Tx Antenna to the point that the signal received by the submarine hits the water.

$cos\theta_i = \frac{R_{alt}}{R_{tot}}$ -> $R_{tot} = \frac{R_{alt}}{cos\theta_i} = \frac{(10{,}000\ m)}{\cos 60^0}$ = 20,000 meters = 20km

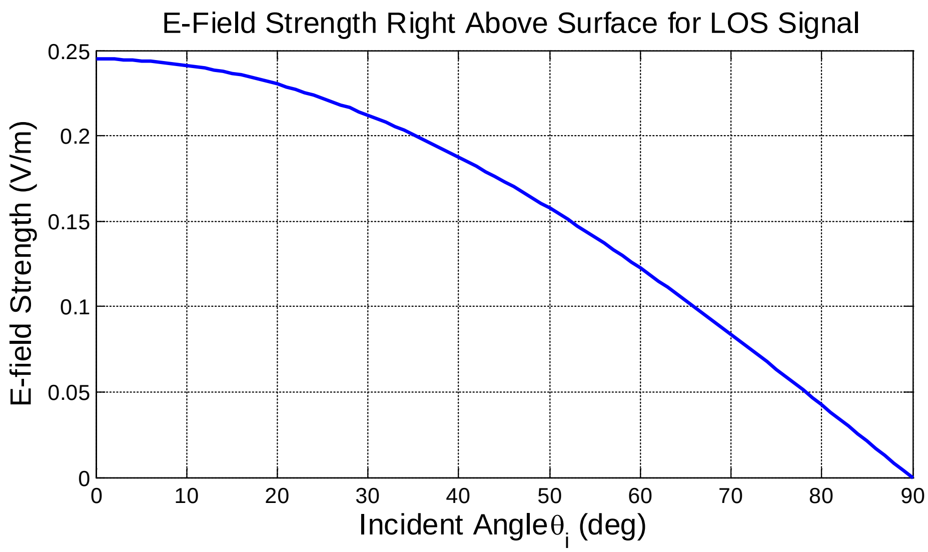


Next step is to scale the E-field strength at the surface of the water based on the new distance:

P_D= P_t/ A_{SA} = P_t/ (4*pi*R^2) = (200,000 W)/($1.257 * 10^9\ m^2$)

$$P_D = \frac{P_t}{A_{SA}} = \frac{P_t}{4\pi R_{tot}^2} = \frac{E^2}{\eta_{air}}$$

$$E_0^i = \sqrt{\frac{\eta_{air} P_t}{4\pi R_{tot}^2}} = \sqrt{\frac{(377\Omega)(200{,}000\ W)}{4\pi(20{,}000\ m)^2}} = 0.122 \text{ V/m}$$



$$|E^t| = \left|\tau E_0^i e^{-\alpha d}\right|$$

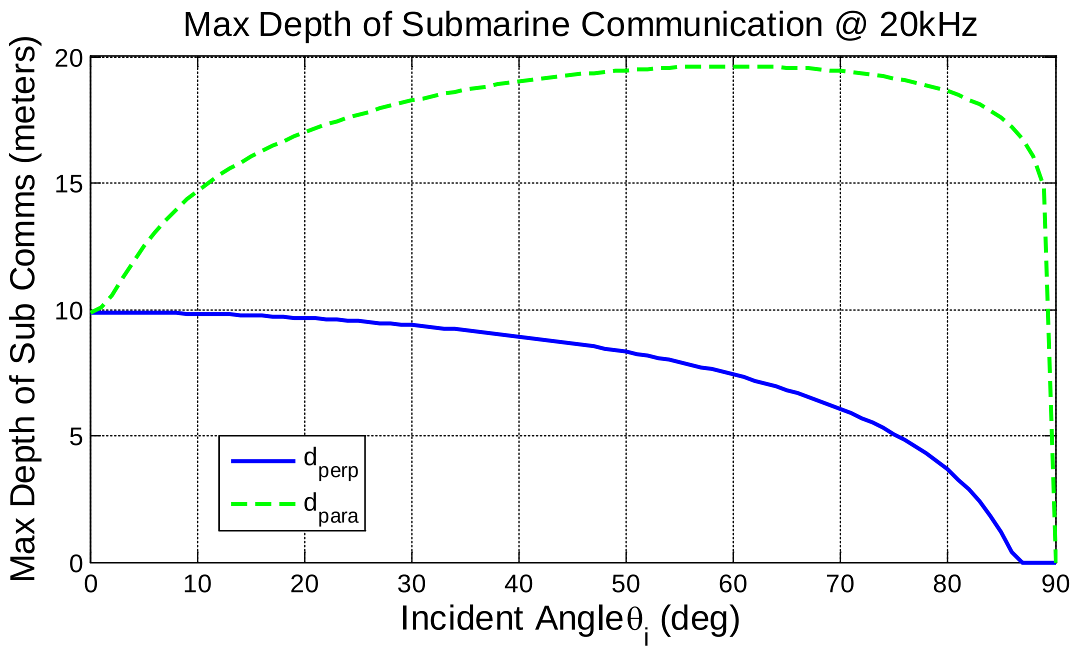
$$E^t = |\tau| E_0^i e^{-\alpha d} \text{ -> } \ln\frac{E_t}{|\tau| E_0^i} = -\alpha d$$

Max Depth of Communication to Submarine:

$$d = -\frac{1}{\alpha}\ln\frac{E_t}{|\tau| E_0^i}$$

$$d_\perp = -\frac{1}{\alpha}\ln\frac{E_t}{|\tau_\perp| E_0^i} = -\frac{1}{(0.562\ \text{Np/m})}\ln\frac{10^{-6}\ \text{V/m}}{|3.7268e{-}04 + j3.7254e{-}04|(0.122\ \text{V/m})} = 7.42 \text{ meters}$$

$$d_{||} = -\frac{1}{\alpha}\ln\frac{E_t}{\left|\tau_{||}\right| E_0^i} = -\frac{1}{(0.562\ \text{Np/m})}\ln\frac{10^{-6}\ \text{V/m}}{|0.5007 + j0.0007|(0.122\ \text{V/m})} = 19.62 \text{ meters}$$



As the plot above demonstrates, to achieve communication from the surveillance aircraft to the submarine with the submarine at a maximum depth, a LOS angle of 60 degrees off of zenith from the aircraft to the submarine should be used with a Tx signal that has the E-field in parallel polarization with respect to the surface of the ocean. Of course, in real life, the seawater is not completely flat (sometimes very choppy) and the seawater properties do vary with temperature. Choppy seas will definitely degrade this performance and a number of studies have been done to characterize the reflectivity of seawater at a number of different sea and temperature conditions.

Question #2:

Disclaimer: For security reasons due to my employment at Northrop Grumman, I am not able to speculate on or have the appearance of speculating on the radar detectability of any military platforms. As an example, Northrop Grumman builds the APG-83 SABR AESA for the F-16 so you can see why this may be relevant. For the purposes of this exam, I will refer to a purely hypothetical fighter jet as the F-XY, which will be assumed to have similar general characteristics to the F-16. Thank you for your understanding. While this may seem frivolous, I want to play it safe and not have to answer any questions about it later.

Assumptions:

- Assuming that the antenna Tx loss only applies on transmit and not on receive

Parameter	**Value**	**Unit**
Frequency	3	GHz
Antenna Gain	38	dBi
Receiver Noise Figure (Rx NF)	6	dB
Receiver Bandwidth (Rx BW)	750	kHz
System Losses	13	dB
Antenna Transmit Loss (L_{Ant_Tx})	2	dB
$SNR_{Detection\_Min}$	8.2	dB

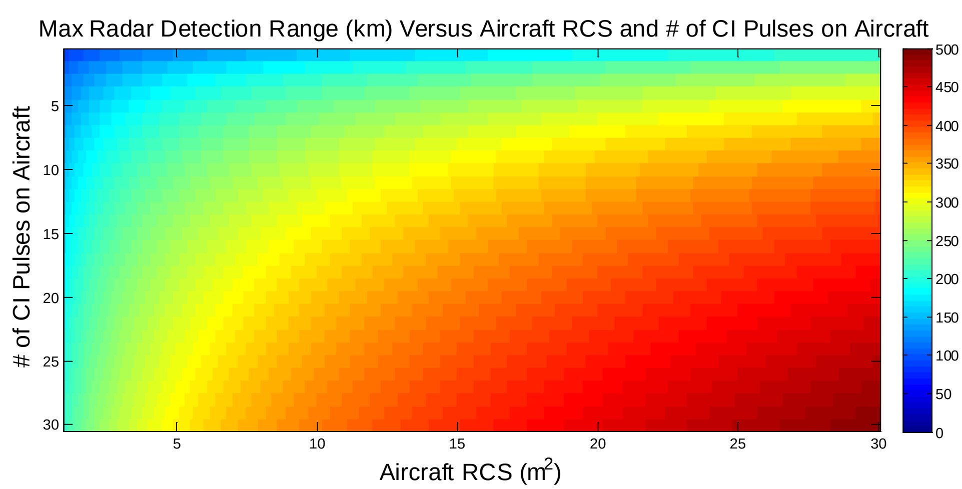
Dear Air Force General X,

The purpose of this memo is to inform you on the capabilities of the adversarial 9SXX radar system located near the enemy facility of interest and to inform you on the potential threat that it poses to our F-XY attack aircraft. My goal is to give you and your command team the relevant information necessary to successfully complete your mission consisting of intelligence, surveillance, reconnaissance, and strike (ISR&T) operations on the targeted enemy ground facility while minimizing risks to our military assets and most importantly protecting the warfighter. Given information provided to us through our human and signals intelligence services on the relevant system parameters of the 9SXX radar, I have run trade studies to give you an understanding of the range at which you can expect the 9SXX radar to begin detecting our F-XY fighter aircraft under various conditions.

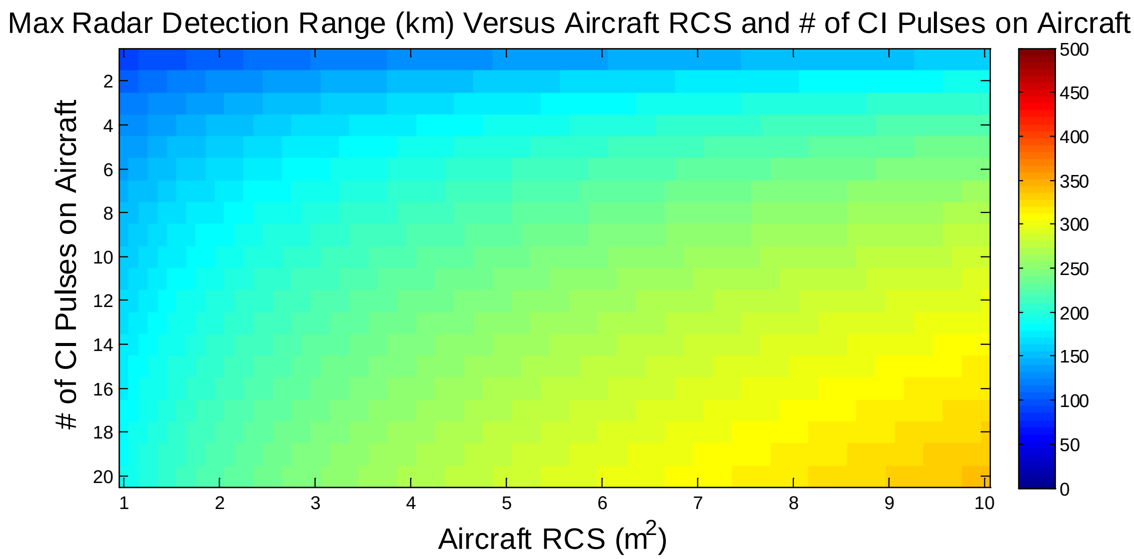
This enemy early warning radar is an S-band search and track radar operating at approximately 3GHz. Given the state of current technology, this is likely an active electronically scanning array (AESA) mobile air defense radar and can be easily transported on a ground vehicle or ship and can come on-line quickly from remote locations. These mobile radar systems are often accompanied by mobile surface-to-air missile (SAM) launchers, creating a real threat for our fighter. Intelligence suggests that the peak power of this 9SXX radar system is roughly 800kW of peak transmitted power. The Chinese JY-8 mobile 3D air surveillance C-band radar has a similar peak transmitted power level and can track dozens of target simultaneously so your command needs to be prepared for this sort of sophisticated mobile threat during your mission.

The ability for this threat radar to detect our F-XY fighter jets is largely dependent on the effective radar cross section (RCS) of our fighter, the number of simultaneous radar pulses they can get on our fighter, their unknown signal processing capabilities, the weather, and our jamming capabilities. The enemy radar's search volume, pulse repetition frequency (PRF) of the radar, exact antenna beamwidths, and other parameters are unknown so I will walk you through the results of the parametric studies that I have done to help you understand the range of possible detection ranges for our F-XY fighter against the enemy 9SXX radar.

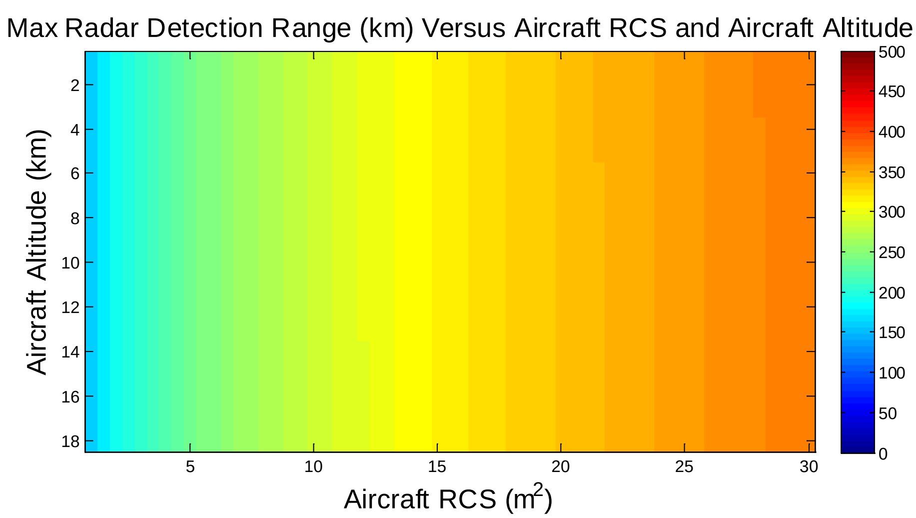
Given my assumptions and available intelligence information, we can expect the 9SXX radar to detect our F-XY fighter jet from a range of approximately 140km with a single pulse on target, 250km with a reasonable 10 pulses on the target using coherent integration (CI) gain, and roughly 300km with a reasonable number of 20 pulses on the target. The radar may only be able to get one or a few pulses on our F-XY aircraft during search mode, however, if there is a quick angular dependent spike in the RCS of our aircraft due to glint, the enemy radar may be able to go into track mode and reasonably put at least 10-20 pulses on our fighter, which effectively doubles the detection range of this 9SXX radar system. Additionally, the RCS of our aircraft is both angle and frequency dependent, however, I have assumed the RCS to be $6m^2$ in the range numbers stated above. In addition, the RCS of our aircraft is dependent on a number of factors including the armament and pod configurations on the wings, the types of radar absorbing coatings that have been applied to the aircraft, and the configuration of antennas located around the aircraft. All of these factors contribute to unwanted scattering and specular reflections that can temporarily increase the RCS of the aircraft. Given more detailed information about the specifics of our F-XY aircraft and the preferred armament and pod configuration of the aircraft for this mission, our team can run electromagnetic (EM) simulations to determine more accurate frequency and angle dependent numbers for the RCS of the F-XY at the appropriate classification level. For now, I have run a parametric study of detection range versus number of pulses on target and aircraft RCS to show you the range of expected values in the chart below:



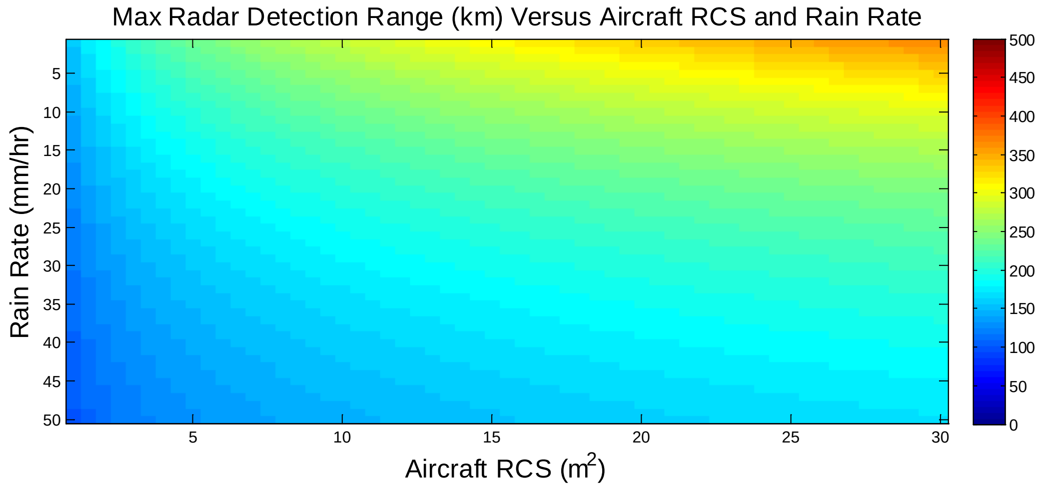
To zoom-in on the primary region of interest:



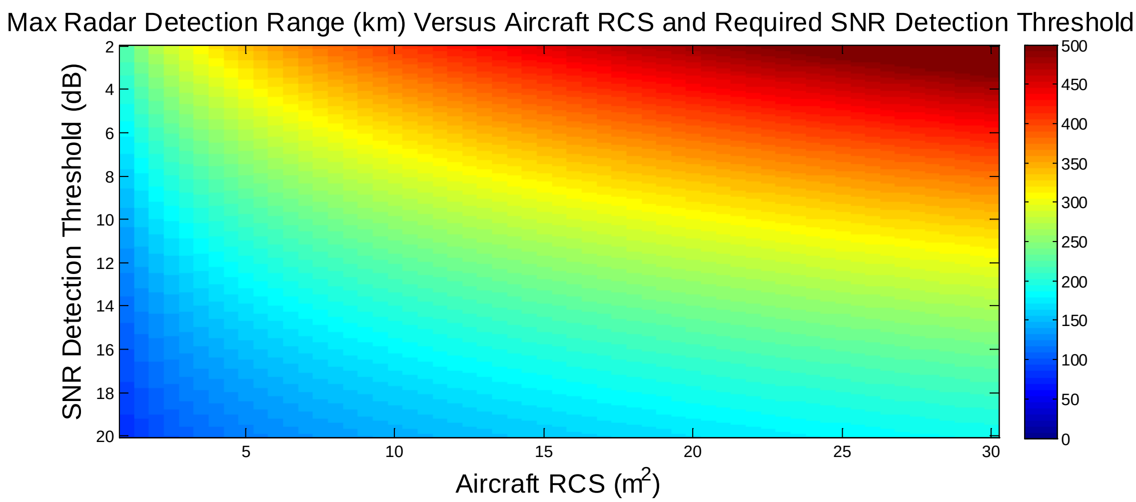
Given the data in the two charts above, our estimates show that it is reasonable to get within 200km of the 9SXX early warning search radar in the F-XY fighter without detection, however, your command and pilots should be prepared to be detected within 300km in reasonable fair weather conditions. For the rest of the plots in this memo, assume the # of CI pulses to be ten unless explicitly stated. Note that this is LOS range from the aircraft and the radar and altitude of the aircraft must be taken into consideration. From the plot below going up to 18km (~60,000ft) in altitude, the altidude of the aircraft has a neglible effect on the maximum detectable range of the radar:



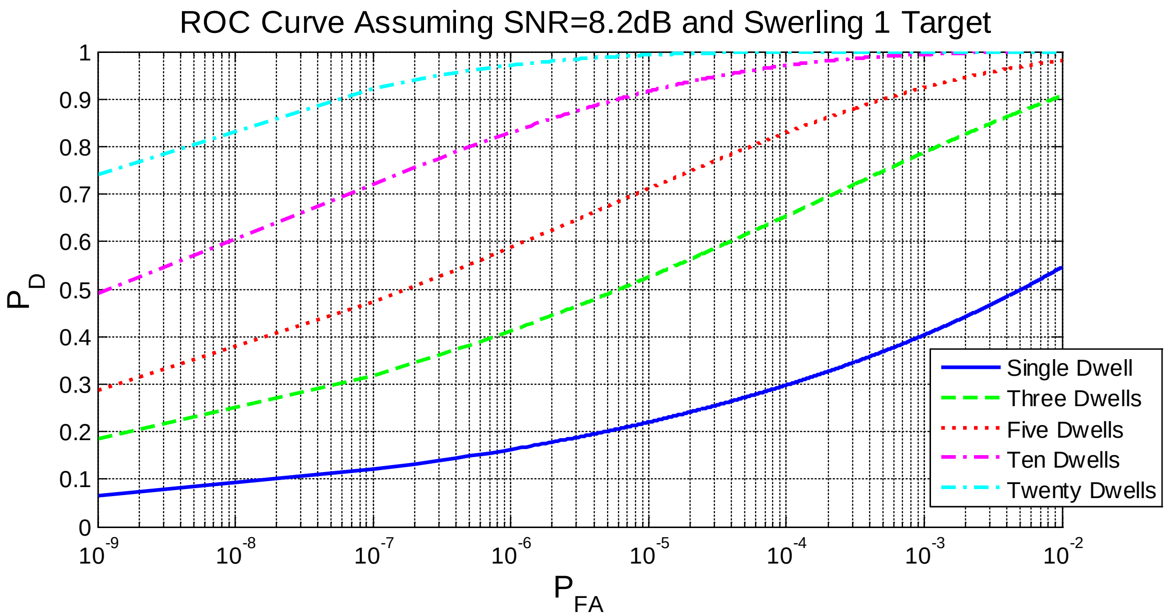
It should also be noted that performing your mission in rain will degrade the performance of your own sensors, however, it also has the potential to significantly decrease the maximum range at which the 9SXX radar can detect the F-XY aircraft shown in the chart below:



Currently, the estimated minimum signal-to-noise (SNR) threshold for target detection of the 9SXX radar is assumed to be 8.2dB, which is used as the detection threshold in the charts above, however, the required SNR detection threshold could change with environmental conditions, clutter level, jamming, or improved signal processing software algorithms. Below is a chart that shows the range of maximum radar detection range versus the aircraft RCS and SNR detection thresholds:



A ROC chart is shown below to understand the probability of detection (P_D) versus probability of false alarm (P_{FA}) for the fighter jet at the expected SNR threshold of 8.2dB, which will be seen at the maximum range of detection. From these results, we can see that a lot of pulse dwells are required to be put on the target for the enemy radar to achieve a reasonable P_D and an a reasonable P_{FA}.



In the table below, I made estimates for several other relevant radar parameters that may be useful to your team in mission planning:

Parameter	Value
Estimated Radar Pulse Repetition Frequency (PRF)	~500 Hz (Low PRFs are typically used for long-range surveillance and search radars)
Effective Radar Antenna Aperture Size	5 m^2
Approx 3dB Antenna Beamwidth	3.1 degrees
Range Resolution	2 km
Unambiguous Range	300 km
Unambiguous Doppler Velocity	12.5 m/s

*Note: Some of these ambiguities can likely be overcome using signal processing techniques

Lastly, all of this radar detection analysis has been done in the absence of jamming. In many real combat scenarios, stand-off jamming using a jamming pod or multi-function AESA radar is used to either blast the radar with random noise, which effectively raises the noise floor and decreases the effective SNR of the radar and decreases the range of maximum detection. For the F-XY fighter aircraft, it is likely that we have more sophisticated jamming techniques utilizing digital RF memory (DRFM) and advanced electronic warfare, electronic attack, and electronic protection (EW/EA/EP) algorithms. These EW techniques will allow either one or multiple fighter aircraft to jam the target while the other aircraft in the group go in undetected for the surveillance or the kinetic or non-kinetic attack. Additional background info on the effectiveness of jamming can be found in Appendix F of this report.

In conclusion, through the various parametric trades, calculations, and analysis in this report, you can see that a number of factors, including environmental factors, determine the range at which our F-XY fighter jet will first be detected by the 9SXX radar system protecting the enemy facility of interest. Flying-in with a low RCS configuration will do a lot to reduce the radar detectability of the fighter aircraft and increase the maximum range of radar detection. Given reasonable estimates and the operation of the search mode on the 9SXX radar, your F-XY aircraft could reasonably get within 150-200km without being detected by the enemy radar, however, using more conservative estimates, you should prepare for potential detection within 300km. It only takes one RCS spike at a given angle and the enemy radar can switch into track mode, use coherent integration by putting more radar pulses on the aircraft and increase the detectable range to potentially ~350km or more. Given current technologies and military capabilities, stand-off jamming techniques can be used to temporarily trick the enemy radar or flood it with random noise, which will allow the aircraft to travel much closer to the target on an attack run without being detected by the enemy radar. Thank you very much for taking the time to read this report. Hopefully, you were able to gain relevant knowledge to enable your team to successfully accomplish your mission while minimizing risk to the warfighter and our military assets. Please reach out to me if you need additional radar detection modeling or want to model and simulate more sophisticated mission scenarios.

Thank you very much,

John Hodge

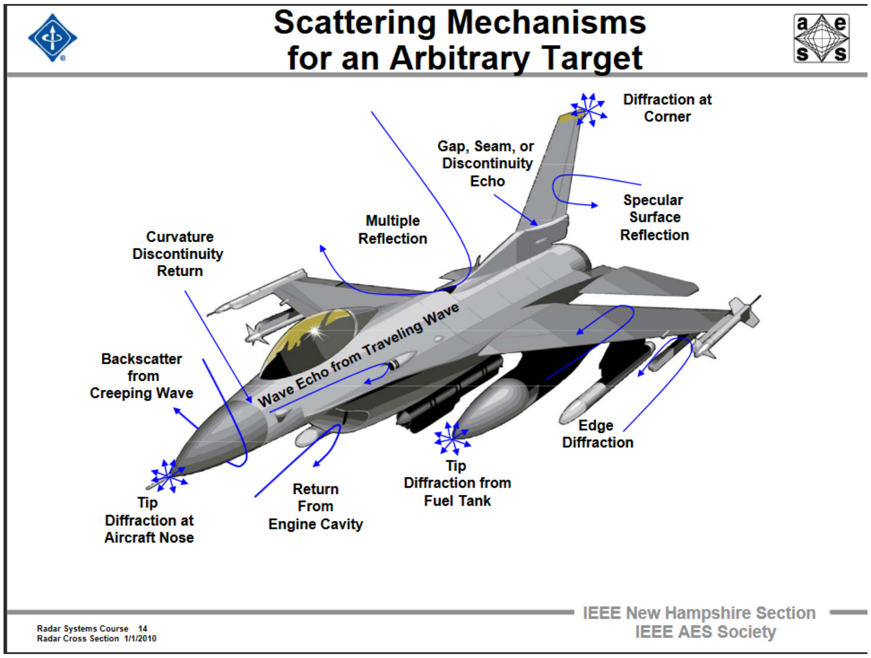
XYZ Radar System Engineer and Dection Analyst

Radar Performance Blake Chart (Implementing Radar Range Equation):

Numerator					
Quantity	**Unit**	**Linear**	**dB - Gain**	**dB - Losses**	
Pt	W	800000	59.0		
Gt	dBi		38.0		
Gr	dBi		38.0		
lambda^2	m^2	0.01	-20.0		
RCS	m^2	6	7.8		
# of Pulses		10	10.0		
Denominator					
(4*pi)^3		1984.40171		33.0	
k	w/Hz*K	1.38E-23		-228.6	
To	K	290		24.6	
B	Hz	7.50E+05		58.8	
NF	dB			6.0	
Ls	dB			13.0	
L_Tx_Ant	dB			2.0	
SNR	dB			8.2	
Totals			132.8	-83.1	
R^4 (dB)					215.9

R^4 (m^4)	3.85721E+21
Range (m)	249211.6
Range (km)	249.2
Range (Nm)	134.6
Range (miles)	154.9

From MIT LL Radar Systems Course Lecture 7:

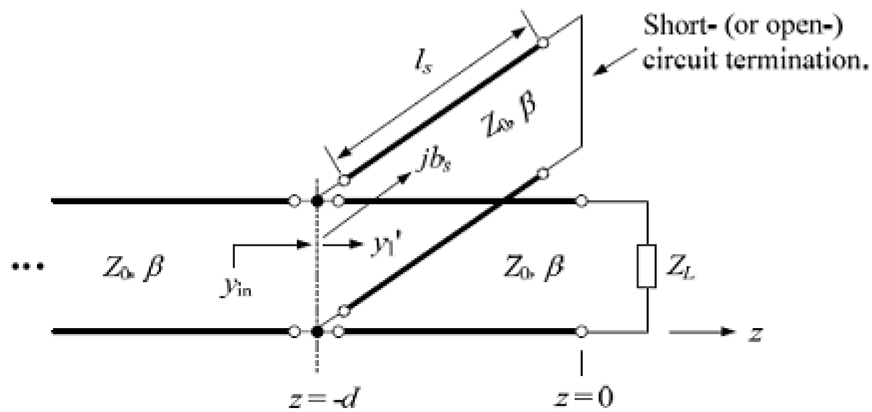


Picture of Similar Adversarial Radar System:

Question #3: Transmission Line Single-Stub Impedance Matching

Quantity	Value	Units
Z_0	50	Ohms
Z_L	40 – j10	Ohms
f	800	MHz
v_p	c	m/s

Diagram of a Single-Stub RF Impedance Matching Circuit of a Load



f = 800 MHz = 8.0e8 Hz -> λ = 0.37m

$\omega = 2\pi f = 2\pi(8.0\text{e}8\text{ Hz})$ = 5.027e9 rad/s

$$Z_L = 40 - j10 = R + \frac{1}{j\omega C}$$

Initial Reflection Coef:

$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(40 - \text{j}10) - 50}{(40 - \text{j}10) + 50}$ = -0.098 - j0.122; -> $|\Gamma_L|$ = 0.156; -> $|\Gamma_L|^2$ = 0.0244 = -16.1dB

Determine Equivalent RF Circuit Values for Load:

R = 40 Ohms; C = 19.89e-12 F = 19.89pF

Normalized Impedance of Load:

$$z_L = \frac{Z_L}{Z_0} = \frac{40 - j10\ \Omega}{50\ \Omega} = 0.8 - j0.2$$

Normalized Admittance of Load:

$$y_L = \frac{1}{z_L} = \frac{1}{0.8 - j0.2} = 1.1765 + j0.2941$$

Input impedance of a lossless transmission line is defined by the telegrapher's equation as follows:

$$Z_{in}(l) = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

Where l is the length of the transmission line and $\beta = \frac{2\pi}{\lambda}$. For this lossless transmission line case, the propagation constant of the wave is purely imaginary such that: $\gamma = j\beta$

*Note: The Smith charts that I used to solve this problem are located in Appendix G.

Additionally, the impedance of short and open circuit stubs is defined as:

Short Stub: $Z_{in}(l) = jZ_0 \tan(\beta l)$

Open Stub: $Z_{in}(l) = -jZ_0 \cot(\beta l)$

Short-Circuit Parallel Stub:

For the short circuit case, it is easier to work in terms of normalized admittances. The input admittance of a stub is always imaginary (inductive if negative, capacitive if positive).

Use math or Smith chart to find a normalized admittance with a real part of 1 using a constant admittance magnitude circle. Alternatively stated, we need to find the points where the unitary conductance circle intersects the constant |Γ(d)| circle on the Smith chart.

These two intersections on the unitary conductance circle are:

y_1 = 1 – j0.325 -> Need to match w/ +j0.325

y_2 = 1 + j0.325 -> Need to match w/ -j0.325

The distance (d) for each stub is determined by the distance in terms of wavelengths that we must rotate around the Smith chart from y_L in order to reach the two intersection points

d_1 = 0.363 λ – 0.182 λ = 0.181 λ = 0.181*(0.37m) = 0.067m

d_2 = (0.5 λ – 0.182 λ) + 0.137 λ = 0.455 λ = 0.455*(0.37m) = 0.168m

To calculate the length of the stub we must find the necessary stub impedance to add to y1 to make the resulting admittance purely real. As stated in Pozar ME Section 5.2, the length of a short-circuited stub that gives this susceptance can be found on the Smith chart by starting at y = ∞ (the short circuit) and moving along the outer edge of the chart (g = 0) toward the generator to the +j0.325 and -j0.325 points respectively.

l_1 = 0.250 λ + 0.051 λ = 0.301 λ = 0.301*(0.37m) = 0.111m

$$l_2 = 0.450\ \lambda - 0.250\ \lambda = 0.20\ \lambda = 0.20*(0.37m) = 0.074m$$

Open-circuit parallel single-stub:

For the open-circuit parallel single-stub case, we will begin on the opposite side of the Smith chart at $y = 0$ (the open circuit) and moving along the outer edge of the chart ($g = 0$) toward the generator to the $+j0.325$ and $-j0.325$ points respectively.

$$l_1 = 0.051\ \lambda = 0.051*(0.37m) = 0.019m$$

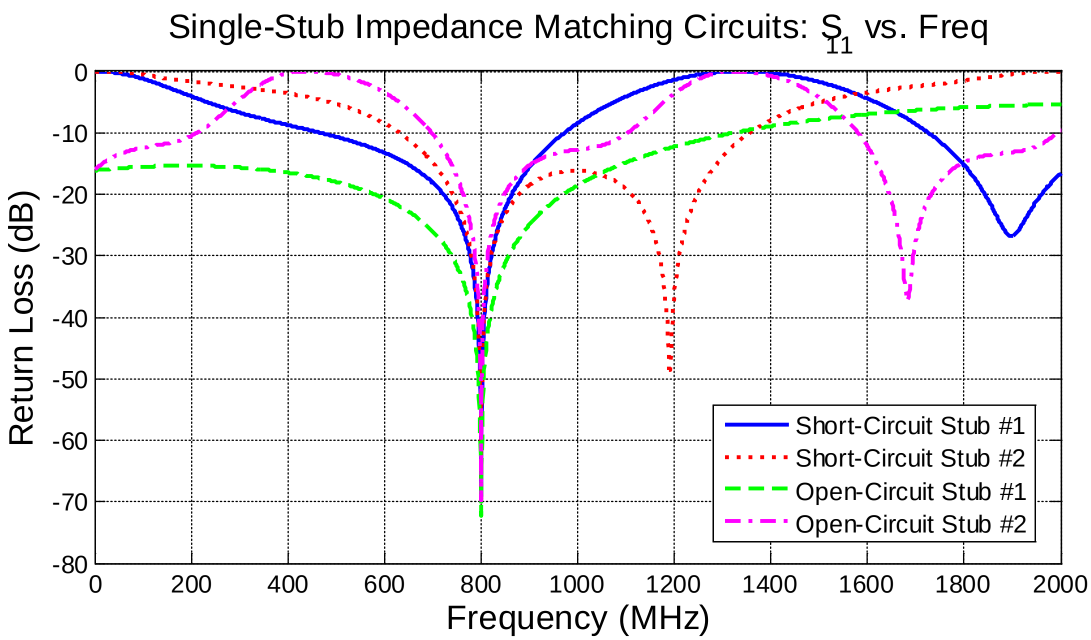
$$l_2 = 0.50\ \lambda - 0.051\ \lambda = 0.449\ \lambda = 0.449*(0.37m) = 0.166m$$

Results Using Smith Chart:

Variable	Parallel Short-Circuit Stub (m)	Parallel Open-Circuit Stub (m)
d_1	0.067	0.067
d_2	0.168	0.168
l_1	0.111	0.019
l_2	0.074	0.166

Used Gradient Optimizer to Fine-Tune Smith Chart Values for Best Match at 800 MHz:

Variable	Parallel Short-Circuit Stub (m)	Parallel Open-Circuit Stub (m)
d_1	0.0689	0.0689
d_2	0.1720	0.1720
l_1	0.1120	0.0183
l_2	0.0755	0.1692



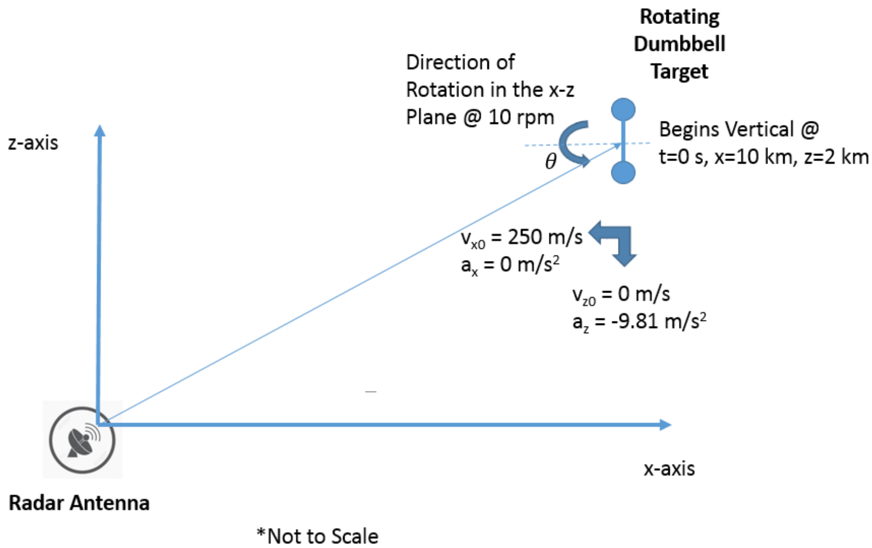
The optimal design really depends on the design requirements of a given circuit or system. For a product design, the engineering team must weigh the RF performance vs. the size, weight, power-handling, temperature-handing, and cost constraints of the given application, platform, and mission. In addition, available technologies, materials, and manufacturing tolerances also drive the design decision between potential design variations. If bandwidth w/ better than 30dB match is the only design requirement, open-circuit stub #1 would be the optimal design choice. This design also has the smallest combined line length (l+d) making it also the smallest design of these four design variations. Given that it is best in both BW @ 800 MHz and has the shortest combined line length, this would probably be the best choice for many applications. For some applications, a short circuit stub may be less prone to leakage of EM radiation and maybe somewhat easier to realize, which could make one of the short-circuit stub implementations a better choice for that specific application.

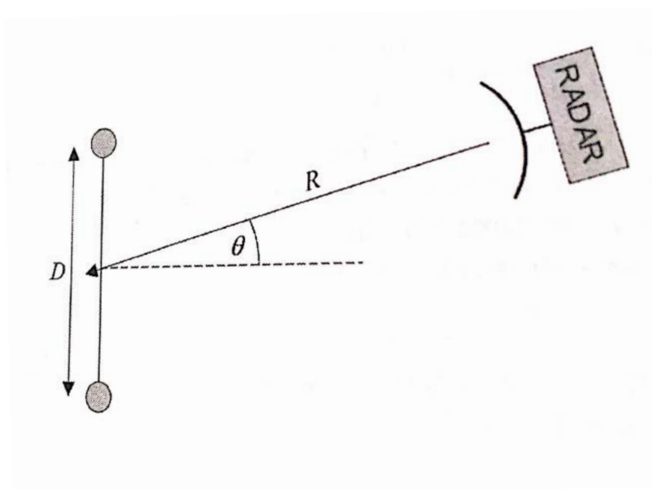
Open-Circuit Series Stub:

I do not believe that it was asked for, however, to play it safe, I also calculated the lengths for series open-circuit stubs using the second Smith chart in Appendix G.

Value	**Design #1**	**Design #2**
d (λ)	0.210	0.434
d (meters)	0.078	0.161
y	1 + j*0.325 -> Need -0.325j	1 – j*0.325 -> Need -0.325j
l (λ)	0.199	0.301
l (meters)	0.074	0.111

Question #4: RCS of Spinning Dumbbell w/ Consisting of Two Small Rotating PEC Spheres





Parameter	Value	Units
Diameter of Each Sphere	1	cm
Length of Rod Connecting Spheres	15	cm
Initial Target Height	2	km
Initial Horizontal Velocity	250	m/s
Angular Velocity of Target	10	rpm
Radar Frequency	1	GHz

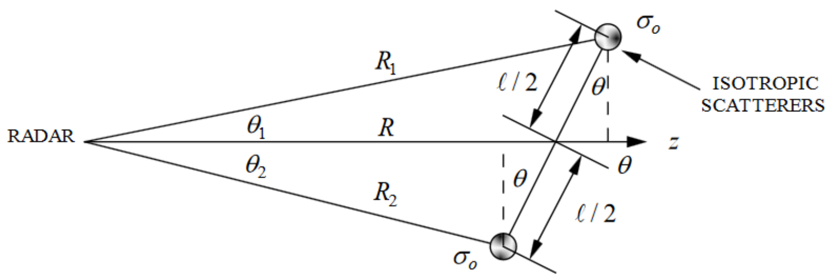
Assumptions:

- Both spheres are PEC
- Rod connecting spheres has no RF interaction or scatterings
- Assuming that the acceleration is -9.81 m/s^2
- Assuming that atmosphere has no effect on the motion of the target (as stated in problem) resulting in no aerodynamic drag or other atmospheric forces outside of drag acting on the velocity and motion of the spinning target
- Assuming that the ground is perfectly flat (as stated in problem)
- Radar perfectly tracks the target (as stated in problem) and the antenna beamwidth is wide enough to capture the entire spinning target
- Target rotates at a steady rate (as stated in problem) and remain rotating in the vertical plane throughout the entire trajectory of the target until it hits the ground
- Assuming radar antenna is located at the origin of the coordinate system such that x=0, y=0, z=0
- Assuming that initial position of the spinning target in the relative coordinate system is x=10km, y=0, z=2km
- Assuming that the sky noise temperature and background clutter are either negligible or constant over the field of view of the target along its trajectory and will not have any effect of the radar's relative received power as it tracks the target
- Assuming that the center of mass of the target is perfectly midway between the two spheres and that the spheres are rotating around this center of mass
- Radar is a monostatic radar using the same antenna at the same fixed location at the origin for both Tx and Rx
- Target will be within the radar systems required SNR and dynamic range throughout the entire trajectory of the target until it hits the ground
- Assuming that the radar properly accounts for the Doppler shift in received power and that the electronics have a steady response over the band of interest, such that Doppler shift of the received signal of the target does not affect the received power
- Assuming that the spheres are vertically oriented at t = 0 and that the angular rotation of the spheres is in the direction that the top sphere initially rotations towards the radar while the bottom sphere rotates away from the radar
- The distance between the centers of the two spheres is d = 15cm, meaning that the connecting rod extends to the center of the spheres [In Appendix E, I add resulting plots for the case where d = 16 cm (15 cm rod + 2*0.5cm sphere radius). Was not completely sure how the geometry of this hypothetical target should be interpreted, however, results are similar]

Diagram:

As outlined in the Complex Target lecture by Jenn (below; more info in Appendix B) as well as Richards Fundamentals of Radar Signal Processing (FoRSP) section 2.2.5 on pg. 68, and in the Radar cross section lectures by Fuhs, the problem of RCS from two closely spaced isotropic scattering spheres rotated at an angle of theta relative to the observer is outlined below:

Consider the RCS obtained from two isotropic scatterers (approximated by spheres).



Law of cosines:

$$R_1 = \sqrt{R^2 + (\ell/2)^2 - 2R(\ell/2)\cos(\theta + \pi/2)} = R\sqrt{1 + (\ell/2R)^2 + 2(\ell/2R)\sin\theta}$$
$$R_2 = \sqrt{R^2 + (\ell/2)^2 - 2R(\ell/2)\cos(\theta - \pi/2)} = R\sqrt{1 + (\ell/2R)^2 - 2(\ell/2R)\sin\theta}$$

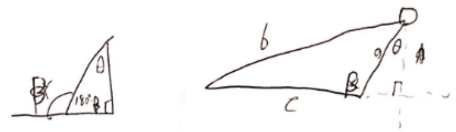
Let $\alpha = \ell\sin\theta/R$ and note that

$$(1 \pm \alpha)^{1/2} = 1 \pm \frac{1}{2}\alpha \mp \underbrace{\frac{3}{8}\alpha^2 \pm \cdots}_{\text{NEGLECT SINCE } \alpha << 1}$$

Below I go through the law of cosines derivation used for this problem:

Q4

α



$$180 = (180-\beta) + 90 + \theta$$

$$\beta = \theta + 90^\circ = \theta + \frac{\pi}{2}$$

$$b^2 = a^2 + c^2 - 2ac \cdot \cos\beta$$

$$R_1^2 = \left(\frac{l}{2}\right)^2 + R^2 - 2\left(\frac{l}{2}\right)(R)\cdot\cos(\beta)$$

$$R_1^2 = \left(\frac{l}{2}\right)^2 + R^2 - Rl\cdot\cos\left(\theta + \frac{\pi}{2}\right)$$

$$R_1 = \sqrt{\left(R^2 + \left(\frac{l}{2}\right)^2 - 2R\left(\frac{l}{2}\right)\cdot\cos\left(\theta+\frac{\pi}{2}\right)\right)}$$

$$R_1 = R\sqrt{1 + \left(\frac{l}{2R}\right)^2 - \frac{l}{R}\cdot\underbrace{\cos\left(\theta+\frac{\pi}{2}\right)}_{=-\sin\theta}}$$

$$R_1 = R\sqrt{1 + \underbrace{\left(\frac{l}{2R}\right)^2}_{\text{Term is very small since } R>>l \text{ and can be neglected}} + \frac{l}{R}\cdot\sin(\theta)} = R\sqrt{1 + \underbrace{\frac{l}{R}\cdot\sin\theta}_{\alpha}}$$

$$R_1 = R\sqrt{1+\alpha} = R\cdot(1+\alpha)^{1/2} = R\left(1 + \frac{\alpha}{2} \underbrace{- \frac{\alpha^2}{8} + \frac{\alpha^3}{16} \cdots}_{\text{can neglect since } \alpha<<1}\right)$$

Series expansion yields

$$R_1 \approx R\left(1+\frac{\alpha}{2}\right) = R + \frac{l}{2}\cdot\sin\theta$$

$$R_1 \approx R + \frac{l}{2}\sin\theta\ ;\ R_2 \approx R - \frac{l}{2}\sin\theta$$

$$\sigma_{eff} = 4\pi R^2|E_s|^2$$

Doing math that will be used for the scattered E-field calculation:

$$2\pi f\left(t - \frac{2R_1(\theta)}{c}\right) = 2\pi f t - (2\pi f)\left(\frac{2R_1(\theta)}{c}\right) = 2\pi f t - \left(\frac{4\pi f R_1(\theta)}{c}\right)$$

$$= 2\pi f t - \left(\frac{4\pi f\left(R + \frac{l}{2}\sin\theta\right)}{c}\right) = 2\pi f t - \frac{4\pi f R}{c} - \frac{(4\pi f)\left(\frac{l}{2}\sin\theta\right)}{c}$$

$$= 2\pi f t - \frac{4\pi f R}{c} - \frac{2\pi f l\sin\theta}{c} = 2\pi f t - \frac{4\pi f R}{c} - \frac{2\pi l\sin\theta}{\lambda}$$

$$= 2\pi f t - \frac{4\pi f R}{c} - kl\sin\theta = 2\pi f\left(t - \frac{2R}{c}\right) - kl\sin\theta$$

Next:

$$e^{2\pi f\left(t-\frac{2R_1(\theta)}{c}\right)} = e^{j2\pi f(t-\frac{2R}{c})}e^{-jkl\sin\theta}$$

$$e^{2\pi f\left(t-\frac{2R_2(\theta)}{c}\right)} = e^{j2\pi f(t-\frac{2R}{c})}e^{+jkl\sin\theta}$$

$$e^{2\pi f\left(t-\frac{2R_1(\theta)}{c}\right)} + e^{2\pi f\left(t-\frac{2R_2(\theta)}{c}\right)} = e^{j2\pi f(t-\frac{2R}{c})}e^{-jkl\sin\theta} + e^{j2\pi f(t-\frac{2R}{c})}e^{+jkl\sin\theta}$$
$$= e^{j2\pi f\left(t-\frac{2R}{c}\right)}(e^{-jkl\sin\theta} + e^{+jkl\sin\theta}) = e^{j2\pi f\left(t-\frac{2R}{c}\right)}(2\cos(kl\sin\theta))$$

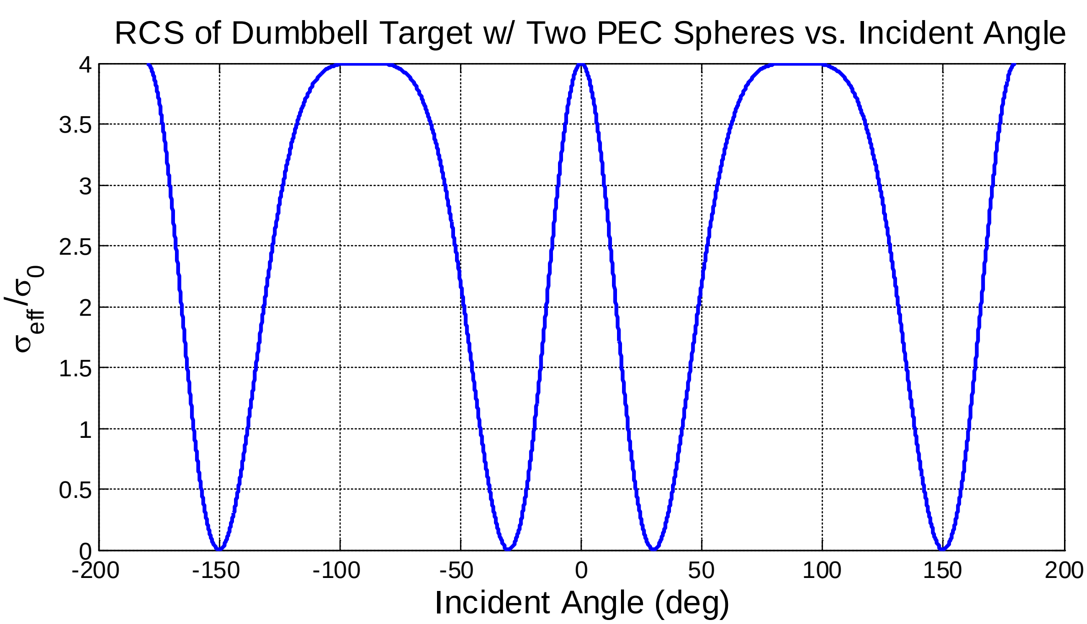
Scattered E-field from the two sphere system can be described as:

$$E_s(t) = (\sqrt{\sigma_0}e^{j2\pi f\left(t-\frac{2R_1(\theta)}{c}\right)} + \sqrt{\sigma_0}e^{j2\pi f\left(t-\frac{2R_2(\theta)}{c}\right)})/\sqrt{4\pi R^2}$$
$$= \sqrt{\sigma_0}e^{j2\pi f\left(t-\frac{2R}{c}\right)}\left(e^{-\frac{j2\pi fl\sin\theta}{c}} + e^{+\frac{j2\pi fl\sin\theta}{c}}\right)/\sqrt{4\pi R^2}$$
$$= 2\sqrt{\sigma_0}e^{j2\pi f\left(t-\frac{2R}{c}\right)}\cos(\frac{2\pi fl\sin\theta}{c})\,/\sqrt{4\pi R^2}$$

$$|E_s|^2 = \frac{4\sigma_0\cos(\frac{2\pi fl\sin\theta}{c})^2}{4\pi R^2} = \frac{\sigma_0\cos(\frac{2\pi fl\sin\theta}{c})^2}{\pi R^2} = \frac{\sigma_0\cos(\frac{2\pi l\sin\theta}{\lambda})^2}{\pi R^2} = \frac{\sigma_0\cos(kl\sin\theta)^2}{\pi R^2}$$

$$\sigma_{eff} = 4\pi R^2|E_s|^2 = (4\pi R^2)\left(\frac{\sigma_0\cos(kl\sin\theta)^2}{\pi R^2}\right) = 4\sigma_0\cos(kl\sin\theta)^2$$

Plotting this equation:



The effective RCS of the dumbbell target oscillates are the reflection phase of the two spherical constructively and destructively interferes. The maximum RCS scattering is four times that of the single PEC sphere because 2x the E-field is reflected back when both reflections are completely in-phase and the reflected power is related to the square of the E-field resulting in 4x the max scattered power for two spheres compared to a single scattering sphere. This is explained further in a problem from Professor Ruohoniemi's ECE 5635 Radar Course shown in Appendix A.

Scattering between the two spheres is at a max when:

$$kl \sin\theta = \pm n\pi \text{ (where n is an integer)}$$

Scattering between the two spheres is at a minimum and destructive interferes when:

$$kl \sin\theta = \pm n\frac{\pi}{2} \text{ (when n is an integer)}$$

The rotating dumbbell target consisting of two PEC spheres will cause deep fades in the received power of a radar when the relative phase between the two isotropic scatters are destructively interfering.

Calculation of the RCS for the Single PEC Sphere (σ_0):

$$f = 1\ GHz = 10^9\ Hz; \rightarrow \lambda = 0.3 \text{ meters}$$

Radius of single PEC sphere: d = 1cm = 0.01m -> a = r_{sphere} = d/2 = 0.005m

$$ka = \left(\frac{2\pi}{\lambda}\right)(0.005m) = \left(\frac{2\pi}{0.3m}\right)(0.005m) = 0.10472$$

$\frac{a}{\lambda} = \frac{0.005m}{0.3m} = 0.0167$ << 0.1 -> Rayleigh Scattering Region

Unfortunately, the standard RCS formula for a sphere in the optical region σ = π a^2 does not apply due to the small electric size of the sphere. According to Jin in the ECE 5105/5106 textbook Theory and Computation of Electromagnetic Fields in in Section 7.4.3 on pg. 273: the Rayleigh scattering region applies to the cross-section of spheres with a/λ < 0.1. According to Jin in equation 7.4.49, the small-argument approximation for the RCS of the PEC sphere can be found using:

$$\lim_{\frac{a}{\lambda}\to 0} \sigma_{3D} \approx \frac{9\lambda^2}{4\pi}(ka)^6 = \frac{9(0.3m)^2}{4\pi}\left(\left(\frac{2\pi}{0.3m}\right)(0.005m)\right)^6 = 8.5 * 10^{-8}\ m^2 \propto \frac{1}{\lambda^4}$$

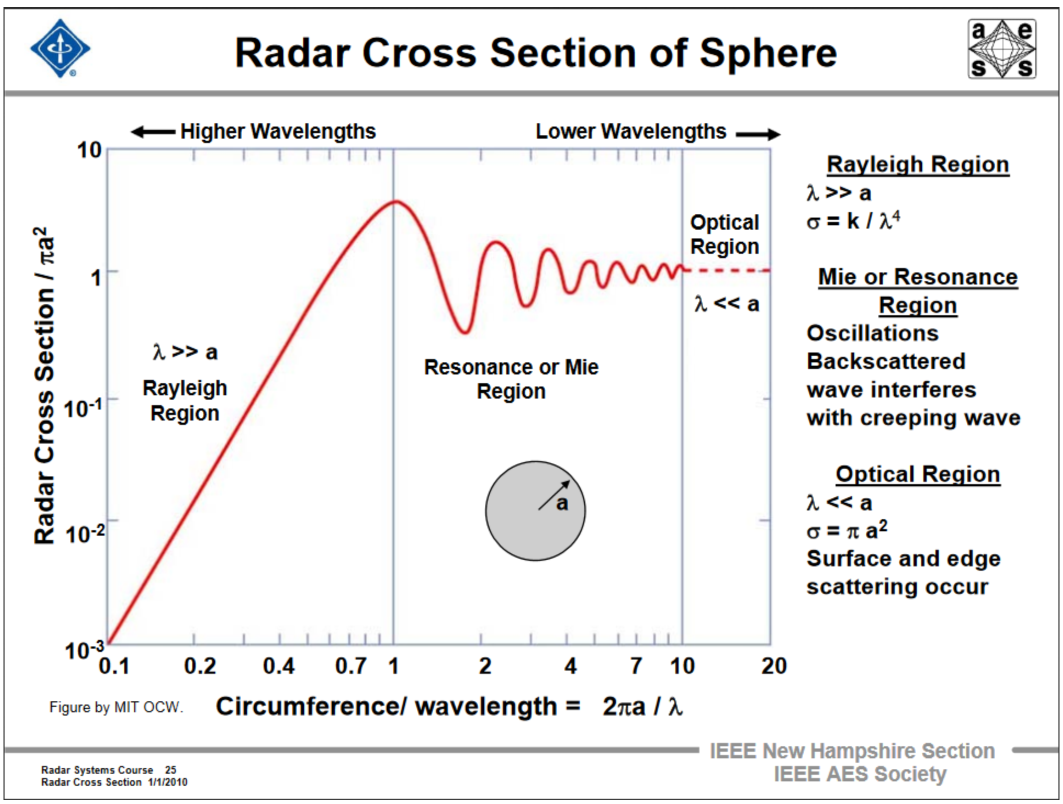
Using a script that I developed to calculate the RCS of a PEC sphere using multipole spherical wave expansion for a HW problem in Professor Xu's 5106 course implanting the equations shown in the slides in Appendix C from Professor Xu's ECE 5106 courses notes on spherical wave scattering analysis.

Using the Matlab script:

RCS of a single sphere with radius of a = 0.005m at f = 1 GHz: σ = 3.1566e-8 m^2 = -75.0 dBsm

As a point of reference, the RCS of an insect is ~-50dBsm according to Skolnik meaning that the RCS of a single sphere in this problem is quite miniscule. This answer is of the same order of magnitude as the approximation and should be close to an exact solution because it is taking into account the spherical wave scattering. I will use this RCS value as the RCS of the single PEC small sphere.

The plot below from the MIT Radar Systems Course Lecture 7 demonstrates the RCS of a sphere through the Rayleigh, Mie, and Optical scattering regions and shows how the RCS of the sphere is highly dependent on its electrical size compared to wavelength:



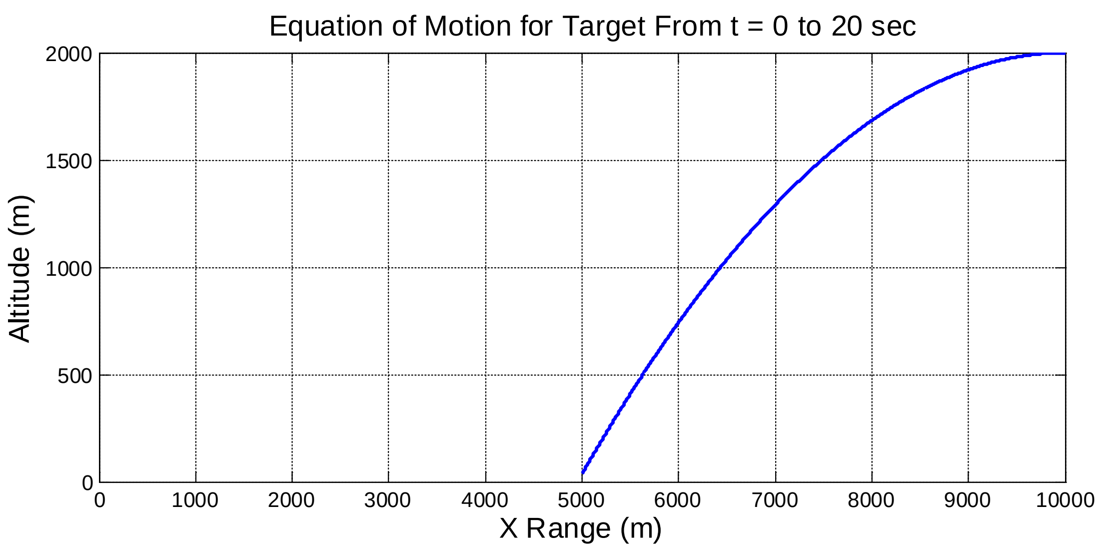
Question 4 Part A Continued: RCS of the Rotating Dumbbell Target Between

Angular Frequency of Spinning Target: ω = 10 rpm = 0.1667 rev/s = 60 deg/s = 1.047 rad/s

Equation of Motion for Target Starting at t=0 w/ Initial Height of 2000m in the z-direction and Initial Velocity of 250 m/s in the x-direction:

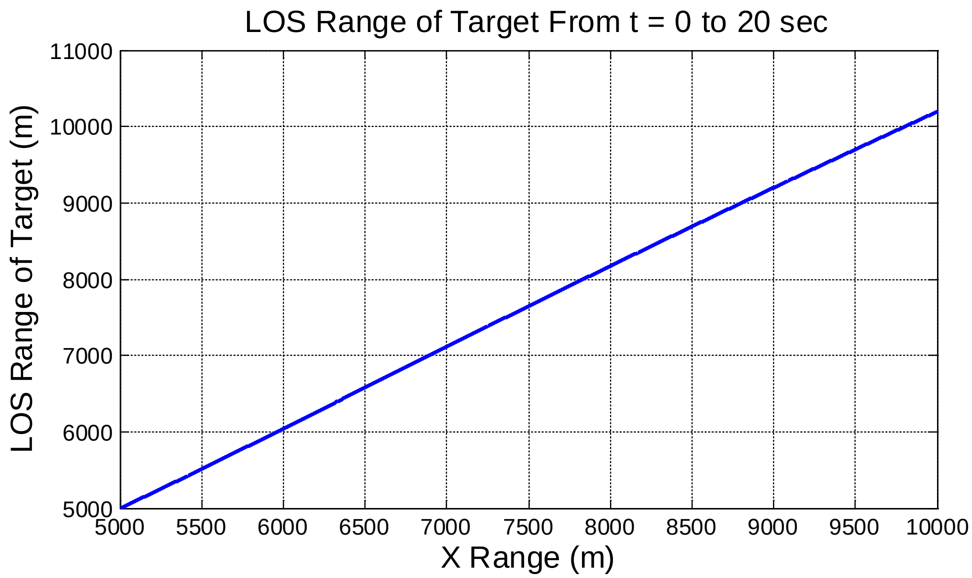
$$pos = (v_{horizontal})(t)\,\overrightarrow{e_x} + (h_0 - \frac{1}{2}gt^2)\,\overrightarrow{e_z}$$
$$= \left(250\frac{m}{s}\right)(t)\,\overrightarrow{e_x} + (2000 - \frac{1}{2}\left(-9.81\frac{m}{s^2}\right)t^2)\,\overrightarrow{e_z}$$

Since there it is assumed that there is no atmospheric or drag forces acting on the spinning target, we can assume that the target maintains the horizontal velocity of 250 m/s throughout its entire trajectory before hitting the ground. We use basic kinematics to plot the equation of motion for the target from t = 0 to t = 20:



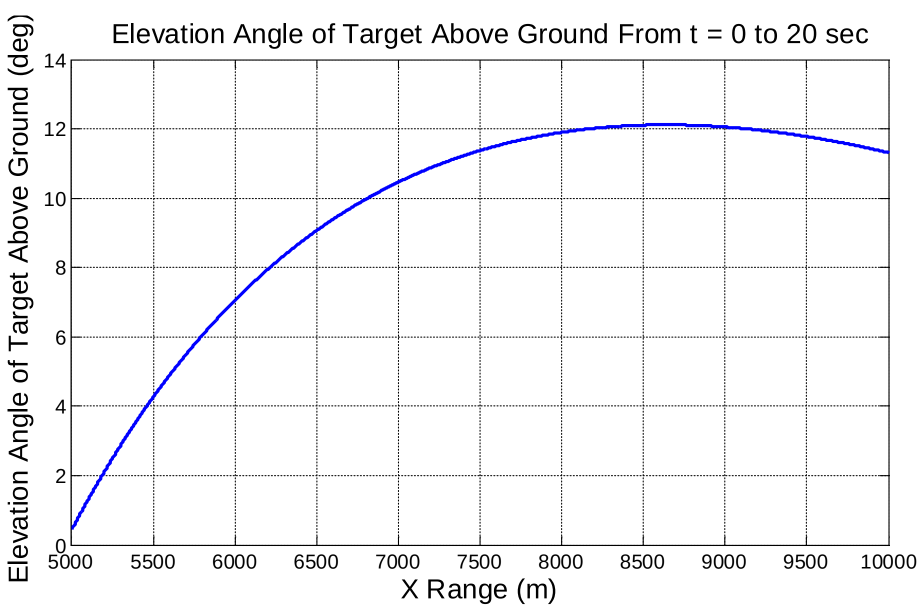
Knowing the x-position and y-position of the target at each point in time during its trajectory allows for the calculation of the line-of-sight (LOS) distance between the radar antenna and the target using Pythagorean Theorem. This range is extremely important because this is the range used in the radar equation and affects the power received in part B of the problem.

$$R = LOS\ distance = \sqrt{(x(t))^2 + (z(t))^2}$$



Additionally, it is important to know the relative angle between the orientation of the target and the radar antenna. The first step in this calculation is knowing the elevation angle of the target above the ground throughout the target's trajectory. This is easily done using the following equation:

$$\tan\theta_{radar} = \frac{z(t)}{x(t)} \rightarrow \theta_{radar} = \tan^{-1}\frac{z(t)}{x(t)}$$



It is a little unintuitive at first glance because the target is falling towards the ground, however, the angle of the target above the ground to the radar antenna actually increases for the first several second. This is because the target is initially falling towards the ground very slowly while already having a large constant velocity towards the target. For these first few seconds, the radar antenna must slightly increase the elevation pointing of the beam to perfectly track the target. The target is also rotating within its own frame of reference at a constant angular velocity of 10 rpm which is 60 deg/s. The angle between the vertically rotating PEC spheres as observed by the radar antenna is a combined function of both the angle of the target above the ground as well as the relative angle of the two spheres as they rotate. There is a propagation delay between the time at which the signal hits the rotating spheres and the time at which we receive the signal however, maximum propagation delay time (10km/3e8m/s = 3.33e-5 seconds) is rather negligible for the purposes of this problem because that time delay is very small compared to the velocities of the objects in this problem. There will be a slight time delay between the instantaneous position and orientation of the target and power received from the target, however, this is not important to this problem, especially since we are plotting the variation of power received by the radar as a function of horizontal (x-axis) distance to the target.

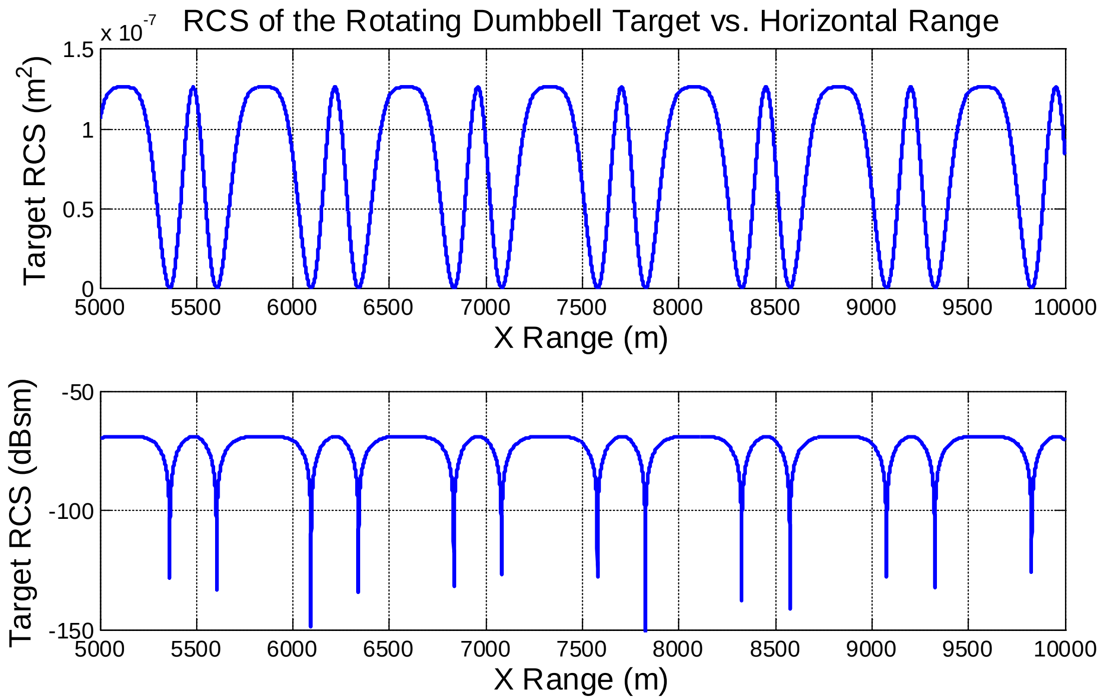
Relative angle between the radar antenna and the connecting rod is defined by the equation:

$$\theta(t) = \tan^{-1}\frac{z(t)}{x(t)} - \omega_{target}t$$

$$\theta(t=0) = \tan^{-1}\frac{z(t=0)}{x(t=0)} - \omega_{target}(0) = \tan^{-1}\frac{2{,}000m}{10{,}000m} = 11.3\ deg$$

The initial angle between the radar and the vector perpendicular to the vertically oriented connection rod of the target is 11.3 deg. Since I specified that the angular rotation is such that the top sphere initially comes towards the point of observation at the radar antenna, the angle between the radar and the vector perpendicular to the connecting rod should be 0 deg roughly ~0.2 seconds after the spinning target is launched. My equations of motion and angle of orientation are oriented in a frame of reference where this will be true.

Next, we will calculate the variation in the RCS of the target (from part A) over time as it rotates and plot the variation of the RCS against the X-range of the target:



Question #4 Part B: Plot Variation in Power Received by Radar Rx as a Function of Horizontal Distance

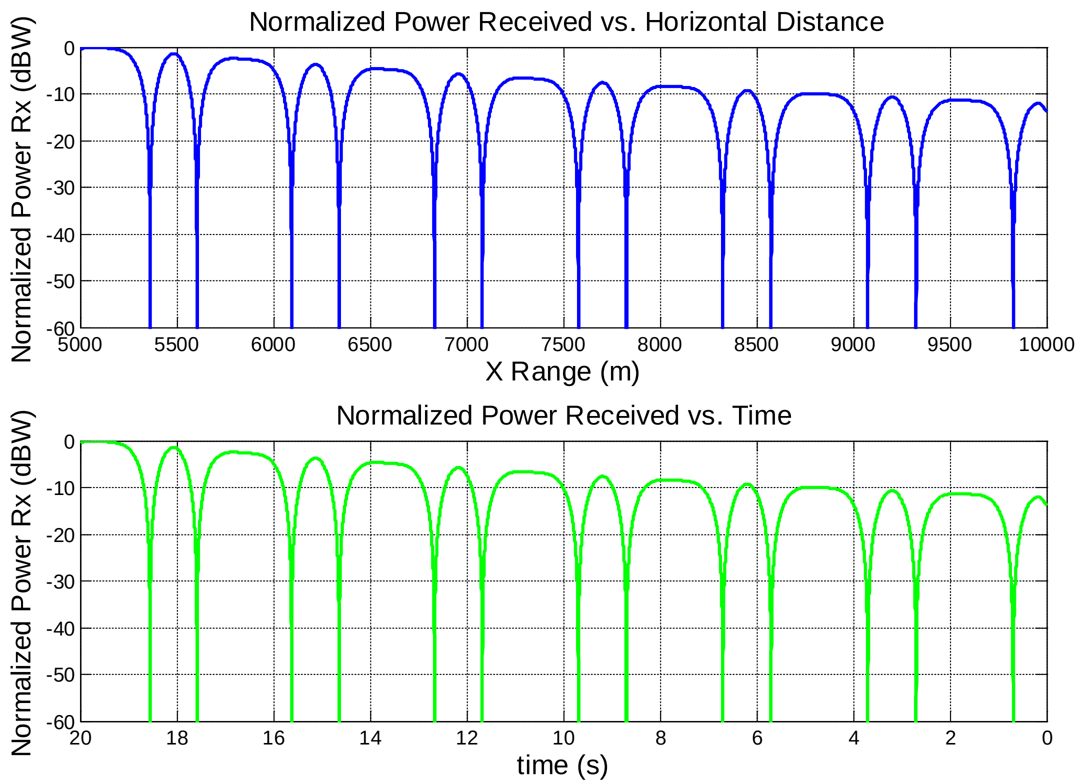
Since we are only concerned with the variation in relative power levels at the radar receiver, we can choose nominal value for the other components in the radar system. To calculate power received at the radar, the following equation from Richard POMR eq. 2.8 is used:

$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma}{(4\pi)^3 R^4}$$

Since the target RCS and a function of position and time, we will write the equation as (for the problem, we care about the power received, RCS, and range as a function of position over the target trajectory, however, position is actually a function of time):

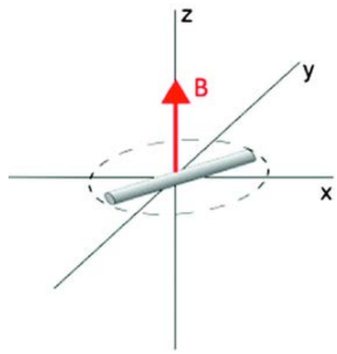
$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma(t)}{(4\pi)^3 R(t)^4} = \frac{P_t G_t G_r \lambda^2 \sigma(x,z)}{(4\pi)^3 R(x,z)^4}$$

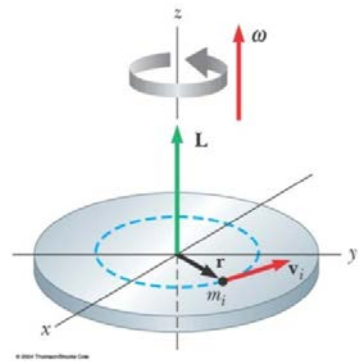
Assuming P_t = 100 kW and $G_t = G_r$ = 30 dBi and then normalizing, the following plots for the normalized power received by the radar from the rotating dumbbell target over horizontal position and time are given below:



As noted during Part A of this question, the rotating target causes the Rx signal of the dumbbell target to fluctuate and go into deep fades at the angles where the relative phase difference between the two PEC spherical scatters destructive interferes. At angles when the scattered waves from the two spheres are perfectly in phase, a peak in the Rx signal of the radar is given. From the plots above, it is shown that the peaks in power received by the radar increase over time of the trajectory as the target approaches the radar antenna. This significant increase in signal strength occurs due to the $1/R^4$ path loss term in the radar equation.

Question #5: Radiated Fields and Power From Rotating Bar Magnet





Initially, I tried following an example from Griffiths in which a rotating electric dipole is treated as the superposition of two oscillating electric dipole that are out of phase by 90 degrees. I tried to do this for the magnetic case, using two oscillating magnetic dipoles 90 degrees out of phase that were approximated by integrating the current around an extremely small loop, however, I do not think that this formulation gave me the appropriate information to calculate the reactive near-fields. After e-mailing Professor Manteghi, I decided to take an alternative approach utilizing the Larmor formula and Liénard-Wiechert potentials, which can be used to calculate the fields and power radiated of an arbitrary moving charger as outlined in Chapter 14 of Jackson.

Original Approach using equations from Chapter 11 of Griffiths:

$$\boldsymbol{m} = m_0\left[\cos\left(\omega\left(t-\frac{r'}{c}\right)\right)\hat{x} + \sin\left(\omega\left(t-\frac{r'}{c}\right)\right)\hat{y}\right]$$

$$r' = \sqrt{r^2 + \left(\frac{L}{2}\right) - 2r\left(\frac{L}{2}\right)\cos\psi} = \sqrt{r^2 + \left(\frac{L}{2}\right) - rL\sin\theta\cos\varphi'}$$

$$\cos\psi = \sin\theta\cos\varphi'$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{|\boldsymbol{r}|^3}$$

$$\boldsymbol{B}\,(\boldsymbol{r}) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{\nabla}\times\left(\frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{|\boldsymbol{r}|^3}\right) = \frac{\mu_0}{4\pi}\left(\frac{3\boldsymbol{r}(\boldsymbol{m}\cdot\boldsymbol{r})}{|\boldsymbol{r}|^5} - \frac{\boldsymbol{m}}{|\boldsymbol{r}|^3}\right)$$

$$\psi(\boldsymbol{r}) = \frac{\boldsymbol{m}\cdot\boldsymbol{r}}{|\boldsymbol{r}|^3}$$

Using this approach, I was able to calculate the Radiate Power for the Rotating Magnetic Dipole to be:

$$\langle P_{magnetic} \rangle = \frac{\mu_0 m_0^2 \omega_0^4}{6\pi c^3}$$

This result is twice the value of a single oscillating magnetic dipole as stated in Griffiths equation 11.40 on pg. 454:

$$\langle P_{magnetic} \rangle = \frac{\mu_0 m_0^2 \omega_0^4}{12\pi c^3}$$

Unfortunately, it seems that this approach uses some far-field assumptions to give of the radiated fields, however, does not give us sufficient information for the reactive near-field.

Advice From Professor Manteghi:

What you described about Griffith's method shows that he is using far-field approximation. Basically close to the rotating dipole you cannot make that assumption. I can point you to Larmor Potential which formulate the radiation from charge with arbitrary motion and then use duality. People in the field of astronomy have faced rotating magnet problem for the case of pulsars. You can check their works. They originally called it the Liénard-Wiechert Potentials.

Assumptions:

- Rotation (angular velocity) of bar magnet is slow enough that we can assume $\gamma \cong 1$ meaning that this is a non-relativistic problem
- Assuming that the bar magnetic has a negligible thickness

In Jackson Elctrodynamics Section 14.1 on pg.663 on Liénard-Wiechert Potentials and Fields for a Point Charge, the following quantities are defined:

$\hat{n}$ is a unit vector in the direction of $\vec{x} - \vec{r}(\tau)$ (where $\vec{x}$ is the observation point) and $\vec{\beta} = \frac{\vec{v}(\tau)}{c}$

where $\vec{v}(\tau) = (\frac{d}{dt})\vec{r}(\tau)$

$$R = |\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(\tau)|$$

Scalar Potential: $\Phi(\vec{\mathbf{x}}, \mathrm{t}) = \left[\frac{e}{\left(1-\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{n}}\right)R}\right]_{ret}$

Vector Potential: $\mathbf{A}(\vec{\mathbf{x}}, \mathrm{t}) = \left[\frac{e\vec{\boldsymbol{\beta}}}{\left(1-\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{n}}\right)R}\right]_{ret}$

Position of two opposite charge of a rotating dipole of length a:

$$\vec{\boldsymbol{r}}(t) = \left(\frac{a}{2}\right)[\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t)] + \left(\frac{-a}{2}\right)[\hat{x}\cos(\omega_0 t + \pi) + \hat{y}\sin(\omega_0 t + \pi)]$$
$$= \left(\frac{a}{2}\right)[\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t)] + \left(\frac{-a}{2}\right)[-\hat{x}\cos(\omega_0 t) - \hat{y}\sin(\omega_0 t)]$$
$$= \left(\frac{a}{2}\right)[\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t)] + \left(\frac{a}{2}\right)[\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t)]$$
$$= (a)[\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t)]$$

Using this result for our position vector:

$$\vec{\boldsymbol{r}}(t) = a[\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t)]$$

$$\vec{\boldsymbol{v}}(t) = \left(\frac{d}{dt}\right)\vec{r}(t) = (a\omega_0)[-\hat{x}\sin(\omega_0 t) + \hat{y}\cos(\omega_0 t)]$$

$$\vec{\boldsymbol{\beta}} = \frac{\vec{v}(\tau)}{c} = \left(\frac{a\omega_0}{c}\right)[-\hat{x}\sin(\omega_0 t) + \hat{y}\cos(\omega_0 t)] \text{ (velocity term)}$$

$$\dot{\boldsymbol{\beta}} = \left(\frac{d}{dt}\right)\vec{\boldsymbol{\beta}} = \left(\frac{-a\omega_0^2}{c}\right)[\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t)] \text{ (acceleration term)}$$

Observation Point: $\vec{\boldsymbol{x}} = z\hat{z}$

*Note: This observation point is independent of the charge's position, since it is always at the same distance from a point on the z-axis.

Additionally: $\vec{\boldsymbol{x}} = z\hat{z} = (r\ \cos\theta)\hat{z}$ since $z = r\ \cos\theta$

$$\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(\tau) = z\hat{z} - a[\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t)] = z\hat{z} - \hat{x}(a)\cos(\omega_0 t) - \hat{y}(a)\sin(\omega_0 t)$$

$$R = |\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(\tau)| = \sqrt{z^2 + a^2}$$

$$\hat{\boldsymbol{n}} = \frac{z\hat{z} - \hat{x}(a)\cos(\omega_0 t) - \hat{y}(a)\sin(\omega_0 t)}{\sqrt{z^2 + a^2}}$$

$$\vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}} = \left(\frac{z\hat{z} - \hat{x}(a)\cos(\omega_0 t) - \hat{y}(a)\sin(\omega_0 t)}{\sqrt{z^2+a^2}}\right) \cdot \left(\left(\frac{a\omega_0}{c}\right)[-\hat{x}\sin(\omega_0 t) + \hat{y}\cos(\omega_0 t)]\right) =$$
$$\left(\frac{a\omega_0}{c\sqrt{z^2+a^2}}\right)[(a)\cos(\omega_0 t)\sin(\omega_0 t) - (a)\cos(\omega_0 t)\sin(\omega_0 t)] = 0$$

$$\Phi(\vec{\mathbf{x}}, \mathrm{t}) = \left[\frac{e}{(1\ -\ 0)\ R}\right]_{ret} = \left[\frac{e}{\sqrt{z^2 + a^2}}\right]_{ret}$$

$$\mathbf{A}(\vec{\mathbf{x}}, \mathrm{t}) = \left[\frac{e\vec{\boldsymbol{\beta}}}{\left(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}}\right) R}\right]_{ret} = \left[\frac{e(\frac{a\omega_0}{c})[-\hat{x}\sin(\omega_0 t) + \hat{y}\cos(\omega_0 t)]}{(1 - 0)\sqrt{z^2 + a^2}}\right]_{ret}$$

$$= \left[(\frac{ea\omega_0}{c\sqrt{z^2 + a^2}})[-\hat{x}\sin(\omega_0 t) + \hat{y}\cos(\omega_0 t)]\right]_{ret}$$

$$\mathbf{B}(\vec{\mathbf{x}}, \mathrm{t}) = \nabla\, x\, \mathbf{A}(\vec{\mathbf{x}}, \mathrm{t})$$

$$\mathbf{E}(\vec{\mathbf{x}}, \mathrm{t}) = -\nabla\Phi(\vec{\mathbf{x}}, \mathrm{t}) - \frac{1}{c}\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{A}(\vec{\mathbf{x}}, \mathrm{t})$$

Using these equations for the vector and scalar potentials, I was able to calculate the following expressions for the E-field and the B-field, however, from reading the literature, I realized that she results may not be correct because the fields are retarded in time due to the motion of the charge and the d/dt may not take this into account. These terms should be evaluated at the time $t_r = t - {R(\tau)}/{c}$. Since R, the distance of the rotating charge and the observation, is constant over time given our choice of observation point, I think that we may be able to take the simple route, and directly use the scalar and vector potentials to calculate the fields in the reactive near-zone, however, I am not completely sure.

$$\mathbf{B}(\vec{\mathbf{x}}, \mathrm{t}) = \nabla\, x\, \mathbf{A}(\vec{\mathbf{x}}, \mathrm{t}) = \left(\frac{ae\omega_0 z}{c(z^2+a^2)^{3/2}}\right)(\hat{x}\sin(\omega_0 t) + \hat{y}\cos(\omega_0 t))$$

$$-\nabla\Phi(\vec{\mathbf{x}}, \mathrm{t}) = \left(\frac{d}{dz}\right)\left(\frac{e}{\sqrt{z^2 + a^2}}\right)\hat{z} = \frac{ez}{(z^2 + a^2)^{3/2}}\hat{z}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{A}(\vec{\mathbf{x}}, \mathrm{t}) = -\left(\frac{ea\omega_0^2}{c^2\sqrt{z^2+a^2}}\right)(\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t))$$

$$\mathbf{E}(\vec{\mathbf{x}}, \mathrm{t}) = -\nabla\Phi(\vec{\mathbf{x}}, \mathrm{t}) - \frac{1}{c}\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{A}(\vec{\mathbf{x}}, \mathrm{t}) = \left(\frac{ez}{(z^2+a^2)^{\frac{3}{2}}}\right)\hat{z} + \left(\frac{ea\omega_0^2}{c^2\sqrt{z^2+a^2}}\right)(\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t))$$

To take a more rigorous approach, I also attempt to calculate the near-fields using the more general approach using the Liénard–Wiechert potential equations (14.13-14.14) in Section 14.1 of Jackson as follows:

$$\mathbf{E}(\mathbf{r}, t) = q\left(\frac{\mathbf{n} - \boldsymbol{\beta}}{\gamma^2(1 - \boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right)_{\mathrm{ret}} + \frac{q}{c}\left(\frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right)_{\mathrm{ret}}$$

and

$$\mathbf{B} = \mathbf{n} \times \mathbf{E},$$

Since the magnetic rod is rotating very slowly compared to the speed of light, we can assume $\gamma = 0$. as noted above for this nonrelativistic accelerated charge problem. As Jackson notes, this equation naturally splits the fields into "velocity fields," which are independent of acceleration and "acceleration fields," which are linearly dependent on $\dot{\boldsymbol{\beta}}$. Jackson also notes the velocity fields are essentially static fields, which fall off as R^{-2}, whereas the acceleration fields are typical radiation fields, both E and H being transverse for the radius vector and varying as R^{-1}.

I will type the results in this document and also attach my handwritten calculations for full mathematical detail. Note that even with my choice of observation point, this is quite a length and messy calculation with many terms that much cancel out. There is a chance that I did not choose the more convenient observation point, missed a simplification, or made a simple error such as a sign error somewhere in the calculation, however, I believe I am taking the correct approach and should yield the correct answer if all of my math is correct.

The first step is to calculate $\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}$:

$$\hat{\boldsymbol{n}} = \frac{z\hat{z} - \hat{x}(a)\cos(\omega_0 t) - \hat{y}(a)\sin(\omega_0 t)}{\sqrt{z^2 + a^2}}$$

$$\vec{\boldsymbol{\beta}} = \frac{\vec{v}(\tau)}{c} = \left(\frac{a\omega_0}{c}\right)[-\hat{x}\sin(\omega_0 t) + \hat{y}\cos(\omega_0 t)] \text{ (velocity term)}$$

$$(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}})_x = \frac{-a\cos(\omega_0 t)}{\sqrt{z^2 + a^2}} + \left(\frac{a\omega_0}{c}\right)\sin(\omega_0 t)$$

$$(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}})_y = \frac{-a\sin(\omega_0 t)}{\sqrt{z^2 + a^2}} + \left(\frac{a\omega_0}{c}\right)\cos(\omega_0 t)$$

$$(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}})_z = \frac{z}{\sqrt{z^2 + a^2}}$$

Above we calculated that $\vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}} = \mathbf{0}$, which hopefully will conveniently simply the rest of our calculation. I will start by evaluated the first half of the expression, which is referred to as the velocity fields using the following equation:

$$\mathbf{E}_v(\vec{\mathbf{x}}, \mathrm{t}) = e\left[\frac{\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}}{\gamma^2(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^3 R^2}\right]_{ret} = \left(\frac{e}{R^2}\right)\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right) = \left(\frac{e}{(z^2 + a^2)}\right)\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)$$

$$R = |\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(\tau)| = \sqrt{z^2 + a^2}$$

$$\text{x-component:} = \left(\frac{e}{(z^2+a^2)}\right)\left(\frac{-a\cos(\omega_0 t)}{\sqrt{z^2+a^2}} + \left(\frac{a\omega_0}{c}\right)\sin(\omega_0 t)\right)$$

$$\text{y-component:} = \left(\frac{e}{(z^2+a^2)}\right)\left(\frac{-a\sin(\omega_0 t)}{\sqrt{z^2+a^2}} + \left(\frac{a\omega_0}{c}\right)\cos(\omega_0 t)\right)$$

$$\text{z-component: :} = \left(\frac{e}{(z^2+a^2)}\right)\left(\frac{z}{\sqrt{z^2+a^2}}\right) = \left(\frac{ez}{(z^2+a^2)^{3/2}}\right)$$

My next step is to calculate the acceleration fields term for the second half of the E-field equation as follows:

$$\mathbf{E}_{\boldsymbol{a}}(\vec{\mathbf{x}}, \mathrm{t}) = \frac{e}{c}\left[\frac{\hat{\boldsymbol{n}}\; x \left\{\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right) \boldsymbol{x}\, \dot{\boldsymbol{\beta}}\right\}}{\gamma^2 (1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^3 R}\right]_{ret} = \left(\frac{e}{cR}\right)\left[\hat{\boldsymbol{n}}\; x \left\{\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right) \boldsymbol{x}\, \dot{\boldsymbol{\beta}}\right\}\right]$$

$$= \left(\frac{e}{c\sqrt{z^2 + a^2}}\right)[\hat{\boldsymbol{n}}\; x \{\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right) \boldsymbol{x}\, \dot{\boldsymbol{\beta}}\}]$$

To calculate this term, we must first take the vector cross product of $\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)\boldsymbol{x}\, \dot{\boldsymbol{\beta}}$ where

$\dot{\boldsymbol{\beta}} = (\frac{d}{dt})\vec{\boldsymbol{\beta}} = (\frac{-a\omega_0^2}{c})[\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t)]$ (acceleration term) as follows:

$$\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)\boldsymbol{x}\, \dot{\boldsymbol{\beta}}$$

$$A_x = \left[\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)\boldsymbol{x}\, \dot{\boldsymbol{\beta}}\right]_x = -\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)_z \dot{\boldsymbol{\beta}}_y = (\frac{za\omega_0^2}{c\sqrt{z^2+a^2}})\ \sin(\omega_0 t)$$

$$A_y = \left[\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)\boldsymbol{x}\, \dot{\boldsymbol{\beta}}\right]_y = -\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)_z \dot{\boldsymbol{\beta}}_x = -(\frac{za\omega_0^2}{c\sqrt{z^2+a^2}})\ \cos(\omega_0 t)$$

$$A_z = \left[\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)\boldsymbol{x}\, \dot{\boldsymbol{\beta}}\right]_z = \left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)_x \dot{\boldsymbol{\beta}}_y - \left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)_y \dot{\boldsymbol{\beta}}_x = \Bigg(\Bigg(\frac{-a\cos(\omega_0 t)}{\sqrt{z^2+a^2}} +$$

$$\left(\frac{a\omega_0}{c}\right)\sin(\omega_0 t)\Bigg)\left(\left(\frac{-a\omega_0^2}{c}\right)\sin(\omega_0 t)\right)\Bigg) - \Bigg(\left(\frac{e}{(z^2+a^2)}\right)\Bigg(\frac{-a\sin(\omega_0 t)}{\sqrt{z^2+a^2}} +$$

$$\left(\frac{a\omega_0}{c}\right)\cos(\omega_0 t)\Bigg)\left(\left(\frac{-a\omega_0^2}{c}\right)\cos(\omega_0 t)\right)\Bigg) = (\frac{-a^2\omega_0^3}{c^2})$$

Given these results, it is now possible to evaluate the term $\hat{\boldsymbol{n}}\; x \{\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)\boldsymbol{x}\, \dot{\boldsymbol{\beta}}\}$:

$$[\hat{\boldsymbol{n}}\; x \{\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)\boldsymbol{x}\, \dot{\boldsymbol{\beta}}\}]_x = n_y A_z - n_z A_y = \left(\frac{a^3\omega_0^2}{c^2\sqrt{z^2+a^2}}\right)\sin(\omega_0 t) + \left(\frac{az^2\omega_0^2}{c\,(z^2+a^2)}\right)\cos(\omega_0 t)$$

$$[\hat{\boldsymbol{n}}\; x \{\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)\boldsymbol{x}\, \dot{\boldsymbol{\beta}}\}]_y = n_z A_x - n_x A_z = -\left(\frac{a^3\omega_0^2}{c^2\sqrt{z^2+a^2}}\right)\cos(\omega_0 t) + \left(\frac{az^2\omega_0^2}{c\,(z^2+a^2)}\right)\sin(\omega_0 t)$$

$$[\hat{\boldsymbol{n}}\; x \{\left(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}\right)\boldsymbol{x}\, \dot{\boldsymbol{\beta}}\}]_z = n_x A_y - n_y A_x = \left(\frac{a^2 z\omega_0^2}{c\,(z^2 + a^2)}\right)$$

These results are now inserted back into:

$$\mathbf{E}_{\boldsymbol{a}}(\vec{\mathbf{x}},\mathrm{t}) = \frac{e}{c}\left[\frac{\hat{\boldsymbol{n}}\,x\,\{(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\,\boldsymbol{x}\,\dot{\boldsymbol{\beta}}\}}{\gamma^2(1-\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{n}})^3 R}\right]_{ret} = \left(\frac{e}{c\sqrt{z^2+a^2}}\right)[\hat{\boldsymbol{n}}\,x\,\{(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\,\boldsymbol{x}\,\dot{\boldsymbol{\beta}}\}]$$

x-component:

$$= \left(\frac{e}{c\sqrt{z^2+a^2}}\right)[\hat{\boldsymbol{n}}\,x\,\{(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\boldsymbol{x}\,\dot{\boldsymbol{\beta}}\}]_x$$

$$= \left(\frac{e}{c\sqrt{z^2+a^2}}\right)\left(\left(\frac{a^3\omega_0^2}{c^2\sqrt{z^2+a^2}}\right)\sin(\omega_0 t) + \left(\frac{az^2\omega_0^2}{c\,(z^2+a^2)}\right)\cos(\omega_0 t)\right)$$

y-component:

$$= \left(\frac{e}{c\sqrt{z^2+a^2}}\right)[\hat{\boldsymbol{n}}\,x\,\{(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\boldsymbol{x}\,\dot{\boldsymbol{\beta}}\}]_y$$

$$= \left(\frac{e}{c\sqrt{z^2+a^2}}\right)\left(-\left(\frac{a^3\omega_0^2}{c^2\sqrt{z^2+a^2}}\right)\cos(\omega_0 t) + \left(\frac{az^2\omega_0^2}{c\,(z^2+a^2)}\right)\sin(\omega_0 t)\right)$$

z-component:

$$= \left(\frac{e}{c\sqrt{z^2+a^2}}\right)[\hat{\boldsymbol{n}}\,x\,\{(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\boldsymbol{x}\,\dot{\boldsymbol{\beta}}\}]_z = \left(\frac{e}{c\sqrt{z^2+a^2}}\right)\left(\frac{a^2z\omega_0^2}{c\,(z^2+a^2)}\right) = \left(\frac{ea^2z\omega_0^2}{c^2\,(z^2+a^2)^{3/2}}\right)$$

Now that we have calculated both the velocity fields and acceleration fields for the rotating electric charge problem, we will add the two terms together to calculate the electric, which encompasses the near-fields, since we have not made any far-field only assumptions to this point:

$$\mathbf{E}(\vec{\mathbf{x}},\mathrm{t}) = \mathbf{E}_{\boldsymbol{v}}(\vec{\mathbf{x}},\mathrm{t}) + \mathbf{E}_{\boldsymbol{a}}(\vec{\mathbf{x}},\mathrm{t}) = e\left[\frac{\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}}}{\gamma^2(1-\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{n}})^3 R^2}\right]_{ret} + \frac{e}{c}\left[\frac{\hat{\boldsymbol{n}}\,x\,\{(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\,\boldsymbol{x}\,\dot{\boldsymbol{\beta}}\}}{\gamma^2(1-\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{n}})^3 R}\right]_{ret}$$

x-component:

$$[\mathbf{E}(\vec{\mathbf{x}},\mathrm{t})]_{\boldsymbol{x}} = \left(\frac{ae}{c^2}\right)\left[\left(\frac{\omega_0(a^2\omega_0+c)}{(z^2+a^2)}\right)\sin(\omega_0 t) + \left(\frac{\omega_0^2 z^2 - c^2}{(z^2+a^2)^{3/2}}\right)\cos(\omega_0 t)\right]$$

y-component:

$$[\mathbf{E}(\vec{\mathbf{x}},\mathrm{t})]_{\boldsymbol{y}} = \left(\frac{ae}{c^2}\right)\left[-\left(\frac{\omega_0(a^2\omega_0+c)}{(z^2+a^2)}\right)\cos(\omega_0 t) + \left(\frac{\omega_0^2 z^2 - c^2}{(z^2+a^2)^{3/2}}\right)\sin(\omega_0 t)\right]$$

z-component:

$$[\mathbf{E}(\vec{\mathbf{x}},\mathrm{t})]_{\boldsymbol{z}} = \frac{ez(a^2\omega_0^2 + c^2)}{c^2\,(z^2+a^2)^{3/2}}$$

Now knowing the result for the electric field generated by the rotating electric charge in the near-field, it is now possible to calculate the **B** field using equation 14.13 from Jackson as follows:

$$\boldsymbol{B} = [\widehat{\boldsymbol{n}}\ x\ \boldsymbol{E}]_{ret}$$

x-component:

$$B_x = \left(\frac{ae}{c^2}\right)\left[\left(\frac{z\omega_0(a^2\omega_0 + c)}{(z^2+a^2)^{3/2}}\right)\cos(\omega_0 t) - \left(\frac{\omega_0^2 z}{(z^2+a^2)}\right)\sin(\omega_0 t)\right]$$

y-component:

$$B_y = \left(\frac{ae}{c^2}\right)\left[\left(\frac{z\omega_0(a^2\omega_0 + c)}{(z^2+a^2)^{3/2}}\right)\sin(\omega_0 t) + \left(\frac{\omega_0^2 z}{(z^2+a^2)}\right)\cos(\omega_0 t)\right]$$

z-component:

$$B_z = \left(\frac{ae}{c^2}\right)\left(\frac{a\omega_0(a^2\omega_0 + c)}{(z^2+a^2)^{3/2}}\right)$$

Now that we have calculated the E and B fields in the near-reactive zone for the rotating electric dipole problem, we can now use the duality theorem to convert our results from electric charge to magnetic charge for the fields insert constants for the rotating bar magnet in this problem. Using duality, we will make the following transformations using the duality transformations listed in Griffiths Electrodynamics equation 7.68 with $\alpha = 90\ deg$ as follows:

$$\vec{E}' = c\vec{B}$$

$$\vec{B}' = -\frac{\vec{E}}{c}$$

$q_m' = -cq_e \rightarrow\ m' = -ce$ (to keep our notation consistent)

Taking the fields from the rotating electric charge above, we can apply these duality transformations and calculate the fields for the rotating magnetic charge:

$$\left(\frac{ae}{c^2}\right) \rightarrow c\left(\frac{a(-m/c)}{c^2}\right) = -\left(\frac{a\,m}{c^2}\right)$$

Convert B-Field to E-Field:

x-component:

$$E_x' = cB_x = -\left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{a\,m}{c^2}\right)\left[\left(\frac{z\omega_0(a^2\omega_0 + c)}{(z^2 + a^2)^{3/2}}\right)\cos(\omega_0 t) - \left(\frac{\omega_0^2 z}{(z^2 + a^2)}\right)\sin(\omega_0 t)\right]$$

y-component:

$$E_y' = cB_y = -\left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{a\,m}{c^2}\right)\left[\left(\frac{z\omega_0(a^2\omega_0 + c)}{(z^2 + a^2)^{3/2}}\right)\sin(\omega_0 t) + \left(\frac{\omega_0^2 z}{(z^2 + a^2)}\right)\cos(\omega_0 t)\right]$$

z-component:

$$E_z' = cB_z = -\left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{a\,m}{c^2}\right)\left(\frac{a\omega_0(a^2\omega_0 + c)}{(z^2 + a^2)^{3/2}}\right)$$

where $z = \ (r\ \cos\theta)$ and $a = L/2$

Similarly, convert E-Field to B-Field:

$$\left(\frac{ae}{c^2}\right) \rightarrow -\frac{1}{c}\left(\frac{a(-m/c)}{c^2}\right) = \left(\frac{a\,m}{c^4}\right)$$

Note that $B_x = [\mathbf{B}(\vec{\mathbf{x}}, \mathrm{t})]_x$ and vice versa.

x-component:

$$B_x' = -\frac{1}{c}[\mathbf{E}(\vec{\mathbf{x}}, \mathrm{t})]_x = \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{a\,m}{c^4}\right)\left[\left(\frac{\omega_0(a^2\omega_0 + c)}{(z^2 + a^2)}\right)\sin(\omega_0 t) + \left(\frac{\omega_0^2 z^2 - c^2}{(z^2 + a^2)^{3/2}}\right)\cos(\omega_0 t)\right]$$

y-component:

$$B_y' = -\frac{1}{c}[\mathbf{E}(\vec{\mathbf{x}}, \mathrm{t})]_y = \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{a\,m}{c^4}\right)\left[-\left(\frac{\omega_0(a^2\omega_0 + c)}{(z^2 + a^2)}\right)\cos(\omega_0 t) + \left(\frac{\omega_0^2 z^2 - c^2}{(z^2 + a^2)^{3/2}}\right)\sin(\omega_0 t)\right]$$

z-component:

$$B_z' = -\frac{1}{c}[\mathbf{E}(\vec{\mathbf{x}}, \mathrm{t})]_z = \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{mz(a^2\omega_0^2 + c^2)}{c^4\,(z^2 + a^2)^{3/2}}\right)$$

where $z = \ (r\ \cos\theta)$ and $a = L/2$

Next, need to relate the strength of the permanent magnet B_0 and length L to the magnetic moment value of *m*:

$$\oint \vec{B} \cdot dl = \mu_0 I_{enc}$$

Griffiths equation 5.87 states that the magnetic field of a magnetic dipole can be written as:

$$\boldsymbol{B}_{dip}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\boldsymbol{m} \cdot \hat{r})\hat{r} - \boldsymbol{m}]$$

Where $\boldsymbol{m} = m_0\widehat{m}$

$$B_0 = \frac{\mu_0}{4\pi}\frac{2m}{r^3} = \frac{\mu_0}{4\pi}\frac{2m}{\left(\frac{L}{2}\right)^3} \rightarrow m = \frac{\pi B_0 L^3}{4\mu_0}$$

Calculate Time-Averaged Radiated Power:

Luckily, the radiated power calculation for the spinning bar magnet is a bit easier than that the near field calculation thanks to Larmor's formula found in section 14.2 of the Jackson textbook. According to Jackson, in the far-field and then the particle's velocity is small compared to the speed of light (nonrelativistic), the acceleration E-field reduces to:

$$\boldsymbol{E_a} = \frac{e}{c}\left[\frac{\widehat{\boldsymbol{n}} \times (\widehat{\boldsymbol{n}} \times \dot{\boldsymbol{\beta}})}{R}\right]_{ret}$$

Poynting Vector: $\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{H} = \frac{c}{4\pi}|\boldsymbol{E_a}|^2\widehat{\boldsymbol{n}}$

When converted to power radiated per solid angle:

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}|R\boldsymbol{E_a}|^2 = \frac{e^2}{4\pi c}\left|\widehat{\boldsymbol{n}} \times (\widehat{\boldsymbol{n}} \times \dot{\boldsymbol{\beta}})\right|^2$$

By assuming that θ theta is the angle between the acceleration $\dot{\boldsymbol{v}}$ *and* $\widehat{\boldsymbol{n}}$ this equation can be re-written as:

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}|\,\dot{\boldsymbol{v}}|^2 \sin^2\theta$$

When integrated over the solid angle, this equation neatly becomes:

$$P = \frac{2}{3}\frac{e^2}{c^3}|\,\dot{\boldsymbol{v}}|^2$$

I will demonstrate calculating the by integrating over the solid angle and also by using the simplified formula above:

$$\vec{\boldsymbol{r}}(t) = a[\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t)]$$

$$\vec{\boldsymbol{v}}(t) = \left(\frac{d}{dt}\right)\vec{\boldsymbol{r}}(t) = (a\omega_0)[-\hat{x}\sin(\omega_0 t) + \hat{y}\cos(\omega_0 t)]$$

$\vec{\boldsymbol{\beta}} = \frac{\vec{\boldsymbol{v}}(\tau)}{c}$ = $(\frac{a\omega_0}{c})[-\hat{x}\sin(\omega_0 t) + \hat{y}\cos(\omega_0 t)]$ (velocity term)

$\dot{\boldsymbol{\beta}}$ = $(\frac{d}{dt})\vec{\boldsymbol{\beta}}$ = $(\frac{-a\omega_0^2}{c})[\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t)]$ (acceleration term)

$$\dot{\boldsymbol{v}}(t) = \left(\frac{d}{dt}\right)\vec{\boldsymbol{v}}(t) = (-a\omega_0^2)[\hat{x}\cos(\omega_0 t) + \hat{y}\sin(\omega_0 t)]$$

$$|\dot{\boldsymbol{v}}(t)|^2 = (a\omega_0^2)^2[\cos^2(\omega_0 t) + \sin^2(\omega_0 t)] = a^2\omega_0^4$$

$$P = \frac{2}{3}\frac{e^2}{c^3}|\,\dot{\boldsymbol{v}}|^2 = \frac{2}{3}\frac{e^2a^2\omega_0^4}{c^3} = \frac{2}{3}\frac{(-m/c)^2\left(\frac{L}{2}\right)^2\omega_0^4}{c^3} = \frac{m^2L^2\omega_0^4}{6c^5} = \frac{\left(\frac{\pi B_0 L^3}{4\mu_0}\right)^2 L^2\omega_0^4}{6c^5} = \frac{\pi^2 B_0^2 L^8 \omega_0^4}{96c^5\mu_0^2}$$

$$m = \frac{\pi B_0 L^3}{4\mu_0}$$

Alternatively, since radiated power should be independent of phi since the geometry is azimuthally symmetric, we can choose a more simple $\hat{\boldsymbol{n}}$ the is in the x-z plane, such as:

$$\hat{\boldsymbol{n}} = \cos\theta\,\hat{z} + \sin\theta\,\hat{x}$$

$$(\hat{\boldsymbol{n}} \times \dot{\boldsymbol{\beta}}) = \left(\frac{-a\omega_0^2}{c}\right)(\hat{y}\cos\theta\cos(\omega_0 t) + \sin(\omega_0 t)\,(\sin\theta\,\hat{z} - \cos\theta\,\hat{x})\,)$$

$$\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \dot{\boldsymbol{\beta}}) = \left(\frac{-a\omega_0^2}{c}\right)(\hat{x}(-\cos^2\theta\cos(\omega_0 t)) + \hat{y}(-\sin(\omega_0 t)) + \hat{z}(\cos\theta\sin\theta\cos(\omega_0 t))$$

$$\left|\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \dot{\boldsymbol{\beta}})\right|^2 = \left(\frac{-a\omega_0^2}{c}\right)^2 \left((\cos^4\theta\cos^2(\omega_0 t)) + \sin^2(\omega_0 t) + \cos^2\theta\sin^2\theta\cos^2(\omega_0 t)\right)$$
$$= \left(\frac{a^2\omega_0^4}{c^2}\right)(\cos^2(\omega_0 t)\cos^2\theta + \sin^2(\omega_0 t))$$

$$\boldsymbol{E_a} = \frac{e}{c}\left[\frac{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \dot{\boldsymbol{\beta}})}{R}\right]_{ret}$$
$$= \left(\frac{e}{cR}\right)\left(\frac{-a\omega_0^2}{c}\right)(\hat{x}(-\cos^2\theta\cos(\omega_0 t)) + \hat{y}(-\sin(\omega_0 t)) + \hat{z}(\cos\theta\sin\theta\cos(\omega_0 t))$$

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}|R\boldsymbol{E_a}|^2 = \frac{e^2}{4\pi c}\left|\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \dot{\boldsymbol{\beta}})\right|^2 = \left(\frac{e^2}{4\pi c}\right)\left(\frac{a^2\omega_0^4}{c^2}\right)(\cos^2(\omega_0 t)\cos^2\theta + \sin^2(\omega_0 t))$$

$$= \left(\frac{e^2a^2\omega_0^4}{4\pi c^3}\right)(\cos^2(\omega_0 t)\cos^2\theta + \sin^2(\omega_0 t))$$

Take the time average of the power density function:

$$\langle\frac{dP}{d\Omega}\rangle = \left(\frac{e^2a^2\omega_0^4}{4\pi c^3}\right)\left(\frac{1}{2}\cos^2\theta + \frac{1}{1}\right) = \left(\frac{e^2a^2\omega_0^4}{8\pi c^3}\right)(\cos^2\theta + 1)$$

Time average of $\cos^2(\omega_0 t)$ and $\sin^2(\omega_0 t)$ are both ½.

Now we integrate this quantity over the solid angle:

$$P = \iint_{\Phi=0,\theta=0}^{\Phi=2\pi,\theta=\pi}\left(\frac{e^2a^2\omega_0^4}{8\pi c^3}\right)(\cos^2\theta + 1)\sin\theta\,d\theta\,d\Phi$$

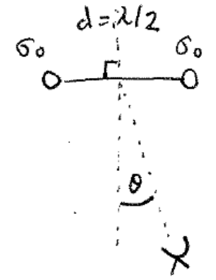
$$P = \left(\frac{e^2a^2\omega_0^4}{8\pi c^3}\right)(2\pi)\int_{\theta=0}^{\pi}(\cos^2\theta + 1)\sin\theta\,d\theta\,d\Phi = \left(\frac{e^2a^2\omega_0^4}{8\pi c^3}\right)(2\pi)\left(\frac{8}{3}\right) = \frac{2}{3}\frac{e^2a^2\omega_0^4}{c^3}$$

This results achieved by integrating the power radiated over solid angle gives us the same result for time-averaged radiated power as the direct equation from Jackson, which we calculated earlier in this problem.

Appendix A:

From Professor Ruohoniemi's ECE 5635 Radar Systems Course:

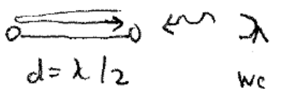
Problem #6 A radar target consists of two identical spheres connected by a rod with no radar reflectivity. The RCS for each individual sphere is σ_o and the RCS for the target is σ_r. If the length of the rod is $d = \lambda/2$ where λ is the radar wavelength, describe (with a plot and a few words of explanation and by referring to the concept of *interference*) the variation of σ_r with θ, where θ is the angle between the perpendicular to the rod drawn through its midpoint and the direction to the radar.



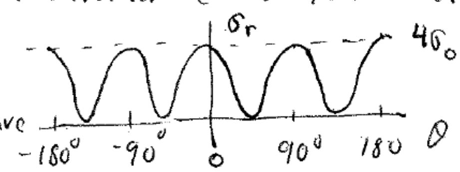
σ_r represents the RCS of the two individual spheres joined by a rod, each with its own RCS, σ_o.

Consider $\theta = 0°$, i.e., the radar looks broadside to the rod. The reflections from the two spheres add up in phase (constructively interfere) and the voltage of the waveform doubles, and the power quadruples. Hence $\sigma_r(0°) = 4\sigma_o$.

Consider $\theta = 90°$. With two way propogation, the path difference is λ, and again we have constructive interference and $\sigma_r(90°) = 4\sigma_o$.



For some intermediate angle, we have destructive interference and $\sigma_r \approx 0$. Consequently, we have



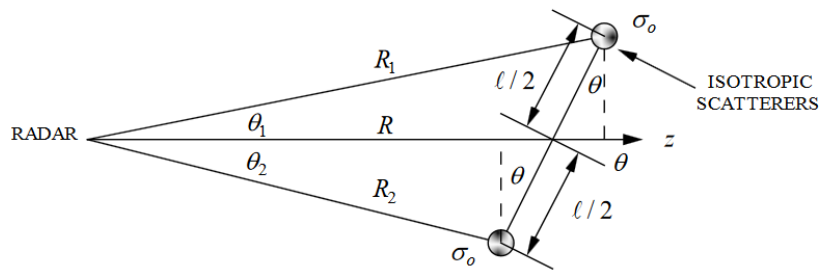
Appendix B:

From Jenn lecture on Complex Targets:
http://faculty.nps.edu/jenn/EC4630/ComplexTargetsV2.pdf

Naval Postgraduate School *Department of Electrical & Computer Engineering* *Monterey, California*

Array of Isotropic Scatterers (1)

Consider the RCS obtained from two isotropic scatterers (approximated by spheres).



Law of cosines:

$$R_1 = \sqrt{R^2 + (\ell/2)^2 - 2R(\ell/2)\cos(\theta + \pi/2)} = R\sqrt{1 + (\ell/2R)^2 + 2(\ell/2R)\sin\theta}$$
$$R_2 = \sqrt{R^2 + (\ell/2)^2 - 2R(\ell/2)\cos(\theta - \pi/2)} = R\sqrt{1 + (\ell/2R)^2 - 2(\ell/2R)\sin\theta}$$

Let $\alpha = \ell \sin\theta / R$ and note that

$$(1 \pm \alpha)^{1/2} = 1 \pm \frac{1}{2}\alpha \mp \underbrace{\frac{3}{8}\alpha^2 \pm \cdots}_{\text{NEGLECT SINCE } \alpha << 1}$$

AY2011 6

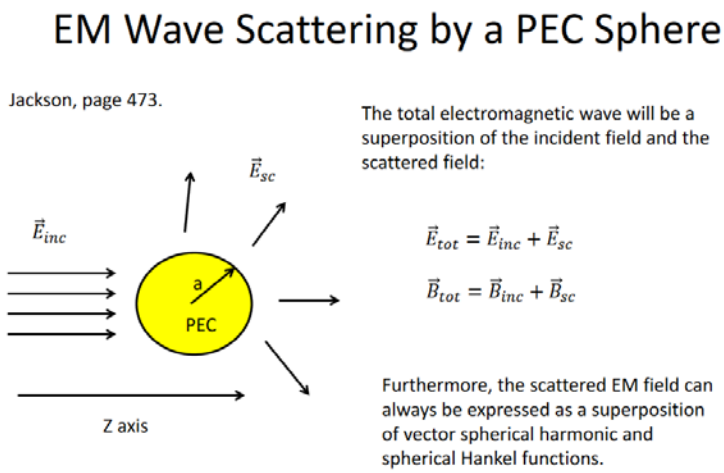
Keeping the first two terms in each case leads to the approximate expressions $R_1 \approx R + (\ell/2)\sin\theta$ and $R_2 \approx R - (\ell/2)\sin\theta$. Total effective RCS of the two spheres is:

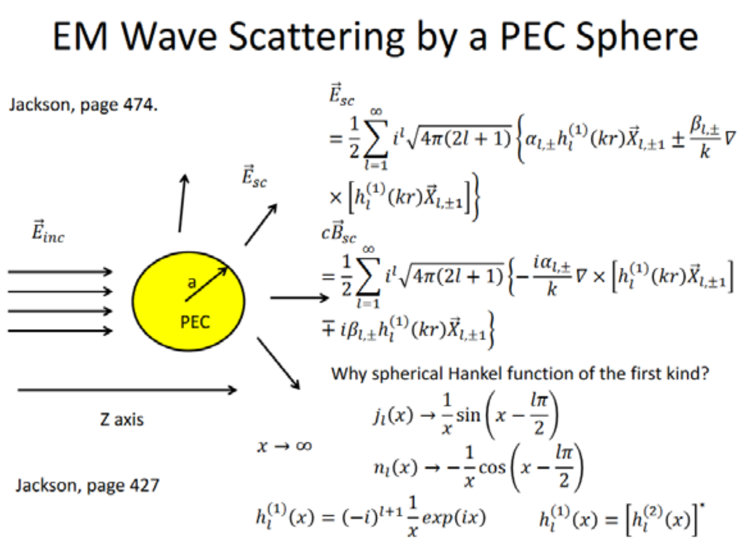
$$\sigma_{\text{eff}} = 4\pi R^2 |E_s|^2 = 4\pi R^2 \left| E_o\left(e^{-j2k\frac{\ell}{2}\sin\theta} + e^{j2k\frac{\ell}{2}\sin\theta}\right) \right|^2 = 4\sigma_o \cos^2(k\ell\sin\theta)$$

where $k = 2\pi/\lambda$ and $|E_o| = \sqrt{\sigma_o / 4\pi R^2} = |Ae^{-jkR}/R|$. A is the scattering amplitude of the element. This can easily be extended to N spheres of various amplitudes.

Appendix C:

Using a script that I developed to calculate the RCS of a PEC sphere using multipole spherical wave expansion for a HW problem in Professor Xu's 5106 course implanting the equations shown in the slides below from Professor Xu's ECE 5106 courses notes on spherical wave scattering analysis from 04/04/16:





EM Wave Scattering by a Sphere

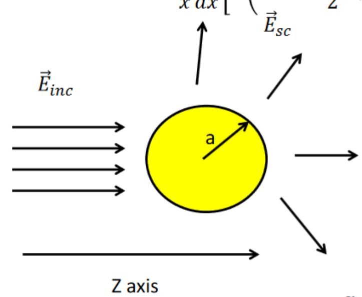
Incident Field: $$\vec{E}_{inc} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ j_l(kr)\vec{X}_{l,\pm 1} \pm \frac{1}{k}\nabla \times \left[j_l(kr)\vec{X}_{l,\pm 1} \right] \right\}$$

Scattered Field: $$\vec{E}_{sc} = \frac{1}{2}\sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ \alpha_{l,\pm} h_l^{(1)}(kr)\vec{X}_{l,\pm 1} \pm \frac{\beta_{l,\pm}}{k}\nabla \times \left[h_l^{(1)}(kr)\vec{X}_{l,\pm 1} \right] \right\}$$

Total Field: $$\vec{E}_{tot} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ \left(j_l(kr) + \frac{1}{2}\alpha_{l,\pm} h_l^{(1)}(kr) \right) \vec{X}_{l,\pm 1} \pm \frac{1}{k}\nabla \times \left[\left(j_l(kr) + \frac{1}{2}\beta_{l,\pm} h_l^{(1)}(kr) \right) \vec{X}_{l,\pm 1} \right] \right\}$$

Scattering by a PEC Sphere

$$\vec{E}_{tot}^{tran} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left\{ \left(j_l(kr) + \frac{1}{2}\alpha_{l,\pm} h_l^{(1)}(kr) \right) \vec{X}_{l,\pm 1} \pm \frac{1}{x}\frac{d}{dx}\left[x\left(j_l(x) + \frac{1}{2}\beta_{l,\pm} h_l^{(1)}(x) \right) \right] \vec{e}_r \times \vec{X}_{l,\pm 1} \right\}$$



Boundary Conditions: $x = ka$

$$j_l(x) + \frac{1}{2}\alpha_{l,\pm} h_l^{(1)}(x) = 0$$

$$\frac{d}{dx}\left[x\left(j_l(x) + \frac{1}{2}\beta_{l,\pm} h_l^{(1)}(x) \right) \right] = 0$$

$$\alpha_{l,\pm} = -\frac{2 j_l(x)}{h_l^{(1)}(x)} \qquad \beta_{l,\pm} = -\frac{2\frac{d}{dx}[x j_l(x)]}{\frac{d}{dx}\left[x h_l^{(1)}(x) \right]}$$

Scattering / Absorption Cross-section

$$P_{sc} = \frac{\pi}{2\mu_0 c k^2} \sum_{l=1}^{\infty} (2l+1)\,[|\alpha_l|^2 + |\beta_l|^2] \qquad \vec{S} = \frac{1}{\mu_0 c}\vec{e}_z$$

$$\sigma_{sc} = \frac{P_{sc}}{|\vec{S}|} = \frac{\pi}{2k^2} \sum_{l=1}^{\infty} (2l+1)\,[|\alpha_l|^2 + |\beta_l|^2]$$

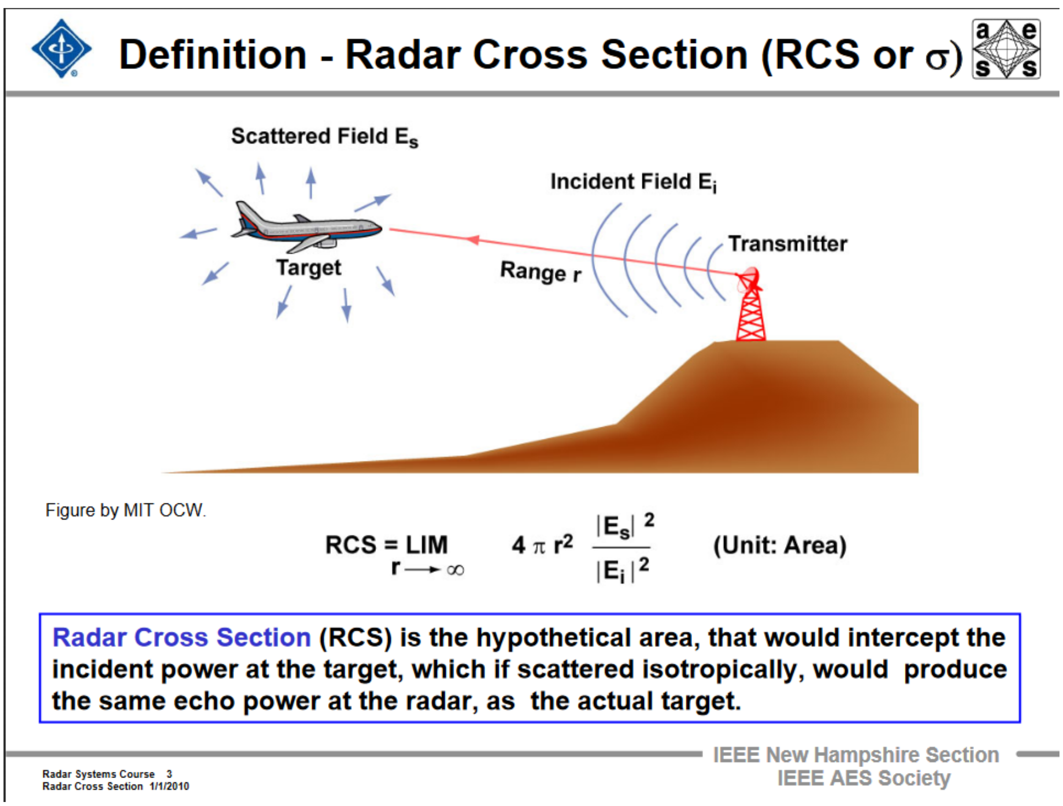
Note that the unit of scattering/absorption cross-section is area (i.e., m^2).
Thus, they really represent the size of the scattering object.

$$P_{abs} = -\int d\Omega \frac{1}{2} Re[\vec{E}_{tot} \times \vec{H}^{*}_{tot}] \cdot \vec{e}_r$$

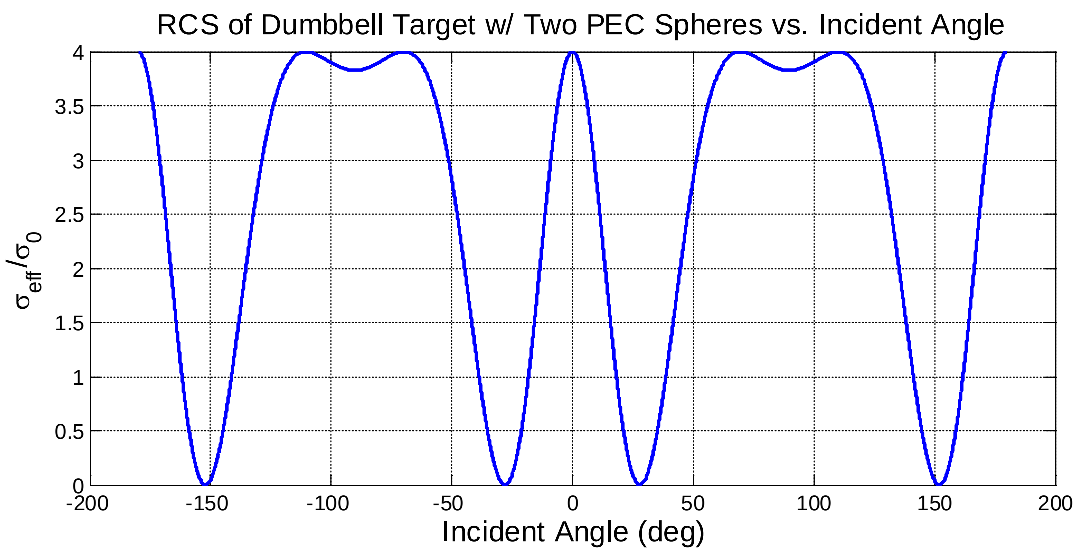
$$\sigma_{abs} = \frac{P_{abs}}{|\vec{S}|} = \frac{\pi}{2k^2} \sum_{l=1}^{\infty} (2l+1)\,[2 - |\alpha_l + 1|^2 - |\beta_l + 1|^2]$$

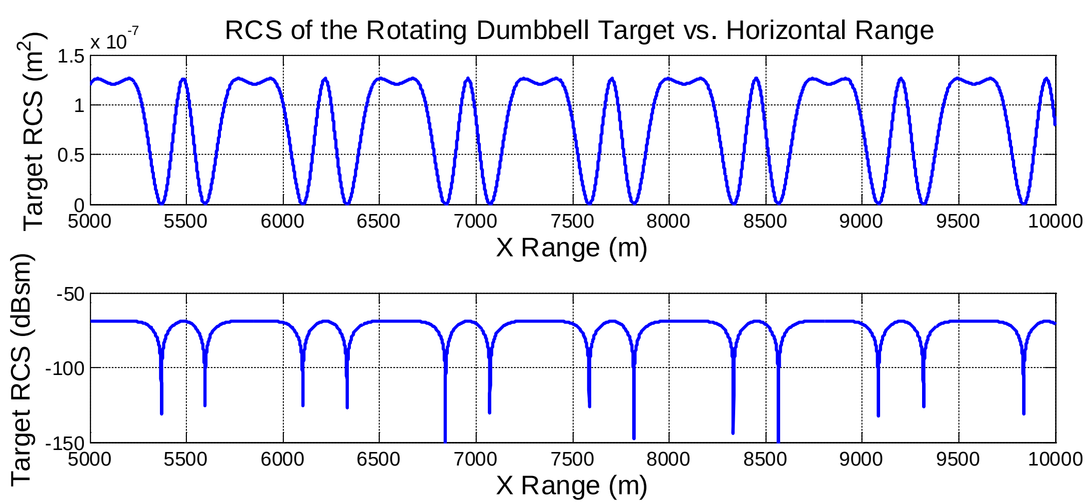
Appendix D:

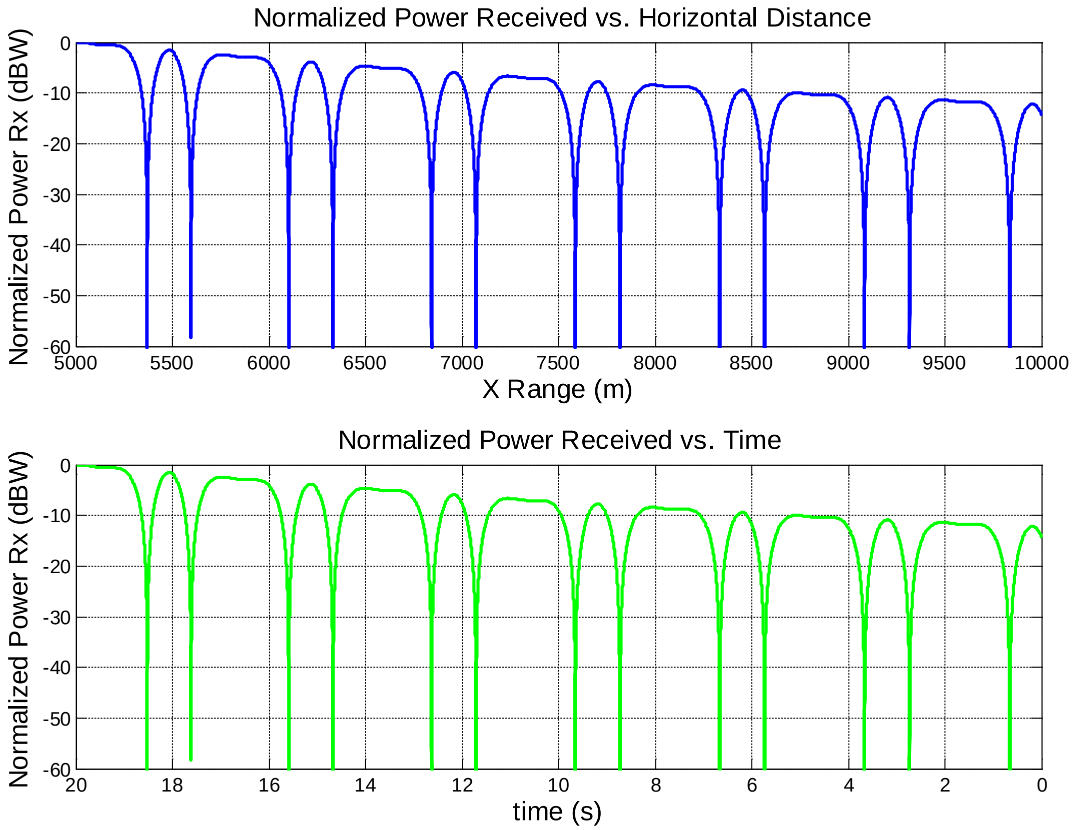
From MIT LL Radar Systems Course Lecture 7



Appendix E: Assuming the distance between the two connected spheres in question #4 is actually d = 16cm instead of d = 15cm







Appendix F: Additional Info on Radar Jamming

Naval Postgraduate School *Microwave Devices & Radar* *Distance Learning*

Jammer Burnthrough Range (2)

The signal-to-jam ratio (SJR) is

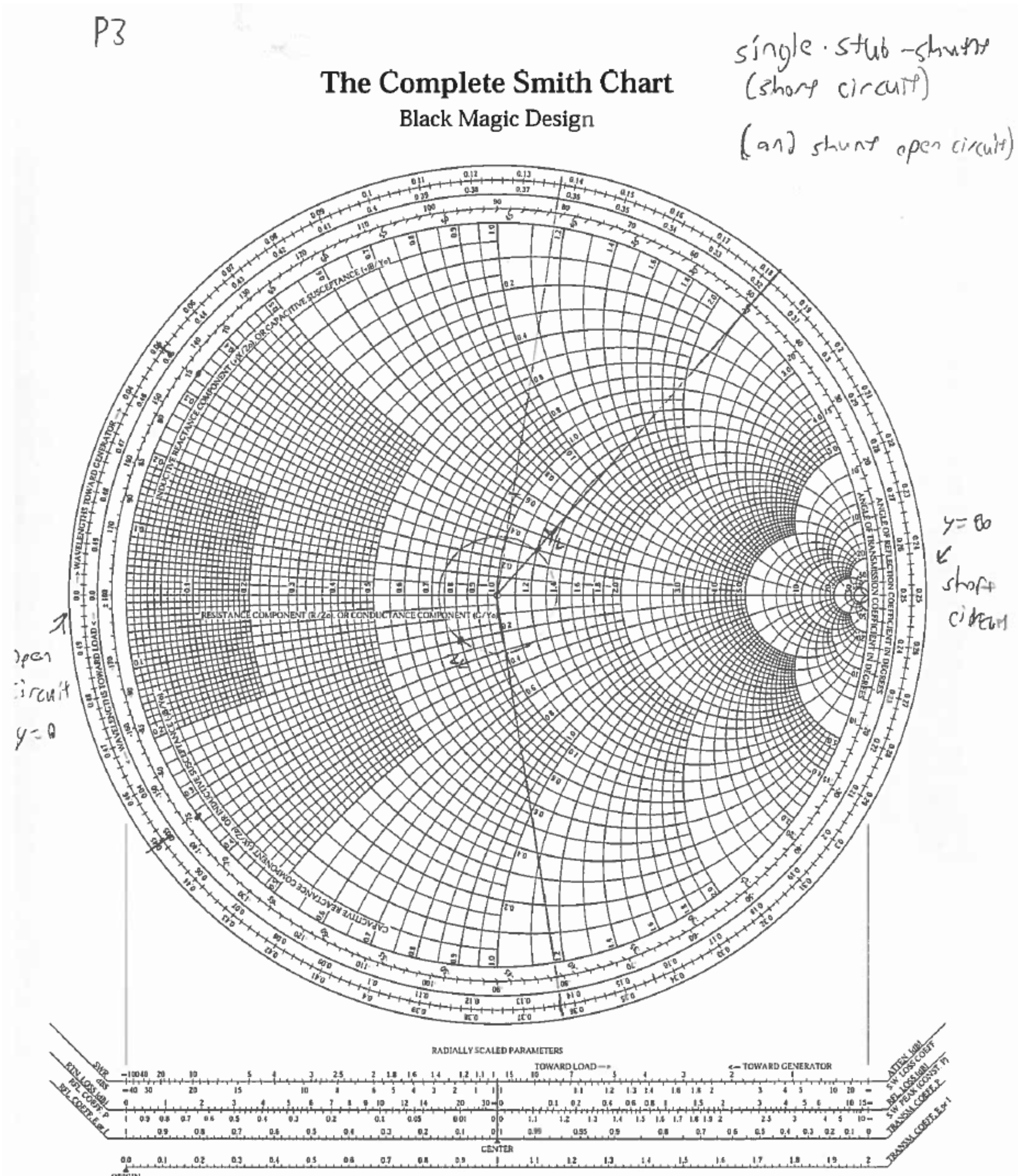
$$\boxed{\mathrm{SJR} = \frac{S}{J} = \frac{P_r}{P_{r,J}} = \left(\frac{P_t G_o}{P_J G_J}\right)\left(\frac{R_J^2}{R^4}\right)\left(\frac{\sigma}{4\pi}\right)\left(\frac{G_o}{G(\theta)}\right)}$$

The burnthrough range for the jammer is the range at which its signal is equal to the target return (SJR=1).

Important points:

1. R_J^2 vs R^4 is a big advantage for the jammer.
2. G vs $G(\theta)$ is usually a big disadvantage for the jammer. Low sidelobe radar antennas reduce jammer effectiveness.
3. Given the geometry, the only parameter that the jammer has control of is the ERP ($P_J G_J$).
4. The radar knows it is being jammed. The jammer can be countered using waveform selection and signal processing techniques.

Appendix G: Smith Charts Used for Solving Question 3

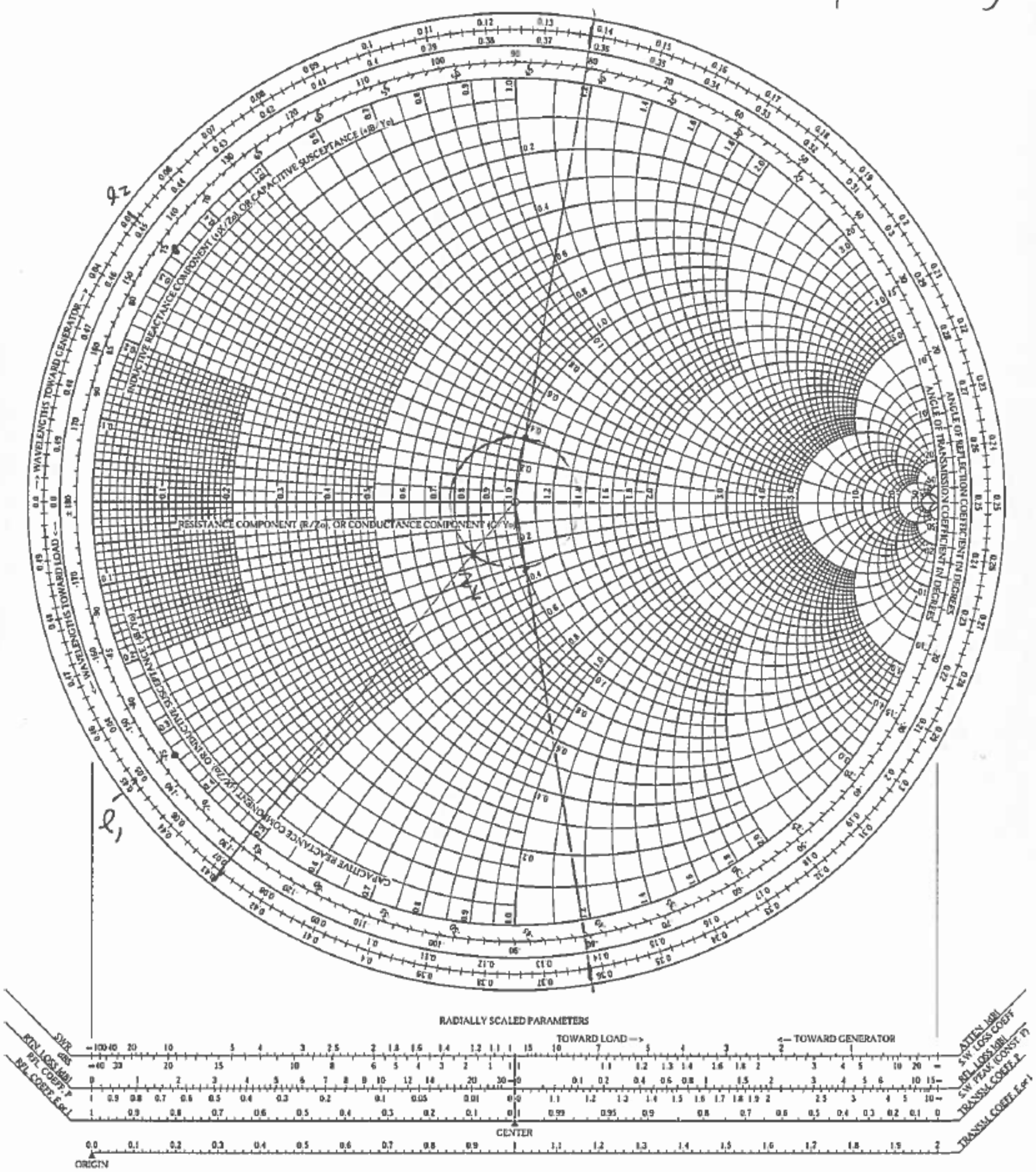


P3

Single-stub series tuning (open circuit)

The Complete Smith Chart

Black Magic Design



Appendix H: List of Textbook Title Acronyms and References Cited:

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