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ECE 5106 – EM Fields II: Final Exam

May 6th, 2016

Whispering Gallery Mode (WGM) Analysis in a Spherical Dielectric Resonator at THz Frequencies

Key Parameters:

Parameter	Value
n_{co}	1.4
n_{cl}	1.1
μ	μ_0= 4*π*1e-7 H/m
Sphere Radius: a	15e-6 m
Constraint	L=m
Mode Considered	TE mode
λ_{min}	0.93e-6 m
λ_{max}	0.98e-6 m
ε_{r_co}	1.96
ε_{r_cl}	1.21
Z_{co}	269.093 Ω
Z_{cl}	342.482 Ω
ω_{low}	1.9234e15 rad/s
ω_{high}	2.0268e15 rad/s

* Note: Parameters Not Given in Assignment Will Be Derived in Parts A and B below

Part A:

Calculated minimum and maximum allowable frequencies:

λ_{min} = 0.93 μm = 0.93e-6 m -> f_{high} = 3.2258e14 Hz -> ω_{high} = 2πf = 2.0268e15 rad/s

λ_{max} = 0.98 μm = 0.98e-6 m -> f_{low} = 3.0612e14 Hz -> ω_{low} = 2πf = 1.9234e15 rad/s

Allowable Wavelength Range: 0.93 μm < λ < 0.98 μm

Allowable Frequency Range: 3.0612e14 Hz < f < 3.2258e14 Hz

Allowable Angular Frequency Range: 1.9234e15 rad/s < ω < 2.0268e15 rad/s

Calculated permittivity and impedance for core and cladding materials:

$$n^2 = \varepsilon_r \mu_r \text{ -> } n = \sqrt{\varepsilon_r \mu_r} \text{ -> } \varepsilon_r = \frac{n^2}{\mu_r}$$

$$\text{Core Region: } \varepsilon_{r\_core} = \frac{(1.4)^2}{(1.0)} = 1.96$$

$$\text{Cladding Region: } \varepsilon_{r\_cladding} = \frac{(1.1)^2}{(1.0)} = 1.21$$

$$Z = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}}$$

Free Space Impedance Calculation (Sanity Check):

$$Z = \sqrt{\frac{(1.0)(4\pi{*}1e{-}7\ H/m)}{(1.0)(8.854e{-}12\ F/m)}} = 376.73\ \Omega \approx 377\ \Omega$$

$$\text{Core Region Impedance} = Z_{core} = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_{r\_core}}} = \sqrt{\frac{(1.0)(4\pi{*}1e{-}7\ H/m)}{(1.96)(8.854e{-}12\ F/m)}} = 269.093\ \Omega$$

$$\text{Cladding Region Impedance} = Z_{cl} = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_{r\_cl}}} = \sqrt{\frac{(1.0)(4\pi{*}1e{-}7\ H/m)}{(1.21)(8.854e{-}12\ F/m)}} = 342.482\ \Omega$$

Calculate Wavenumber k-vectors for Core and Cladding Regions:

$$k^2 = \omega^2 \varepsilon\mu = \omega^2 (\mu_0 \mu_r)(\varepsilon_0 \varepsilon_r)$$

$$c = \frac{1}{\sqrt{(\varepsilon_0 \mu_0)}}\ ;\ \frac{1}{c} = \sqrt{(\varepsilon_0 \mu_0)}$$

$$k = \omega\sqrt{\varepsilon\mu} = \omega\sqrt{(\mu_0 \mu_r)(\varepsilon_0 \varepsilon_r)} = \omega\sqrt{(\varepsilon_0 \mu_0)}\sqrt{(\varepsilon_r \mu_r)} = \frac{\omega n}{c}$$

$$\text{Core Region k-vector: } k_{core} = \frac{\omega n_{core}}{c} = \frac{(\omega)(1.4)}{(3e8\ \frac{m}{s})}$$

$$\text{Core Region k-vector: } k_{cladding} = \frac{\omega n_{cladding}}{c} = \frac{(\omega)(1.1)}{(3e8\ \frac{m}{s})}$$

Whispering Gallery Mode in Spherical Resonator Conditions:

$$n_{co} > n_{cl} \text{ -> } (n_{co} = 1.4) > (n_{cl} = 1.1)$$

$$a >> \lambda \text{ -> } (a = 15\ \mu m) >> (\lambda = 0.93\ \mu m \text{ to } 0.98\ \mu m)$$

Steps to Analyze WGM Modes in Spherical Resonator:

- Separately find solution in core region (r<a) and cladding region (r>a)
- Classify modes as either TE or TM modes
- Match boundary conditions at the core and cladding interface (r=a)
- Impose appropriate boundary conditions at the orgins (r=0) and at infinity (r -> ∞)

General Spherical Wave Solutions:

TE and TM Solutions

TE Solutions:

$$\vec{E}^{TE}_{lm} = Z a^{TE}_{lm} g_l(kr)\vec{X}_{lm}(\theta,\phi)$$

$$\vec{H}^{TE}_{lm} = \frac{j}{k} a^{TE}_{lm} \nabla \times \left[g_l(kr)\vec{X}_{lm}(\theta,\phi)\right]$$

TM Solutions:

$$\vec{H}^{TM}_{lm} = a^{TM}_{lm} f_l(kr)\vec{X}_{lm}(\theta,\phi)$$

$$\vec{E}^{TM}_{lm} = -j\frac{Z}{k} a^{TM}_{lm} \nabla \times \left[f_l(kr)\vec{X}_{lm}(\theta,\phi)\right]$$

All solutions can be separated into TE and TM components. So taken together, we have

$$\vec{E} = Z\sum_{l,m}\left\{a^{TE}_{lm} g_l(kr)\vec{X}_{lm}(\theta,\phi) - \frac{j}{k} a^{TM}_{lm}\nabla \times \left[f_l(kr)\vec{X}_{lm}(\theta,\phi)\right]\right\}$$

$$\vec{H} = \sum_{l,m}\left\{\frac{j}{k} a^{TE}_{lm}\nabla \times \left[g_l(kr)\vec{X}_{lm}(\theta,\phi)\right] + a^{TM}_{lm} f_l(kr)\vec{X}_{lm}(\theta,\phi)\right\}$$

Definition of Spherical Bessel J and Hankel Functions:

$$\text{Spherical Bessel J Function: } j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+\frac{1}{2}}(x)$$

Spherical Hankel Function of the First Kind:

$$h_l^{(1)}(x) = j_l(x) + i n_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+\frac{1}{2}}(x) + i\sqrt{\frac{\pi}{2x}}Y_{l+\frac{1}{2}}(x)$$

TE Modes in Dielectric Core Region (r < a):

$$\vec{E}_{trans\_co} = A j_l(k_{co} r)\vec{X}_{lm}(\theta, \Phi)$$

$$\vec{H}_{trans\_cl} = -\frac{iA}{Z_{co}}\left[j_{l-1}\left(k_{co} r\right) - \frac{l}{k_{co} r} j_l\left(k_{co} r\right)\right]\left[\vec{e}_r \times \vec{X}_{lm}(\theta, \Phi)\right]$$

Utilizing relation: $\frac{d}{dx}\left[xj_l(x)\right] = \; xj_{l-1}(x) - \; lj_l(x)$

TE Modes in Dielectric Cladding Region (r > a):

$$\vec{E}_{trans\_co} = Bh_l^{(1)}(k_{cl}r)\vec{X}_{lm}(\theta, \Phi)$$

$$\vec{H}_{trans\_cl} = \; -\frac{iB}{Z_{cl}}\left[h_{l-1}^{(1)}\left(k_{cl}r\right) - \; \frac{l}{k_{cl}r}h_l^{(1)}\left(k_{cl}r\right)\right]\left[\vec{e}_r \times \vec{X}_{lm}(\theta, \Phi)\right]$$

Characteristic Equation for Solving Frequency of WGM Mode in Spherical Cavity Resonator by Matching Fields Across Interface (r = a):

$$\frac{1}{Z_{co}}\frac{j_{l-1}\left(k_{co}a\right) - \frac{l}{k_{co}a}j_l\left(k_{co}a\right)}{j_l(k_{co}a)} = \frac{1}{Z_{cl}}\frac{h_{l-1}^{(1)}\left(k_{cl}a\right) - \frac{l}{k_{cl}a}h_l^{(1)}\left(k_{cl}a\right)}{h_l^{(1)}(k_{cl}a)}$$

*Note: A, B, $\vec{X}_{lm}(\theta, \Phi)$, and $\left[\vec{e}_r \times \vec{X}_{lm}(\theta, \Phi)\right]$ cancel out of equation

Where: $k_{co} = \frac{\omega n_{co}}{c}$ and $k_{cl} = \frac{\omega n_{cl}}{c}$

*Key insight from the characteristic equation: We have all variable except for the ω in the k-vector so we must find the ω that sets both sides of the characteristic equation equal.

For purposes of processing in MATLAB, the characteristic equation will be split into the left-hand side (LHS) and right-hand side (RHS) as follows:

$$LHS = \frac{1}{Z_{co}}\frac{j_{l-1}\left(k_{co}a\right) - \frac{l}{k_{co}a}j_l\left(k_{co}a\right)}{j_l(k_{co}a)}$$

$$RHS = \frac{1}{Z_{cl}}\frac{h_{l-1}^{(1)}\left(k_{cl}a\right) - \frac{l}{k_{cl}a}h_l^{(1)}\left(k_{cl}a\right)}{h_l^{(1)}(k_{cl}a)}$$

To calculate the resonant frequency, we want to find the ω where RHS = LHS

To do this, I will find this frequency point by calculating the zero-crossing in MATLAB as follows y_test_abs = abs(LHS - RHS)

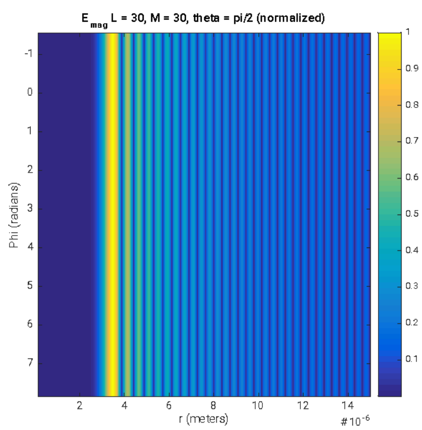
y_test_abs_db = $10|LHS \; - \; RHS|$

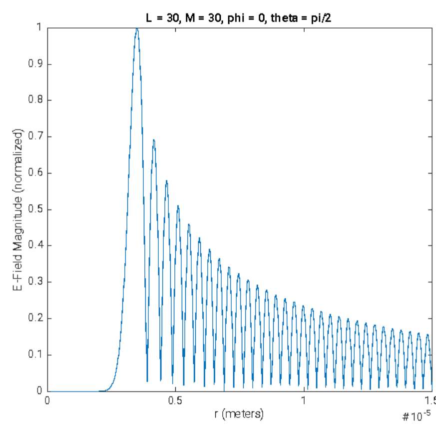
* **Important Note:** After using this technique which gave me a pretty good answer based on the difference of stored electrical and magnetic energy values in Part B, I alternatively used the method demonstrated in HW 3 Prob 3 of finding the frequency of points where the |det[M(ω)]|=0 in search of a more precise solution. I believe the |det[M(ω)]|=0 method gives a more accurate answer. I also detail this alternative below beginning on Page 12. I updated my solution for Part B based on the resonant frequency found using |det[M(ω)]|=0.

Steps we must now follow:

- Find appropriate l (mode number; I will also refer to l as L to avoid confusion in notation w/ 1) that satisfies characteristic equation
- Verify for solution (mode number L and angular frequency ω) that magnitude of field is non-zero for core region (0 < r < a)

Try L = M = 30:

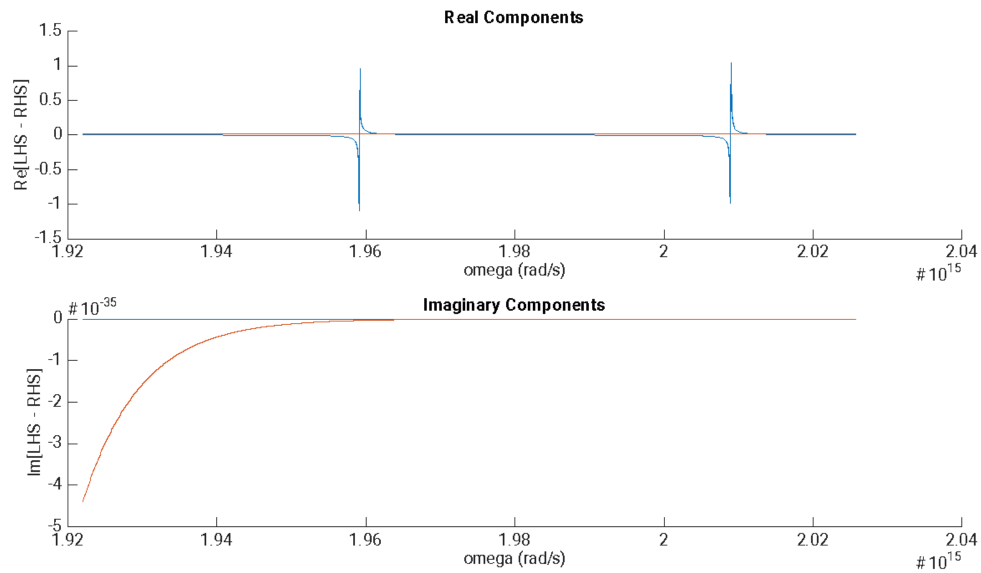


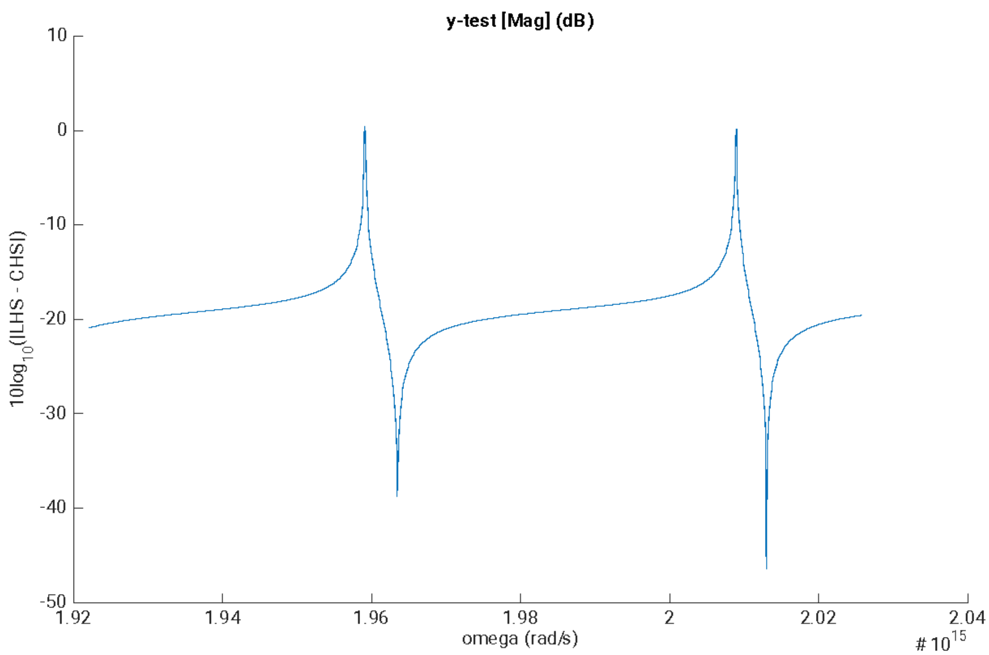


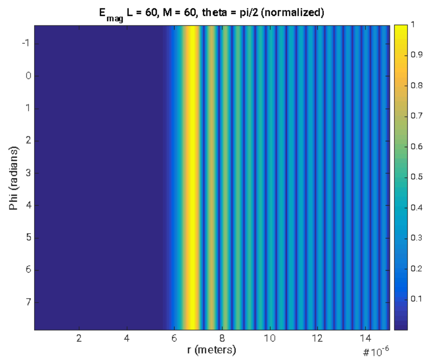
*Note: Field is not non-zero from 0 < r < a. Need to try greater value of L

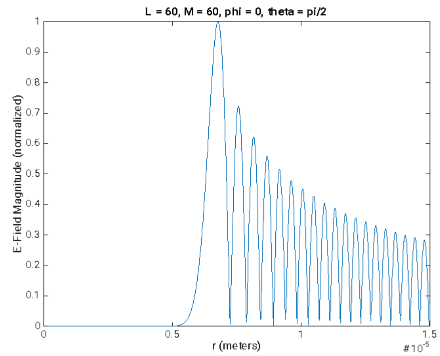
```
Frequency: w = 2.016344474620612e+15; L = 30;
```

Try L = M = 60:


Real Components
Re[LHS - RHS]
omega (rad/s)
10^15
Imaginary Components
10^-35
Im[LHS - RHS]
omega (rad/s)
10^15


y-test [Mag] (dB)
10log10(|LHS - CHS|)
omega (rad/s)
10^15

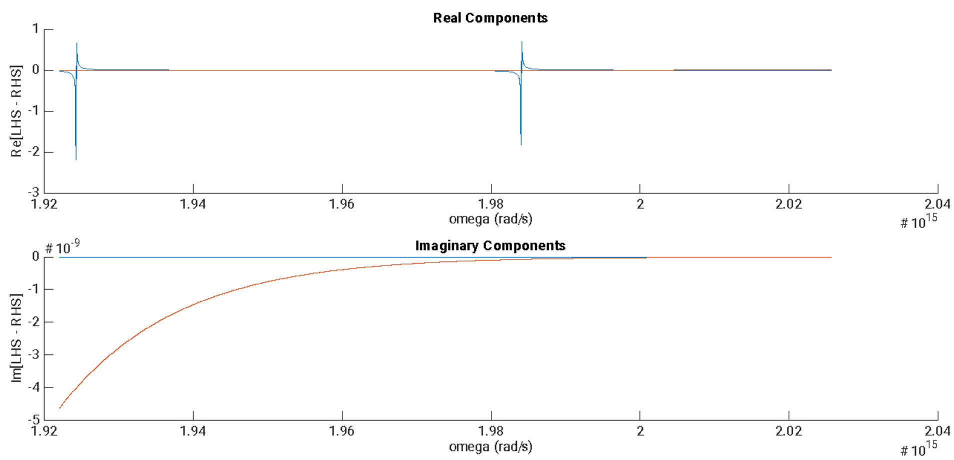


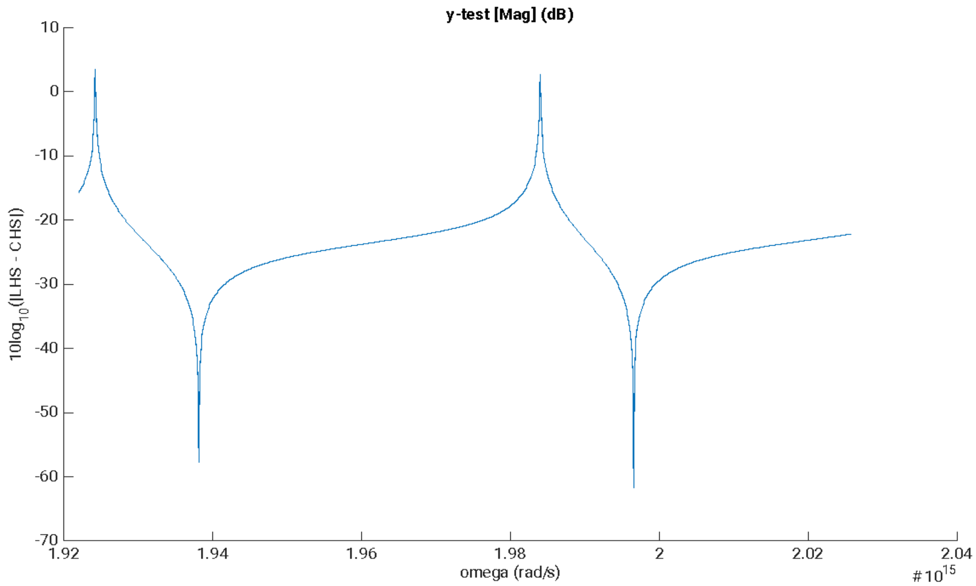


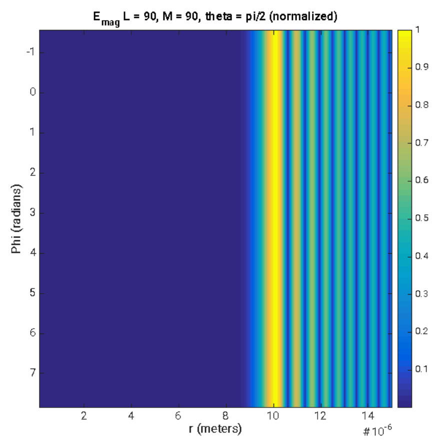
*Note: Field is not non-zero from 0 < r < a. Need to try greater value of L

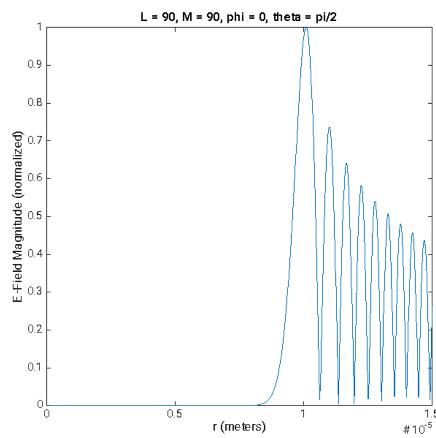
```
Frequency: w = 2.013031066546643e+15; L = 60;
```

Try L = M = 90:





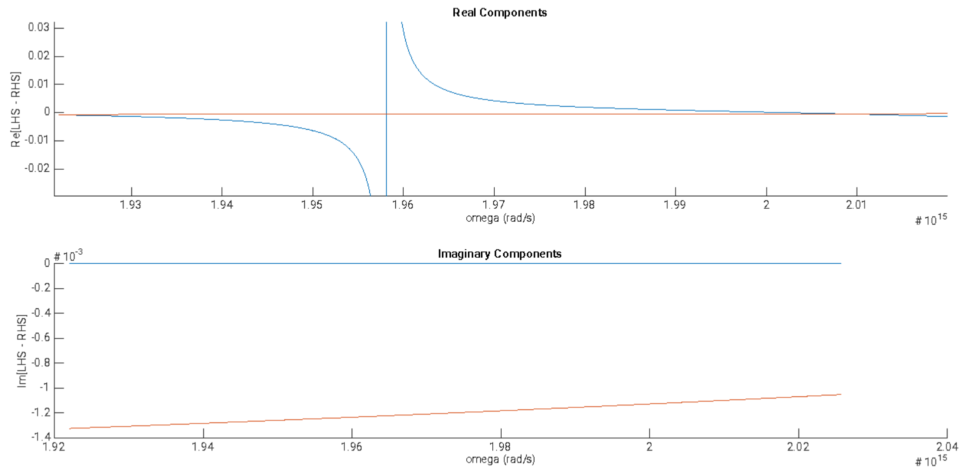


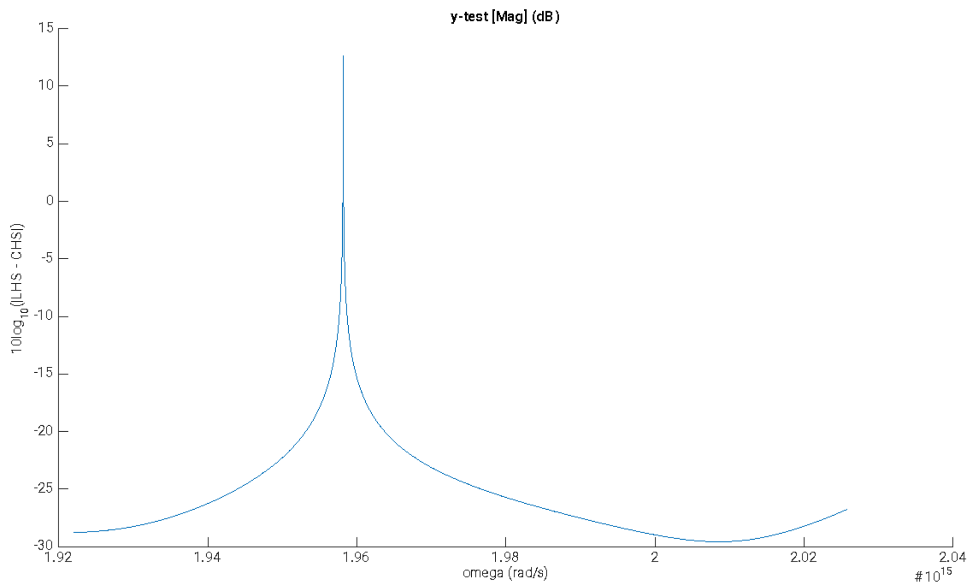


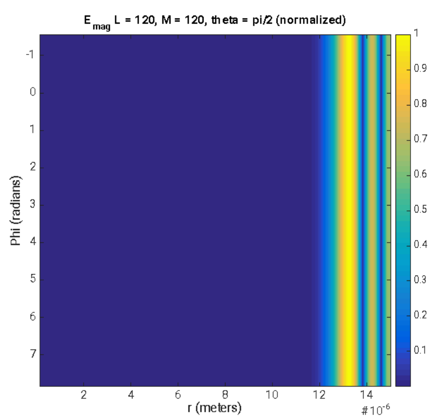
*Note: Field is not non-zero from 0 < r < a. Need to try greater value of L

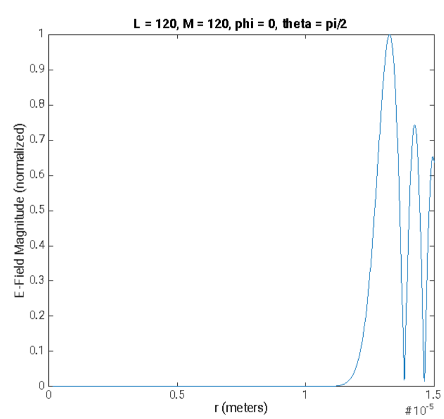
```
w = 1.996539707900641e+15; L = 90;
```

Try L = M = 120:





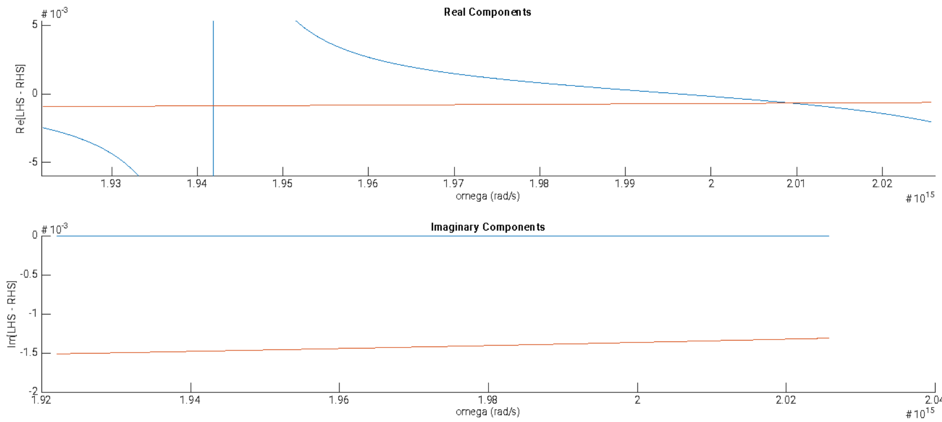


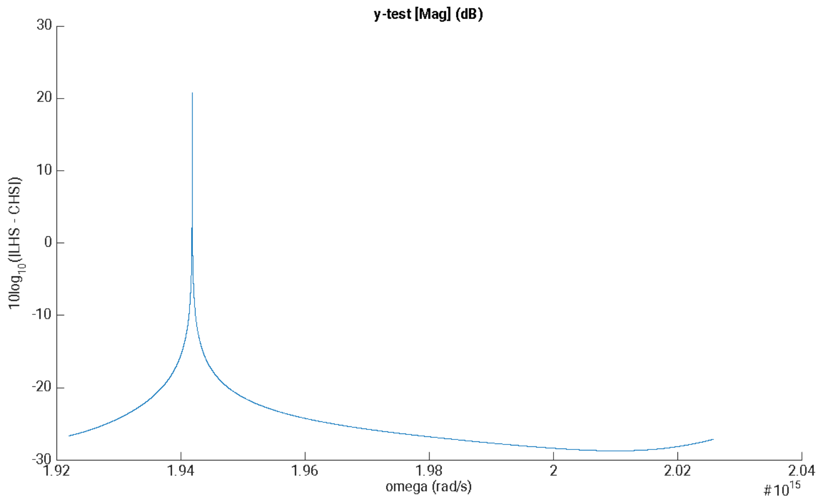


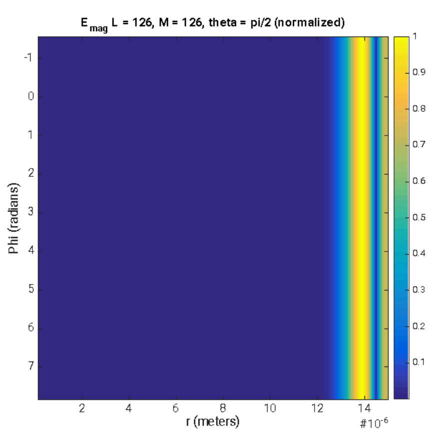
*Note: Field is not non-zero from 0 < r < a. Need to try greater value of L

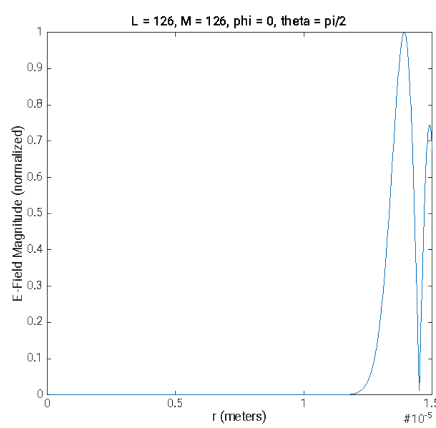
```
w = 2.008562734632123e+15; L = 120;
```

Try L = M = 126:





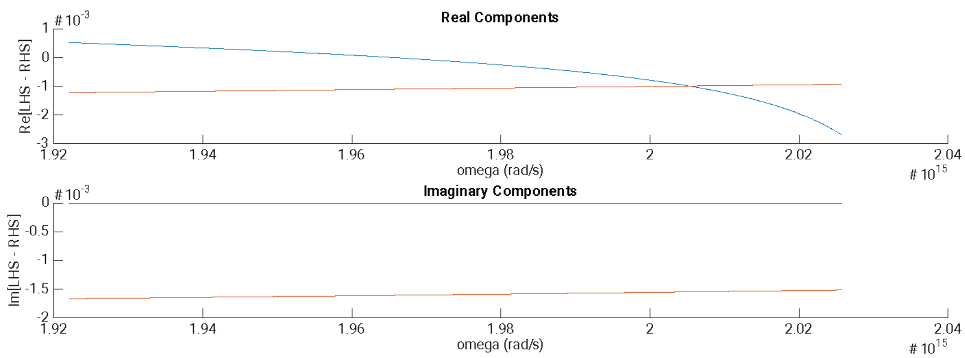


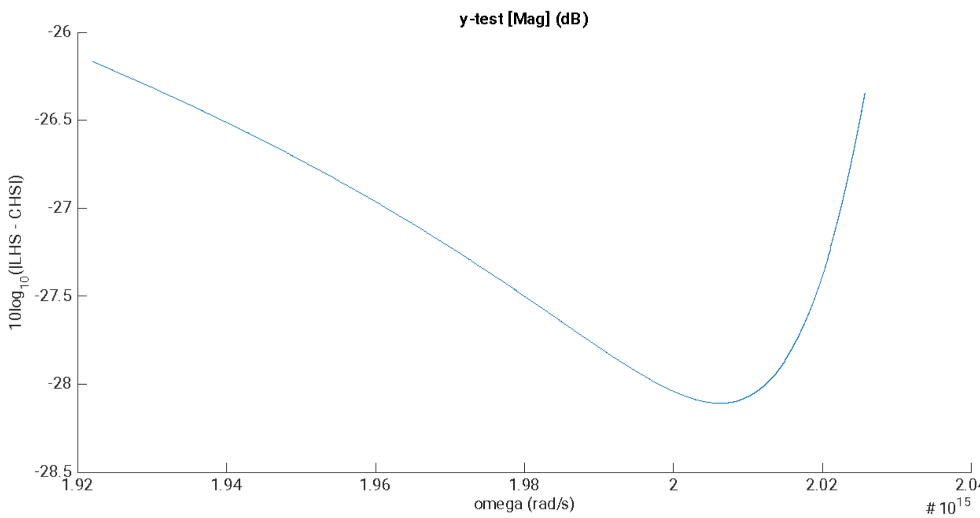


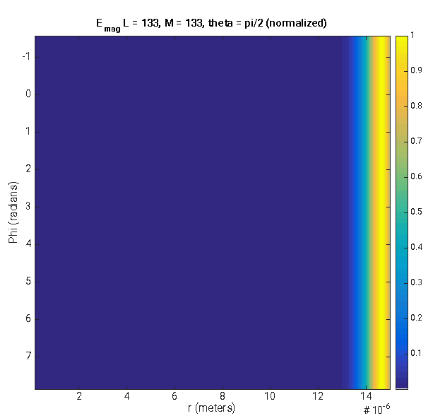
*Note: Field is not non-zero from 0 < r < a. Need to try greater value of L

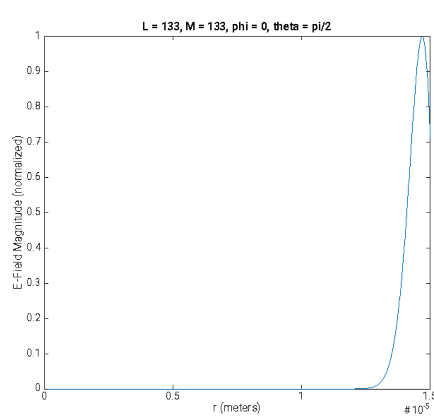
```
w = 2.010113691602917e+15; L = 126;
```

Try L = M = 133:


Real Components
Re[LHS - RHS]
omega (rad/s)
Imaginary Components
Im[LHS - RHS]
omega (rad/s)


y-test [Mag] (dB)
10log10(|LHS - CHS|)
omega (rad/s)


E mag L = 133, M = 133, theta = pi/2 (normalized)
Phi (radians)
r (meters)


L = 133, M = 133, phi = 0, theta = pi/2
E-Field Magnitude (normalized)
r (meters)

*Note: Field is non-zero from 0 < r < a. Appears to be an appropriate mode number of L = 133 (however, not the only possible mode number).

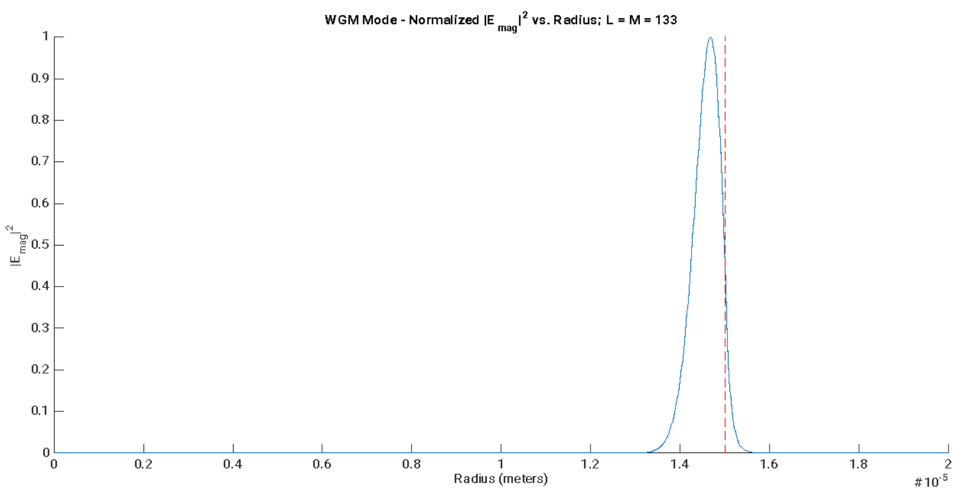
```
w = 2.006292768594956e+15; L = 133;
```

Calculate A/B Ratio at Interface (r = a):

Using a = 15e-6m, L = 133, and ω = 2.0063e+15 rad/s:

$$\frac{A}{B} = \frac{h_l^{(1)}(k_{cl}a)}{j_l(k_{co}a)} = 0.0956 + 1.0142i \ \text{[Calculated Using MATLAB]}$$

E-Field Magnitude In Core and Cladding Regions of Dielectric Spherical Resonator:



*****Alternative Method of Calculating Resonant Mode Frequency for Part A:**

E-Field Continuity at Interface (r=a):

$$Aj_l\left(k_{co}a\right) = Bh_l^{(1)}(k_{cl}a)$$

H-Field Continuity at Interface (r=a):

$$\left(\frac{A}{Z_{co}}\right)\left[j_{l-1}\left(k_{co}a\right) - \frac{l}{k_{co}a}j_l\left(k_{co}a\right)\right] = \left(\frac{B}{Z_{cl}}\right)\left[h_{l-1}^{(1)}\left(k_{cl}a\right) - \frac{l}{k_{cl}a}h_l^{(1)}\left(k_{cl}a\right)\right]$$

In which we can form equations:

$$A j_l(k_{co}a) - B h_l^{(1)}(k_{cl}a) = 0$$

$$\left(\frac{A}{Z_{co}}\right)[j_{l-1}(k_{co}a) - \frac{l}{k_{co}a} j_l(k_{co}a)] - \left(\frac{B}{Z_{cl}}\right)[h_{l-1}^{(1)}(k_{cl}a) - \frac{l}{k_{cl}a} h_l^{(1)}(k_{cl}a)] = 0$$

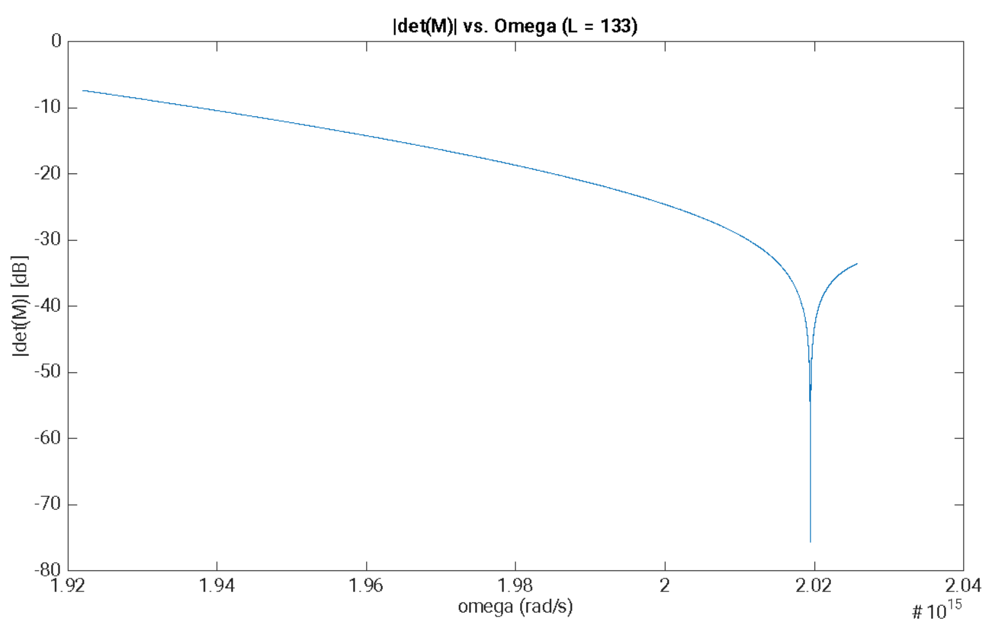
ω dependence is imbedded within the k-vector as such:

$$k_{co} = \frac{\omega n_{co}}{c} \text{ and } k_{cl} = \frac{\omega n_{cl}}{c}$$

From these two equations and three unknowns (A, B, and ω) we can solve for the resonant frequency (ω) by creating a two-by-two matrix from the two equations above and find the value of ω that satisfies |det[M(ω)] = 0, similar to the method used in HW 3 Problem 3 of this course.

$$\begin{bmatrix} j_l(k_{co}a) & -h_l^{(1)}(k_{cl}a) & \left(\frac{1}{Z_{co}}\right)[j_{l-1}(k_{co}a) - \frac{l}{k_{co}a} j_l(k_{co}a)] & -\left(\frac{1}{Z_{cl}}\right)[h_{l-1}^{(1)}(k_{cl}a) - \frac{l}{k_{cl}a} h_l^{(1)}(k_{cl}a)] \end{bmatrix} [A\ B] = 0$$

$$M = \begin{bmatrix} j_l(k_{co}a) & -h_l^{(1)}(k_{cl}a) & \left(\frac{1}{Z_{co}}\right)[j_{l-1}(k_{co}a) - \frac{l}{k_{co}a} j_l(k_{co}a)] & -\left(\frac{1}{Z_{cl}}\right)[h_{l-1}^{(1)}(k_{cl}a) - \frac{l}{k_{cl}a} h_l^{(1)}(k_{cl}a)] \end{bmatrix}$$



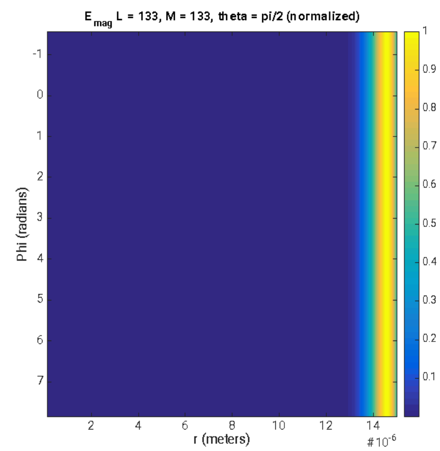
ω = 2.019454682449744e+15 rad/s; L = 133;

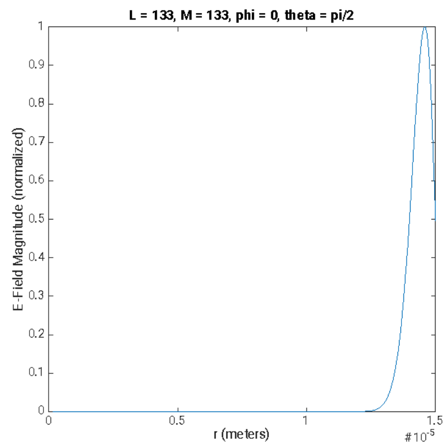
Result: Using this matrix determinate method of solving for the resonant frequency, I was able to find a presumably more accurate solution of ω = 2.0195e+15 rad/s at mode number L = 133.

Cavity Resonance at L=133:

ω = 2.0195e+15 rad/s -> f= ω/(2π) = 3.2141e+14 Hz -> λ = c/f = 9.3338e-7 m

λ_{res} = 0.9334μm





Analysis: At this frequency and mode number, the magnitude of the E-field is non-zero for 0 < r <a, which meets our requirements and the field is concentrated near the edge of the spherical cavity as desired.

Part B:

Assumption: Assume that the maximum value of the complex electric field amplitude is 1 V/m

Normalize E-field amplitude from Part A to maximum amplitude of 1 V/m such that:

$$\vec{E}\left(\vec{r}, t\right) = \vec{E}\left(\vec{r}\right)e^{-i\omega t}; \left(\left|\vec{E}\left(\vec{r}\right)\right|\right) \ = \ 1$$

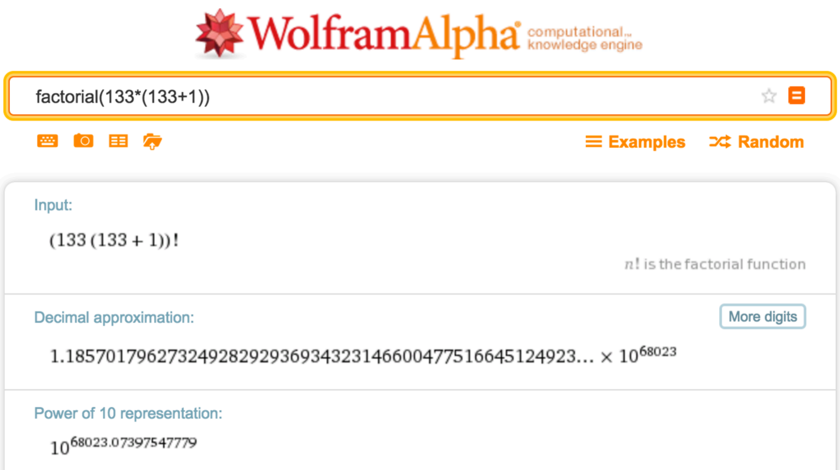
Note: $\vec{E}_{trans\_co} \ = \ Aj_l(k_{co}r)\vec{X}_{lm}(\theta, \Phi)$, however, $\vec{X}_{lm}(\theta, \Phi)$ has no radial dependence so it can be dropped from our normalized E-field equation. This is especially convenient because MATLAB cannot calculate a factorial greater than (2*85)! easily due to the magnitude of the number and the bits of precision storing that number. For L=133, there is a $Y_{lm}\left(\theta \ = \frac{\pi}{2}, \Phi, \ L \ = \ 133, \ M \ = \ L\right)$ inside of $\vec{X}_{lm}(\theta \ = \frac{\pi}{2}, \Phi, \ L \ = \ 133, \ M \ = \ L)$, which requires calculating a factorial(L*(L+1)).

Example of MATLAB numerical limitations for vector spherical harmonics at high mode order numbers:

```
>> L = 133;
>> Y = factorial(L*(L+1))
```

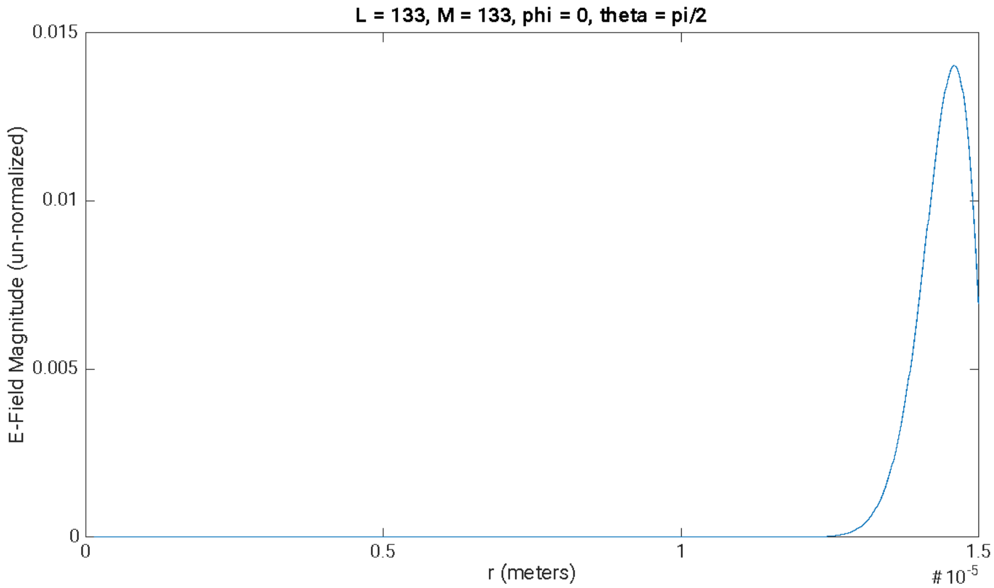
Result in MATLAB: Y = Inf;

Actual Result:



This numerical calculation issue in MATLAB can be dealt with by using a normalized associated Legendre function instead of the standard associated Legendre function, however, since we are normalizing the fields anyways, it is more convenient to drop the $\vec{X}_{lm}(\theta, \Phi)$ from our normalized E-field equation since it does not have r-dependence and will cancel out after the field is normalized to max of 1 V/m.

To normalize the E-field, $\vec{E}_{trans\_co} = j_l(k_{co}r)$ is plotted from 0 < r < a as shown below:



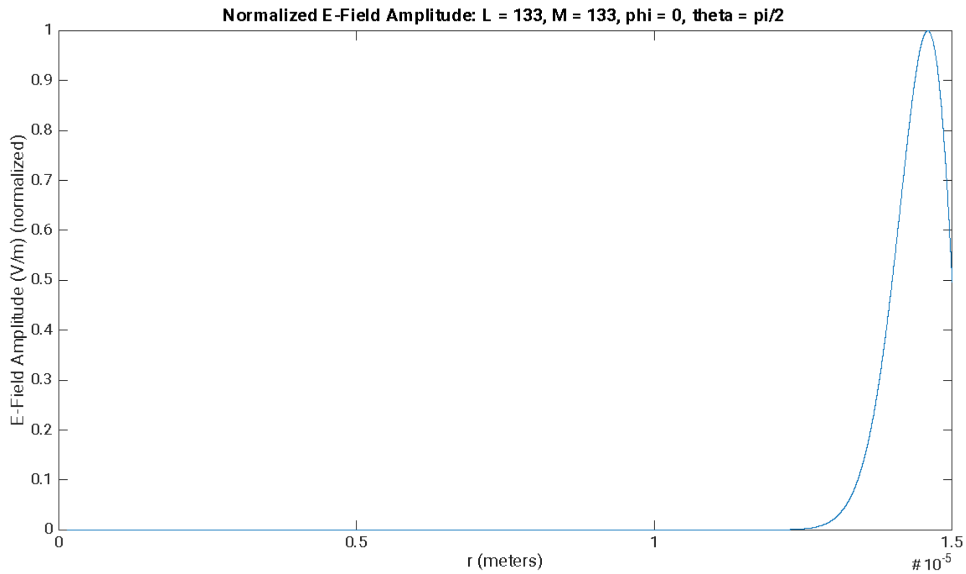
Max: 0.014021136331607 V/m = 0.0140 V/m at r = 14.589µm

Maximum value of this plot is max($|\vec{E}(\vec{r})|$) = 0.0140 V/m at r = 14.589µm.

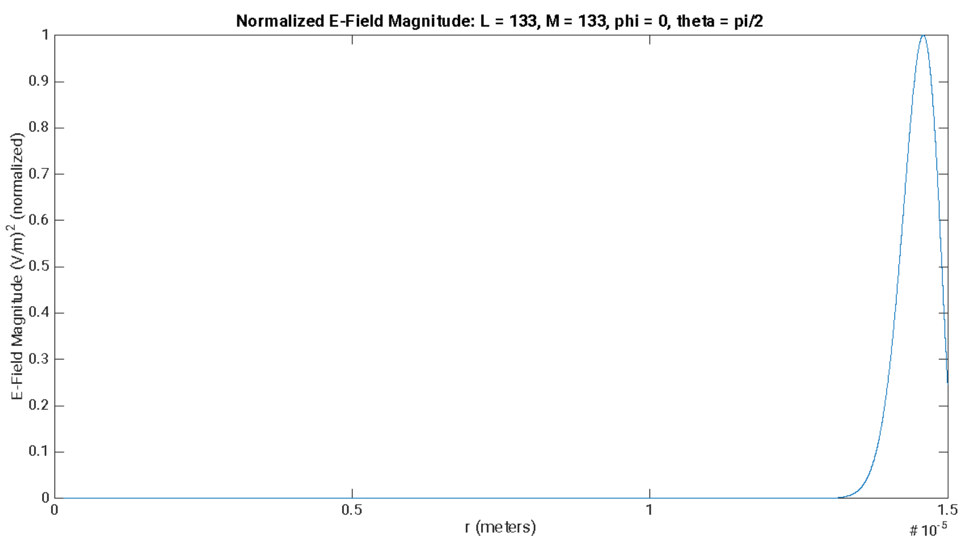
To normalize the maximum value of the E-field within the core region to 1 V/m, the constant A will be set to:

$$A = 1/\max(|\vec{E}(\vec{r})|) = 1/0.0140 = 71.3209$$

Applying this constant of A to $\vec{E}_{trans\_co} = Aj_l(k_{co}r) = (71.3209)j_l(k_{co}r)$ over the range of 0 < r < a gives the normalized E-field with a maximum of 1 V/m:



With Magnitude $\left|\vec{E}\left(\vec{r}\right)\right|^2$:



Objectives:

- Calculate total time-averaged energy stored in the WGM from Part A
- Separately list the energy stored in the form of the electric field and magnetic field

Definition of Electric and Magnetic Time-Average Stored Energy:

<u>Instantaneous Stored Energy</u>

Instantaneous Electric Energy Density: $w_e = \frac{1}{2}\varepsilon E^2$

Instantaneous Magnetic Energy Density: $w_m = \frac{1}{2}\mu H^2$

Total Instantaneous Stored Energy Density (energy per unit volume):
$$w = w_e + w_m = \frac{1}{2}\left[\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}\right] = \frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2$$

Instantaneous Stored Electrical Energy: $W_e = \iiint_V w_e \, dV = \frac{1}{2}\iiint_V \varepsilon E^2 \, dV$

Instantaneous Stored Magnetic Energy: $W_m = \iiint_V w_m \, dV = \frac{1}{2}\iiint_V \mu H^2 \, dV$

Total Instantaneous Stored Energy: $W = W_e + W_m = \frac{1}{2}\iiint_V \varepsilon E^2 \, dV + \frac{1}{2}\iiint_V \mu H^2 \, dV$

<u>Time-Averaged Stored Energy</u>

Time-Averaged Definition: $\langle \vec{A} \cdot \vec{B} \rangle = \frac{1}{2} Re[\vec{A} \cdot \vec{B}^*]$

Time-Averaged Electric Energy Density: $w_e = \frac{1}{2} Re[\frac{1}{2}\varepsilon E^2] = \frac{1}{4}\varepsilon\left|\vec{E}\right|^2$

Time-Averaged Magnetic Energy Density: $w_m = \frac{1}{2} Re[\frac{1}{2}\mu H^2] = \frac{1}{4}\mu\left|\vec{H}\right|^2$

Total Time-Averaged Stored Energy Density (energy per unit volume):
$$w = w_e + w_m = \frac{1}{4} Re\left[\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}\right] = \frac{1}{4}\varepsilon\left|\vec{E}\right|^2 + \frac{1}{4}\mu\left|\vec{H}\right|^2$$

Time-Averaged Stored Electrical Energy: $W_e = \iiint_V w_e \, dV = \frac{1}{4}\iiint_V \varepsilon\left|\vec{E}\right|^2 dV$

Time-Averaged Stored Magnetic Energy: $W_m = \iiint_V w_m \, dV = \frac{1}{4}\iiint_V \mu\left|\vec{H}\right|^2 dV$

Total Time-Averaged Stored Energy: $W = W_e + W_m = \frac{1}{4}\iiint_V \varepsilon\left|\vec{E}\right|^2 dV + \frac{1}{4}\iiint_V \mu\left|\vec{H}\right|^2 dV$

Note: At the resonant frequency -> $w_e = w_m$

Calculate Time-Averaged Energy Density for Spherical TE Mode:

Core TE E-Field: $\vec{E}_{trans\_co} = Aj_l(k_{co}r)\vec{X}_{lm}(\theta, \Phi)$

Time-Averaged Stored Electrical Energy Density:

$$w_e = \frac{1}{4}\varepsilon\left|\vec{E}\right|^2 = \frac{\varepsilon}{4}|A|^2 j_l(k_{co}r)^2\left|\vec{X}_{lm}(\theta, \Phi)\right|^2$$

*Key Insight: Use the orthogonality property of the vector spherical harmonic to simply integral and remove the $\left|\vec{X}_{lm}(\theta, \Phi)\right|^2$ term from the numerical computation.

Orthogonality Property of Vector Spherical Harmonics: $\int d\Omega\, \vec{X}_{lm} \cdot \vec{X}^*_{l'm'} = \delta_{ll'}\delta_{mm'}$

This means that for L=m and integrated over the entire sphere:

$$\int \vec{X}_{lm} \cdot \vec{X}^*_{lm}\, d\Omega = \int \left|\vec{X}_{lm}(\theta, \Phi)\right|^2 d\Omega = \delta_{ll}\delta_{mm} = 1$$

For this reason, the vector spherical harmonic can be removed from the numerical computation of stored energy since we know the analytical value of its contribution.

Time-Averaged Stored Electrical Energy:

$$W_e = \iiint_V w_e\, dV = \iiint_V \frac{\varepsilon}{4}|A|^2 j_l(k_{co}r)^2\left|\vec{X}_{lm}(\theta, \Phi)\right|^2 dV$$

where $dV = r^2 sin(\phi)d\phi d\theta dr$

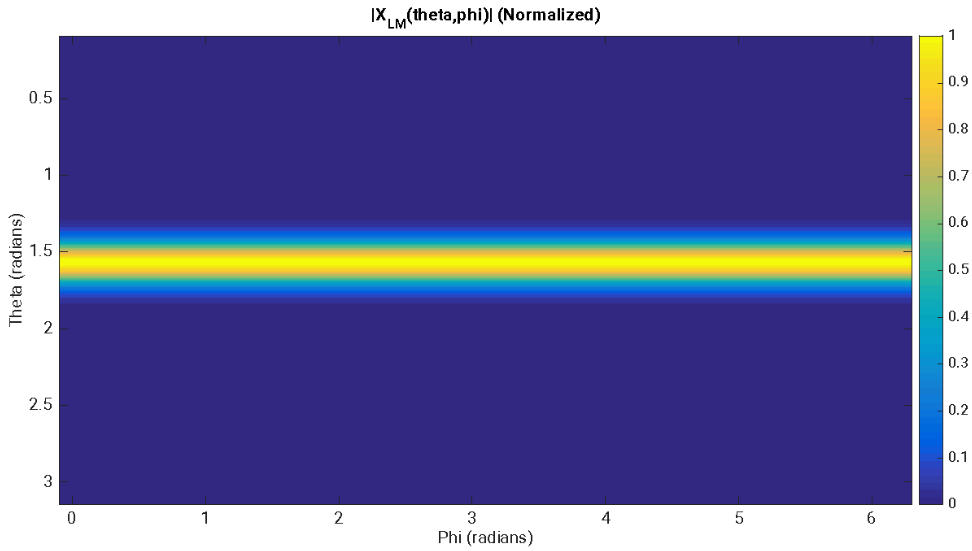
$$W_e = \iiint_V w_e\, dV = \frac{\varepsilon}{4}|A|^2 \iint_0^a j_l(k_{co}r)^2 r^2\left|\vec{X}_{lm}(\theta, \Phi)\right|^2 dr d\Omega = \frac{\varepsilon}{4}|A|^2 \int_0^a j_l(k_{co}r)^2 r^2 dr$$

Normalize maximum value of the complex electric field amplitude to 1 V/m:

Need to create: $\left(\left|\vec{E}\right|\right) = 1$ by setting A = $1/\left(\left|j_l(k_{co}r)\vec{X}_{lm}(\theta, \Phi)\right|\right)$

Core TE E-Field: $\vec{E}_{trans\_co} = Aj_l(k_{co}r)\vec{X}_{lm}(\theta, \Phi)$

*Note: $\vec{X}_{lm}(\theta, \Phi)$ is at a max at θ=π/2 and Φ=0 and has no r dependence so it can be dropped from the normalized E-field because it would cancel out anyways



$|\vec{X}_{lm}(\theta, \Phi)|$ is maximized at $\theta = \pi/2$ radians

Normalized Core TE E-Field: $\vec{E}_{trans\_co} = Aj_l(k_{co}r)$ from 0 < r < a; A = 71.3209

Used trapz command in MATLAB to numerically evaluate and integrate:

$$|A|^2 \int_0^a j_l(k_{co}r)^2 r^2 dr = |71.3209|^2 \int_0^{15\mu m} j_{133}(k_{co}r)^2 r^2 dr = 1.5477\text{e-}16$$

$$W_e = \iiint_V w_e\, dV = \frac{\varepsilon}{4}\left|\vec{E}\right|^2 dV = \left(\frac{\varepsilon}{4}\right)\left(|71.3209|^2 \int_0^{15e-6m} j_{133}(k_{co}r)^2 r^2 dr\right)$$

$$W_e = \left(\frac{\varepsilon_0\varepsilon_r}{4}\right)(1.5477\text{e-}16) = \frac{(1.96)\left(8.854e-12\frac{F}{m}\right)}{4}(1.5477\text{e-}16) = 6.7146\text{e-}28 \text{ W}$$

Calculate Time-Averaged Magnetic Stored Energy for Spherical TE Mode:

$$\vec{H}_{tr\_co} = -\frac{iA}{Z_{co}}\left[j_{l-1}\left(k_{co}r\right) - \frac{l}{k_{co}r}j_l\left(k_{co}r\right)\right]\left[\vec{e}_r \times \vec{X}_{lm}(\theta, \Phi)\right]$$

Time-Averaged Stored Magnetic Energy: $W_m = \iiint_V w_m\, dV = \frac{1}{4}\iiint_V \mu\left|\vec{H}\right|^2 dV$

$$W_m = \frac{1}{4}\iiint \mu\left|\vec{H}\right|^2 r^2 sin(\phi) d\phi d\theta dr$$

$$\vec{H} = \frac{A\sqrt{l(l+1)}}{k_{co}Z_{co}r} j_l\left(k_{co}r\right) Y_{lm}(\theta, \Phi)\vec{e}_r - \frac{iA}{Z_{co}}\left[j_{l-1}\left(k_{co}r\right) - \frac{l}{k_{co}r} j_l\left(k_{co}r\right)\right]\left[\vec{e}_r \times \vec{X}_{lm}(\theta, \Phi)\right]$$

$$\left|\vec{H}_r\right|^2 = \frac{|A|^2 l(l+1)}{(k_{co}Z_{co})^2 r^2} j_l\left(k_{co}r\right)^2 \left|Y_{lm}(\theta, \Phi)\right|^2$$

Can also use orthogonally to demonstrate: $\int d\Omega\, \vec{Y}_{lm} \cdot \vec{Y}^*_{l'm'} = \delta_{ll'}\delta_{mm'}$

This means that for L=m and integrated over the entire sphere:

$$\int \vec{Y}_{lm} \cdot \vec{Y}^*_{lm}\, d\Omega = \int \left|\vec{Y}_{lm}(\theta, \Phi)\right|^2 d\Omega = \delta_{ll}\delta_{mm} = 1$$

$$W_{m\,r} = \frac{\mu_0}{4} \iiint \left|\vec{H}_r\right|^2 r^2 dr = \frac{\mu_0}{4} \frac{|A|^2 l(l+1)}{(k_{co}Z_{co})^2} \int_0^a j_l\left(k_{co}r\right)^2 dr$$

Evaluate the integral using MATLAB trapz command: $\int_0^a j_{133}\left(k_{co}r\right)^2 dr$ = 1.4492e-10

$$W_{m\,r} = \frac{\mu_0}{4} \int_0^a \left|\vec{H}_r\right|^2 r^2 dr = \frac{\mu_0}{4} \frac{|71.3209|^2 (133)(133+1)}{((9.3627e{+}06\ \frac{rad}{m})(269.093\ \Omega))^2} (1.4492\text{e-}10) = 6.5022\text{e-}28 \text{ W}$$

Input interpretation:

$$\left(\frac{1}{4} \times \frac{4\pi}{10^7}\right) \times \frac{71.3209^2 \times 133 \times 134}{(9.3627 \times 10^6 \times 269.093)^2} \times 1.4492 \times 10^{-10}$$

Result:

$6.50221\ldots \times 10^{-28}$

$$\left|\vec{H}_{trans}\right|^2 = \left(\frac{A}{Z_{co}}\right)^2 \left[j_{l-1}\left(k_{co}r\right) - \frac{l}{k_{co}r} j_l\left(k_{co}r\right)\right]^2 \left[\vec{e}_r \times \vec{X}_{lm}(\theta, \Phi)\right]^2$$

It can be shown that: $\left|\vec{e}_r \times \vec{X}_{lm}(\theta, \Phi)\right|^2 = \left|\vec{X}_{lm}(\theta, \Phi)\right|^2$ where

$$\left|\vec{X}_{lm}(\theta, \Phi)\right|^2 = \left|\vec{e}_r \times \vec{X}_{lm}(\theta, \Phi)\right|^2 = \frac{1}{l(l+1)} \left[\left(\frac{dY_{lm}}{d\theta}\right)^2 + \left(\frac{1}{sin\theta}\right)^2 \left(\frac{dY_{lm}}{d\Phi}\right)^2\right]$$

Such that: $\int \left|\vec{X}_{lm}(\theta,\Phi)\right|^2 d\Omega = \int \left|\vec{e}_r \times \vec{X}_{lm}(\theta,\Phi)\right|^2 d\Omega = \delta_{ll}\delta_{mm}$ = 1 when integrating over a sphere.

$$W_{m\,tr} = \frac{\mu_0}{4}\int_0^a \left(\left|\vec{H}_{trans}\right|^2 r^2 dr\right)\left(\int \left|\vec{e}_r \times \vec{X}_{lm}(\theta,\Phi)\right|^2 d\Omega\right) = \frac{\mu_0}{4}\iiint \left|\vec{H}_{trans}\right|^2 r^2 dr$$

$$W_{m\,tr} = \frac{\mu_0}{4}\int_0^a \left(\frac{A}{Z_{co}}\right)^2 \left[j_{l-1}\left(k_{co}r\right) - \frac{l}{k_{co}r} j_l\left(k_{co}r\right)\right]^2 \left|\vec{e}_r \times \vec{X}_{lm}(\theta,\Phi)\right|^2 r^2 sin(\phi) d\phi d\theta dr$$

$$W_{m\,tr} = \frac{\mu_0}{4}\left(\frac{A}{Z_{co}}\right)^2 \int_0^a \left[j_{l-1}\left(k_{co}r\right) - \frac{l}{k_{co}r} j_l\left(k_{co}r\right)\right]^2 r^2 dr$$

Evaluate the integral using MATLAB trapz command: $\int_0^a \left[j_{133-1}\left(k_{co}r\right) - \frac{l}{k_{co}r} j_{133}\left(k_{co}r\right)\right]^2 r^2 dr$ = 2.5407e-23

$$W_{m\,tr} = \frac{\mu_0}{4}\left(\frac{A}{Z_{co}}\right)^2 (2.3972e\text{-}23) = \frac{\mu_0}{4}\left(\frac{71.3209}{269.093\ \Omega}\right)^2 (2.5407e\text{-}23) = 5.6070e\text{-}31\ W$$

Total time-averaged magnetic stored energy:

$$W_m = W_{m\_r} + W_{m\_tr} = 6.5022e\text{-}28\ W + 5.6070e\text{-}31\ W = 6.5078e\text{-}28\ W$$

Total time-averaged electric stored energy (calculated above):

$$W_e = 6.7146e\text{-}28\ W$$

Total time-averaged electrical AND mechanical stored energy inside of spherical resonator cavity for WGM mode:

$$\mathbf{W = W_e + W_m = 6.7146e\text{-}28\ W + 6.5078e\text{-}28\ W = 1.3222\text{-}27\ W}$$

Analysis: At the resonant frequency, we are supposed to have the condition that:

$$W_e = W_m$$

Percent Difference Between $W_e = W_m$:

% Difference = (6.7146e-28 W - 6.5078e-27 W)/(6.5078e-27 W)*100 = 3.18%

My calculation has slightly different values for electric and magnetic stored energy, however, the two values are pretty close (differing by 3.18%), which is pretty good by my estimation. I suspect the difference most likely lies in numerical rounding error of the many calculations required to achieve these numbers. I am pleased to see that the numbers are over the same order of magnitude and within a few percent of each other.