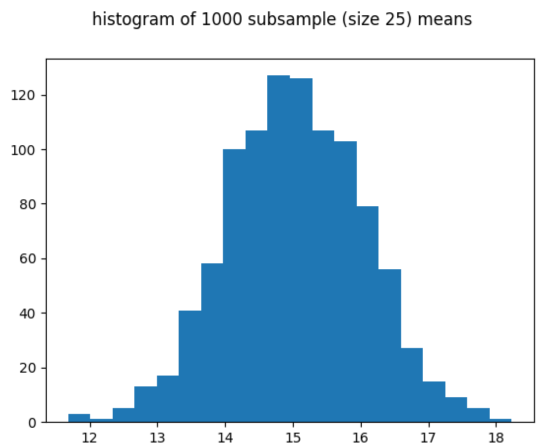
Problem 1

Part 1

calculated mean: 15.04153263 (same as the true mean)

calculated standard deviation: 5.025375677636071, while the true standard deviation is 0.158995850202947. The computed std by numpy is much larger than the true std.

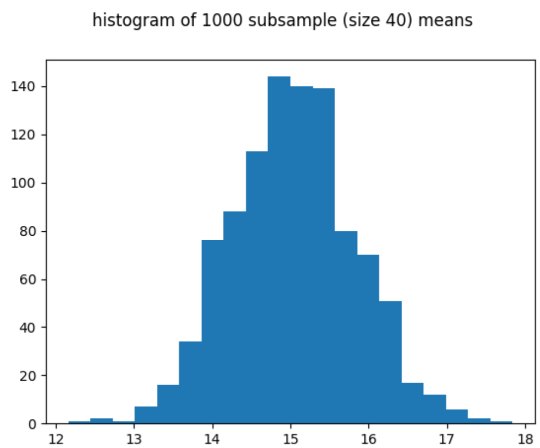
Part 3



Part 4

The distribution of the means of 1000 samples is close to normal distribution. The mean of the distribution is around the true mean (15.04153263).

Part 5



Similar to the 25-size one in part 3, the distribution of the means here is also close to normal distribution and its mean is around the true mean. However, in the size-40 one, there are more samples having means close to the true mean (more samples around the center of the graph). Thus, we can get the conclusion that taking larger sized samples will get the sample means closer to the true mean.

Part 6

for the first 25 examples:

Sample mean 14.565156192

Sample standard error: 0.9362242966547015

Confidence interval: (12.632884212773156, 16.497428171226844)

Since the true mean value is 15.04153263, it falls into the 0.95 confidence interval.

Problem 2

Part a

The sum of all possible outcomes of these 2 fair dices:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Let x_i be the outcome of the sum = i, Θ_i be the probability of the sum = i

$\Theta_2 = 1/36$

$\Theta_3 = 2/36 = 1/18$

$\Theta_4 = 3/36 = 1/12$

$\Theta_5 = 4/36 = 1/9$

$\Theta_6 = 5/36$

$\Theta_7 = 6/36 = 1/6$

$\Theta_8 = 5/36$

$\Theta_9 = 4/36 = 1/9$

$\Theta_{10} = 3/36 = 1/12$

$\Theta_{11} = 2/26 = 1/18$

$\Theta_{12} = 1/36$

Part b

The expected value E(x) = 2* Θ_2 + 3* Θ_3 +4* Θ_4 +5* Θ_5 +6* Θ_6 +7* Θ_7 +8* Θ_8 +9* Θ_9 +10* Θ_{10} +11* Θ_{11} +12* Θ_{12} = 2*(1/36) + 3*(1/18) + 4*(1/12) + 5*(1/9) + 6*(5/36) + 7*(1/6) + 8*(5/36) + 9*(1/9) + 10*(1/12) + 11*(1/18) + 12*(1/36) = 7

Part c

The probability of seeing the outcome of 4 in a single trial is Θ_4 = 1/12

The probability of NOT seeing the outcome of 4 in a single trial is 1-1/12 = 11/12

Thus, the probability of NOT seeing 4 in all 5 trials: $(11/12)^5$ = 161051/248832 ≈ 0.6472

The probability of seeing odd-sum outcomes in a single trial is $\Theta_3 + \Theta_5 + \Theta_7 + \Theta_9 + \Theta_{11}$ = 1/18 + 1/9 + 1/6 + 1/9 + 1/18 = ½

Thus, the probability of seeing odd-sum outcomes in all 5 trials: $(1/2)^5$ = 1/32 = 0.03125

Problem 3

(a) Take the integral over its possible values on the interval [a,b]:

$$\int_a^b \frac{1}{b-a}dx = \frac{1}{b-a}(b-a) = 1$$

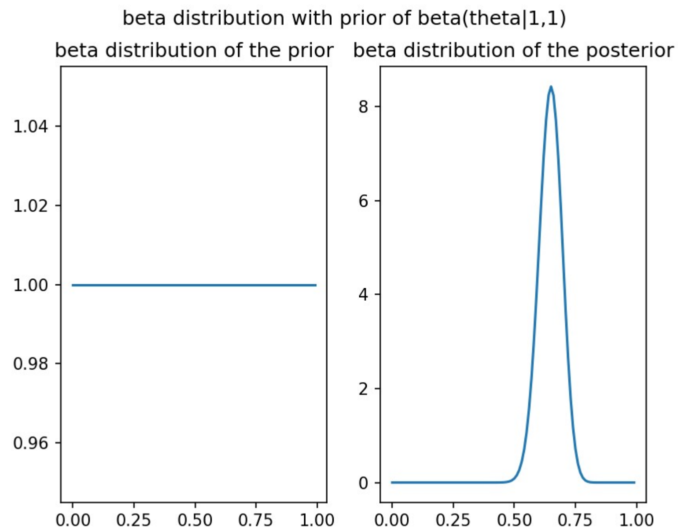
So we verified that the uniform distribution is properly normalized.

(b) E(x) = $\int_a^b x\frac{1}{b-a}dx = \frac{1}{2(b-a)}(b^2-a^2) = \frac{(b+a)(b-a)}{2(b-a)} = \frac{b+a}{2}$

Problem 4

(a) The ML estimate of theta is 0.65.

(b)



(c) the posterior: Beta(Θ | 66, 36) = $\frac{\Gamma(66+36)}{\Gamma(66)\Gamma(36)}\Theta^{65}(1-\Theta)^{35}$

- calculate the MAP estimate of theta

Take the log of it, we get:

$\log(\frac{\Gamma(66+36)}{\Gamma(66)\Gamma(36)}) + \log(\Theta^{65}) + \log((1-\Theta)^{35})$

$= \log(\frac{\Gamma(66+36)}{\Gamma(66)\Gamma(36)}) + 65\log(\Theta) + 35\log(1-\Theta)$

Take the derivative and set it to 0 to find Θ that maximizes the posterior:

$0 + \frac{65}{\theta} - \frac{35}{1-\theta} = 0$ => $65(1-\theta) = 35\Theta$ => $\Theta = 0.65$

So $\Theta_{MAP} = 0.65$

(or use the equation: $\Theta_{MAP} = \frac{N_1+a-1}{N_1+N_2+a+b-2}$

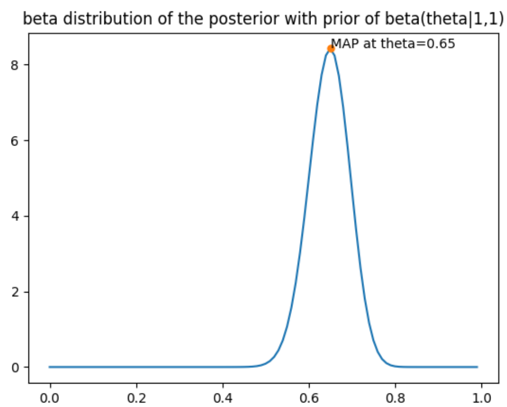
Here $N_1 = 65$, $a = 1$, $N_2 = 35$, $b = 1$,

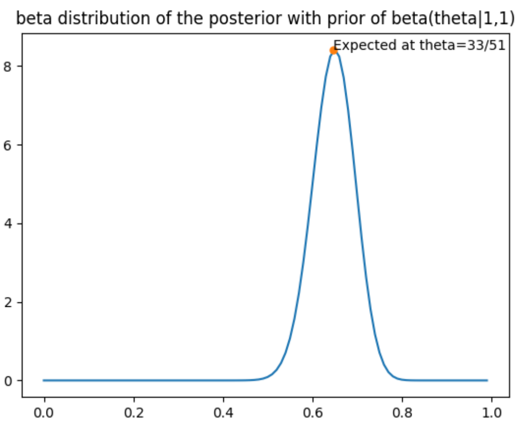
$\Theta_{MAP} = \frac{65+1-1}{65+35+2-2} = 65/100 = 0.65$)

- calculate the expected value of theta

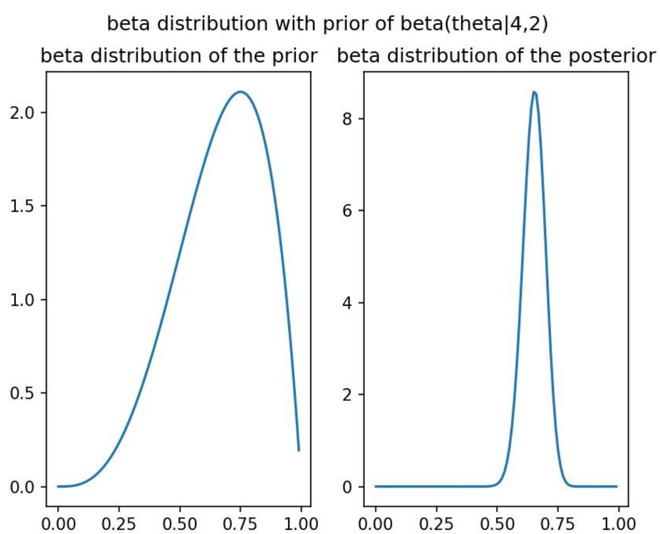
$\mu = E[\theta] = \int_0^1 \theta \frac{\Gamma(66+36)}{\Gamma(66)\Gamma(36)}\theta^{65}(1-\theta)^{35} = \frac{\Gamma(66+36)}{\Gamma(66)\Gamma(36)}\int_0^1 \theta^{66}(1-\theta)^{35} = \frac{\Gamma(102)}{\Gamma(66)\Gamma(36)}\frac{\Gamma(67)\Gamma(36)}{\Gamma(67+36)}$

$= \frac{101!}{65!}\frac{66!}{102!} = \frac{66}{102} = \frac{33}{51}$





(d)



the posterior: Beta(θ | 69, 37) = $\frac{\Gamma(69+37)}{\Gamma(69)\Gamma(37)}\Theta^{68}(1-\Theta)^{36}$

- calculate the MAP estimate of theta

Take the log of it, we get:

$\log(\frac{\Gamma(69+37)}{\Gamma(69)\Gamma(37)}) + \log(\Theta^{68}) + \log((1-\Theta)^{36})$

$= \log(\frac{\Gamma(69+37)}{\Gamma(69)\Gamma(37)}) + 68\log(\Theta) + 36\log(1-\Theta)$

Take the derivative and set it to 0 to find Θ that maximizes the posterior:

$0 + \frac{68}{\theta} - \frac{36}{1-\theta} = 0$ => $68(1-\theta) = 36\Theta$ => $\Theta = 68/104 = 17/26$

So $\Theta_{MAP} = 17/26$

(or use the equation: $\Theta_{MAP} = \frac{N_1+a-1}{N_1+N_2+a+b-2}$

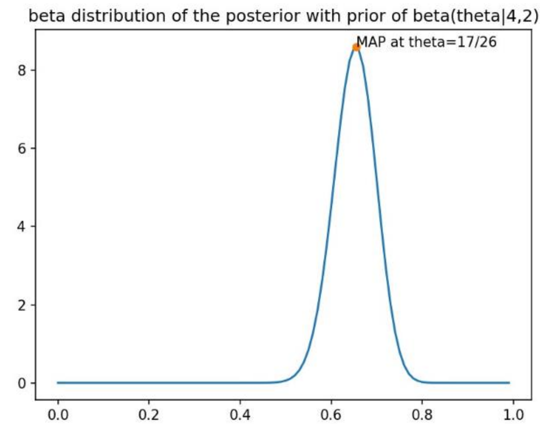
Here $N_1 = 65$, $a = 4$, $N_2 = 35$, $b = 2$,

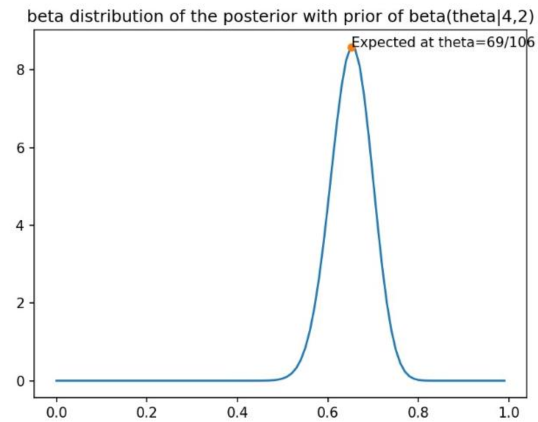
$\Theta_{MAP} = \frac{65+4-1}{65+35+6-2} = 68/104 = 17/26$)

- calculate the expected value of theta

$\mu = E[\theta] = \int_0^1 \theta \frac{\Gamma(69+37)}{\Gamma(69)\Gamma(37)} \theta^{68}(1-\theta)^{36} = \frac{\Gamma(69+37)}{\Gamma(69)\Gamma(37)} \int_0^1 \theta^{69}(1-\theta)^{36} = \frac{\Gamma(106)}{\Gamma(69)\Gamma(37)} \frac{\Gamma(70)\Gamma(37)}{\Gamma(107)}$

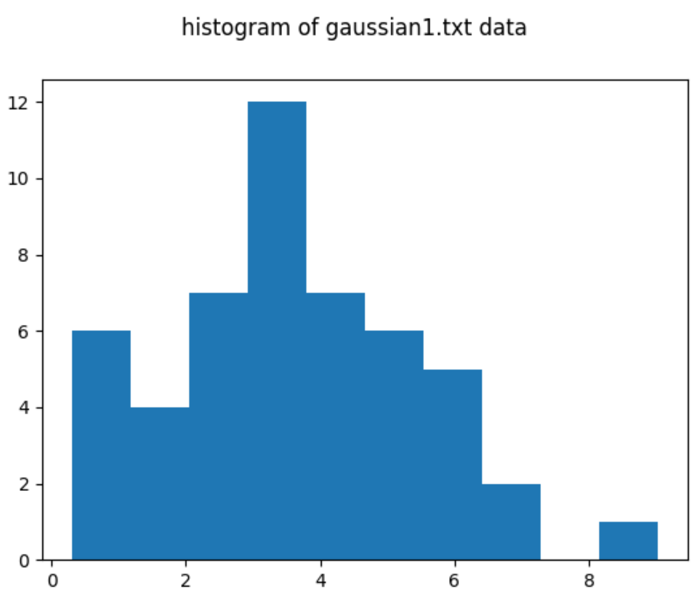
$= \frac{105!}{68!} \frac{69!}{106!} = \frac{69}{106}$





Problem 5

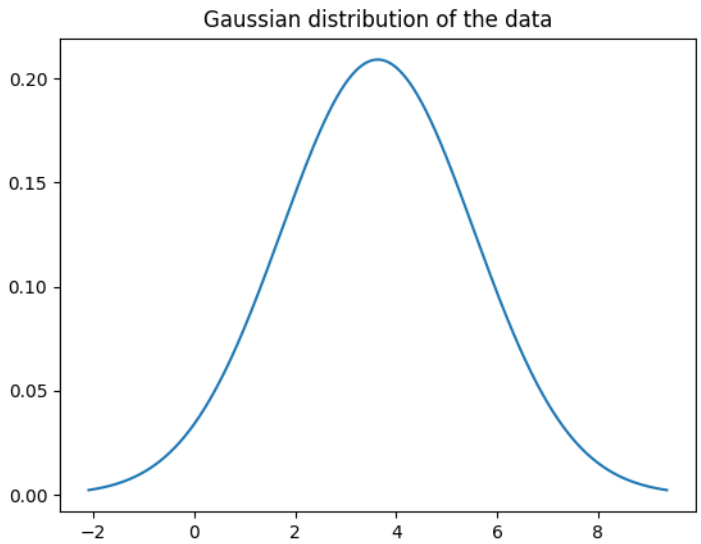
(a)



(b)

ML estimate of mean is $\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$ = 3.6377316957999994

ML estimate of unbiased variance is $\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu)^2$ = 3.6414284586746737



Problem 6

Part 1

Since the n samples are independent, the likelihood function is $\prod_{i=1}^{n} e^{-\lambda} \frac{1}{x_i!} \lambda^{x_i}$

Take the log of it, we get the log likelihood function:

$\sum_{i=1}^{n} \log\left(e^{-\lambda} \frac{1}{x_i!} \lambda^{x_i}\right)$

= $\sum_{i=1}^{n} \log(e^{-\lambda})$ + $\sum_{i=1}^{n} \log\left(\frac{1}{x_i!}\right)$ + $\sum_{i=1}^{n} \log(\lambda^{x_i})$

= n(-λ) - $\sum_{i=1}^{n} \log(x_i)$ + $\log(\lambda) \sum_{i=1}^{n} x_i$

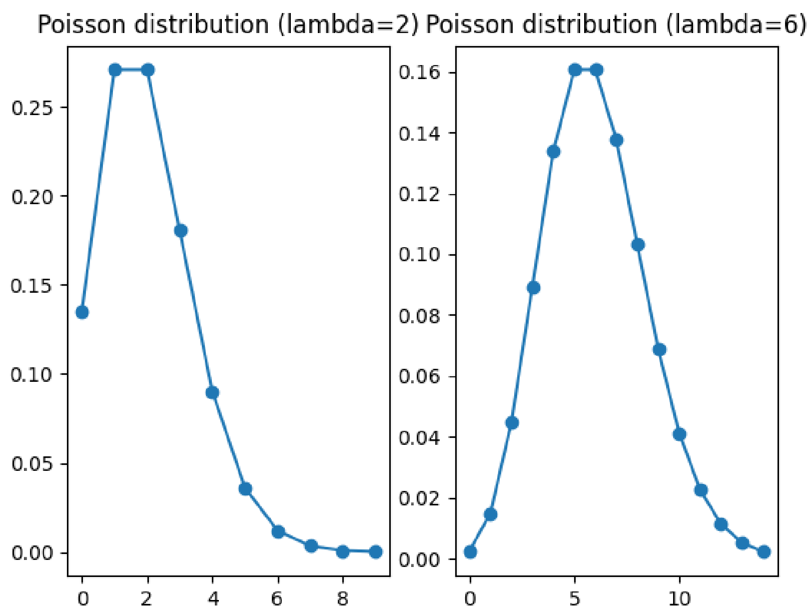
Take its derivative and set it to 0 to find the ML estimate of λ:

-n – 0 + $\frac{\sum_{i=1}^{n} x_i}{\lambda}$ = 0

=> the ML estimate of $\lambda = \frac{1}{n}\sum_{i=1}^{n} x_i$

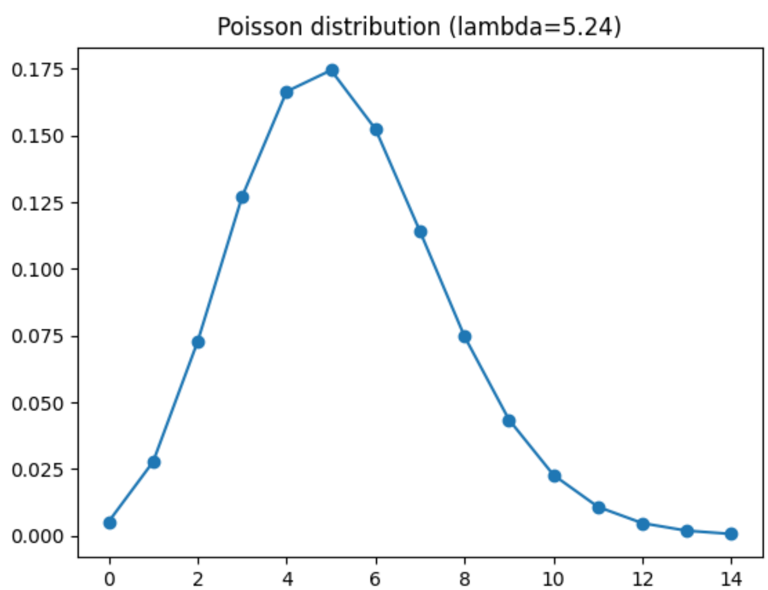
Part 2

(a)

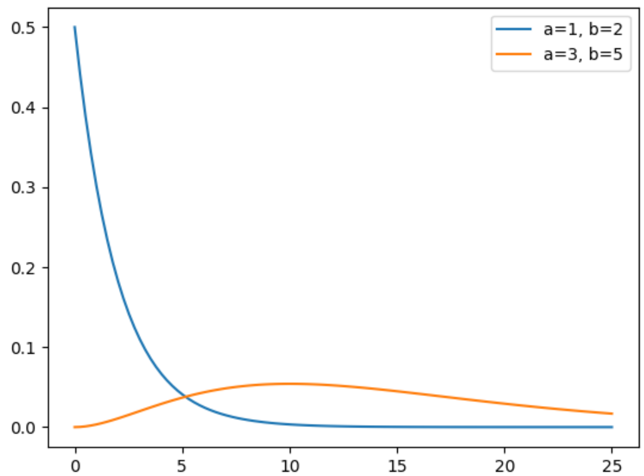


(b)

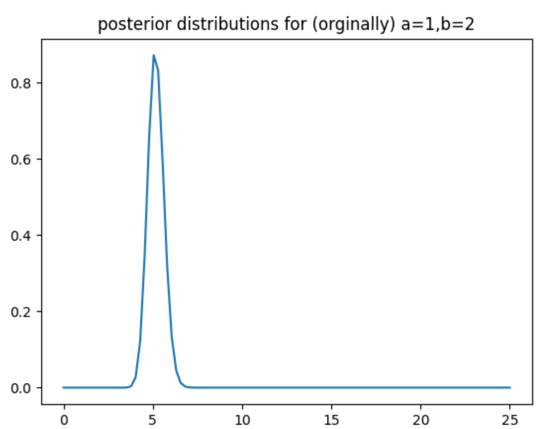
Use $\lambda = \frac{1}{n}\sum_{i=1}^{n} x_i$ we derived from part 1, here the ML estimate of $\lambda = 5.24$

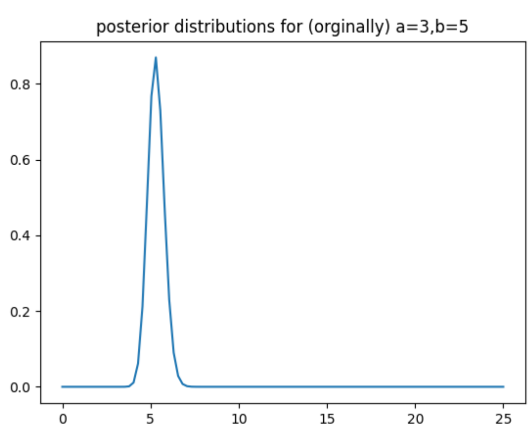


(c)


a=1, b=2
a=3, b=5

(d)


posterior distributions for (orginally) a=1,b=2


posterior distributions for (orginally) a=3,b=5