Problem 1. Conditional independence

Proof:

P(A|B) = $\frac{P(A,B)}{P(B)}$

Conditioning on C, we get P(A|B,C) = $\frac{P(A,B|C)}{P(B|C)}$

We already know that P(A,B|C) = P(A|C)P(B|C),

Thus we obtain that P(A|B,C) = $\frac{P(A,B|C)}{P(B|C)}$

$= \frac{P(A|C)P(B|C)}{P(B|C)}$

= P(A|C) QED

Problem 2. Bayesian belief networks

Part a

P(E = F, B = F, A = F, J = T, M = F)

= P(E=F)P(B=F)P(A=F|E=F,B=F)P(J=T|A=F)P(M=F|A=F)

Part b

Substitute the probabilities, we get

P(E = F, B = F, A = F, J = T, M = F)

= P(E=F)P(B=F)P(A=F|E=F,B=F)P(J=T|A=F)P(M=F|A=F)

= (0.998)(0.999)(0.999)(0.05)(0.99)

≈ 0.0493

Problem 3. Bayesian belief networks

Part a (blind solution)

P(B=T, E=T) = $\sum_{a\in T,F}\sum_{c\in T,F}\sum_{d\in T,F,X}\sum_{f\in T,F} P(A=a,B=T,C=c,D=d,E=T,F=f)$

$$= \sum_{a\in T,F}\sum_{c\in T,F}\sum_{d\in T,F,X}\sum_{f\in T,F} P(A=a)P(B=T)P(C=c)P(D=d|A=a,B=T,C=c)P(E=T|C=c)P(F=f|D=d)$$

Computational Cost:

Number of additions = $2^3(3) - 1 = 23$

Number of multiplications: $2^3(3)(5) = 120$

Part b (interleaving sums and products)

P(B=T, E=T)

$$= \sum_{a\in T,F} \sum_{c\in T,F} \sum_{d\in T,F,X} \sum_{f\in T,F} P(A=a)P(B=T)P(C=c)P(D=d|A=a,B=T,C=c)P(E=T|C=c)P(F=f|D=d)$$

$$= \sum_{a\in T,F} \sum_{c\in T,F} \sum_{d\in T,F,X} P(A=a)P(B=T)P(C=c)P(D=d|A=a,B=T,C=c)P(E=T|C=c) \sum_{f\in T,F} P(F=f|D=d)$$

$$= \sum_{a\in T,F} \sum_{c\in T,F} P(A=a)P(B=T)P(C=c)P(E=T|C=c) \sum_{d\in T,F,X} P(D=d|A=a,B=T,C=c) \sum_{f\in T,F} P(F=f|D=d)$$

$$= \sum_{a\in T,F} P(A=a) \sum_{c\in T,F} P(B=T)P(C=c)P(E=T|C=c) \sum_{d\in T,F,X} P(D=d|A=a,B=T,C=c) \sum_{f\in T,F} P(F=f|D=d)$$

$$= P(B=T) \sum_{a\in T,F} P(A=a) \sum_{c\in T,F} P(C=c)P(E=T|C=c) \sum_{d\in T,F,X} P(D=d|A=a,B=T,C=c) \sum_{f\in T,F} P(F=f|D=d)$$

Computational Cost:

Number of additions = 9

Number of multiplications = 16

Comparing the number of additions and multiplications computational cost in these two methods, we can see that the interleaving one decreases the cost to a great extent thus making the computation more efficient.

Problem 4. Pneumonia diagnosis

Part a

When pneumonia is true:

	True	False
Fever	0.9	0.1
Paleness	0.7	0.3
Cough	0.9	0.1
High WBC	0.8	0.2

When pneumonia is false:

	True	False
Fever	0.6	0.4
Paleness	0.5	0.5
Cough	0.1	0.9
High WBC	0.5	0.5

Part b

P(Pneumonia=T | Fever=T, Paleness=F, Cough=T, HighWBC=F)

$$= \frac{P(\text{Pneumonia} = \text{T}, \text{Fever} = \text{T}, \text{Paleness} = \text{F}, \text{Cough} = \text{T}, \text{HighWBC} = \text{F})}{P(\text{Fever} = \text{T}, \text{Paleness} = \text{F}, \text{Cough} = \text{T}, \text{HighWBC} = \text{F})}$$

$$= \frac{P(Pneumonia = T)P(\text{ Fever = T |Pneumonia = T})\text{P(Paleness = F|Pneumonia = T)P(Cough = T|Pneumonia = T)P(HighWBC = F|Pneumonia = T)}}{\sum_{p\in T,F} P(Pneumonia = p)P(Fever = T|Pneumonia = p)\, P(Paleness = F|Pneumonia = p)P(Cough = T|Pneumonia = p)P(HighWBC = F|Pneumonia = p)}$$

$$= \frac{(0.02)(0.9)(0.3)(0.9)(0.2)}{(0.02)(0.9)(0.3)(0.9)(0.2)+(0.98)(0.6)(0.5)(0.1)(0.5)}$$

≈ 0.062

Part c

P(Pneumonia = T | Fever = T, Cough =T)

$$= \frac{P(\text{Pneumonia} = \text{T}, \text{Fever} = \text{T}, \ \text{Cough} = \text{T})}{\sum_{p\in T,F} P(\text{Pneumonia} = \text{p}, \text{Fever} = \text{T}, \text{Cough} = \text{T})}$$

$$= \frac{\text{P(Pneumonia=T)P(Fever=T |Pneumonia=T)P(Cough = T|Pneumonia = T)}}{\sum_{\text{p}\in\text{T,F}} \text{P(Pneumonia=p)P(Fever = T|Pneumonia = p)P(Cough=T|Pneumonia=p)}}$$

$$= \frac{(0.02)(0.9)(0.9)}{(0.02)(0.9)(0.9)+(0.98)(0.6)(0.1)}$$

= 0.216

Part d

Computed probabilities:

Fever	Paleness	Cough	HighWBC	P(pneumonia|symptoms)
1	0	1	0	0.06202143950995407
1	0	1	-1	0.1418563922942207
0	1	-1	0	0.002849002849002848 7
1	-1	-1	0	0.012096774193548388
-1	0	1	-1	0.09926470588235294