Notes On Ising Model

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1 Abstract

The Ising Model, developed by Dr. Ernst Ising, is a mathematical model of ferromagnetism in statistical physics. Dr. Ernst Ising gave the analytical solution in one dimension. The result shows the phase trasition point is at $T = 0$. While people believed that here is no phase transition point in two dimensions, the result given by Dr. Onsager showed phase transition point exists indeedly. After that, other methods were developed like mean-field theory, renormalization group and so on. They all indicate two-dimension Ising Model has phase transition. In this article, I will introduce these thoeries and perform some numerical methods such as Metropolis algorithm and cluster algorithm.

2 Ising Model

In Ising Model, each atom can adopt two states, corresponding to $S = \{-1, 1\}$ where S represents the spin. The spin interaction are dependent of the coupling parameter J_{ij} between the adjacent atoms. For the sake of simplification, I set $J_{ij} = J$, which indicate the system is isotropic. As a result, the Hamiltonian of ferromagnetic Ising Model is:

$$H = -J \sum_{<ij>} S_i S_j, \tag{1}$$

where $J > 0$ and $S_i \in \{1, -1\}$.

The partition function is given by:

$$\mathcal{Z} = \sum_i \exp\{-\beta E_i\} = \sum_{S_0} \sum_{S_1} \dots \sum_{S_N} \exp\{-\beta H\}, \tag{2}$$

in the canonical ensemble, free energy can be written as:

$$\mathcal{F} = -kTln(\mathcal{Z}). \tag{3}$$

At the same time, energy has the form:

$$U = \frac{Tr(exp\{-\beta H\})}{\mathcal{Z}}, \tag{4}$$

As a result, heat capacity is:

$$C_v = \frac{\partial U}{\partial T} = \frac{\overline{E^2} - \overline{E}^2}{kT^2}, \tag{5}$$

Similarly, magnetic susceptibility can be obtained.

$$\kappa = \frac{\partial M}{\partial B} = \frac{\overline{M^2} - \overline{M}^2}{kT}. \tag{6}$$

As above, Ising Model is described briefly. Next, I will introduce some analytical methods.

3 Mean Field Theory

3.1 Bragg-Williams approximation

4 matropolic algorithm

5 Cluster Method