1 The Prove of Marshall's Theory

Lemma 1. *The eigenstates of Heisenberg model could be labelled by:* $\Phi = |S_{tot}, M \ldots >$*,where* $M = -S_{tot}, \ldots, S_{tot}$ *and* $S_{tot} \leq \mathcal{N}S$.

Proof. It's easy to vertify that the Hamitonian of Heisenberg model commutes with S_{tot}^2 and S_{tot}^z, what's more, S_{tot}^2 also commutes with S_{tot}^z. Aa a result, S_{tot} and M are good quantum numbers to label eigenstates. □

For the antiferromagnet, it's meaningful to rewrite the Hamitonian as: $\mathcal{H} = \sum_{i\in A, j\in B} |J_{ij}| S_i^z S_j^z + \frac{1}{2}\sum_{i\in A, j\in B} |J_{ij}|(S_i^+ S_j^- + S_i^- S_j^+)$. It's convenient to rotate the spin axes on sublattice B about the z axis, which amounts to the unitary transformation:$i \in B, S_i^+ \to -\tilde{S}_i^+, S_i^- \to -\tilde{S}_i^-, S_i^z \to \tilde{S}_i^z$. Under this transformation, the new basic eigenstates has a relation with former eigenstates.

$$|S, m_i >_i \to |S, \tilde{m}_i >_i = \begin{cases} |S, m_i >_i, & i \in A, \\ (-1)^{S+m_i} |S, m_i >_i & i \in B. \end{cases} \tag{1}$$

In the subspace where the eigenvalue of S_{tot}^z is M, the eigenstates could be written as:$\Phi^M = \sum_\alpha f_\alpha^M \tilde{\phi}_\alpha^M$, in which $\tilde{\phi}_\alpha^M = \prod_{\sum_i \tilde{m}_i^\alpha = M} |S, \tilde{m}_i^\alpha >_i$.

It's convenient to rewitten Hamiltonian with the operators after transfoemation. $\mathcal{H} = \sum_{i\in A, j\in B} |J_{ij}| S_i^z \tilde{S}_j^z - \frac{1}{2}\sum_{i\in A, j\in B} |J_{ij}|(S_i^+ \tilde{S}_j^- + S_i^- \tilde{S}_j^+) = \mathcal{H}_{zz} + \mathcal{H}_{xy}$. The most meaningful difference is that we get a minus on the second term. In the M sector, $\tilde{\phi}_\alpha^M$ is an eigenstate of $\mathcal{H}_{zz}$. For the sake of simplifying, we just ignore M. $\mathcal{H}_{zz}\tilde{\phi}_\alpha = e_\alpha \tilde{\phi}_\alpha$.Furthermore, $< \tilde{\phi}_\alpha | \mathcal{H}_{xy} | \tilde{\phi}_\beta >= -|K_{\alpha\beta}| \leq 0$.

Lemma 2. *The operator* $-H_{xy}$ *is irreducible.*

Proof. The lemma is equivalent to if we can get every basic state from one state in the subspace sector M. The operator set $\{S_i^+ S_j^- | i, j \in N\}$ could get every state. The meaning of these operators is to take an particle from j to i, so every state could be gotten by these operates. What's differnce, in our condition, the operator set is $\{S_i^+ S_j^-, S_i^- S_j^+ | i \in A, j \in B\}$. The swap between the points in the same sublattice is forbidden. Fortunately, it's convenient to construct them. The swap between $m, n \in A$ can be constructed by $S_m^+ S_i^- S_i^+ S_n^-$ and the S_i^+ is chosen where $\tilde{m}_i < S$(for sector $M < S$, this is feasible). So, the operator $-H_{xy}$ is irreducible. □

Theorem 1 (Marshall's Theorem 1)**.** *See it on page 53.*

Proof. If $\tilde{\Phi}$ is a ground state, so $f_\alpha e_\alpha - \sum_\beta |K_{\alpha\beta}| f_\beta = E f_\alpha$. If we establish another state, which is $\tilde{\Phi}_1 = |f_\alpha| \tilde{\phi}_\alpha$. The energy of new state will be: $\sum_\alpha |f_\alpha|^2 e_\alpha - \sum_{\alpha\beta} |K_{\alpha\beta}| |f_\alpha| |f_\beta| \leq \sum_\alpha |f_\alpha|^2 e_\alpha - \sum_{\alpha\beta} |K_{\alpha\beta} f_\alpha f_\beta = E$. But as we assumed, E is the groud energy. As a result, $\tilde{\Phi}_1$ is also a ground state.Then the equation from $\mathcal{H}\tilde{\Phi}_1$ is $|f_\alpha| e_\alpha - \sum_\beta |K_{\alpha\beta}| f_\beta| = E |f_\alpha|$. Two important equations come out.

$$\begin{cases} f_\alpha (e_\alpha - E) = \sum_\beta |K_{\alpha\beta}| f_\beta, \\ |f_\alpha| (e_\alpha - E) = \sum_\beta |K_{\alpha\beta}| |f_\beta|. \end{cases} \tag{2}$$

The thing I want to emphasize here is that for some α, β, $|K_{\alpha\beta}| = 0$. As a result, it's not strcit enough to get $f_\alpha \geq 0$. It's more difficult than it seems to be. From the equation above, we can just get that if $f_\alpha \neq 0$ then $f_\beta \geq 0$ where $|K_{\alpha\beta}| \neq 0$. From the lemma.2 and the fact $e_\alpha - E > 0$, $f_\alpha \neq 0$ for any α, or all $f_\alpha = 0$. What's more, f_α have the same phase. □

Lemma 3. *In every M sector, the ground state is nondegenerate.*

Proof. If here is another ground state, a ground state which satisfy $< \tilde{\Phi}_2 | \tilde{\Phi}_1 >= 0$ must exist. Acoording to the Marshaal's theorem 1, this condition can't exist. □

Lemma 4. *For a special condition where Hamiltonian is $\mathcal{H} = |J| \sum_{i \in A, j \in B} S_i S_j = |J| S_{tot,A} S_{tot,B}$, the ground state of M sector is $|M, M, \ldots >$.*

Proof. $2S_{tot,A}S_{tot,B} = S_{tot}^2 - S_{tot,A}^2 - S_{tot,B}^2 = S(S+1) - S_A(S_A+1) - S_B(S_B+1)$, $S_{tot,A}^2, S_{tot,B}^2$ commute with $\{S_{tot}^2, S_{tot}^z, \mathcal{H}\}$.To minimum the energy, S should be as small as possible, which is $S = M$. □

Theorem 2 (Marshall's Theorem 2)**.** *See it on page 53.*

Proof. Since Hamiltonian in lemma.4 is a special condition of Heisenberg model, the ground state of M sector should obey the Marshall's theorem 1. Because the eigenstate of Hamiltonian has the form $|S, M, \ldots >$(lemma.1), to satisfy the Marshall's theorem 1, the Heisenberg model's ground state in sector M has the form $|M, M, \ldots >$. For sector 0, ground state is $|0, 0, \ldots >$. □

Above all,Marshall's theorems are proven.

2 Exercise

The ground state of infinite biparticle Hamiltonian is $|0, 0, S_{tot,A} = \mathcal{N}S/2, S_{tot,B} = \mathcal{N}S/2 >$, but I have no idea how to transform this state into basis set.

For ferromagnet, it's the same to get the result. For sector $M < S$, we can prove that f_α have the same phase.

The ground state in setor M has the form $|\mathcal{N}S, M \ldots >$. To find the ground state of the Hamiltonian, we can just compare the ground states in different sector M.

$$< \mathcal{N}S, M ... | \mathcal{H} | \mathcal{N}S, M ... >= \sum_\alpha e_\alpha |f_\alpha|^2 - \sum_{\alpha\beta} |K_{\alpha\beta}| |f_\alpha| |f_\beta| \tag{3}$$

Failed.