$$\left(D^2-2D+1\right)y=6x^2e^{-x}$$

Find General solution

- 對於一個非齊次線性常微分方程式，我們可以使用以下幾個步驟與工具找到通解

1. 找到 y_h.
   - 一元二次方程
   - Reduction of Order
   - Euler-Cauchy Equation
2. 找到 y_p.
   - Method of Undetermined Coefficients
   - Method of Variation of Parameters
   - 逆運算子法
3. 找到General solution.

$$y = y_h + y_p$$

1. 找到 y_h

- 一元二次方程式解：$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = y_1, y_2$

- Reduction of Order
  - y_1 is given basis, the 2^{nd} basis $y_2 = \int \frac{1}{y^2} e^{-\int pdx} dx$

$$then \quad y = c_1 y_1 + c_2 \int \frac{1}{y^2} e^{-\int pdx} dx$$

1. 找到 y_h

- Euler-Cauchy Equation $(x^2y''+axy'+by=0)$

  let $y=x^m, y'=mx^{m-1}, y''=m(m-1)x^{m-2}$

  → $[m(m-1)+am+b]x^m=0$, $m=\dfrac{(1-a)\pm\sqrt{(1-a)^2-4ac}}{2}$

  - 兩相異實根 $(m=m_1,m_2)$:

    $$y=c_1x^{m_1}+c_2x^{m_2}$$

  - 一個實數重根 $\left(m=\dfrac{1-a}{2},\dfrac{1-a}{2}\right)$: $y=c_1x^m+c_2x^m\ln x$
  - 共軛複數根 $(m=\alpha\pm\beta i)$

    $$y=Ax^{\alpha}\cos(\beta\ln x)+Bx^{\alpha}\sin(\beta\ln x)$$

1. 找到 y_h

- 令非齊次線性常微分 ODE 變成齊次線性常微分 ODE ，以方便我們尋找 y_h

$$let \quad r(x)=0$$

$$\rightarrow (D^2-2D+1)y=0$$

$$\begin{array}{cc} D & -1 \\ D & -1 \\ \hline -D-D= & -2D \end{array} \quad \Rightarrow D=1,1$$

- 對於 Double Root ，我們可以用 Reduction of Order 來發現 $y_2 = xy_1$

$$\therefore y_h = c_1 e^x + c_2 x e^x$$

2. 找到 y_p

– Method of Undetermined Coefficients

- Basic rule

$r_{(x)}$	y_p
ke^{rx}	ce^{rx}
$kx^n{}_{n=0,1,2....}$	$K_n x^n + K_{(n-1)} x^{(n-1)} ... + K_1 x + K_0$
$\left.\begin{array}{c} k\cos\omega x \\ k\sin\omega x \end{array}\right\}$	$K\cos\omega x + M\sin\omega x$
$\left.\begin{array}{c} ke^{\alpha x}\cos\omega x \\ ke^{\alpha x}\sin\omega x \end{array}\right\}$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

2. 找到 y_p

- Method of Undetermined Coefficients
  - Modification rule

    假設的 y_p 有出現和 y_h 一樣的函數，則我們假設的函數要再乘 x or x^2 倍
  - Sum rule

    $r(x)$有什麼函數，y_p 就有什麼函數

    Ex: $r(x) = e^{0.5x} + 40\cos 10x - 190\sin 10x$

    $$then \quad y_p = ce^{rx} + K\cos\omega x + M\sin\omega x$$

2. 找到 y_p

- Method of Variation of Parameters

$$
\begin{aligned}
y_h &= c_1 y_1 + c_2 y_2 \\
y_p &= u y_1 + v y_2 \\
&= y_1 \int \frac{W_1}{W} r\,dx + y_2 \int \frac{W_2}{W} r\,dx
\end{aligned}
$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2 \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$$

2. 找到 y_p

- 這個例題中，$r(x)$是個較為複雜的函數，我們難以使用Method of Undetermined Coefficients 來對 y_p 做假設，因此在這邊我們使用 Method of Variation of Parameters 會是個比較輕鬆的選擇。

$$
\begin{aligned}
y_p =& e^x \int \frac{-xe^x}{e^{2x}} \times 6x^2 e^{-x} dx + xe^x \int \frac{e^x}{e^{2x}} \times 6x^2 e^{-x} dx \\
=& 6e^x \left(\int xe^{-2x} dx + x \int x^2 e^{-2x} dx \right) \\
=& 6e^x \left[-\frac{1}{2}\left(x+\frac{1}{2}\right)e^{-2x} - \frac{1}{2} x\left(x^2+x+\frac{1}{2}\right)e^{-2x} \right] \\
=& -3e^{-x}\left[x+\frac{1}{2}+x\left(x^2+x+\frac{1}{2}\right)\right] \\
=& -3e^{-x}\left(x^3+x^2+\frac{3}{2}x+\frac{1}{2}\right)
\end{aligned}
$$

3. 找到General solution.

$$\because y = y_h + y_p$$

$$\therefore y = c_1e^x + c_2xe^x - 3(x^3 + x^2 + \frac{3}{2}x + \frac{1}{2})e^{-x}$$

Thank you for listening

Work sheet link：
https://github.com/ZiMiTan01/Note/tree/master/Markdown/112-1學期/Eng-Math