
$$(D^2 - 2D + 1)y = 6x^2 e^{-x}$$

Find General solution



- 對於一個非齊次線性常微分方程式，我們可以使用以下幾個步驟與工具找到通解

1. 找到 y_h .

- 一元二次方程
- Reduction of Order
- Euler-Cauchy Equation

2. 找到 y_p .

- Method of Undetermined Coefficients
- Method of Variation of Parameters
- 逆運算子法

3. 找到 General solution.

$$y = y_h + y_p$$



1. 找到 y_h

- 一元二次方程式解：
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = y_1, y_2$$

- Reduction of Order

- y_1 is given basis, the 2nd basis $y_2 = \int \frac{1}{y_1^2} e^{-\int p dx} dx$

then
$$y = c_1 y_1 + c_2 \int \frac{1}{y_1^2} e^{-\int p dx} dx$$

1. 找到 y_h

- Euler-Cauchy Equation ($x^2 y'' + axy' + by = 0$)

$$\text{let } y = x^m, y' = mx^{m-1}, y'' = m(m-1)x^{m-2}$$

$$\rightarrow [m(m-1) + am + b]x^m = 0, \quad m = \frac{(1-a) \pm \sqrt{(1-a)^2 - 4ac}}{2}$$

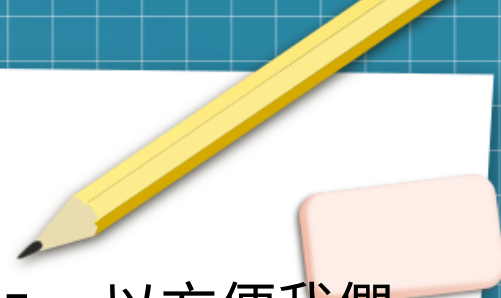
- 兩相異實根 ($m = m_1, m_2$):

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

- 一個實數重根 ($m = \frac{1-a}{2}, \frac{1-a}{2}$) : $y = c_1 x^m + c_2 x^m \ln x$

- 共軛複數根 ($m = \alpha \pm \beta i$)

$$y = Ax^\alpha \cos(\beta \ln x) + Bx^\alpha \sin(\beta \ln x)$$



1. 找到 y_h

- 令非齊次線性常微分 ODE 變成齊次線性常微分 ODE，以方便我們尋找 y_h

$$\text{let } r(x)=0$$

$$\rightarrow (D^2 - 2D + 1)y = 0$$

$$\begin{array}{r} D \qquad \qquad -1 \\ D \qquad \qquad -1 \\ \hline -D - D = -2D \end{array}$$

$$\Rightarrow D = 1, 1$$

- 對於 Double Root，我們可以用 Reduction of Order 來發現 $y_2 = xy_1$

$$\therefore y_h = c_1 e^x + c_2 x e^x$$

2. 找到 y_p

– Method of Undetermined Coefficients

- Basic rule

$r(x)$	y_p
ke^{rx}	ce^{rx}
$kx^n \quad n=0,1,2,\dots$	$K_n x^n + K_{(n-1)} x^{(n-1)} \dots + K_1 x + K_0$
$\left. \begin{array}{l} k \cos \omega x \\ k \sin \omega x \end{array} \right\}$	$K \cos \omega x + M \sin \omega x$
$\left. \begin{array}{l} ke^{\alpha x} \cos \omega x \\ ke^{\alpha x} \sin \omega x \end{array} \right\}$	$e^{\alpha x} (K \cos \omega x + M \sin \omega x)$

2. 找到 y_p

– Method of Undetermined Coefficients

- Modification rule

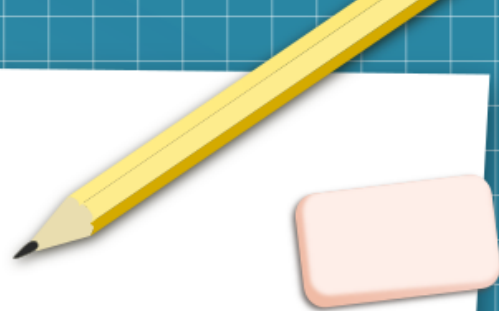
假設的 y_p 有出現和 y_h 一樣的函數，則我們假設的函數要再乘 x or x^2 倍

- Sum rule

$r(x)$ 有什麼函數， y_p 就有什麼函數

Ex: $r(x) = e^{0.5x} + 40 \cos 10x - 190 \sin 10x$

$$\text{then } y_p = ce^{rx} + K \cos \omega x + M \sin \omega x$$



2. 找到 y_p

– Method of Variation of Parameters

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u y_1 + v y_2$$

$$= y_1 \int \frac{W_1}{W} r dx + y_2 \int \frac{W_2}{W} r dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2$$

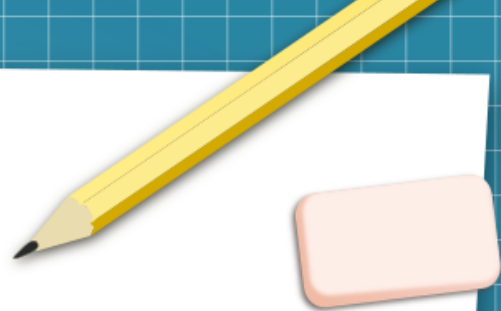
$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$$

2. 找到 y_p

- 這個例題中， $r(x)$ 是個較為複雜的函數，我們難以使用 Method of Undetermined Coefficients 來對 y_p 做假設，因此在這邊我們使用 Method of Variation of Parameters 會是個比較輕鬆的選擇。

$$\begin{aligned} y_p &= e^x \int \frac{-xe^x}{e^{2x}} \times 6x^2 e^{-x} dx + xe^x \int \frac{e^x}{e^{2x}} \times 6x^2 e^{-x} dx \\ &= 6e^x \left(\int xe^{-2x} dx + x \int x^2 e^{-2x} dx \right) \\ &= 6e^x \left[-\frac{1}{2} \left(x + \frac{1}{2} \right) e^{-2x} - \frac{1}{2} x \left(x^2 + x + \frac{1}{2} \right) e^{-2x} \right] \\ &= -3e^{-x} \left[x + \frac{1}{2} + x \left(x^2 + x + \frac{1}{2} \right) \right] \\ &= -3e^{-x} \left(x^3 + x^2 + \frac{3}{2}x + \frac{1}{2} \right) \end{aligned}$$

3. 找到 General solution.



$$\therefore y = y_h + y_p$$

$$\therefore y = c_1 e^x + c_2 x e^x - 3\left(x^3 + x^2 + \frac{3}{2}x + \frac{1}{2}\right)e^{-x}$$



Thank you for listening

Work sheet link :
[https://github.com/ZiMiTan01/Note/
tree/master/Markdown/112-1學期/Eng-
Math](https://github.com/ZiMiTan01/Note/tree/master/Markdown/112-1學期/Eng-Math)



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