Eng-Math Problem sloving

先備知識

- 對於一個非齊次線性常微分方程式，我們可以使用以下幾個步驟與工具找到通解
  i. 找到齊次解 $y_h \Rightarrow let \quad r(x) = 0$
    - 一元二次方程 $(\lambda^2 + a\lambda + b) = 0$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \lambda_1, \lambda_2$$

      - 兩相異實根 $(\lambda = \lambda_1, \lambda_2)$

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

      - 一個實數重根 $(\lambda = -\frac{a}{2}, -\frac{a}{2})$

$$y = c_1 e^{-\frac{a}{2}x} + c_2 x e^{-\frac{a}{2}x}$$

      - 共軛複數根 $(\lambda = \alpha \pm \beta i)$

$$y = Ae^{-\frac{a}{2}x} \cos \omega + Be^{-\frac{a}{2}x} \sin \omega$$

> 有一個給定的方程式 $f(x) = y'' + ay' + b = 0$ ，如何求 y 呢？
> $\quad\Rightarrow$ 令 y 是一個微分後不會改變的函數，這樣就可以使用一元二次方程式
> $\quad\Rightarrow$ 指數函數和三角函數是微分前後都不會改變的數
>
> $$let \quad \begin{cases} y(x) = e^{\lambda x} \\ y'(x) = \lambda e^{\lambda x} \\ y''(x) = \lambda^2 e^{\lambda x} \end{cases}$$
>
> $$then \quad \begin{aligned} &\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0 \\ &\Rightarrow (\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \end{aligned}$$
>
> $$\therefore \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

    - Reduction of Order
      y_1 is given basis, the 2^{nd} basis y_2 is

$$y_2 = \int \frac{1}{y^2} e^{-\int p dx} dx$$

$$\therefore y = c_1 y_1 + c_2 \int \frac{1}{y^2} e^{-\int p dx} dx$$

- Euler-Cauchy Equation $(x^2y'' + axy' + by = 0)$

$$let \quad y = x^m, y' = mx^{m-1}, y'' = m(m-1)x^{m-2}$$
$$\Rightarrow [m(m-1) + am + b]x^m = 0, \quad m = \frac{(1-a) \pm \sqrt{(1-a)^2 - 4ac}}{2}$$

  - 兩相異實根 $(m = m_1, m_2)$

$$y = c_1x^{m1} + c_2x^{m2}$$

  - 一個實數重根 $(m = \dfrac{1-a}{2}, \dfrac{1-a}{2})$

$$y = c_1x^m + c_2x^m lnx$$

  - 共軛複數根 $m = \alpha \pm \beta i$

$$y = Ax^\alpha \cos(\beta \ln x) + Bx^\alpha \sin(\beta \ln x)$$

ii. 找到非齊次解 y_p .

- Method of Undetermined Coefficients
  - Basic rule

$r_{(x)}$	y_p
ke^{rx}	ce^{rx}
$kx^n{}_{n=0,1,2....}$	$K_nx^n + K_{(n-1)}x^{(n-1)}... + K_1x + K_0$
$\left.\begin{matrix} k\cos\omega x \\ k\sin\omega x \end{matrix}\right\}$	$K\cos\omega x + M\sin\omega x$
$\left.\begin{matrix} ke^{\alpha x}\cos\omega x \\ ke^{\alpha x}\sin\omega x \end{matrix}\right\}$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

  - Modification rule
    假設的 y_p 有出現和 y_h 一樣的函數，則我們假設的函數要再乘 x or x^2 倍
  - Sum rule
    $r(x)$ 有什麼函數， y_p 就有什麼函數

$$\begin{aligned} r(x) =& e^{0.5x} + 40\cos 10x - 190\sin 10x \\ & then \quad y_p = ce^{rx} + K\cos\omega x + M\sin\omega x \end{aligned}$$

- Method of Variation of Parameters

$$\begin{aligned} y_p =& uy_1 + vy_2 \\ =& y_1 \int \frac{W_1}{W} r dx + y_2 \int \frac{W_2}{W} r dx \end{aligned}$$

> Assume

$$y_p=y_{p1}+y_{p2}=uy_1+vy_2$$

> $\because$ 基底們是線性獨立的，而線性獨立的函數比值為某一函數
>
> $$\therefore y_p=u_{(x)}y_1+v_{(x)}y_2$$

$$\begin{cases} y_h=c_1y_1+c_2y_2 \\ \\ y_p=uy_{p1}+vy_{p2} \begin{cases} y_p'=uy_{p1}'+vy_{p2}'+u'y_{p1}+vy_{p2}' & ,u'y_{p1}+vy_{p2}'=0 \\ y_p''=u'y_{p1}'+uy_{p1}''+v'y_{p2}'+vy_{p2}'' \end{cases} \end{cases}$$

y_p , y_p' , y_p'' 代回ODE

$$\therefore u'y'+v'y'=r$$

解聯立方程式

$$\begin{cases} u'y'+v'y'=r \\ u'y_{p1}+vy_{p2}'=0 \end{cases} \Rightarrow \begin{aligned} u' &= \frac{-y_2r}{y_1y_2'-y_2y_1'}=\frac{-y_2r}{W(y_1,y_2)} \\ v' &= \frac{y_1r}{y_1y_2'-y_2y_1'}=\frac{y_1r}{W(y_1,y_2)} \end{aligned}$$

u' , v' 積分

$$u=\int\frac{-y_2r}{W}dx \qquad v=\int\frac{y_1r}{W}dx$$

找到 y_p

$$\begin{aligned} \therefore y_p &= uy_1+vy_2 \\ &= y_1\int\frac{W_1}{W}rdx+y_2\int\frac{W_2}{W}rdx \end{aligned}$$

- 逆運算子法

iii. 找到General solution.

$$y=y_h+y_p$$

$(D^2-2D+1)y=6x^2e^{-x}$

1. Find y_h

$$\begin{aligned} let \quad r_{(x)} &= 0 \\ &\Rightarrow (D^2-2D+1)y=0 \end{aligned}$$

> 令這個非齊次線性常微分ODE變成齊次線性常微分ODE，以方便我們尋找 y_h

$$\begin{array}{cc} D & -1 \\ D & -1 \\ \hline -D-D= & -2D \end{array} \qquad \Rightarrow D=1,1$$

$$\therefore y_h = c_1e^x + c_2xe^x$$

> 對於Double Root，我們可以用Reduction of Order來發現：
>
> $$\begin{aligned} &\text{Base}_2 = x \times \text{Base}_1 \\ &\Rightarrow y_2 = xy_1 \end{aligned}$$

2. Find y_p
   這個例題中， $r_{(x)}$ 是個較為複雜的函數，我們難以使用 Method of Undetermined Coefficients 來對 y_p 做假設，因此在這邊我們使用 Method of Variation of Parameters 會是個比較輕鬆的選擇。

> - For a 2^{nd} order non-homogeneous ODE
>
> $$\begin{aligned} y_p =& uy_1 + vy_2 \\ =& y_1\int\frac{W_1}{W}rdx + y_2\int\frac{W_2}{W}rdx \end{aligned}$$
>
> - Wronskian Method
>
> $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2 \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$$

$$\begin{aligned} \therefore y_p =& e^x\int\frac{-xe^x}{e^{2x}}\times 6x^2e^{-x}dx + xe^x\int\frac{e^x}{e^{2x}}\times 6x^2e^{-x}dx \\ =& 6e^x(\int xe^{-2x}dx + x\int x^2e^{-2x}dx) \\ =& 6e^x[-\frac{1}{2}(x+\frac{1}{2})e^{-2x} - \frac{1}{2}x(x^2+x+\frac{1}{2})e^{-2x}] \\ =& -3e^{-x}[x+\frac{1}{2}+x(x^2+x+\frac{1}{2})] \\ =& -3e^{-x}(x^3+x^2+\frac{3}{2}x+\frac{1}{2}) \end{aligned}$$

> - Wronskian Method
>
> $$W = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = e^{2x} \quad W_1 = \begin{vmatrix} 0 & xe^x \\ 1 & (x+1)e^x \end{vmatrix} = -xe^x \quad W_2 = \begin{vmatrix} e^x & 0 \\ e^x & 1 \end{vmatrix} = e^x$$
>
> - $\int xe^{-2x}dx$
>
> $$let \begin{cases} u = x \quad , \quad du = dx \\ dv = e^{-2x}dx \quad , \quad v = -\frac{1}{2}e^{-2x} \end{cases}$$

$$\because \int udv = uv - \int vdu$$
$$\therefore \int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}dx$$
$$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$
$$= -\frac{1}{2}(x + \frac{1}{2})e^{-2x}$$

- $\int x^2e^{-2x}dx$

$$let \begin{cases} u = x^2 \quad , \quad du = 2x \\ dv = e^{-2x}dx \quad , \quad v = -\frac{1}{2}e^{-2x} \end{cases}$$

$$\because \int udv = uv - \int vdu$$
$$\therefore \int x^2e^{-2x}dx = -\frac{1}{2}x^2e^{-2x} - \int -\frac{1}{2} \times 2xe^{-2x}dx$$
$$= -\frac{1}{2}x^2e^{-2x} - \int -xe^{-2x}dx$$
$$= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$
$$= -\frac{1}{2}(x^2 + x + \frac{1}{2})e^{-2x}$$

3. General Solution: $y = y_h + y_p$

$$\because y = y_h + y_p$$
$$\therefore y = c_1e^x + c_2xe^x - 3(x^3 + x^2 + \frac{3}{2}x + \frac{1}{2})e^{-x}$$