CS544 - Homework 2

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1 Problem Formulation

The total variation denoising(TVD) method for image denoising is formulated as follows:

$$\underset{x}{\text{minimize}} \quad \frac{1}{2}\|x-y\|_2^2 + \alpha\|Dx\|_1$$

where x is the denoised image and y is the original image, D is the difference matrix, and λ is the regularization parameter. We can then reformulate it with a constraint equation to be amenable to ADMM:

$$\begin{aligned} \underset{x,z}{\text{minimize}} \quad & \frac{1}{2}\|x-y\|_2^2 + \alpha\|z\|_1 \\ & s.t. Dx = z \end{aligned}$$

The general form of ADMM is:

$$\begin{aligned} \underset{x,z}{\text{minimize}} \quad & f(x) + g(z) \\ \text{s.t.} \quad & Ax + Bz = c \end{aligned}$$

The iterates of ADMM are defined by the augmented Lagrangian,

$$L_\rho(x, z, y) = f(x) + g(z) + \lambda^T(Ax + Bz - c) + (\rho/2)\|Ax + Bz - c\|_2^2,$$

and consist of,

$$\begin{aligned} x^{k+1} &= \underset{x}{\text{argmin}}\, L_\rho(x, z^k, \lambda^k), \\ z^{k+1} &= \underset{z}{\text{argmin}}\, L_\rho(x^{k+1}, z, \lambda^k), \\ \lambda^{k+1} &= \lambda^k + \rho(Ax^{k+1} + Bz^{k+1} - c), \\ \rho^{k+1} &= \beta\rho^k \end{aligned}$$

where β is a hyper-parameter. In our case, the ADMM objective function and update function are as follows:

$$L(x, z, \lambda, \rho) = \frac{1}{2}\|x-y\|_2^2 + \alpha\|z\|_1 + \lambda(Dx - z) + \frac{\rho}{2}\|Dx - z\|_2^2 \tag{1}$$

$$x^{k+1} = (I + \rho D^T D)^{-1}(y + \rho D(z^k - u^k)) \tag{2}$$

$$\begin{aligned} z^{k+1} &= \begin{cases} Dx^{k+1} + u^k - \alpha & \text{if } Dx^{k+1} + u^k > \alpha/\rho, \\ 0 & \text{otherwise}, \\ Dx^{k+1} + u^k + \alpha & \text{if } Dx^{k+1} + u^k < -\alpha/\rho. \end{cases} \\ u^{k+1} &= u^k + Dx^{k+1} - z^{k+1} \\ \rho^{k+1} &= 2.0 * \rho^k \end{aligned}$$

where $u = \frac{\lambda}{\rho}$.
We use two ways to update the x_k, one way is to directly compute the inverse matrix, and another way is to use Conjugate Gradient Decent.

2 Experiments

2.1 Implement TVD for 64 x 64 Intensity Images

We implement our alternating direction method of multipliers (ADMM [4]) approach for image total variance denoising (TVD) following the methods discussed above.

Data Preparation

In our empirical evaluation, we apply our method to a diverse dataset comprising 5 distinct images [1, 6, 8, 7, 5] sourced from the Internet. This dataset spans a wide range of content categories, including animal photography, natural landscapes, artistic sketches, and computer-generated imagery. This selection strategy ensures a comprehensive assessment of our method's performance across varied image types.

For the data preprocessing, we first load the images and resize them to the scale of 64x64 pixels. Then, we transformed these images into grayscale to maintain focus on luminance variations. We add IID normal gaussian noise to each pixel of the image. Specifically, we generated noise with a zero mean and a unit standard deviation. We then normalized it to achieve a standard deviation of $\delta = 0.05$ and added the noise onto the original images. This procedure allowed us to add a controlled level of noise to each image, thereby generating pairs of original and corresponding noisy images.

ADMM Implementation

Our method is implemented following the approach described in Section 1. We have chosen the following hyper-parameters: the learning rate (α) is set to 0.075, and the initial value of ρ is 2.0. We limit the number of iterations to a maximum of 100. The criteria for stopping the algorithm is set when the tolerance level reaches 1×10^{-7}. For the Conjugate Gradient Descent part of our implementation, we establish a tolerance threshold of 1×10^{-5}. Similar to the previous setting, the number of iterations for this process is also capped at 100.

In terms of the inverse of $I + \rho D^T D$ in 2, we derive two different methods for computation.

A numerical approach involves directly applying the Conjugate Gradient (CG) method to solve the equation $Ax = b$, thereby eliminating the need to compute the inverse of the target matrix. In this context, the matrix $I + \rho D^T D$ is positive semi-definite, making the application of the CG method particularly efficient.

Another way is we could try to apply eigen decomposition to compute the inverse for $I + \rho D^T D$, let's let $A = I + \rho D^T D$ and $B = D^T D$, Then B can be factorized as $B = Q \Lambda Q^{-1}$, where Q is the square n × n matrix whose ith column is the eigenvector q_i of B, and Λ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues. Then A^{-1} can be represent as $Q(\Lambda + I)^{-1} Q^T$. Since $\Lambda + I$ is a diagonal matrix, its inverse can be straightforwardly determined by calculating the reciprocal of each element on its diagonal. We could precompute and save the Q before ADMM optimization iteration to optimize performance.

In our experiments, we compare both of these two methods. Despite the second method being more theoretically satisfying, we observed that the Conjugate Gradient (CG) method consistently outperformed it in terms of efficiency. This discrepancy can be attributed to several factors inherent to the practical application and operational dynamics of the CG method. Primarily, the CG method's efficiency in sparse matrix operations and its ability to rapidly converge to a solution without necessitating the explicit computation of matrix inverses play pivotal roles in its superior performance. In comparison, although the eigen decomposition could save some resource in compute matrix inversion, it still has large computation overhead in matrix multiplication process.

Results

The effectiveness of our denoising method is vividly illustrated in Figure 1, showcasing a substantial decrease in the Root Mean Square Error (RMSE) between the original and denoised images. This comparison underscores not only a quantitative enhancement, but also reflects in image quality. Post-denoising, the images emerge significantly less noisy and more polished.

2.2 Color Images Denoising

We apply our approach to three channels of the color image independently, and the result is shown in Fig.2. Our analysis of denoising results for RGB images indicates a notable success in noise reduction across the Red, Green, and Blue channels, leading to visibly enhanced image quality. However, it is observed that the average

Root Mean Square Error (RMSE) for the denoised RGB images is slightly higher than that achieved when applying denoising solely to intensity (grayscale) images. This discrepancy in RMSE can be attributed to the inherent complexity of color image data.

In RGB images, noise reduction must be balanced across three channels, each contributing to the overall error measurement, whereas intensity images consolidate information into a single channel, potentially simplifying the denoising task. Moreover, the process of denoising RGB images independently by channel can introduce minor color balance shifts or fail to exploit inter-channel correlations effectively, which might slightly elevate the RMSE.

Regarding processing speed, denoising RGB images is inherently more resource-intensive and time-consuming than working with intensity images due to the tripled data volume and the need to maintain color integrity across three channels. This increased computational demand underscores the trade-off between achieving higher-quality color image denoising and the efficiency of the process.

2.3 Compare with Opensource Implementation

We further compare our method to public opensourced ADMM implementation for TVD (Total Variance Denoising). We chose the denoising technique provided by the Scientific Computational Imaging Code (SCICO [2]). The experiment result is shown in 3. The results, as evidenced by both visual assessment and the Root Mean Square Error (RMSE) metric, demonstrate our method has compariable performance as SCICO [2] implementations. Specifically, the SCICO's result tends to be ambiguous while our result can maintain the contour of the image. However, the SCICO implementation exhibits a faster processing time, averaging 2.67s, compared to 21.54s in our ADMM implmentations. The enhanced speed of SCICO's implementation could be attributed to the use of more efficient optimization techniques (e.g. JAX [3]) for acceleration.

3 Conclusion

In conclusion, this course project successfully demonstrates the application and effectiveness of the Total Variation Denoising (TVD) method using the Alternating Direction Method of Multipliers (ADMM) for image denoising. Through rigorous experimentation on both grayscale and color images, the project showcases the method's ability to significantly reduce noise while preserving the essential features and contours of the images. The comparison with an open-source implementation, SCICO, reveals that while the developed method achieves comparable denoising performance, it operates at a slower processing speed.

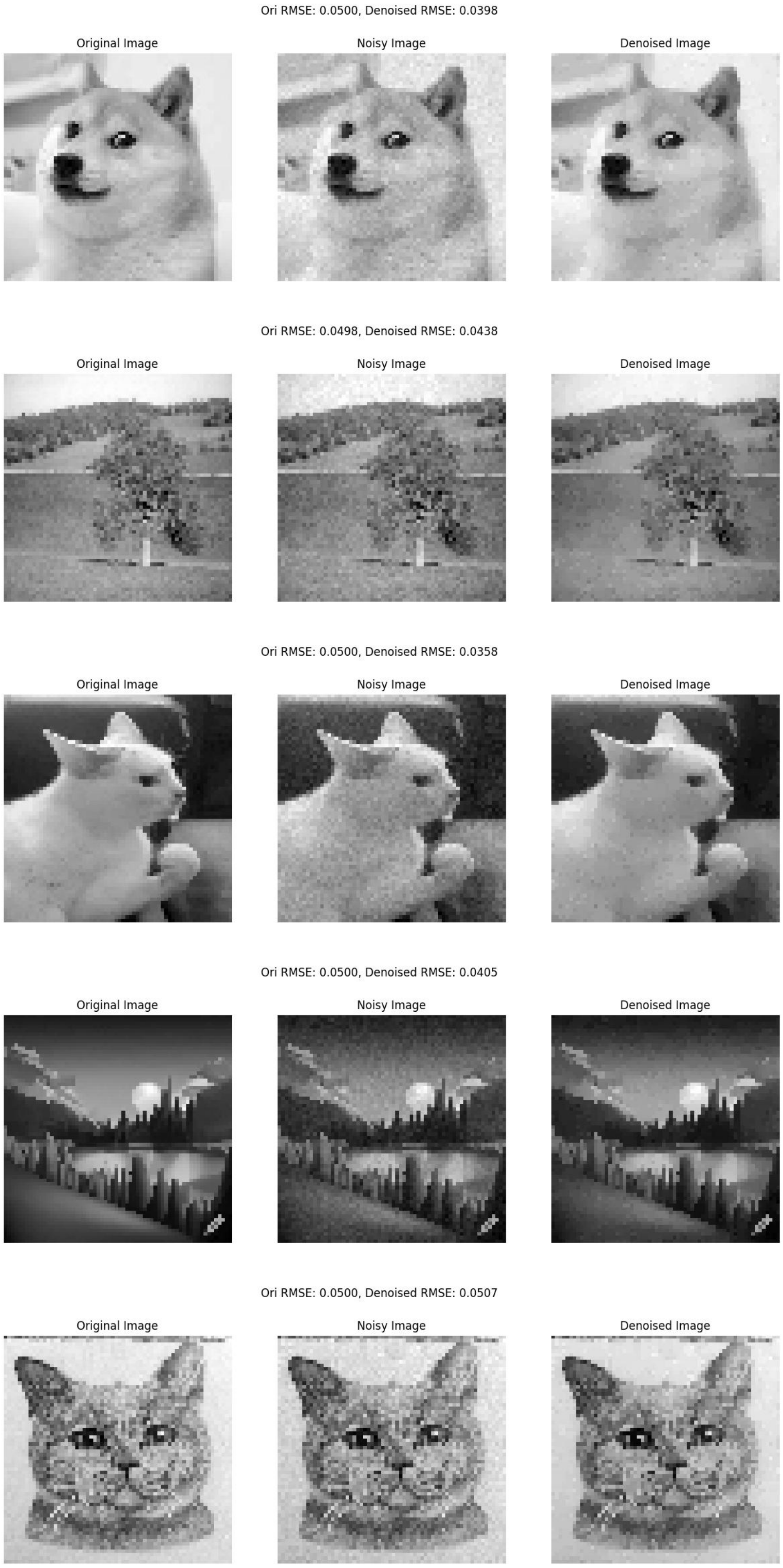

Figure 1: Denoising intensity images

Figure 2: Denoising RGB images

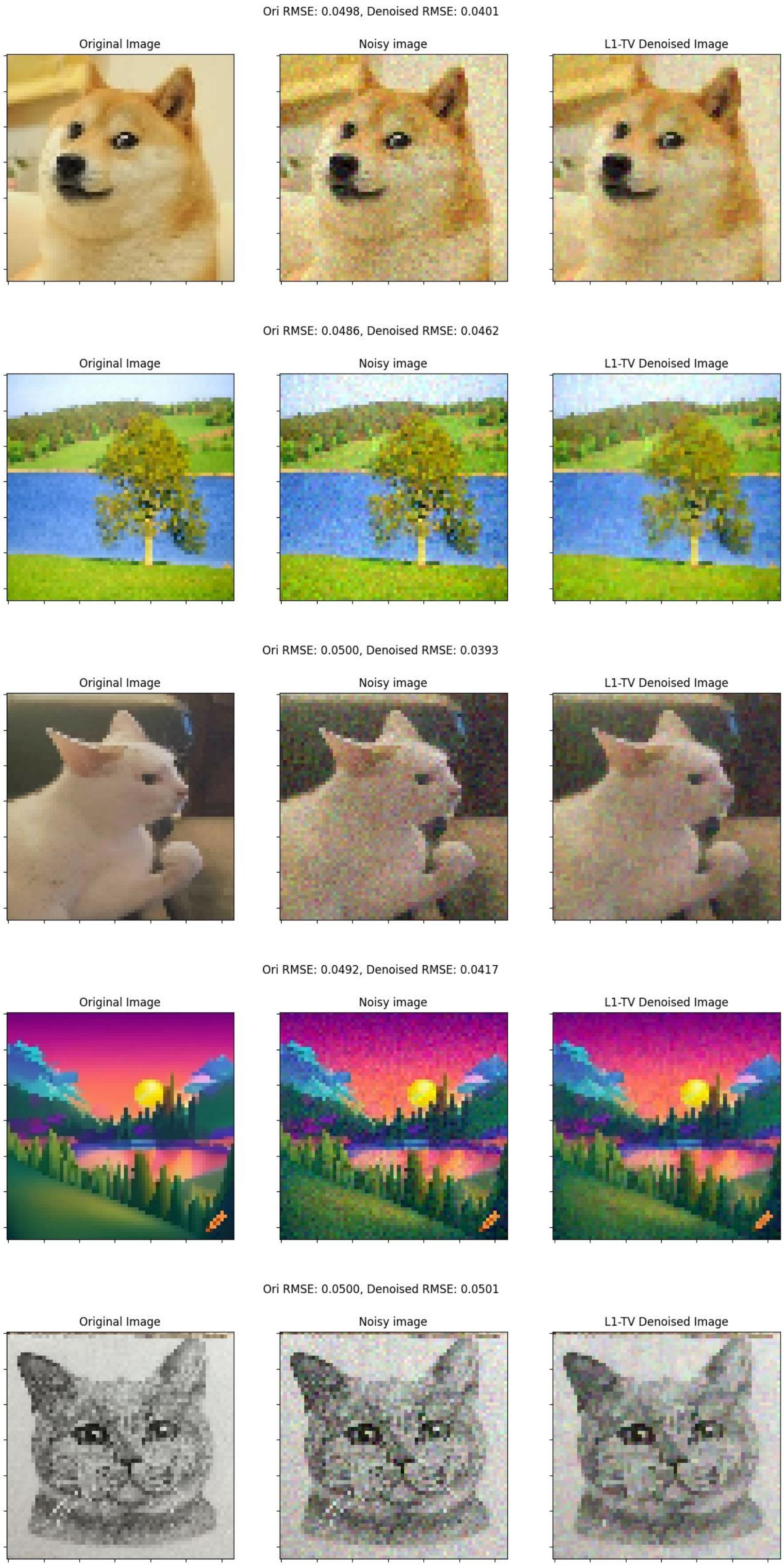

Figure 3: Denoising RGB images with the implementation of SCICO [2]

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