

數位影像處理 DIP Homework Chapter 4_1 (100 pts)

1. If a discrete sequence ($M=4$) is $\{f(0), f(1), f(2), f(3)\} = \{4, 2, 1, 5\}$, please find its DFT $F(u)$? and count how many multiplications and additions in the $M=4$ DFT? (30)

Discrete Fourier Transform

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}, \quad u = 0, 1, 2, \dots, M-1$$

1. $M=4$ $\{f(0), f(1), f(2), f(3)\} = \{4, 2, 1, 5\}$ find $F(u)$

$$F(0) = f(0)e^0 + f(1)e^0 + f(2)e^0 + f(3)e^0 = 4 + 2 + 1 + 5 = 12$$

$$F(1) = f(0)e^0 + f(1)e^{-j\pi/2} + f(2)e^{-j\pi} + f(3)e^{j3\pi/2} = 4 - 2j - 1 + 5j = 3 + 3j$$

$$F(2) = f(0)e^0 + f(1)e^{-j\pi} + f(2)e^{-j2\pi} + f(3)e^{-j3\pi} = 4 - 2 - 1 - 5 = -2$$

$$F(3) = f(0)e^0 + f(1)e^{j3\pi/2} + f(2)e^{j\pi} + f(3)e^{j\pi/2} = 4 + 2j - 1 - 5j = 3 - 3j$$

Ans: $\{F(0), F(1), F(2), F(3)\} = \{12, 3+3j, -2, 3-3j\}$

multiplications:
 $4 \times 4 = 16$ 次

additions: $4 \times (4-1) = 12$ 次

以 code 驗證

```
2 import numpy as np
3
4 x = np.array([4, 2, 1, 5])
5 print(np.fft.fft(x))
```

```
PS C:\Users\USER> & "C:/Program Files/Python37/python."
[12.+0.j 3.+3.j -2.+0.j 3.-3.j]
```

2. If a discrete sequence ($M=8$) is $\{f(0), f(1), f(2), f(3), f(4), f(5), f(6), f(7)\} = \{4, 2, 1, 5, 4, 2, 1, 5\}$, please find its DFT $F(u)$? and count how many multiplications and additions in the $M=8$ DFT? (30)

2.

$M=8$

$\{f(0), f(1), f(2), f(3), f(4), f(5), f(6), f(7)\} = \{4, 2, 1, 5, 4, 2, 1, 5\}$

additions = $4 \times (4-1) = 12$

$F(0) = f(0) + f(1) + f(2) + \dots + f(7) = 24$

$F(1) = f(0) + f(1)e^{-j2\pi/8} + f(2)e^{-j4\pi/8} + f(3)e^{-j6\pi/8} + f(4)e^{-j8\pi/8} + f(5)e^{-j10\pi/8} + f(6)e^{-j12\pi/8} + f(7)e^{-j14\pi/8}$

$= 4 + 2 \cdot \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) + 1 \cdot j + 5 \cdot \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) + 4 \cdot (-1) + 2 \cdot \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) + 1 \cdot (-j) + 5 \cdot \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) = 0$

$F(2) = f(0) + f(1)e^{-j4\pi/8} + f(2)e^{-j8\pi/8} + f(3)e^{-j12\pi/8} + f(4)e^{-j16\pi/8} + f(5)e^{-j20\pi/8} + f(6)e^{-j24\pi/8} + f(7)e^{-j28\pi/8}$

$= 4 + 2 \cdot (-j) + 1 \cdot (-1) + 5 \cdot j + 4 \cdot 1 + 2 \cdot (-j) + 1 \cdot (-1) + 5 \cdot j = 6 + 6j$

$$F(3) = f(0) + f(1)e^{-j\frac{6\pi}{8}} + f(2)e^{-j\frac{12\pi}{8}} + f(3)e^{-j\frac{18\pi}{8}} + f(4)e^{-j\frac{24\pi}{8}} \\ + f(5)e^{-j\frac{30\pi}{8}} + f(6)e^{-j\frac{36\pi}{8}} + f(7)e^{-j\frac{42\pi}{8}}$$

$$= 4 + 2 \cdot \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) + 1 \cdot (-1) + 5 \cdot \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) + 4 \cdot (-1) + 2 \cdot \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) + 1 \cdot (-1) \\ + 5 \cdot \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) = 0$$

$$F(4) = f(0) + f(1)e^{-j\frac{8\pi}{8}} + f(2)e^{-j\frac{16\pi}{8}} + f(3)e^{-j\frac{24\pi}{8}} + f(4)e^{-j\frac{32\pi}{8}} + f(5)e^{-j\frac{40\pi}{8}} \\ + f(6)e^{-j\frac{48\pi}{8}} + f(7)e^{-j\frac{56\pi}{8}} = 4 + 2 \cdot (-1) + 1 \cdot (-1) + 5 \cdot (-1) + 4 \cdot (-1) + 2 \cdot (-1) + 1 \cdot (-1) \\ + 5 \cdot (-1) = -4$$

$$F(5) = f(0) + f(1)e^{-j\frac{10\pi}{8}} + f(2)e^{-j\frac{20\pi}{8}} + f(3)e^{-j\frac{30\pi}{8}} + f(4)e^{-j\frac{40\pi}{8}} + f(5)e^{-j\frac{50\pi}{8}} \\ + f(6)e^{-j\frac{60\pi}{8}} + f(7)e^{-j\frac{70\pi}{8}} = 4 + 2 \cdot \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) + 1 \cdot (-1) + 5 \cdot \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) \\ + 4 \cdot (-1) + 2 \cdot \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) + 1 \cdot (-1) + 5 \cdot \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) = 0$$

$$F(6) = f(0) + f(1)e^{-j\frac{12\pi}{8}} + f(2)e^{-j\frac{24\pi}{8}} + f(3)e^{-j\frac{36\pi}{8}} + f(4)e^{-j\frac{48\pi}{8}} \\ + f(5)e^{-j\frac{60\pi}{8}} + f(6)e^{-j\frac{72\pi}{8}} + f(7)e^{-j\frac{84\pi}{8}} = 4 + 2 \cdot j + 1 \cdot (-1) + 5 \cdot (-j) + 4 \cdot (-1) \\ + 2 \cdot (-j) + 1 \cdot (-1) + 5 \cdot (-j) = 6 - 6j$$

$$F(7) = 4 + 2 \cdot \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) + 1 \cdot (-1) + 5 \cdot \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) + 4 \cdot (-1) + 2 \cdot \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) + 1 \cdot (-1) \\ + 5 \cdot \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) = 0$$

Ans: $\{F(0), F(1), F(2), F(3), F(4), F(5), F(6), F(7)\}$

$$= \{24, 0, 6+6j, 0, -4, 0, 6-6j, 0\}$$

multiplications = $8^2 = 64$ 次

additions = $8 \times (8-1) = 56$ 次

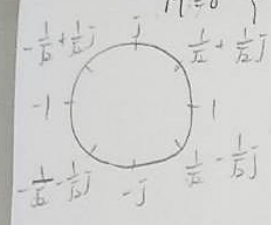
以 code 驗證

```
2 import numpy as np
3
4 x = np.array([4, 2, 1, 5, 4, 2, 1, 5])
5 print(np.fft.fft(x))
```

```
PS C:\Users\USER> & "C:/Program Files/Python37/python.exe" c:/Users
[24.+0.j 0.+0.j 6.+6.j 0.+0.j -4.+0.j 0.+0.j 6.-6.j 0.+0.j]
```

3. If a discrete sequence ($M=8$) is $\{f(0), f(1), f(2), f(3), f(4), f(5), f(6), f(7)\} = \{4, -2, 1, -5, 4, -2, 1, -5\}$, please find its DFT $F(u)$? and count how many multiplications and additions in the $M=8$ DFT? (30)

3. $M=8$ $\{f(0), f(1), f(2), f(3), f(4), f(5), f(6), f(7)\} = \{4, -2, 1, -5, 4, -2, 1, -5\}$



$F(0) = f(0) + f(1) + \dots + f(7) = -4$ ✖
 $F(1) = 4 + -2 \cdot (\frac{1}{8} - \frac{j}{8}) + 1 \cdot -j + -5 \cdot (-\frac{1}{8} - \frac{j}{8}) + 4 \cdot -1 + -2 \cdot (\frac{1}{8} + \frac{j}{8}) + 1 \cdot j + -5 \cdot (\frac{1}{8} - \frac{j}{8}) = 0$ ✖
 $F(2) = 4 \cdot 1 + -2 \cdot -j + 1 \cdot -1 + -5 \cdot j + 4 \cdot 1 + -2 \cdot -j + 1 \cdot -1 + -5 \cdot j = 6 - 6j$ ✖
 $F(3) = 4 \cdot \frac{1}{8} + -2 \cdot (\frac{1}{8} - \frac{j}{8}) + 1 \cdot j + -5 \cdot (\frac{1}{8} - \frac{j}{8}) + 4 \cdot -1 + -2 \cdot (\frac{1}{8} + \frac{j}{8}) + 1 \cdot -j + -5 \cdot (\frac{1}{8} + \frac{j}{8}) = 0$
 $F(4) = 4 \cdot 1 + -2 \cdot -1 + 1 \cdot 1 + -5 \cdot -1 + 4 \cdot 1 + -2 \cdot -1 + 1 \cdot 1 + -5 \cdot -1 = 24$ ✖
 $F(5) = 4 \cdot \frac{1}{8} + -2 \cdot (-\frac{1}{8} + \frac{j}{8}) + 1 \cdot -j + -5 \cdot (-\frac{1}{8} + \frac{j}{8}) + 4 \cdot -1 + -2 \cdot (\frac{1}{8} - \frac{j}{8}) + 1 \cdot j + -5 \cdot (\frac{1}{8} - \frac{j}{8}) = 0$ ✖
 $F(6) = 4 \cdot j + -2 \cdot j + 1 \cdot -1 + -5 \cdot -j + 4 \cdot 1 + -2 \cdot j + 1 \cdot -1 + -5 \cdot -j = 6 + 6j$ ✖
 $F(7) = 4 \cdot \frac{1}{8} + -2 \cdot (\frac{1}{8} + \frac{j}{8}) + 1 \cdot j + -5 \cdot (-\frac{1}{8} + \frac{j}{8}) + 4 \cdot (-1) + -2 \cdot (-\frac{1}{8} - \frac{j}{8}) + 1 \cdot -j + -5 \cdot (\frac{1}{8} - \frac{j}{8}) = 0$ ✖

$\{F(0), F(1), F(2), F(3), F(4), F(5), F(6), F(7)\} = \{-4, 0, 6-6j, 0, 24, 0, 6+6j, 0\}$

multiplications $= 8^2 = 64$ ✖
 additions $= 8 \times (8-1) = 56$ ✖

以 code 驗證

```
2 import numpy as np
3
4 x = np.array([4, -2, 1, -5, 4, -2, 1, -5])
5 print(np.fft.fft(x))
```

```
PS C:\Users\USER> & "C:/Program Files/Python37/python.exe" c:/User
[-4.+0.j  0.+0.j  6.-6.j  0.+0.j 24.+0.j  0.+0.j  6.+6.j  0.+0.j]
```


4. Please state the relation between the question (1), (2), and (3) ?
(10)

- (2) 與 (1) 數值一樣，然而 (2) 的 DFT 結果，數值為 (1) 的兩倍。因(2) 取樣頻率為(1)的兩倍，因此根據 Fourier Transform 中 Time Transformation 的觀念(如下圖)，當 a 為 $1/2$ ，也就是取樣頻率變 2 倍，如同(2)的情況，則做 DFT 的結果，值應為原(1)的兩倍，且在(1)兩倍的位置才出現值。

Time Transformation

$$f(t) \rightarrow F(\omega)$$
$$f(at - t_0) \rightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right) e^{-j\frac{\omega}{a}t_0}$$

- (3) 是 (2) 做 Frequency shifting，將(2)的 data 在 time domain $*e^{-jn\pi}$ ，因此在奇數位的 data 會變為負值，做此處理的結果是 DFT 的結果會往右移動 π 。(原始 data 的長度是 2π)