ENGINEERING MATHEMATICS

New Programs Students

Master

Mathematics - 1

1

Center

For Education

مدينه نصر- شارع النزهة - عمارات الشركه السعوديه - امام البوابه الرائيسيه لدارالدفاع الجوى - بجوار سوبر ماركت مكه مول

للأستعلام : ٠١٠٠٤٣٤١٧١٣ / ٠١١٥٢٠٨٨٩٥٦

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Basic Results and Concepts

I. General Information

1) Greek Letters

α	alpha	β	beta	γ	gamma	δ	delta
θ	theta	ϕ	Phi	ε	epsilon	ψ	Psi
ω	omega	λ	lambda	η	eta	σ	Sigma
π	Pi	ρ	rho	Δ	Cap. delta	Σ	Cap. Sigma
μ	mu	ζ	zeta	τ	tau	Γ	Cap. gamma

2) Some Notations

$\in$	belongs to	$\notin$	doesn't belong to	$\cup$	union	$\cap$	intersection

3) Useful Data

$e \simeq 2.7183$	$\pi \simeq 3.1416$	$\sqrt{2} \simeq 1.4142$	$\sqrt{3} \simeq 1.732$

II. Algebra

1) Factoring Formulas

$a^2-b^2=(a-b)(a+b)$	$a^3-b^3=(a-b)(a^2+ab+b^2)$	$a^3+b^3=(a+b)(a^2-ab+b^2)$

2) Product Formulas

$(a\pm b)^2=a^2\pm 2ab+b^2$	$(a-b)^3=a^3-3a^2b+3ab^2-b^3$	$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

3) Powers

$a^m a^n=a^{m+n}$	$\frac{a^m}{a^n}=a^{m-n}$	$(ab)^m=a^m b^m$	$\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$	$(a^m)^n=a^{mn}$	$a^{-m}=\frac{1}{a^m}$	$a^0=1,\ a\neq 0$

4) Roots

$\sqrt[n]{ab}=\sqrt[n]{a}\sqrt[n]{b}$	$\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}},\ b\neq 0$	$\sqrt[n]{a^m}=(\sqrt[n]{a})^m=a^{\frac{m}{n}}$	$(\sqrt[n]{a})^n=a$	$\frac{1}{\sqrt{a}\pm\sqrt{b}}=\frac{\sqrt{a}\mp\sqrt{b}}{a-b}$

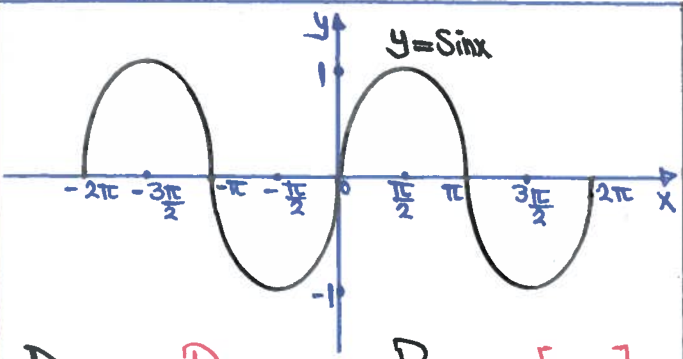
5) Equations

$ax+b=0$	$x=-\frac{b}{a}$	$ax^2+bx+c=0$	$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

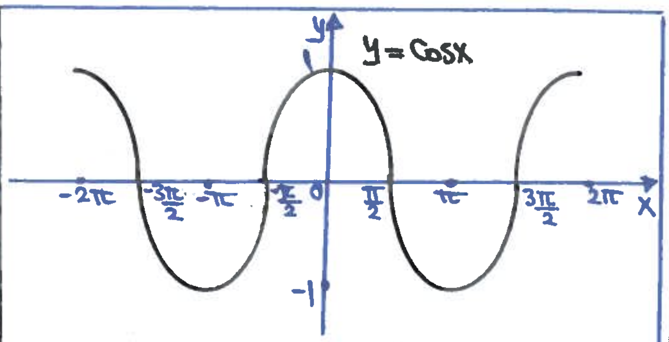
6) Inequalities

Inequality	Interval Notation	Inequality	Interval Notation
$a\leq x\leq b$	$[a,b]$	$a<x\leq b$	$]a,b]$
$a\leq x<b$	$[a,b[$	$a<x<b$	$]a,b[$
$-\infty<x\leq b,\ x\leq b$	$]-\infty,b]$	$-\infty<x<b,\ x<b$	$]-\infty,b[$
$a\leq x<\infty,\ x\geq a$	$[a,\infty[$	$a<x<\infty,\ x>a$	$]a,\infty[$

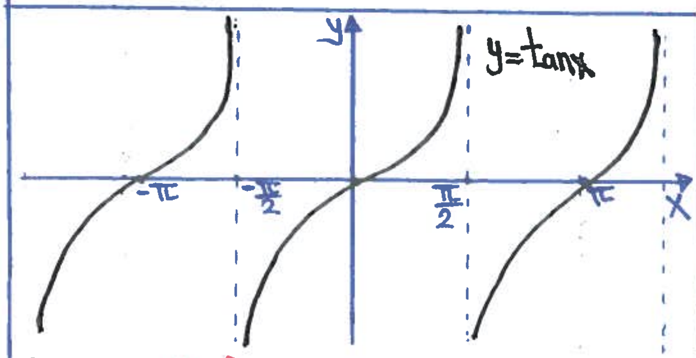
III) Trigonometric Functions



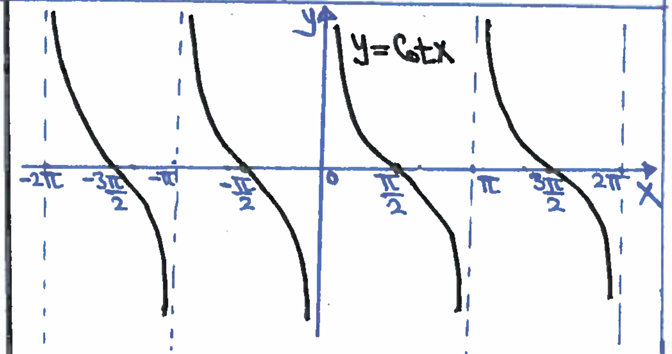
- Domain = R
- Range = $[-1, 1]$
- Zeros = $\{n\pi\}$
- Period = 2π



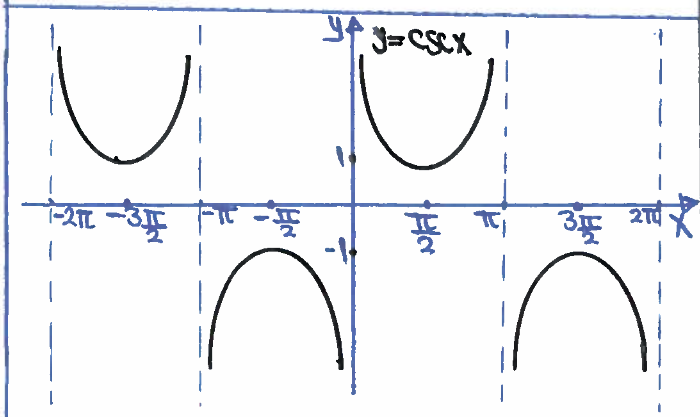
- Domain = R
- Range = $[-1, 1]$
- Zeros = $\{(2n+1)\frac{\pi}{2}\}$
- Period = 2π



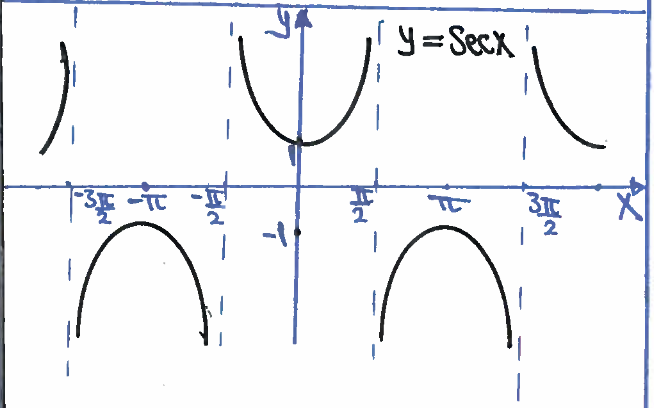
- Domain = $R - \{(2n+1)\frac{\pi}{2}\}$
- Range = R
- Zeros = $\{n\pi\}$
- Period = π



- Domain = $R - \{n\pi\}$
- Range = R
- Zeros = $\{(2n+1)\frac{\pi}{2}\}$
- Period = π



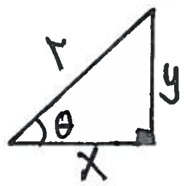
- Domain = $R - \{n\pi\}$
- Range = $R -]-1, 1[$
- Zeros = ϕ
- Period = 2π



- Domain = $R - \{(2n+1)\frac{\pi}{2}\}$
- Range = $R -]-1, 1[$
- Zeros = ϕ
- Period = 2π

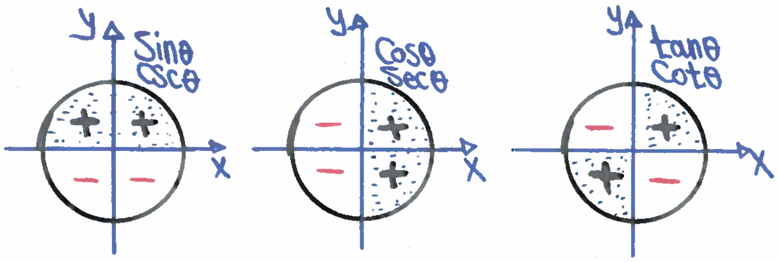
1) Trigonometric Indentities

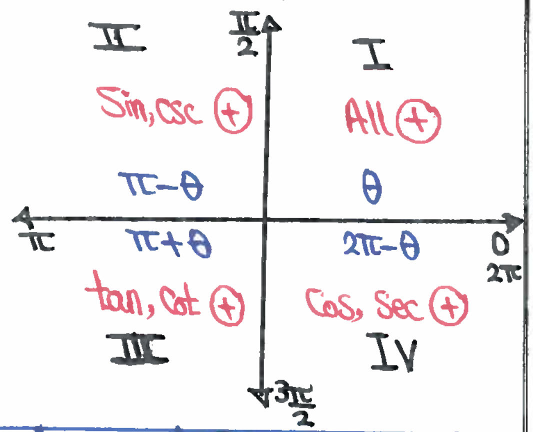
$\sin\theta = \frac{y}{r}$	$\cos\theta = \frac{x}{r}$	$\tan\theta = \frac{y}{x}$
$\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$	$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$	$\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$



$r^2 = x^2 + y^2$

2) Signs of Trigonometric Functions





3) Trigonometric Functions of Common Angles

θ^{rad}	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	-1	0
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	-1	0	1
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0	$-\infty$	0

4) Most Important Formulas

$\sin^2\theta+\cos^2\theta=1$	$\tan^2\theta+1=\sec^2\theta$	$\cot^2\theta+1=\csc^2\theta$	$\tan\theta=\frac{\sin\theta}{\cos\theta}$	$\cot\theta=\frac{\cos\theta}{\sin\theta}$

5) Addition and Subtraction Formulas

$\sin(\alpha\pm\beta)=\sin\alpha.\cos\beta\pm\cos\alpha.\sin\beta$	$\cos(\alpha\pm\beta)=\cos\alpha.\cos\beta\mp\sin\alpha.\sin\beta$	$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha.\tan\beta}$

6) Double, Half and Multiple Angles Formulas

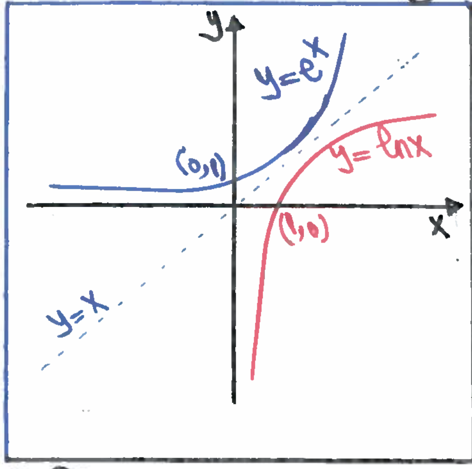
$\sin 2\alpha=2\sin\alpha.\cos\alpha$	$\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}$	$\sin 3\alpha=3\sin\alpha-4\sin^3\alpha$
$\cos 2\alpha= \begin{cases}\cos^2\alpha-\sin^2\alpha\\ 1-2\sin^2\alpha\\ 2\cos^2\alpha-1\end{cases}$	$\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}$	$\cos 3\alpha=4\cos^3\alpha-3\cos\alpha$
$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$	$\tan\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$	$\tan 3\alpha=\frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}$

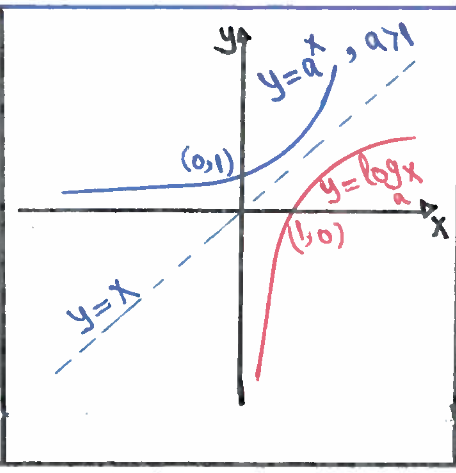
7) Powers of Trigonometric Functions

$\sin^2\alpha=\frac{1}{2}(1-\cos 2\alpha)$	$\sin^3\alpha=\frac{1}{4}(3\sin\alpha-\sin 3\alpha)$	$\sin^4\alpha=\frac{1}{8}(\cos 4\alpha-4\cos 2\alpha+3)$
$\cos^2\alpha=\frac{1}{2}(1+\cos 2\alpha)$	$\cos^3\alpha=\frac{1}{4}(3\cos\alpha+\cos 3\alpha)$	$\cos^4\alpha=\frac{1}{8}(\cos 4\alpha+4\cos 2\alpha+3)$

$\sin A\cos B=\frac{1}{2}[\sin(A-B)+\sin(A+B)]$	$\cos A\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$	$\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$

IV) Exponential and Logarithmic Functions





- Domain of e^x is $\mathbb{R}$
- Range of e^x is $]0,\infty[$
- Domain of $\ln x$ is $]0,\infty[$
- Range of $\ln x$ is $\mathbb{R}$
- $e^{\infty}=\infty$ • $e^{-\infty}=0$
- $\ln\infty=\infty$ • $\ln 0^+=-\infty$
- $e^0=1$ • $\ln 1=0$

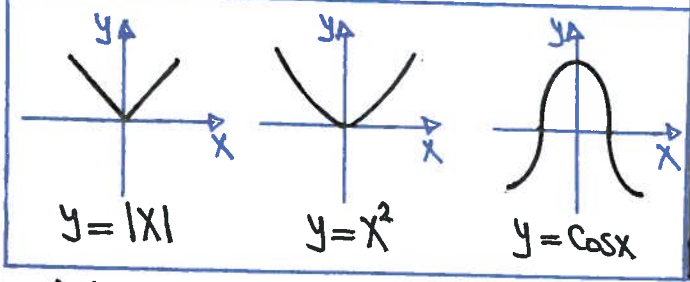
1) Properties of Exponential and Logarithmic Functions

$\log_e x=\ln x$	$\log_a a^x = x$	$\log_b a=\dfrac{\ln a}{\ln b}$	$\ln e^x = x$
$a^{\log_a x}=x$	$e^{\ln x}=x$	$\ln(x\cdot y)=\ln x+\ln y$	$\ln\left(\frac{x}{y}\right)=\ln x-\ln y$
$\ln\left(\frac{xy}{z}\right)=\ln x+\ln y-\ln z$	$\ln x^y = y\ln x$	$\ln(x+y)\neq \ln x+\ln y$	$(\ln x)^y\neq y\ln x$

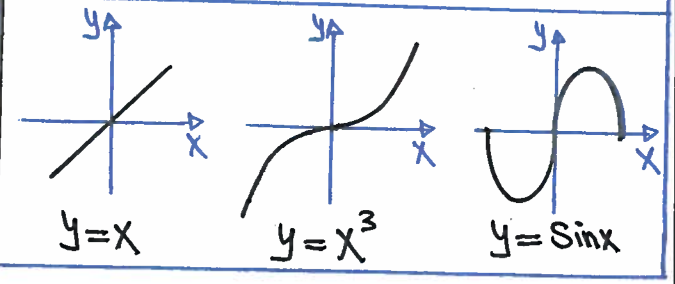
Domain of:
- $\ln u$ is $u>0$
- $\sqrt{u}$ is $u\geq 0$
- $\dfrac{1}{\sqrt{u}}$ is $u>0$
- e^u is the domain of u

V) Even and Odd Functions

Even $f(x)=f(-x)$



Odd $f(-x)=-f(x)$



VI) Absolute Function

$$|x-a|=\begin{cases}(x-a) & \text{if } x\geq a\\ -(x-a) & \text{if } x<a\end{cases}$$

- Slope of the line joining two points (x_1,y_1) and (x_2,y_2) is: $\dfrac{y_2-y_1}{x_2-x_1}=m$
- Equation of straight line with slope m and c-intercept is: $y=mx+c$
- Equation of straight line joining two points is: $y-y_1=m(x-x_1)$

Limits Techniques

$\frac{0}{0}$

(To Cancel Zero Factor)

1) Factorize up and down.

OR

2) Multiply by the conjugate.

OR

3) $\lim\limits_{x \to a} \dfrac{x^m - a^m}{x^n - a^n} = \dfrac{m}{n}(a)^{m-n}$

(From trigonometric fns.)

1) $\lim\limits_{x \to 0} \dfrac{\sin ax}{bx} = \dfrac{a}{b}$

2) $\lim\limits_{x \to 0} \dfrac{\tan ax}{bx} = \dfrac{a}{b}$

3) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

4) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2} = \dfrac{1}{2}$

$\frac{\infty}{\infty}$

(∞ term)

- Divide up and down by the term in the denominator that produces ∞.

1) $\lim\limits_{x \to \infty} \dfrac{1}{x^n} = 0$

2) $\lim\limits_{x \to \infty} x^n = \infty$

3) $\lim\limits_{x \to \infty} c = c$

Reduction Formulas

$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$	$\tan(-x) = -\tan x$	$\cot(-x) = -\cot x$
$\sin(\frac{\pi}{2}-x) = \cos x$	$\cos(\frac{\pi}{2}-x) = \sin x$	$\tan(\frac{\pi}{2}-x) = \cot x$	$\cot(\frac{\pi}{2}-x) = \tan x$
$\sin(\frac{\pi}{2}+x) = \cos x$	$\cos(\frac{\pi}{2}+x) = -\sin x$	$\tan(\frac{\pi}{2}+x) = -\cot x$	$\cot(\frac{\pi}{2}+x) = -\tan x$
$\sin(\pi-x) = \sin x$	$\cos(\pi-x) = -\cos x$	$\tan(\pi-x) = -\tan x$	$\cot(\pi-x) = -\cot x$
$\sin(\pi+x) = -\sin x$	$\cos(\pi+x) = -\cos x$	$\tan(\pi+x) = \tan x$	$\cot(\pi+x) = \cot x$
$\sin(\frac{3\pi}{2}-x) = -\cos x$	$\cos(\frac{3\pi}{2}-x) = -\sin x$	$\tan(\frac{3\pi}{2}-x) = \cot x$	$\cot(\frac{3\pi}{2}-x) = -\tan x$
$\sin(\frac{3\pi}{2}+x) = -\cos x$	$\cos(\frac{3\pi}{2}+x) = \sin x$	$\tan(\frac{3\pi}{2}+x) = -\cot x$	$\cot(\frac{3\pi}{2}+x) = -\tan x$
$\sin(2\pi-x) = -\sin x$	$\cos(2\pi-x) = \cos x$	$\tan(2\pi-x) = -\tan x$	$\cot(2\pi-x) = -\cot x$

Solved Examples

Example 1 Evaluate the following limits:

① $\lim_{x\to 1} \frac{x^2+x-2}{x-1}$

② $\lim_{x\to -4} \frac{2x+8}{x^2+x-12}$

③ $\lim_{x\to 1} \frac{x^3-1}{x^2-1}$

④ $\lim_{x\to 2} \frac{x^3-8}{x^2-4}$

⑤ $\lim_{u\to 1} \frac{u^4-1}{u^3+5u^2-6u}$

⑥ $\lim_{v\to 2} \frac{v^2+2v-8}{v^4-16}$

⑦ $\lim_{x\to -4} \frac{x^2-x-20}{x^3+64}$

⑧ $\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2}$

⑨ $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x}$

⑩ $\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x^2-49}$

Solution

① $\lim_{x\to 1} \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}} = \lim_{x\to 1} (x+2) = \boxed{3}$

② $\lim_{x\to -4} \frac{2\cancel{(x+4)}}{\cancel{(x+4)}(x-3)} = \lim_{x\to -4} \frac{2}{(x-3)} = \boxed{\frac{-2}{7}}$

③ $\lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)} = \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \boxed{\frac{3}{2}}$

④ $\lim_{x\to 2} \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2} \frac{x^2+2x+4}{x+2} = \boxed{3}$

⑤ $\lim_{u\to 1} \frac{(u^2+1)\cancel{(u-1)}(u+1)}{u(u+6)\cancel{(u-1)}}$

$= \lim_{u\to 1} \frac{(u^2+1)(u+1)}{u(u+6)} = \boxed{\frac{4}{7}}$

⑥ $\lim_{v\to 2} \frac{\cancel{(v-2)}(v+4)}{\cancel{(v-2)}(v+2)(v^2+4)}$

$= \lim_{v\to 2} \frac{(v+4)}{(v+2)(v^2+4)} = \boxed{\frac{3}{16}}$

⑦ $\lim_{x\to -4} \frac{\cancel{(x+4)}(x-5)}{\cancel{(x+4)}(x^2-4x+16)}$

$= \lim_{x\to -4} \frac{(x-5)}{(x^2-4x+16)} = \boxed{\frac{-3}{16}}$

⑧ $\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^2(x-3)} \cdot \frac{\sqrt{x+6}+x}{\sqrt{x+6}+x}$

$= \lim_{x\to 3} \frac{x+6-x^2}{x^2(x-3)(\sqrt{x+6}+x)}$

$= \lim_{x\to 3} \frac{-\cancel{(x-3)}(x+2)}{x^2\cancel{(x-3)}(\sqrt{x+6}+x)} = \boxed{\frac{-5}{54}}$

⑨ $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x} \cdot \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}$

$= \lim_{x\to 0} \frac{1-(1-x^2)}{x(1+\sqrt{1-x^2})} = \lim_{x\to 0} \frac{x^2}{x(1+\sqrt{1-x^2})}$

$= \lim_{x\to 0} \frac{x}{1+\sqrt{1-x^2}} = \boxed{0}$

⑩ $\lim_{x\to 7} \frac{\sqrt{x+2}-3}{(x-7)(x+7)} \cdot \frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}$

$= \lim_{x\to 7} \frac{\cancel{x+2-9}}{\cancel{(x-7)}(x+7)(\sqrt{x+2}+3)}$

$= \lim_{x\to 7} \frac{1}{(x+7)(\sqrt{x+2}+3)} = \boxed{\frac{1}{84}}$

Example (2) Evaluate the following limits :

① $\lim\limits_{x\to 2} \dfrac{\tan(x-2)}{x^2-4}$

② $\lim\limits_{x\to 1} \dfrac{\sin(x-1)}{x^2+x-2}$

③ $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sqrt{x+9}-3}$

④ $\lim\limits_{x\to 1} \dfrac{\tan(x^3-1)}{x-1}$

⑤ $\lim\limits_{x\to 1} (x-1)\tan\left(\dfrac{\pi x}{2}\right)$

⑥ $\lim\limits_{x\to \frac{1}{3}} \dfrac{\sqrt{3}-\tan \pi x}{3x-1}$

Solution

① $\lim\limits_{x-2\to 0} \dfrac{\tan(x-2)}{(x-2)(x+2)} = \lim\limits_{x-2\to 0} \dfrac{\tan(x-2)}{(x-2)} \cdot \dfrac{1}{(x+2)}$

$= (1) \cdot \dfrac{1}{4} = \boxed{\dfrac{1}{4}}$

② $\lim\limits_{x-1\to 0} \dfrac{\sin(x-1)}{(x-1)(x+2)} = \lim\limits_{x-1\to 0} \dfrac{\sin(x-1)}{(x-1)} \cdot \dfrac{1}{(x+2)}$

$= (1) \cdot \dfrac{1}{3} = \boxed{\dfrac{1}{3}}$

③ $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sqrt{x+9}-3} \cdot \dfrac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$

$= \lim\limits_{x\to 0} \dfrac{\sin 3x}{x} \cdot (\sqrt{x+9}+3)$

$= (3) \cdot (\sqrt{0+9}+3) = \boxed{18}$

④ $\lim\limits_{x-1\to 0} \dfrac{\tan(x^3-1)}{(x-1)(x^2+x+1)} \cdot (x^2+x+1)$

$= \lim\limits_{x\to 1} \dfrac{\tan(x^3-1)}{(x^3-1)} \cdot (x^2+x+1)$

$= (1) \cdot (3) = \boxed{3}$

⑤ let $y = x-1 \Rightarrow x = y+1$

$\therefore \lim\limits_{y\to 0} y \tan\left(\dfrac{\pi}{2}y + \dfrac{\pi}{2}\right)$

$= \lim\limits_{y\to 0} -y \cot\left(\dfrac{\pi}{2}y\right)$

$= -\lim\limits_{y\to 0} \dfrac{y}{\tan(\frac{\pi}{2}y)}$

$= -\lim\limits_{y\to 0} \dfrac{(\frac{\pi}{2}y)}{\tan(\frac{\pi}{2}y)} \cdot \dfrac{2}{\pi}$

$= -(1)\left(\dfrac{2}{\pi}\right) = \boxed{\dfrac{-2}{\pi}}$

⑥ let $y = x-\dfrac{1}{3} \Rightarrow x = y+\dfrac{1}{3}$

$\therefore \lim\limits_{y\to 0} \dfrac{\sqrt{3}-\tan(\pi y+\frac{\pi}{3})}{3y}$

$= \lim\limits_{y\to 0} \dfrac{\sqrt{3}-\left(\dfrac{\tan \pi y+\sqrt{3}}{1-\sqrt{3}\tan \pi y}\right)}{3y} \cdot \dfrac{1-\sqrt{3}\tan \pi y}{1-\sqrt{3}\tan \pi y}$

$= \lim\limits_{y\to 0} \dfrac{(\cancel{\sqrt{3}}-3\tan \pi y)-(\tan \pi y+\cancel{\sqrt{3}})}{3y(1-\sqrt{3}\tan \pi y)}$

$= \lim\limits_{y\to 0} \dfrac{-4\tan \pi y}{3y} \cdot \dfrac{1}{1-\sqrt{3}\tan \pi y}$

$= \left(-4\,\dfrac{\pi}{3}\right) \cdot \left(\dfrac{1}{1-0}\right) = \boxed{\dfrac{-4\pi}{3}}$

Example 3 Evaluate the following limits:

① $\lim\limits_{x\to\infty} \dfrac{5x^2-3x+6}{2x-7x^2}$

② $\lim\limits_{x\to\infty} \dfrac{x^5-2x^2}{x^4+3x^3-1}$

③ $\lim\limits_{x\to\infty} \dfrac{5x-6}{\sqrt{9x^2+7}}$

④ $\lim\limits_{x\to\infty} \dfrac{2x^3+\sqrt{x^4+5}}{3x^4+2x^3+10}$

⑤ $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x}-x\right)$

⑥ $\lim\limits_{x\to\infty} \left(\sqrt{x^2-x+10}-\sqrt{x^2+x+1}\right)$

Solution

① $\lim\limits_{x\to\infty} \dfrac{5-\frac{3}{x}+\frac{6}{x^2}}{\frac{2}{x}-7} = \dfrac{5-0+0}{0-7} = \boxed{\dfrac{-5}{7}}$

② $\lim\limits_{x\to\infty} \dfrac{x-\frac{2}{x^2}}{1+\frac{3}{x}-\frac{1}{x^4}} = \dfrac{\infty-0}{1+0-0} = \boxed{\infty}$

③ $\lim\limits_{x\to\infty} \dfrac{5-\frac{6}{x}}{\sqrt{9+\frac{7}{x^2}}} = \dfrac{5-0}{\sqrt{9+0}} = \boxed{\dfrac{5}{3}}$

④ $\lim\limits_{x\to\infty} \dfrac{\frac{2}{x}+\sqrt{\frac{1}{x^4}+\frac{5}{x^8}}}{3+\frac{2}{x}+\frac{10}{x^4}} = \dfrac{0}{3+0+0} = \boxed{0}$

⑤ $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x}-x\right)\cdot\dfrac{\left(\sqrt{x^2+x}+x\right)}{\left(\sqrt{x^2+x}+x\right)}$

$= \lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+x}+x} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{x}}+1}$

$= \dfrac{1}{\sqrt{1+0}+1} = \boxed{\dfrac{1}{2}}$

⑥ $\lim\limits_{x\to\infty} \left(\sqrt{x^2-x+10}-\sqrt{x^2+x+1}\right)\cdot\dfrac{\sqrt{x^2-x+10}+\sqrt{x^2+x+1}}{\sqrt{x^2-x+10}+\sqrt{x^2+x+1}}$

$= \lim\limits_{x\to\infty} \dfrac{-2x}{\sqrt{x^2-x+10}+\sqrt{x^2+x+1}}$

$= \lim\limits_{x\to\infty} \dfrac{-2}{\sqrt{1-\frac{1}{x}+\frac{10}{x^2}}+\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} = \dfrac{-2}{\sqrt{1}+\sqrt{1}} = \boxed{-1}$

$\frac{dy}{dx} = y'$ (Rules of Differentiation) $\frac{d^2y}{dx^2} = y''$

$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x) = 1$
$\frac{d}{dx}(ax+b) = a$	$\frac{d}{dx}(ax^2+bx+c) = 2ax+b$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(x^{-n}) = \frac{-n}{x^{n+1}}$
$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$	$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
$\frac{d}{dx}(\sin u) = u' \cos u$	$\frac{d}{dx}(\cos u) = -u' \sin u$
$\frac{d}{dx}(\tan u) = u' \sec^2 u$	$\frac{d}{dx}(\cot u) = -u' \csc^2 u$
$\frac{d}{dx}(\sec u) = u' \sec u \cdot \tan u$	$\frac{d}{dx}(\csc u) = -u' \csc u \cdot \cot u$
$\frac{d}{dx}(e^u) = u' e^u$	$\frac{d}{dx}(a^u) = u' a^u \ln a$
$\frac{d}{dx}(\ln u) = \frac{u'}{u}$	$\frac{d}{dx}(\log_a u) = \frac{u'}{u \ln a}$
$\frac{d}{dx}(u)^n = n u^{n-1} \cdot u'$	$\frac{d}{dx}(\sqrt{u}) = \frac{u'}{2\sqrt{u}}$
$\frac{d}{dx}(u \cdot v) = u'v + uv'$	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$

Parametric Differentiation

$x = f(t)$ & $y = g(t)$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}$	$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(y')}{\frac{dx}{dt}}$	$\frac{d^3y}{dx^3} = \frac{\frac{d}{dt}(y'')}{\frac{dx}{dt}}$; if $\frac{dx}{dt} \neq 0$

Implicit Differentiation

$\frac{d}{dx}(x^2+y^2) = 2x+2yy'$	$\frac{d}{dx}(x^2 y) = 2xy + x^2 y'$	$\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{y - xy'}{y^2}$	$\frac{d}{dx}(\ln y) = \frac{y'}{y}$

Given $f(x,y)$ & Point (x_0, y_0)

Tangent Equation: $y - y_0 = y'_p(x - x_0)$	Normal Equation: $y - y_0 = \frac{-1}{y'_p}(x - x_0)$
Horizontal tangent means: $y' = 0$ OR $\frac{dy}{dt} = 0$	Vertical tangent means: $y' = \frac{1}{0}$ OR $\frac{dx}{dt} = 0$
Tangents are Parallel if: $y'_1 = y'_2$	Tangents are normal if: $y'_1 = \frac{-1}{y'_2}$

Example (1) Find the first derivative of the following:

1. $f(x) = \sqrt{x}\,(x^3 + 2x)$
2. $f(t) = \dfrac{t+2}{3t-1}$
3. $f(x) = x^4 + \dfrac{1}{x^4}$
4. $y = x^{1/4}(x+2)$
5. $f(x) = 3x^2 \sec(x)$
6. $f(x) = \dfrac{1+\sec x}{\tan x + \sin x}$
7. $P(r) = \dfrac{1+\sec r}{1-\sec r}$
8. $y = x^2 \tan x + x^3 \cot x$
9. $f(x) = \sqrt{x^3+2}$
10. $f(x) = \sin(x^2) + \sin^2 x$
11. $f(t) = \left(\dfrac{t+2}{3t-1}\right)^{10}$
12. $f(x) = \sin(\cos(\tan x))$
13. $f(x) = \sqrt{\sec x^4}$
14. $f(x) = \dfrac{1}{x^2+4}$
15. $f(x) = x^2 \tan^4 x^2$
16. $f(z) = \sqrt{\sin(\sqrt{z})}$
17. $f(x) = 3x^2 \sec^3(x)$
18. $f(x) = \sec^3(x) \cdot \tan^3(x)$
19. $g(x) = \sec^5(2x)$
20. $h(x) = \sqrt[3]{x + \cot 3x}$
21. $k(t) = \cos^4\left(1 - \dfrac{1}{t}\right)$
22. $g(u) = \dfrac{1}{\sin u + \cos u}$
23. $g(x) = \sec^3(2x) \cdot \tan^3(2x)$
24. $y = (2x-1)^6 \csc^5(3x)$
25. $f(x) = \ln(\sec x)$
26. $y = x^{10} e^{\ln x}$
27. $y = \ln\sqrt{\dfrac{2x+1}{x-3}}$
28. $y = \ln(\cos e^{2x})$
29. $f(x) = 7^{\tan x} + (\tan x)^7$
30. $y = \log_3(x^2+1) + \log(\sin x)$
31. $y = \ln(x^3+4)^3 + \ln\left(\dfrac{1}{x^3+1}\right)$
32. $y = \ln(\sec\sqrt{x}) + \ln(\ln x) + \ln^2(x)$
33. $y = \sin(5^{x^2}) + \pi^{x^3} + 2^{\sqrt{1-x^2}}$
34. $y = x e^{\tan x} + e^{\log x}$
35. $y = \sec(2^{\ln x}) + e^{x^2} \ln x + e^3$
36. $y = \dfrac{\ln x}{2^x} + \dfrac{\log_2 x}{2} + \pi^2$

Solutions

① $f'(x) = \frac{1}{2\sqrt{x}}(x^3+2x) + \sqrt{x}\,(3x^2+2)$

② $f'(t) = \frac{(3t-1)-3(t+2)}{(3t-1)^2} = \frac{-7}{(3t-1)^2}$

③ $f'(x) = 4x^3 - \frac{4}{x^5}$

$f''(x) = 12x^2 + \frac{20}{x^6}$

④ $y' = \frac{1}{4}x^{-3/4}(x+2) + x^{1/4}$

$= \frac{1}{4}\cdot\frac{x+2}{x^{3/4}} + x^{1/4} = \frac{5x+2}{4x^{3/4}}$

⑤ $f'(x) = 6x\sec x + 3x^2\sec x\cdot\tan x$

$= 3x\sec x\,(2 + x\tan x)$

⑥ $f(x) = \frac{1+\sec x}{\sin x\,(1+\sec x)} = \csc x$

$\therefore\ f'(x) = -\csc x\cdot\cot x$

⑦ $\frac{dP}{dr} = \frac{(\sec r\tan r)(1-\sec r) + \sec r\tan r\,(1+\sec r)}{(1-\sec r)^2}$

$= \frac{2\sec r\cdot\tan r}{(1-\sec r)^2}$

⑧ $y' = 2x\tan x + x^2\sec^2 x + 3x^2\cot x - x^3\csc^2 x$

⑨ $f'(x) = \frac{1}{2\sqrt{x^3+2}}(3x^2) = \frac{3x^2}{2\sqrt{x^3+2}}$

⑩ $f'(x) = 2x\cos(x^2) + 2\sin x\cdot\cos x$

⑪ $f'(t) = 10\left(\frac{t+2}{3t-1}\right)^9\cdot\frac{-7}{(3t-1)^2}$

⑫ $f'(x) = \cos(\cos(\tan x))\cdot -\sin(\tan x)\cdot\sec^2 x$

⑬ $f'(x) = \frac{4x^3\sec x^4\cdot\tan x^4}{2\sqrt{\sec x^4}} = \frac{2x^3\sec x^4\cdot\tan x^4}{\sqrt{\sec x^4}}$

⑭ $f'(x) = \frac{-1}{(x^2+4)^2}\cdot(2x) = \frac{-2x}{(x^2+4)^2}$

⑮ $f'(x) = 2x\tan^4 x^2 + 8x^3\tan^3 x^2\cdot\sec^2 x^2$

⑯ $f'(z) = \frac{1}{2\sqrt{\sin\sqrt{z}}}\cdot\cos\sqrt{z}\cdot\frac{1}{2\sqrt{z}} = \frac{\cos\sqrt{z}}{4\sqrt{z}\,\sqrt{\sin\sqrt{z}}}$

⑰ $f'(x) = 6x\sec^3 x + 9x^2\sec^2 x\cdot\sec x\tan x$

$= 3x\sec^3 x\,(2 + 3x\tan x)$

⑱ $f'(x) = 3\sec^2 x\cdot\sec x\tan x\cdot\tan^3 x$

$+ 3\tan^2 x\cdot\sec^2 x\cdot\sec^3 x$

$= 3\sec^3 x\cdot\tan^2 x\,(\sec^2 x + \tan^2 x)$

⑲ $g'(x) = 10\sec^4(2x)\cdot\sec(2x)\cdot\tan(2x)$

$= 10\sec^5(2x)\cdot\tan(2x)$

⑳ $h'(x) = \frac{1}{3}(x+\cot 3x)^{-2/3}\cdot(1 - 3\csc^2 3x)$

$= \frac{1-3\csc^2 3x}{3(x+\cot 3x)^{2/3}}$

㉑ $k'(t) = 4\cos^3\left(1-\frac{1}{t}\right)\cdot -\sin\left(1-\frac{1}{t}\right)\cdot\frac{1}{t^2}$

$= \frac{-4}{t^2}\cos^3\left(1-\frac{1}{t}\right)\sin\left(1-\frac{1}{t}\right)$

㉒ $g'(u) = \frac{-1}{(\sin u+\cos u)^2}\cdot(\cos u - \sin u)$

$= \frac{\sin u - \cos u}{(\sin u + \cos u)^2}$

(23) $g'(x) = 6\sec^2(2x)\cdot\sec(2x)\cdot\tan(2x)\cdot\tan^3(2x) + 6\tan^2(2x)\cdot\sec^2(2x)\cdot\sec^3(2x)$

$= 6\sec^3(2x)\cdot\tan^2(2x)\left[\sec^2(2x)+\tan^2(2x)\right]$

(24) $y' = 12(2x-1)^5\cdot\csc^5(3x) - 15\csc^4(3x)\cdot\csc(3x)\cdot\cot(3x)\cdot(2x-1)^6$

$= 3(2x-1)^5\csc^5(3x)\left[4-5(2x-1)\cot(3x)\right]$

(25) $f'(x) = \frac{1}{\sec x}\cdot(\sec x\tan x) = \tan x$

(26) $y' = e^{10\ln x} + \cancel{x}\cdot e^{10\ln x}\cdot\frac{10}{\cancel{x}}$

$= 11\, e^{\ln x^{10}} = 11x^{10}$

(27) $y = \frac{1}{2}\left[\ln(2x+1) - \ln(x-3)\right]$

$\therefore\ y' = \frac{1}{2}\left[\frac{2}{2x+1} - \frac{1}{x-3}\right]$

(28) $y' = \frac{1}{\cos(e^{2x})}\cdot -\sin(e^{2x})\cdot 2e^{2x}$

$= -2e^{2x}\tan(e^{2x})$

(29) $f'(x) = 7^{\tan x}\cdot\sec^2 x\cdot\ln 7 + 7(\tan x)^6\cdot\sec^2 x$

(30) $y' = \frac{2x}{x^2+1}\cdot\frac{1}{\ln 3} + \frac{\cos x}{\sin x}\cdot\frac{1}{\ln 10}$

(31) $y = 3\ln(x^3+4) - \ln(x^2+1)$

$\therefore\ y' = \frac{9x^2}{x^3+4} - \frac{2x}{x^2+1}$

(32) $y' = \frac{1}{\sec\sqrt{x}}\cdot\sec\sqrt{x}\cdot\tan\sqrt{x}\cdot\frac{1}{2\sqrt{x}} + \frac{1}{x\ln x} + \frac{2\ln x}{x}$

$= \frac{\tan\sqrt{x}}{2\sqrt{x}} + \frac{1}{x\ln x} + \frac{2\ln x}{x}$

(33) $y' = \cos(5^{x^2})\cdot 2x\cdot 5^{x^2}\cdot\ln 5 + 3x^2\cdot\pi^{x^3}\cdot\ln\pi + \frac{-2x}{2\sqrt{1-x^2}}\cdot 2^{\sqrt{1-x^2}}\cdot\ln 2$

(34) $y' = e^{\tan x} + x\, e^{\tan x}\cdot\sec^2 x + e^{\log x}\cdot\frac{1}{x}\cdot\frac{1}{\ln 10}$

(35) $y' = \sec(2^{\ln x})\cdot\tan(2^{\ln x})\cdot 2^{\ln x}\cdot\ln 2\cdot\frac{1}{x} + 2x\, e^{x^2}\ln x + \frac{e^{x^2}}{x}$

(36) $y = 2^{-x}\cdot\ln x + \frac{1}{2}\log_2 x + \pi^2$

$\therefore\ y' = -2^{-x}\ln 2\cdot\ln x + \frac{2^{-x}}{x} + \frac{1}{2}\cdot\frac{1}{x}\cdot\frac{1}{\ln 2}$

Example 2 Find the first derivative of the following:

1. $y = \log_{e^x}(x^2+1)$
2. $y = \log_x(x^2+e^x)$
3. $y = \log_{10}(x^2 \cdot 2^{x^2+1})$
4. $y = \log_3(\sin x) + \ln(\tan x)$
5. $y = e^{\ln(\sin x)+\ln(\cos x)}$
6. $y = \log_{x+1}(x)$
7. $x^2+y^2 = \sin(xy)$
8. $xy = \tan y$
9. $2x - \sqrt{xy} + y^3 = 16$
10. $\tan^3(xy^2+y) = x+2$
11. $x^4 + e^{xy} - y^2 = 20$
12. $x^2+y^2 = \tan(xy)$
13. $y = \sec(2x)\,\ln(\cos x + \sqrt{x^3-1})$
14. $y^2 + 5^{xy} = \log_3(x^2+2)$
15. $y = 2^{\sin 3x} + \sqrt{\sec x^4}$
16. $x^2 + \tan(xy) = e^{\ln(x+y)}$
17. $y = x^2 + \ln(\tan x)\cos^2(x^2)$
18. $y = \ln\left[\frac{(2x+1)^5}{\sqrt{x^2+1}}\right]$
19. $e^{x^2 y} = \tan(x+y)$
20. $e^{x+y} - \ln\left|\frac{x}{y}\right| = 2^x$
21. $y = \sqrt[3]{\frac{(x+1)(x+2)}{(x^2+1)(x^3+2)}}$
22. $y = \frac{(5t+8)^3\,\sqrt[3]{1-9\cos 4t}}{\sqrt{t^2+10t}}$
23. $y = \frac{x^3 \cdot \sin^3 x \cdot e^x}{\sqrt{1+x^2} \cdot \tan^2 x}$
24. $y = \sqrt[3]{\frac{\tan^2(\sec x) \cdot 3^{\sin x}}{\sin^{3/4}(x^2+1) \cdot (\ln x)^{5/2}}}$
25. $2^y = \frac{(x-1)^2(2y+3)}{xy^2}$
26. $y^{\sin x} = x^y$
27. $y^x = 1 - x^{\sin y}$
28. $x^y + y^x = 1$
29. $y = (x+1)^{\tan x}$
30. $y = (\cos x)^{x^3}$
31. $\tan y = (x+y)^{e^{x^2}}$
32. $y = (\sec x + e^{\sqrt{x}})^{\tan x}$
33. $x^y + 2^{\sin x} = 1$
34. $x^y + 1 = y^x$
35. $x^y + y^x = \cos x^2$
36. $x^{\sin y} + y^{\sin x} = 1$

37. $y = (x + e^{x})^{1/x}$

38. $3^{y} \cdot 2^{x} = y^{2}$

39. $y = \log_{5}\left(x^{2}\sqrt[3]{\tan x}\right)$

40. $y = \sqrt[5]{\dfrac{(x^{2}-1)^{3} \cdot x}{\csc^{4} x}}$

41. $y = (x^{2}+1)^{\ln x}$

42. $y = \log_{x^{2}} \sqrt{\cot x}$

43. $(\sin x)^{y} + y^{\sqrt{x}} = 1$

44. $y = \log_{e^{\sqrt{x}}}\left(x^{2}\sqrt{\tan x}\right)$

45. $x^{2} \cdot 3^{y} = \sqrt[x]{y}$

46. $y = (3\sqrt{x} - 1)^{\sin^{2} x}$

47. $y = (\sin 2x)^{\tan^{2} 2x}$

48. $x^{y} - y^{\sin x} = 0$

49. $y = x^{\sin x} + (\cos x)^{x}$

50. $y = \sin(x^{x}) + (\tan x)^{x}$

51. $y = x^{x^{x}}$

52. $x^{3} + y^{3} = xy + e^{x^{2}}$

Solutions

① $y=\dfrac{\ln(x^2+1)}{\ln(e^x)}=\dfrac{\ln(x^2+1)}{x}$

$y'=\dfrac{\frac{2x}{x^2+1}\cdot x-\ln(x^2+1)}{x^2}=\dfrac{\frac{2x^2}{x^2+1}-\ln(x^2+1)}{x^2}$

② $y=\dfrac{\ln(x^2+e^x)}{\ln x}$

$y'=\dfrac{\frac{2x+e^x}{x^2+e^x}\cdot\ln x-\frac{\ln(x^2+e^x)}{x}}{(\ln x)^2}$

③ $y=\log_{10}x^2+\log_{10}2^{x^2+1}=2\log_{10}x+(x^2+1)\cdot\log_{10}2$

$y'=\dfrac{2}{x\ln 10}+2x\cdot\log_{10}2$

④ $y'=\dfrac{\cos x}{\sin x\cdot\ln 3}+\dfrac{\sec^2 x}{\tan x}$

$=\dfrac{1}{\ln 3}\cot x+\dfrac{\sec^2 x}{\tan x}$

⑤ $y=e^{\ln(\sin x)}\cdot e^{\ln(\cos x)}=\sin x\cos x=\frac{1}{2}\sin 2x$

$y'=\cos 2x$

⑥ $y=\dfrac{\ln x}{\ln(x+1)}$

$y'=\dfrac{\frac{\ln(x+1)}{x}-\frac{\ln x}{x+1}}{\ln^2(x+1)}$

⑦ $2x+2yy'=(xy'+y)\cos(xy)$

$2x+2yy'=xy'\cos(xy)+y\cos(xy)$

$y'(2y-x\cos(xy))=y\cos(xy)-2x$

$y'=\dfrac{y\cos(xy)-2x}{2y-x\cos(xy)}$

⑧ $xy'+y=y'\sec^2 y$

$y'(x-\sec^2 y)=-y$

$y'=\dfrac{-y}{x-\sec^2 y}$

⑨ $2-\left(\dfrac{xy'+y}{2\sqrt{xy}}\right)+3y^2y'=0$

$2-\dfrac{xy'}{2\sqrt{xy}}-\dfrac{y}{2\sqrt{xy}}+3y^2y'=0$ $\quad(*2\sqrt{xy})$

$4\sqrt{xy}-xy'-y+6\sqrt{xy}\,y^2y'=0$

$y'=\dfrac{4\sqrt{xy}-y}{x-6\sqrt{xy}\,y^2}$

⑩ $3\tan^2(xy^2+y)\cdot\sec^2(xy^2+y)\cdot(y^2+2xyy'+y')=1$

$y'(2xy+1)\left[3\tan^2(xy^2+y)\cdot\sec^2(xy^2+y)\right]$

$=1-3y^2\tan^2(xy^2+y)\cdot\sec^2(xy^2+y)$

$y'=\dfrac{1-3y^2\tan^2(xy^2+y)\sec^2(xy^2+y)}{3(2xy+1)\tan^2(xy^2+y)\sec^2(xy^2+y)}$

⑪ $4x^3+(xy'+y)e^{xy}-2yy'=0$

$xy'e^{xy}-2yy'=-4x^3-ye^{xy}$

$y'=\dfrac{-4x^3-ye^{xy}}{xe^{xy}-2y}$

⑫ $2x+2yy'=(xy'+y)\sec^2(xy)$

$2yy'-xy'\sec^2(xy)=y\sec^2(xy)-2x$

$y'=\dfrac{y\sec^2(xy)-2x}{2y-x\sec^2(xy)}$

(13) $y' = (2\sec 2x \cdot \tan 2x) \cdot \ln(\cos x + \sqrt{x^3-1})$

$+ \sec 2x \cdot \dfrac{1}{\cos x + \sqrt{x^3-1}} \cdot \left(-\sin x + \dfrac{3x^2}{2\sqrt{x^3-1}}\right)$

(14) $2yy' + (xy'+y) \cdot 5^{xy} \cdot \ln 5 = \dfrac{2x}{(x^2+2) \cdot \ln 3}$

$2yy' + xy' \cdot 5^{xy} \cdot \ln 5 = \dfrac{2x}{(x^2+2)\ln 3} - y \cdot 5^{xy} \cdot \ln 5$

$y' = \dfrac{\dfrac{2x}{(x^2+2)\ln 3} - y \cdot 5^{xy} \cdot \ln 5}{2y + x \cdot 5^{xy} \cdot \ln 5}$

(15) $y' = 2^{\sin 3x} \cdot 3\cos 3x \cdot \ln 2 + \dfrac{4x^3 \sec x^4 \tan x^4}{2\sqrt{\sec x^4}}$

(16) $\because x^2 + \tan(xy) = x + y$

$\therefore 2x + (xy'+y)\sec^2(xy) = 1 + y'$

$xy'\sec^2(xy) - y' = 1 - 2x - y\sec^2(xy)$

$y' = \dfrac{1 - 2x - y\sec^2(xy)}{x\sec^2(xy) - 1}$

(17) $y' = 2x + \dfrac{\sec^2 x}{\tan x} \cdot \cos^2 x^2 - 2x\cos(x^2)\sin(x^2) \cdot \ln(\tan x)$

(18) $y = 5\ln(2x+1) - \frac{1}{2}\ln(x^2+1)$

$y' = \dfrac{10}{2x+1} - \dfrac{x}{x^2+1}$

(19) $(x^2y' + 2xy) \cdot e^{x^2y} = (1+y') \cdot \sec^2(x+y)$

$y'(x^2 e^{x^2y} - \sec^2(x+y)) = \sec^2(x+y) - 2xy\,e^{x^2y}$

$y' = \dfrac{\sec^2(x+y) - 2xy\,e^{x^2y}}{x^2 e^{x^2y} - \sec^2(x+y)}$

(20) $\because e^{x+y} - \ln(x) + \ln(y) = 2^x$

$\therefore (1+y')\,e^{x+y} - \dfrac{1}{x} + \dfrac{y'}{y} = 2^x \ln 2$

$y'\left(e^{x+y} + \dfrac{1}{y}\right) = 2^x \ln 2 - e^{x+y}$

$y' = \dfrac{2^x \ln 2 - e^{x+y}}{e^{x+y} + \dfrac{1}{y}}$

(21) $\ln y = \frac{1}{3}\left[\ln(x+1) + \ln(x+2) - \ln(x^2+1) - \ln(x^2+2)\right]$

$y' = \frac{1}{3}\sqrt[3]{\dfrac{(x+1)(x+2)}{(x^2+1)(x^2+2)}}\left[\dfrac{1}{x+1} + \dfrac{1}{x+2} - \dfrac{2x}{x^2+1} - \dfrac{2x}{x^2+2}\right]$

(22) $\ln y = 3\ln(5t+8) + \frac{1}{3}\ln(1-9\cos 4t) - \frac{1}{2}\ln(t^2+10t)$

$y' = \dfrac{(5t+8)^3\sqrt[3]{1-9\cos 4t}}{\sqrt{t^2+10t}}\left[\dfrac{15}{5t+8} + \dfrac{12\sin 4t}{1-9\cos 4t} - \dfrac{t+5}{t^2+10t}\right]$

(23) $\ln y = 3\ln x + 3\ln(\sin x) + x - \frac{1}{2}\ln(x^2+1) - 2\ln(\tan x)$

$y' = \dfrac{x^3\sin^3 x \cdot e^x}{\sqrt{1+x^2} \cdot \tan^2 x}\left[\dfrac{3}{x} + 3\cot x + 1 - \dfrac{x}{1+x^2} - \dfrac{2\sec^2 x}{\tan x}\right]$

(24) $\ln y = \frac{1}{3}\left[2\ln(\tan \sec x) + \sin x \cdot \ln 3 - \frac{3}{4}\ln \sin(x^2+1) - \frac{5}{2}\ln x\right]$

$y' = \frac{1}{3}\sqrt[3]{\dfrac{\tan^2(\sec x) \cdot 3^{\sin x}}{\sin^{3/4}(x^2+1) \cdot \ln^{5/2} x}}\left[\dfrac{2\sec^2(\sec x) \cdot \sec x \tan x}{\tan(\sec x)} + \ln 3 \cdot \cos x - \dfrac{3}{2}x\cot(x^2+1) - \dfrac{5}{2} \cdot \dfrac{1}{x}\right]$

(25) $y\ln 2 = 2\ln(x-1) + \ln(2y+3) - \ln x - 2\ln y$

$y'\ln 2 = \dfrac{2}{x-1} + \dfrac{2y'}{2y+3} - \dfrac{1}{x} - \dfrac{2y'}{y}$

$y'\ln 2 - \dfrac{2y'}{2y+3} + \dfrac{2y'}{y} = \dfrac{2}{x-1} - \dfrac{1}{x}$

$y' = \dfrac{\dfrac{2}{x-1} - \dfrac{1}{x}}{\ln 2 - \dfrac{2}{2y+3} + \dfrac{2}{y}}$

(26) $\sin x \cdot \ln y = y\ln x \Rightarrow \cos x \ln y + \sin x \cdot \dfrac{y'}{y} = y'\ln x + \dfrac{y}{x}$

$y' = \dfrac{\dfrac{y}{x} - \cos x \cdot \ln y}{\dfrac{\sin x}{y} - \ln x}$

(27) $\because\ y^{x}+x^{\sin y}=1$

let $u=y^{x} \Rightarrow \ln u = x\ln y$

$u' = y^{x}\left(x\frac{y'}{y}+\ln y\right)$

, $v = x^{\sin y} \Rightarrow \ln v = \sin y \ln x$

$v' = x^{\sin y}\left(\frac{\sin y}{x}+y'\cos y \ln x\right)$

$\therefore\ y^{x}\left(\frac{xy'}{y}+\ln y\right)+x^{\sin y}\left(\frac{\sin y}{x}+y'\cos y \ln x\right)=0$

$$y' = -\frac{y^{x}\ln y + x^{\sin y-1}\cdot \sin y}{x y^{x-1}+x^{\sin y}\cdot \cos y\cdot \ln x}$$

(28) let $u=x^{y} \Rightarrow \ln u = y\ln x$

$u' = x^{y}\left(y'\ln x+\frac{y}{x}\right)$

, $v = y^{x} \Rightarrow \ln v = x\ln y$

$v' = y^{x}\left(\ln y + x\frac{y'}{y}\right)$

$\therefore\ x^{y}\left(y'\ln x+\frac{y}{x}\right)+y^{x}\left(\ln y + x\frac{y'}{y}\right)=0$

$$y' = -\frac{y x^{y-1}+y^{x}\ln y}{x^{y}\ln x + x y^{x-1}}$$

(29) $\ln y = \tan x \cdot \ln(x+1)$

$$y' = (x+1)^{\tan x}\left[\frac{\tan x}{x+1}+\sec^2 x\cdot \ln(x+1)\right]$$

(30) $\ln y = x^{3}\ln(\cos x)$

$$y' = (\cos x)^{x^{3}}\left(3x^{2}\ln(\cos x)-x^{3}\tan x\right)$$

(31) $\ln(\tan y) = e^{x^{2}}\ln(x+y)$

$\frac{y'\sec^2 y}{\tan y} = 2x e^{x^{2}}\cdot \ln(x+y)+e^{x^{2}}\cdot\frac{1+y'}{x+y}$

$y'\left(\frac{\sec^2 y}{\tan y}-\frac{e^{x^{2}}}{x+y}\right)=2x e^{x^{2}}\ln(x+y)+\frac{e^{x^{2}}}{x+y}$

$$y' = \frac{2x e^{x^{2}}\ln(x+y)+\frac{e^{x^{2}}}{x+y}}{\frac{\sec^2 y}{\tan y}-\frac{e^{x^{2}}}{x+y}}$$

(32) $\ln y = \tan x\cdot \ln\left(\sec x + e^{\sqrt{x}}\right)$

$$y' = \left(\sec x+e^{\sqrt{x}}\right)^{\tan x}\left[\sec^2 x\cdot \ln\left(\sec x+e^{\sqrt{x}}\right)+\tan x\cdot\frac{\sec x\tan x+\frac{e^{\sqrt{x}}}{2\sqrt{x}}}{\sec x+e^{\sqrt{x}}}\right]$$

(33) let $u=x^{y} \Rightarrow \ln u = y\ln x$

$u' = x^{y}\left(\frac{y}{x}+y'\ln x\right)$

$\therefore\ x^{y}\left(\frac{y}{x}+y'\ln x\right)+2^{\sin x}\cos x\cdot \ln 2 = 0$

$$y' = -\frac{y x^{y-1}+2^{\sin x}\cdot \cos x\cdot \ln 2}{x^{y}\ln x}$$

(34) $\because\ x^{y}-y^{x}=1$

let $u=x^{y} \Rightarrow \ln u = y\ln x$

$u' = x^{y}\left(y'\ln x+\frac{y}{x}\right)$

, $v = y^{x} \Rightarrow \ln v = x\ln y$

$v' = y^{x}\left(\ln y + \frac{xy'}{y}\right)$

$\therefore\ x^{y}\left(y'\ln x+\frac{y}{x}\right)-y^{x}\left(\ln y+\frac{xy'}{y}\right)=0$

$$y' = -\frac{y x^{y-1}-y^{x}\ln y}{x^{y}\ln x - x y^{x-1}}$$

(35) Let $u = x^y \Rightarrow \ln u = y \ln x$

$u' = x^y\left(y'\ln x + \frac{y}{x}\right)$

, $v = y^x \Rightarrow \ln v = x \ln y$

$v' = y^x\left(\ln y + x\frac{y'}{y}\right)$

$\therefore x^y\left(y'\ln x + \frac{y}{x}\right) + y^x\left(\ln y + x\frac{y'}{y}\right) = -2x\sin(x^2)$

$$y' = -\frac{y x^{y-1} + y^x \ln y + 2x\sin(x^2)}{x^y \ln x + x y^{x-1}}$$

(36) Let $u = x^{\sin y} \Rightarrow \ln u = \sin y \ln x$

$u' = x^{\sin y}\left(\frac{\sin y}{x} + y'\cos y \ln x\right)$

, $v = y^{\sin x} \Rightarrow \ln v = \sin x \cdot \ln y$

$v' = y^{\sin x}\left(\cos x \cdot \ln y + \frac{y'}{y}\sin x\right)$

$\therefore x^{\sin y}\left(\frac{\sin y}{x} + y'\cos y \ln x\right) + y^{\sin x}\left(\cos x \cdot \ln y + \frac{y'\sin x}{y}\right) = 0$

$$y' = -\frac{\sin y \cdot x^{\sin y - 1} + y^{\sin x} \cdot \cos x \cdot \ln y}{x^{\sin y} \cdot \cos y \cdot \ln x + \sin x \cdot y^{\sin x - 1}}$$

(37) $\ln y = \frac{1}{x}\ln(x + e^x)$

$$y' = (x+e^x)^{1/x}\left[-\frac{\ln(x+e^x)}{x^2} + \frac{1}{x}\cdot\frac{1+e^x}{x+e^x}\right]$$

(38) $\ln(3^y) + \ln(2^x) = \ln(y^2)$

$\therefore y\ln 3 + x \ln 2 = 2\ln y$

$y'\ln 3 + \ln 2 = \frac{2y'}{y} \Rightarrow y' = \frac{\ln 2}{\frac{2}{y} - \ln 3}$

(39) $\log_5 x^2 + \log_5 \sqrt[3]{\tan x} = y$

$\therefore y = 2\log_5 x + \frac{1}{3}\log_5(\tan x)$

$$y' = \frac{2}{x\ln 5} + \frac{1}{3}\,\frac{\sec^2 x}{\tan x \cdot \ln 5}$$

(40) $\ln y = \frac{1}{5}\left[3\ln(x^2-1) + \ln x - 4\ln(\csc x)\right]$

$$y' = \frac{1}{5}\sqrt[5]{\frac{(x^2-1)^3 x}{\csc^4 x}}\left[\frac{6x}{(x^2-1)} + \frac{1}{x} - 4\,\frac{-\csc x \cdot \cot x}{\csc x}\right]$$

$$y' = \frac{1}{5}\sqrt[5]{\frac{(x^2-1)^3 x}{\csc^4 x}}\left[\frac{6x}{x^2-1} + \frac{1}{x} + 4\cot x\right]$$

(41) $\ln y = \ln x \cdot \ln(x^2+1)$

$$y' = (x^2+1)^{\ln x}\left[\frac{\ln(x^2+1)}{x} + \frac{2x\ln x}{x^2+1}\right]$$

(42) $y = \frac{\ln\sqrt{\cot x}}{\ln x^2} = \frac{\frac{1}{2}\ln(\cot x)}{2\ln x}$

$$y' = \frac{1}{4}\,\frac{\frac{-\csc^2 x}{\cot x}\cdot \ln x - \frac{1}{x}\ln(\cot x)}{(\ln x)^2}$$

(43) Let $u = (\sin x)^y \Rightarrow \ln u = y\ln(\sin x)$

$u' = (\sin x)^y\left(y'\ln(\sin x) + y\cot x\right)$

$v = y^{\sqrt{x}} \Rightarrow \ln v = \sqrt{x}\ln y$

$v' = y^{\sqrt{x}}\left(\frac{\ln y}{2\sqrt{x}} + \sqrt{x}\frac{y'}{y}\right)$

$(\sin x)^y\left(y'\ln(\sin x) + y\cot x\right) + y^{\sqrt{x}}\left(\frac{\ln y}{2\sqrt{x}} + \sqrt{x}\frac{y'}{y}\right) = 0$

$$y' = -\frac{y(\sin x)^{y-1} - y^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\cdot\ln y}{(\sin x)^y \cdot \ln(\sin x) + \sqrt{x}\cdot y^{\sqrt{x}-1}}$$

(44) $y = \frac{\ln(x^2\sqrt{\tan x})}{\ln(e^{\sqrt{x}})} = \frac{2\ln x + \frac{1}{2}\ln(\tan x)}{\sqrt{x}}$

$$y' = \frac{\sqrt{x}\left(\frac{2}{x} + \frac{\sec^2 x}{2\tan x}\right) - \frac{1}{2\sqrt{x}}\left(2\ln x + \frac{1}{2}\ln(\tan x)\right)}{x}$$

(45) $\ln(x^2 \cdot 3^y) = \ln(y^{1/x})$

$2\ln x + y\ln 3 = \frac{1}{x}\ln y$

$\therefore \frac{2}{x} + y'\ln 3 = \frac{-1}{x^2}\ln y + \frac{1}{x}\cdot\frac{y'}{y}$

$$y' = -\frac{\frac{2}{x} + \frac{1}{x^2}\ln y}{\ln 3 - \frac{1}{xy}}$$

(46) $\ln y = \sin^2 x \cdot \ln(3\sqrt{x} - 1)$

$$y' = (3\sqrt{x}-1)^{\sin^2 x}\left[2\sin x \cdot \cos x \cdot \ln(3\sqrt{x}-1) + \sin^2 x \cdot \frac{\frac{3}{2\sqrt{x}}}{3\sqrt{x}-1}\right]$$

(47) $\ln y = \tan^2(2x) \cdot \ln(\sin 2x)$

$$y' = (\sin 2x)^{\tan^2 2x}\left[4\tan(2x)\cdot\sec^2(2x)\cdot\ln(\sin 2x) + \frac{2\cos 2x}{\sin 2x}\cdot\tan^2 2x\right]$$

$$= (\sin 2x)^{\tan^2 2x}\left[4\tan(2x)\cdot\sec^2(2x)\cdot\ln(\sin 2x) + 2\tan 2x\right]$$

(48) $y^{\sin x} = x^y \Rightarrow \sin x \cdot \ln y = y\ln x$

$\cos x \cdot \ln y + \sin x \cdot \frac{y'}{y} = y'\ln x + \frac{y}{x}$

$$y' = \frac{\frac{y}{x} - \cos x \cdot \ln y}{\frac{\sin x}{y} - \ln x}$$

(49) let $u = x^{\sin x} \Rightarrow \ln u = \sin x \cdot \ln x$

$u' = x^{\sin x}\left(\cos x \cdot \ln x + \frac{\sin x}{x}\right)$

, $v = (\cos x)^x \Rightarrow \ln v = x\ln(\cos x)$

$v' = (\cos x)^x(\ln(\cos x) - x\tan x)$

$$y' = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right) + (\cos x)^x(\ln(\cos x) - x\tan x)$$

(50) let $u = x^x \Rightarrow \ln u = x\ln x$

$u' = x^x(\ln x + 1)$

, $v = (\tan x)^x \Rightarrow \ln v = x\ln(\tan x)$

$v' = (\tan x)^x\left(\ln(\tan x) + x\frac{\sec^2 x}{\tan x}\right)$

$$y' = x^x(\ln x + 1) + (\tan x)^x\left[\ln(\tan x) + \frac{x\sec^2 x}{\tan x}\right]$$

(51) $\ln y = x^x \ln x \Rightarrow \ln(\ln y) = \ln(x^x \ln x)$

$\therefore \ln(\ln y) = x\ln x + \ln(\ln x)$

$\frac{y'}{y}\cdot\frac{1}{\ln y} = 1 + \ln x + \frac{1}{x\ln x}$

$y' = y\ln y\left[1 + \ln x + \frac{1}{x\ln x}\right]$

$$\therefore y' = x^{x^x}\cdot x^x \ln x\left[1 + \ln x + \frac{1}{x\ln x}\right]$$

(52) $3x^2 + 3y^2y' = xy' + y + 2xe^{x^2}$

$y'(3y^2 - x) = y + 2xe^{x^2} - 3x^2$

$$y' = \frac{y + 2xe^{x^2} - 3x^2}{3y^2 - x}$$

Example (3)

(1) Find y'' for the function: $x = \sec^3 2\theta$, $y = \tan^3 2\theta$ at $\theta = \frac{\pi}{12}$.

(2) Find $\frac{dy}{dx}$ if: $y = \frac{3}{z^2}$, $z = \tan r^2$, $r = \cos\sqrt{x}$.

(3) Find $\frac{d^3y}{dx^3}$ if $\frac{dy}{dx} = -\tan^2 t$, $x = 2\cos^4 t$

(4) Find the points on the curve: $y = x^{1/3}(8-x)$ for horizontal and vertical tangents.

(5) Find the points on the curve: $f(x) = x^{1/4}(5-x)$ for vertical and horizontal tangents.

(6) Find the equations of tangent and normal to: $x^2+y^2-2x-4y=20$ at $(5,5)$.

(7) Find the tangent equations to $y = x^2-4x+9$ passing through the origin.

(8) Find the tangent equation to $f(x) = \frac{6}{x^2+x-3}$ at the point where the curve intersects the y-axis.

(9) Determine the tangent and normal equations to $y = \frac{x+1}{x-3}$ at $P(2,-3)$, and the tangent equations that make angle $\frac{3\pi}{4}$ with positive direction of x.

(10) Find normal equations to the curve: $y = \frac{\cos x}{1+\sin x}$ on the interval $[0, 2\pi]$ and parallel to the line $2y - x = 7$.

(11) Find the tangent and normal equations to: $\sin(xy) = y$ at $(\frac{\pi}{2}, 1)$.

(12) Find the tangent and normal equations to: $x^2y + \sin y = 2\pi$ at $(1, 2\pi)$.

(13) Find the tangent and normal lines to: $x = t - \sin t$, $y = 1 - \cos t$ at $t = \frac{\pi}{3}$.

(14) Given: $y = \sin^3 t$ & $x = \cos^3 t$. Find the tangent and normal at $t = \frac{\pi}{4}$.

(15) Find the tangent and normal to: $x = 12t - t^3$ & $y = t^2 - 5t$ at $t = 1$.

(16) Find y'' if $x = \csc^3 2\theta$, $y = \cot^3 2\theta$. Then find the tangent and normal at $\theta = \frac{\pi}{4}$.

(17) Find the tangent and normal to: $\tan(xy) = y$ at $(\frac{\pi}{4}, 1)$.

(18) Given: $x = t\sin t$ & $y = t\cos t$. Find the tangent and normal at $t = \pi$.

(19) Find y'' to: $x = 2(\cos t + t\sin t)$ & $y = 2(\sin t - t\cos t)$. Then find the tangent and normal at $t = 0$ & $t = \frac{\pi}{2}$.

(20) Find the tangent and normal to: $y = \frac{3x+1}{x^2-2x}$ at $(1, -4)$.

(21) Find the points on: $y=\sqrt{x^3-8}$ for horizontal and vertical tangents.

(22) Find the points on: $x=3t^2-6t$ & $y=4t^3-2t^2+7$ for horizontal and vertical tangents.

(23) Find y'' at $t=\frac{\pi}{2}$ if: $x=a(t-\sin t)$, $y=a(1-\cos t)$.

(24) Find the equation of the tangent to: $\tan\left(\frac{x-y}{\sqrt{x+8}}\right)=\frac{1}{3}(y-1)-\pi(x-1)$ at $(1,1)$.

(25) Find horizontal and vertical tangents to the curve:

$y=\sin 2t+t-3$ & $x=(6t-\pi)^3$; $t\in[0,\pi]$.

(26) Find the equations of the tangents to $y=3x+\cos 2x$ on $[0,\frac{\pi}{2}]$ that are perpendicular to the line $2x+4y=7$.

(27) Find horizontal and vertical tangents to the curve: $y=x^{1/3}(8-x)$.

(28) Find horizontal and vertical tangents to the curve: $y=(x^2-9)^{1/4}$.

(29) Find horizontal and vertical tangents to the curve: $y=x^{2/3}(x^2-16)$.

(30) Find the first derivative y' if $y=\frac{1}{x}\frac{d^2y}{dx^2}\left(\frac{1}{1+x}\right)$.

Solutions

① $y' = \frac{\dot y}{\dot x} = \frac{6\tan^2 2\theta \cdot \sec^2 2\theta}{6\sec^3 2\theta \cdot \sec 2\theta \tan 2\theta} = \sin 2\theta$

$y'' = \frac{dy'/d\theta}{\dot x} = \frac{2\cos 2\theta}{6\sec^3 2\theta \tan 2\theta} = \frac{1}{3}\,\frac{\cos^4 2\theta}{\tan 2\theta}$

$y''\Big|_{\theta=\frac{\pi}{12}} = \frac{1}{3}\,\frac{9/16}{1/\sqrt{3}} = \frac{3\sqrt{3}}{16}$

② $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dr}\cdot\frac{dr}{dx}$

$= \left(\frac{-6}{z^3}\right)\left(2r\sec^2 r^2\right)\left(\frac{-1}{2\sqrt{x}}\sin\sqrt{x}\right)$

③ $y'' = \frac{dy'/dt}{\dot x} = \frac{-2\tan t\cdot\sec^2 t}{-8\cos^3 t\cdot\sin t} = \frac{1}{4}\sec^6 t$

$y''' = \frac{dy''/dt}{\dot x} = \frac{\frac{6}{4}\sec^5 t\cdot\sec t\tan t}{-8\cos^3 t\cdot\sin t} = \frac{-3}{16}\sec^{10} t$

④ $y = 8x^{1/3} - x^{4/3}$

$y' = \frac{8}{3}x^{-2/3} - \frac{4}{3}x^{1/3} = \frac{8-4x}{3x^{2/3}}$

For horizontal tangent $y'=0$

$8-4x=0 \Rightarrow x=2 \Rightarrow y=6\sqrt[3]{2}$ $\therefore P(2, 6\sqrt[3]{2})$

Equation of horizontal tangent is: $y=6\sqrt[3]{2}$

For vertical tangent $y'=\infty$

$3x^{2/3}=0 \Rightarrow x=0 \Rightarrow y=0$ $\therefore Q(0,0)$

Equation of vertical tangent is: $x=0$

⑤ $y = 5x^{1/4} - x^{5/4}$

$y' = \frac{5}{4}x^{-3/4} - \frac{5}{4}x^{1/4} = \frac{5-5x}{4x^{3/4}}$

For horizontal tangent $y'=0$

$5-5x=0 \Rightarrow x=1,\ y=4$ $\therefore P(1,4)$

For vertical tangent $y'=\infty$

$4x^{3/4}=0 \Rightarrow x=0,\ y=0$ $\therefore Q(0,0)$

⑥ $2x+2yy'-2-4y'=0$

$y' = \frac{2x-2}{4-2y}\Big|_{(5,5)} = -\frac{4}{3}$

Equation of tangent line:

$y-5 = -\frac{4}{3}(x-5) \Rightarrow 4x+3y=35$

Equation of normal line:

$y-5 = \frac{3}{4}(x-5) \Rightarrow 4y-3x=5$

⑦ $y' = 2x-4$

Assume (x_1,y_1) lies on the curve

$y_1 = x_1^2 - 4x_1 + 9 \rightarrow (1)$

Tangent equation at the origin is:

$0-y_1 = (2x_1-4)(0-x_1) \Rightarrow y_1 = 2x_1^2 - 4x_1 \rightarrow (2)$

From (1) & (2): $x_1^2-4x_1+9 = 2x_1^2-4x_1 \Rightarrow x_1^2=9$

$\therefore x_1=\pm 3 \Rightarrow P_1(3,6)$ & $P_2(-3,30)$

Tangent Equations: $y=2x$ and $y=-10x$

⑧ $f'(x) = \frac{-6(2x-1)}{(x^2-x-3)^2}\Big|_{x=0} = \frac{-2}{3}$

The curve intersects the y-axis at $(0,-2)$

Tangent equation is: $y+2 = \frac{-2}{3}(x-0)$

$\therefore 3y+2x+6=0$

⑨ $y' = \frac{(x-3)-(x+1)}{(x-3)^2} = \frac{-4}{(x-3)^2}\Big|_{x=2} = -4$

i)

Tangent Equation: $y+3=-4(x-2) \Rightarrow y+4x=5$

Normal Equation: $y+3=\frac{1}{4}(x-2) \Rightarrow 4y-x=-14$

ii) $m=\tan\left(\frac{3\pi}{4}\right)=-1 = y'$

$\therefore \frac{-4}{(x-3)^2}=-1 \Rightarrow (x-3)^2=4$

$(x-3)=2 \Rightarrow x=5$ $\therefore y=3$	$(x-3)=-2 \Rightarrow x=1$ $\therefore y=-1$
Tangent equation:	Tangent equation:
$y-3=-(x-5) \Rightarrow y+x=8$	$y+1=-(x-1) \Rightarrow y+x=0$

⑩ $y' = \frac{-\sin x(1+\sin x)-\cos x(\cos x)}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$

$m_\perp = \frac{-1}{y'} \Rightarrow 1+\sin x = \frac{1}{2}$

$\therefore \sin x = \frac{-1}{2}$ → $x=\pi+\frac{\pi}{6}=\frac{7\pi}{6} \Rightarrow y=-\sqrt{3}$; $x=2\pi-\frac{\pi}{6}=\frac{11\pi}{6} \Rightarrow y=\sqrt{3}$

Normal Equations: $y+\sqrt{3}=\frac{1}{2}\left(x-\frac{7\pi}{6}\right)$

& $y-\sqrt{3}=\frac{1}{2}\left(x-\frac{11\pi}{6}\right)$

⑪ $(xy'+y)\cdot\cos(xy)=y'$

$y'(x\cos xy-1)=-y\cos xy$

$\therefore y' = \frac{-y\cos xy}{x\cos xy-1}\Big|_{(\frac{\pi}{2},1)} = \text{zero}$

Tangent equation is: $y=1$

Normal equation is: $x=\frac{\pi}{2}$

⑫ $x^2y'+2xy+y'\cos y=0$

$y' = \frac{-2xy}{x^2+\cos y}\Big|_{(1,2\pi)} = -2\pi$

Tangent equation is: $y-2\pi=-2\pi(x-1)$

Normal equation is: $y-2\pi=\frac{1}{2\pi}(x-1)$

⑬ $y' = \frac{\dot{y}}{\dot{x}} = \frac{\sin t}{1-\cos t}$

at $t=\frac{\pi}{3}$ → $y'=\sqrt{3}$; $x=\frac{\pi}{3}-\frac{\sqrt{3}}{2}$; $y=\frac{1}{2}$

Tangent equation: $y-\frac{1}{2}=\sqrt{3}\left(x-\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$

Normal equation: $y-\frac{1}{2}=\frac{-1}{\sqrt{3}}\left(x-\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$

⑭ $y' = \frac{\dot{y}}{\dot{x}} = \frac{-3\cos^2 t\cdot\sin t}{3\sin^2 t\cdot\cos t} = -\tan(t)$

at $t=\frac{\pi}{4}$ → $y'=-1$; $x=\frac{1}{2\sqrt{2}}$; $y=\frac{1}{2\sqrt{2}}$

Tangent equation: $y-\frac{1}{2\sqrt{2}}=-\left(x-\frac{1}{2\sqrt{2}}\right)$

Normal equation: $y-\frac{1}{2\sqrt{2}}=x-\frac{1}{2\sqrt{2}} \Rightarrow y=x$

⑮ $y' = \frac{\dot{y}}{\dot{x}} = \frac{2t-5}{12-3t^2}$

at $t=1$ → $y'=\frac{-1}{3}$; $x=11$; $y=-4$

Tangent equation: $y+4=\frac{-1}{3}(x-11)$

Normal equation: $y+4=3(x-11)$

⑯ $y' = \frac{\dot{y}}{\dot{x}} = \frac{-6\cot^2 2\theta\cdot\csc^2 2\theta}{-6\csc^3 2\theta\cdot\cot 2\theta} = \cos 2\theta$

$y'' = \frac{dy'/d\theta}{\dot{x}} = \frac{-2\sin 2\theta}{-6\csc^3 2\theta\cdot\cot 2\theta} = \frac{1}{3}\sin^4 2\theta\tan 2\theta$

at $\theta=\frac{\pi}{4}$ → $y'=0$; $x=1$; $y=0$

Tangent equation: $y=0$

Normal equation: $x=1$

(17) $(xy'+y)\cdot\sec^2(xy)=y'$

$$y'=\frac{-y\sec^2(xy)}{x\sec^2(xy)-1}\Bigg|_{(\frac{\pi}{4},1)}=\frac{-2}{\frac{\pi}{2}-1}=\frac{-4}{-2+\pi}$$

Tangent equation: $y-1=\frac{-4}{-2+\pi}\left(x-\frac{\pi}{4}\right)$

Normal equation: $y-1=\frac{-2+\pi}{4}\left(x-\frac{\pi}{4}\right)$

(18) $y'=\frac{\dot y}{\dot x}=\frac{\cos t-t\sin t}{\sin t+t\cos t}$

at $t=\pi$: $y'=\frac{1}{\pi}$, $x=0$, $y=-\pi$

Tangent equation: $y+\pi=\frac{1}{\pi}(x-0)$

Normal equation: $y+\pi=-\pi(x-0)$

(19) $y'=\frac{\dot y}{\dot x}=\frac{2(\cancel{\cos t}-\cancel{\cos t}+t\sin t)}{2(\cancel{-\sin t}+\cancel{\sin t}+t\cos t)}=\tan t$

$$y''=\frac{dy'/dt}{\dot x}=\frac{\sec^2 t}{2t\cos t}=\frac{\sec^3 t}{2t}$$

at $t=0$: $y'=0$, $x=2$, $y=0$

Tangent equation: $y=0$

Normal equation: $x=2$

at $t=\frac{\pi}{2}$: $y'=\infty$, $x=\pi$, $y=2$

Tangent equation: $x=\pi$

Normal equation: $y=2$

(20) $y'=\frac{3(x^2-2x)-(3x+1)(2x-2)}{(x^2-2x)^2}\Bigg|_{x=1}=-3$

Tangent equation: $y+4=-3(x-1)$

Normal equation: $y+4=\frac{1}{3}(x-1)$

(21) $y'=\frac{3x^2}{2\sqrt{x^3-8}}$

For horizontal tangent: $y'=0\Rightarrow 3x^2=0$

$\therefore x=0$, $y=\sqrt{-8}$ (rejected)

For vertical tangent: $y'=\infty\Rightarrow x^3-8=0$

$\therefore x=2$, $y=0\Rightarrow P(2,0)$

(22) $y'=\frac{\dot y}{\dot x}=\frac{12t^2-4t}{6t-6}$

For horizontal tangent: $y'=0\Rightarrow 12t^2-4t=0$

$t=0,\ \frac{1}{3}\Rightarrow P_1(0,7)\ \&\ P_2\left(-\frac{5}{3},\frac{187}{27}\right)$

For vertical tangent: $y'=\infty\Rightarrow 6t-6=0$

$t=1\Rightarrow Q(-3,9)$

(23) $y'=\frac{\dot y}{\dot x}=\frac{a\sin t}{a(1-\cos t)}=\frac{\sin t}{1-\cos t}$

$$y''=\frac{dy'/dt}{\dot x}=\frac{\dfrac{\cos t(1-\cos t)-\sin t(\sin t)}{(1-\cos t)^2}}{a(1-\cos t)}=\frac{-1}{a(1-\cos t)^2}$$

$$y''\Big|_{t=\frac{\pi}{2}}=\frac{-1}{a}$$

(24) $\sec^2\left(\frac{x-y}{\sqrt{x+8}}\right)\cdot\left[\frac{(1-y')\cdot\sqrt{x+8}-(x-y)\cdot\frac{1}{2\sqrt{x+8}}}{(x+8)}\right]=\frac{y'}{3}-\pi$

at $(1,1)$:

$$\frac{(1-y')\cdot\sqrt{9}-0}{9}=\frac{y'}{3}-\pi\Rightarrow y'=\frac{3\pi+1}{2}$$

Tangent equation: $y-1=\frac{1+3\pi}{2}(x-1)$

Normal equation: $y-1=\frac{-2}{1+3\pi}(x-1)$

(25) $y' = \frac{\dot{y}}{\dot{x}} = \frac{2\cos 2t + 1}{18(6t-\pi)^2}$

For horizontal tangent: $y' = 0$

$2\cos 2t + 1 = 0 \Rightarrow \cos 2t = \frac{-1}{2}$

$\because t \in [0, \pi] \Rightarrow \therefore 2t \in [0, 2\pi]$

$\therefore 2t = \pi - \frac{\pi}{3}$ or $2t = \pi + \frac{\pi}{3}$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$t = \frac{\pi}{3}$ $\qquad\qquad$ $t = \frac{2\pi}{3}$

$\therefore y = \frac{\sqrt{3}}{2} + \frac{\pi}{3} - 3$ & $y = \frac{\sqrt{3}}{2} + \frac{2\pi}{3} - 3$

For vertical tangents: $y' = \infty$

$6t - \pi = 0 \Rightarrow t = \frac{\pi}{6} \Rightarrow x = 0$

(26) $\because y' = 3 - 2\sin 2x = y'_{\perp} = 2$

$\therefore \sin 2x = \frac{1}{2}$

$\because x \in [0, \frac{\pi}{2}] \Rightarrow \therefore 2x = [0, \pi]$

$\therefore 2x = \frac{\pi}{6}$ or $2x = \pi - \frac{\pi}{6}$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$x = \frac{\pi}{12}$ $\qquad\qquad$ $x = \frac{5\pi}{12}$

$\therefore y = \frac{\pi}{4} + \frac{\sqrt{3}}{2}$ $\qquad$ $y = \frac{5\pi}{4} - \frac{\sqrt{3}}{2}$

Tangent equations are:

$y - \left(\frac{\pi}{4} + \frac{\sqrt{3}}{2}\right) = 2\left(x - \frac{\pi}{12}\right)$

& $y - \left(\frac{5\pi}{4} - \frac{\sqrt{3}}{2}\right) = 2\left(x - \frac{5\pi}{12}\right)$

(27) $y = 8x^{1/3} - x^{4/3}$

$y' = \frac{8}{3}x^{-2/3} - \frac{4}{3}x^{1/3} = \frac{8-4x}{3x^{2/3}}$

For horizontal tangent: $y' = 0$

$8 - 4x = 0 \Rightarrow x = 2 \Rightarrow y = 6\sqrt[3]{2}$

For vertical tangent: $y' = \infty$

$3x^{2/3} = 0 \Rightarrow x = 0$

(28) $y' = \frac{2x}{4(x^2-9)^{3/4}}$

For horizontal tangent: $y' = 0$

$2x = 0$ (rejected)

For vertical tangent: $y' = \infty$

$x^2 - 9 = 0 \Rightarrow x = 3$ or $x = -3$

(29) $y = x^{8/3} - 16x^{2/3}$

$y' = \frac{8}{3}x^{5/3} - \frac{32}{3}x^{-1/3} = \frac{8x^2 - 32}{3x^{1/3}}$

For horizontal tangent: $y' = 0$

$8x^2 - 32 = 0 \Rightarrow x = \pm 2 \Rightarrow y = -19$

For vertical tangent: $y' = \infty$

$3x^{1/3} = 0 \Rightarrow x = 0$

(30) $\because \frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{-1}{(1+x)^2}$

$\therefore \frac{d^2}{dx^2}\left(\frac{1}{1+x}\right) = \frac{2}{(1+x)^3}$

$\therefore y = \frac{1}{x}\left(\frac{2}{(1+x)^3}\right)$

$\therefore \frac{dy}{dx} = \frac{-1}{x^2}\left(\frac{2}{(1+x)^3}\right) + \frac{1}{x}\left(\frac{-6}{(1+x)^4}\right)$

$\frac{dy}{dx} = \frac{-2(1+4x)}{x^2(1+x)^4}$

Example (4)

① If $y=\ln(\cos x)$, show that: $y''+e^{-2y}=0$

② If $y=ae^{-2x}+be^{3x}$, show that: $y''-y'-6y=0$

③ If $y=\ln|\sec x+\tan x|$, show that: $y''=\sec x\tan x$

④ If $y=a\sin(cx)+b\cos(cx)$, show that: $\frac{d^2y}{dx^2}=-c^2y$

⑤ If $x=\sin t$ & $y=\sin(nt)$, show that: $(1-x^2)y''-xy'+n^2y=0$

⑥ If $y=\sec x$, prove that: $y\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^2=y^2(3y^2-2)$

⑦ If $3y^3=2x^2$, prove that: $y\left(\frac{d^2y}{dx^2}\right)+2\left(\frac{dy}{dx}\right)^2=\frac{4}{9y}$.

⑧ Find $\frac{dy}{dx}$ for $y=\log_{\tan x}(2x^{-3}+10)$.

⑨ Find $\frac{dy}{du}$ where $y(u)=\left[\frac{(u+1)(u+2)}{(u^2+1)(u^2+2)}\right]^{1/3}$.

⑩ Find $\frac{dy}{dx}$ if $y=\frac{(x+1)^3\sqrt{x-1}}{(x^2+4)^4\cdot e^{4x}}$.

Solutions

① $y' = \frac{-\sin x}{\cos x} = -\tan x$

$y'' = -\sec^2 x$

$\therefore y'' + e^{-2y} = -\sec^2 x + e^{-2\ln(\cos x)}$

$= -\sec^2 x + e^{\ln(\sec^2 x)}$

$= -\sec^2 x + \sec^2 x = 0$

② $y' = -2a e^{-2x} + 3b e^{3x}$

$y'' = 4a e^{-2x} + 9b e^{3x}$

$y'' - y' - 6y = 4a e^{-2x} + 9b e^{3x} + 2a e^{-2x} - 3b e^{3x} - 6a e^{-2x} - 6b e^{3x}$

$= 0$

③ $y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$

$\therefore y'' = \sec x \cdot \tan x$

④ $y' = ac\cos(cx) - bc\sin(cx)$

$y'' = -ac^2\sin(cx) - bc^2\cos(cx)$

$= -c^2\left[a\sin(cx) + b\cos(cx)\right]$

$\therefore y'' = -c^2 y$

⑤ $y' = \frac{\dot{y}}{\dot{x}} = \frac{n\cos(nt)}{\cos(t)}$

$y'' = \frac{dy'/dt}{\dot{x}} = n\,\frac{-n\cos(t)\sin(nt) + \cos(nt)\sin(t)}{\cos^2(t)} \cdot \frac{1}{\cos t}$

$= \frac{-n^2\sin(nt) + \sin t \cdot \frac{n\cos(nt)}{\cos t}}{\cos^2(t)}$

$\therefore y'' = \frac{-n^2 y + x y'}{1 - \sin^2 t} = \frac{-n^2 y + x y'}{1 - x^2}$

$\therefore (1 - x^2) y'' - x y' + n^2 y = 0$

⑥ $\frac{dy}{dx} = \sec x \cdot \tan x$

$\frac{d^2y}{dx^2} = \sec x \cdot \tan^2 x + \tan x \cdot \sec x \cdot \tan x$

$= \sec x\left(\sec^2 x + \tan^2 x\right)$

L.H.S: $y\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 =$

$\sec^2 x\left(\sec^2 x + \tan^2 x\right) + \left(\sec^2 x \cdot \tan^2 x\right)$

$= \sec^2 x\left(\sec^2 x + 2\tan^2 x\right)$

$= \sec^2 x\left(\sec^2 x + 2\sec^2 x - 2\right)$

$= y^2\left(3y^2 - 2\right) = \text{R.H.S}$

⑦ $9y^2 y' = 4x$

$\therefore 18y(y')^2 + 9y^2 y'' = 4 \quad (\div 9y)$

$\therefore y\left(\frac{d^2y}{dx^2}\right) + 2\left(\frac{dy}{dx}\right)^2 = \frac{4}{9y}$

⑧ $y = \frac{\ln(2x^{-3} + 10)}{\ln(\tan x)}$

$y' = \frac{\ln(\tan x) \cdot \frac{-6x^{-4}}{2x^{-3} + 10} - \ln(2x^{-3} + 10) \cdot \frac{\sec^2 x}{\tan x}}{\left(\ln(\tan x)\right)^2}$

⑨ $\ln y = \frac{1}{3}\left[\ln(u+1) + \ln(u+2) - \ln(u^2+1) - \ln(u^2+2)\right]$

$y' = \frac{1}{3}\left[\frac{(u+1)(u+2)}{(u^2+1)(u^2+2)}\right]^{1/3}\left(\frac{1}{u+1} + \frac{1}{u+2} - \frac{2u}{u^2+1} - \frac{2u}{u^2+2}\right)$

⑩ $\ln y = 3\ln(x+1) + \frac{1}{2}\ln(x-1) - 4\ln(x^2+4) - 4x$

$y' = \frac{(x+1)^3\sqrt{x-1}}{(x^2+4)^4 \cdot e^{4x}}\left[\frac{3}{x+1} + \frac{1}{2(x-1)} - \frac{8x}{x^2+4} - 4\right]$

Exercise (1)

A) Find $\frac{dy}{dx}$ of the following:

1. $y = \tan\left(\sqrt{\sec^3(\cos x)}\right)$
2. $y = e^x + x^e + e^{\frac{1}{x}} + \frac{1}{e^x}$
3. $y = 4^{\tan x} + e^x \cdot x^e + \tan(e^x) + \sec(7^x)$
4. $y = x e^{\cos x} + e^{x\cos x} + \cot\left(e^{\csc x}\right)$
5. $y = \ln(2\sin^2 x + 3\tan x) + \ln(\sin(\ln x))$
6. $y = \ln\sqrt[3]{x + \cos(e^x)} + \log_3(\cos x^2)$
7. $y = \log_2\left[\frac{x^2 e^x}{2\sqrt{x+1}}\right]$
8. $y = (\cos x)^{\sin x} + \cos(x^x)$
9. $x^{\sqrt{y}} = y^{\sqrt{x}}$
10. $y = \frac{\sqrt{1+2x}\,\sqrt[4]{1+4x}\,\sqrt[6]{1+6x}}{\sqrt[3]{1+3x}\,\sqrt[5]{1+5x}\,\sqrt[7]{1+7x}}$ at $x = 0$
11. $3x^2y^3 + 2x^2 + 5y^2 = 3x + 2y + 1$
12. $y e^x + \sin(y^2) + 3x = 5$
13. $e^{xy} + y\sec(x^2) = 2x + 3y - 1$
14. $\tan(xy) - \sec(xy) = x^2$
15. $y = 2t^2 + 3t + 14$ & $x = \sin(t^2)$
16. $y = a(1 - \cos t)$ & $x = a(t - \sin t)$ at $t = \pi$
17. $2\sqrt{x^2 + y^2} - e^{xy} = \sin y$
18. $y = \sin\left((\ln x)^{\cos x}\right) + \pi^{\sqrt{4 - x^2}}$
19. $\ln y = \sin(x^y) - \pi^{\csc x}$
20. $(\sin y)^x = x^{\sin y} + \pi^{x^2}$
21. $f(x) = (\sec x + \tan x)(\sec x - \tan x)$
22. $x^y + y^x = e^x$
23. $y = \ln\sqrt{\frac{\sec^3 x \cdot \ln\sqrt{x}}{2^{\sqrt{x-1}} \cdot \sin 2x}}$
24. $\sin(xy) + 3^{x+y} = \tan(x+y)$
25. $y = (\ln x)^{\frac{1}{\ln x}}$
26. $y = \frac{1}{\sqrt[3]{x}} + e^{-\sin x} + \csc 2x$
27. $y = x^2 + \sin(xy)$
28. $y = (\tan x + 3\sec x^3)^2$
29. $y = \ln(x + \sec(e^{3x})) + (\sin x)^{\sqrt{x}}$
30. $y = \log_5\left(x^2\sqrt[3]{\tan x}\right)$
31. $y = \sqrt[5]{\frac{(x^2-1)^3 \cdot x}{\csc^4 x}}$
32. $y = (x^2+1)^{\ln x}$
33. $y = \log_{x^2}\sqrt{\cot x}$
34. $(\sin x)^y + 5^{\sqrt{x}} = 1$
35. $y = \log_{e^{\sqrt{x}}}\left(x^2\sqrt{\tan x}\right)$
36. $y = (3\sqrt{x} - 1)^{\sin^2 x}$
37. $y = \pi^{\sin x} + (\ln x)^{x+1}$
38. $y = \pi^{\tan x} + (\ln x)^e$
39. $y = 2(\ln x)^{x/2}$
40. $y = \frac{(2x)\cdot 2^x}{\sqrt{x^2+1}}$

B)

1. Find the equation of the tangent to the curve $f(x)=\frac{3}{x^4}+\frac{1}{x}-1$ at $P(-1,1)$.
2. Find the equation of the normal to the curve $f(x)=\sqrt{x}+\frac{2}{\sqrt[3]{x}}$ at $x=1$.
3. Find the equation of the normal to the curve $xy^3+2x^2+y^2=2y+6$ at $P(1,-1)$.
4. If $y=x^2\tan x$, Prove that: $x^2y'-(2x+y)y=x^4$.
5. If $y=a\cos(\ln x)+b\sin(\ln x)$, Prove that: $x^2y''+xy'+y=0$
6. If $y=\frac{\sin x}{1-x^2}$, Prove that: $(1-x^2)y''-4xy'-(1+x^2)y=0$
7. If $y=[x+\sqrt{1+x^2}]^m$, Prove that: $(1+x^2)y''+xy'-m^2y=0$
8. If $y=2\ln(\cot\theta)$ & $x=\tan\theta+\cot\theta$, Prove that: $y'=\tan 2\theta$
9. Find the values of "t" for horizontal and vertical tangents:
   $x=2t^3-9t^2-24t+5$ & $y=\sin t-t$
10. Find y' and y'' at $\theta=\frac{\pi}{4}$ if: $y=4\cos^3\theta$ & $x=4\sin^3\theta$.
11. Find $\frac{d^2y}{dx^2}$ at $t=\pi$ if: $x=e^{t}(\sin t+\cos t)$ & $y=e^{t}(\sin t-\cos t)$
12. If $x^2+2tx-3t=-4$ & $y^3-3t^3=5$, find the tangent equation at $t=1$.
13. If $y\sqrt{t+1}+2t\sqrt{y}=4$ & $x+x^{3/2}=t^2+1$, find $\frac{dy}{dx}$ at $t=0$.
14. Prove that: $yy''=-4a^2$ if $y=a(t-\frac{1}{t})$ & $x=a(t+\frac{1}{t})$, a is constant
15. If $x=\ln(\cot\frac{t}{2})-\cos t$, $y=\sin t$, Prove that: $\frac{y'}{y''}(1+y'^2)=x+\ln(\tan\frac{t}{2})$
16. If $x=e^{\theta}\sin\theta$ & $y=e^{2\theta}$, Find the tangent equation at $\theta=0$.
17. Find y'' at $t=1$ if $y=t+\frac{1}{t}$ & $x=t-\frac{1}{t}$
18. Find the tangent equation of the curve: $x^2-2tx+2t^2=4$ & $2y^3-3t^2=4$ at $t=2$.
19. Find the point on the curve: $x=e^{\sin t}$ & $y=e^{\cos t}$ for horizontal tangent; $0\le t\le 2\pi$
20. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if: $x=\tan 2\theta$ & $y=\sec 2\theta$.
21. If $x=\sin\theta$, $y=\sin(2n+1)\theta$; Prove that: $(1-x^2)(y')^2=(2n+1)^2(1-y^2)$
22. Find the tangent equation of: $y=te^{t}$ & $t=\ln(x-t)$ at $t=0$, then find y''.
23. Find the tangent equation of: $x^2\sin t+2x=2$ & $\sin t-2ty=2t$ at $t=\pi$.
24. Find the value of "c" such that the line: $y=\frac{3}{2}x+6$ is tangent to the curve: $y=c\sqrt{x}$

(25) Find the equation of the tangent line to the curve: $x+\tan\left(\frac{y}{x}\right)=2$ at $\left(1,\frac{\pi}{4}\right)$.

(26) Find $\frac{dy}{dx}$ if: $y=\log_7\left[\frac{\sin x\cdot\cos x}{e^x\cdot 2^x}\right]$

(27) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if: $x=1+\sec t$ & $y=2+3\tan t$

(28) Find $\frac{dy}{dx}$ if: $y=\left[\log_2(x^3+1)\right]^{\cos x}$

(29) Find $\frac{dy}{dx}$ if: $x\cos(\sqrt{y})+e^{xy}=\ln(x^2+y)$.

(30) Find the tangent equation to the curve: $x=\sec^2 t-1$ & $y=\tan t$ at $t=-\frac{\pi}{4}$ & y''.

(31) Find $\frac{dy}{dx}$ if: $y=\left[\sin(x+1)\right]^{\ln x}+\log_2\left(\sqrt{x+2}\right)$

(32) Find $\frac{dy}{dx}$ if: $y^2=\sqrt[3]{\frac{\tan(x^3+1)\cdot\pi^{\csc x^2}}{(x^6+2)\cdot\ln(x^3+1)}}$

(33) If $x=t+\frac{1}{t}$ & $y=t-\frac{1}{t}$, Show that: $y^3\frac{d^2y}{dx^2}+4=0$

(34) Find the value of "c" such that the curve $y=\frac{c}{x+1}$ is tangent to the line passing through the points $(0,3)$ and $(5,-2)$.

(35) Find $\frac{dy}{dx}$ if: $y=3^{x^2}+\left[\log_{10}(x^3+1)\right]^{\cos x}$.

(36) Find $\frac{dy}{dx}$ if: i) $y=5^{\tan x}+x^{\sin x}+e^{\sqrt{x}}$ ii) $y=(\sin x)^{\sqrt{x}}+(\tan x)^{e^x}$

(37) Find the points on the curve $y=x^3-3x$ for horizontal tangent.

(38) Find the points on: $y=t^3-3t$ & $x=t^2-4t+5$ for horizontal and vertical tangents.

(39) Find $\frac{dy}{dx}$ if: i) $y=x^{x^{\sin x}}$ ii) $(\sin x)^y=y^{\sin x}$ iii) $y=\ln\left[x\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}}\right]$

(40) Find $\frac{dy}{dx}$ if: i) $y^2x^{\frac{1}{y}}+\cos\sqrt{y}=e^{1/x}$ ii) $y=(x+1)^x+x^{x+1}$

(41) Find the tangent equation to the curve $f(x)=\int_0^{\sin x}\frac{\cos x\cdot e^t}{\sqrt{4+t^2}}\,dt$ at $x=\pi$.

(42) Find the tangent and normal to $F(x)=\int_1^{x^2}\frac{\sin\left(\frac{\pi}{2}t^2\right)}{e^t-1}\,dt$ at $x=1$

(43) If $f(t)=\int_1^{t\sin t}\frac{\sqrt{1+u^4}}{u}\,du$, $F(x)=\int_1^x f(t)\,dt$. Find $F''\left(\frac{\pi}{2}\right)$.

c) Choose the correct answer :

① If $y = 2(\sqrt{e})^{\ln x}$, then $\frac{dy}{dx} = ------$

a) $\frac{2\ln x}{x^2}$ b) $\frac{2}{x^2}$ c) none d) $\frac{1}{\sqrt{x}}$

② If $3(x^2 + axy - y^2) = 0$, and $\frac{dy}{dx} = \frac{x}{y}$, then $a = -----$

a) $\frac{-3}{2}$ b) $\frac{1}{2}$ c) ∞ d) 0

③ If $y = 20\ln(x^{1/x})$, then $\frac{dy}{dx} = ------$

a) $20\left(\frac{1-\ln x}{x^2}\right)$ b) $20\left(\frac{\ln x - 1}{x^2}\right)$ c) $\frac{20}{x^2}$ d) $\frac{20\ln x}{x^2}$

④ If $x^2 + 2xy - y^2 = 8$, then $\frac{dy}{dx}$ at $(1,1)$ is ----

a) $\frac{1}{2}$ b) $-\frac{3}{2}$ c) ∞ d) zero

⑤ If $y = x^{x^2}$, then $y' = -----$

a) $x^{x^2+1}(1+2\ln x)$ b) $x^2 \cdot x^{x^2-1}$ c) x^{2x} d) $(2x)\cdot x^{x^2}(\ln x)$

⑥ If $e^{f(x)} = 1 + x^2$, then $f'(x) = -----$

a) $\frac{1}{1+x^2}$ b) $\frac{2x}{1+x^2}$ c) $2x(1+x^2)$ d) $2x\ln(1+x^2)$

⑦ If $f(x) = e^x$, then $\ln(f'(2)) = ---$

a) 2 b) zero c) $\frac{1}{e^2}$ d) $2e^2$

⑧ If $\log_a(2^a) = \frac{a}{4}$, then $a = ----$

a) 2 b) 4 c) 8 d) 16 e) 32

⑨ If $\cos(x+y) + \sin(x+y) = \frac{1}{3}$, then $\frac{dy}{dx} = ----$

a) $\frac{\sin(x+y) - \cos(x+y)}{\sin(x+y) + \cos(x+y)}$ b) $\frac{\sin(x+y) - 1}{\cos(x+y) + 1}$ c) -1 d) 1

⑩ $\ln(x-2) < 0$ if and only if ------

a) $x < 3$ b) $0 < x < 3$ c) $2 < x < 3$ d) $x > 2$ e) $x > 3$

⑪ If $x = t^2 - 1$ and $y = 2e^t$, then $\frac{dy}{dx} = -----$

a) $\frac{e^t}{t}$ b) $\frac{2e^t}{t}$ c) $\frac{e^t}{t^2}$ d) e^t

⑫ If $\ln x - \ln \frac{1}{x} = 2$, then $x =$ ----

a) $\frac{1}{e^2}$ b) $\frac{1}{e}$ c) e d) $2e$

⑬ If $y = \ln(x^2+y^2)$, then $\frac{dy}{dx}$ at $(1,0)$ is -----

a) 0 b) $\frac{1}{2}$ c) 1 d) 2

⑭ The domain of $f(x) = \frac{x+3}{xe^x - 2x}$ is ----

a) $R - \{0\}$ b) $R - \{0, 2\}$ c) $R - \{0, \ln 2\}$ d) R

⑮ If $10 \sin x = e^{4y}$, then $y' =$ ----

a) $\frac{5}{2}\cos x$ b) $\frac{1}{4}\frac{\cos x}{\sin x}$ c) $\frac{\cos x}{\sin x}$ d) $\frac{-1}{4}\frac{\sin x}{\cos x}$

⑯ $\frac{d}{dx}\left(e^{\cos\frac{\pi}{3}}\right) =$ -----

a) $-e^{\cos\frac{\pi}{3}} \cdot \sin\frac{\pi}{3}$ b) $e^{\cos\frac{\pi}{3}} \cdot \sin\frac{\pi}{3}$ c) $e^{\sin\frac{\pi}{3}}$ d) Zero

⑰ If $y = \ln(\sec x + \tan x)^6$, then $y' =$ ----

a) $\frac{-6}{\sec x + \tan x}$ b) $\frac{6}{\sec x + \tan x}$ c) $\sec^6 x$ d) $6\sec x$

⑱ If $x = \sin t$ and $\cos t \frac{dy}{dx} = x$, then -----

a) $\cos(t)\frac{d^2y}{dx^2} - \sin t \frac{dy}{dx} = 1$ b) $\cos t \frac{d^2y}{dx^2} - \sin t \frac{dy}{dx} = \sec t$

c) $\cos t \frac{d^2y}{dx^2} - \tan t \frac{dy}{dx} = 1$ d) $\frac{d^2y}{dx^2} - \tan t \frac{dy}{dx} = 1$

⑲ If $x = a(\cos t + \ln \tan\frac{t}{2})$ & $y = a \sin t$, then $\frac{dy}{dx} =$ ----

a) $\sin t$ b) $\cos t$ c) $\tan t$ d) $\cot t$

⑳ If $x = \tan t + \cot t$ & $y = 2\ln(\cot t)$, then $\frac{dy}{dx} =$ ---

a) $\sin 2t$ b) $\cos 2t$ c) $\tan 2t$ d) $\cot 2t$

㉑ The domain of $f(x) = \sqrt{3^x - 3^{-2}} + \frac{1}{\sqrt{e^{-x} - e}}$ is ------

a) $]1, 2[$ b) $[\frac{1}{3}, e[$ c) ϕ d) $[-2, 1[$

㉒ If $f(x) = \ln\sqrt{\frac{x}{5e^x}}$, then $f'(x) =$ ---

a) $\frac{1}{2x} - \frac{1}{2}$ b) $\frac{1}{2x} + \frac{1}{2}$ c) $\frac{-1}{2x} - \frac{\ln 5}{2\sqrt{x}}$ d) $\frac{1}{2x} - \frac{1}{2}e^{x/2}$

(23) If $y = (\sin x)^{x^2} + x$, then $y' = ---$

a) $(\sin x)^{x^2} \cos x \cdot \ln(\sin x) \cdot 2x + 1$ b) $y\left[\frac{x^2}{\sin x} + 2x \ln(\sin x) + \frac{1}{x}\right]$

c) $x^2(\sin x)^{x^2-1}(2x) + (\sin x)^{x^2} \cos x \cdot 2x + 1$ d) $x^2(\sin x)^{x^2-1} \cdot \cos x + (\sin x)^{x^2} \cdot \ln(\sin x) \cdot 2x + 1$

(24) If $x^2y + xy^2 = 3x$, then $\frac{dy}{dx} = ---$

a) $\frac{3}{4xy}$ b) $\frac{-4xy - y^2 + 3}{x^2}$ c) $\frac{3}{2x+2y}$ d) $\frac{-2xy - y^2 + 3}{x^2 + 2xy}$

(25) If $z = \sin(y')$, then $z' = ---$

a) $y' \cos(y'')$ b) $y'' \cos(y')$ c) $y' y'' \cos(y')$ d) $\cos y' + \sin y''$

(26) For $xy = \sin y$, then $\frac{dy}{dx} = ---$

a) $\frac{\cos y - y}{x}$ b) $\frac{xy' + y}{\cos y}$ c) $\frac{\cos x - x}{y}$ d) $\frac{y}{\cos y - x}$

(27) If $y = \sqrt{e^{2\ln x + 2}}$, then $y' = ----$

a) e b) $2x + e^2$ c) $\frac{2x}{\sqrt{x^2+2}}$ d) xe

(28) If $2y = ye^x + 1$, then y' at $x = 0$ is ----

a) 0 b) 1 c) 2 d) 4

(29) If $2y = ye^x + 1$, then y'' at $x = 0$ is ----

a) 0 b) 1 c) 2 d) 3

(30) If $\sqrt{x} + \sqrt{y} = \sqrt{a}$, then $\frac{dy}{dx} = ---$

a) $-\frac{\sqrt{x}}{\sqrt{y}}$ b) $-\frac{1}{2}\frac{\sqrt{y}}{\sqrt{x}}$ c) $-\frac{\sqrt{y}}{\sqrt{x}}$ d) none

(31) The equation of the tangent line to the curve $y = 2x \sin x$ at $p(\frac{\pi}{2}, \pi)$ is---

a) $y = 2x + 2\pi$ b) $y = 2x$ c) $y = -2x + 2\pi$ d) $y = -2x$

(32) The values of "x" for $f(x) = \frac{1}{3}x^3 - x^2 + 3$ to have a horizontal tangents are---

a) 0 b) 0 and 2 c) 0 and 3 d) 3

(33) The second derivative of: $y^3 = xy + 1$ at $p(0,1)$ is ----

a) 0 b) 1 c) 3 d) 1/3 e) none

(34) If $f'(0) = 1$, then $\frac{d}{dx}(f(2ax)) = ----$

a) 2 b) $2a$ c) a d) $\frac{1}{2a}$ e) 0