Chapter 1

Interpolation

1.1 Interpolation

Let $y(x)$ be a continuous function in some interval $[a, b]$, and defined at $n+1$ distinct points $x_0, x_1, \cdots, x_n$ such that $a = x_0 < x_1 < x_2 \cdots < x_n = b$. These points may be equispaced or non equispaced. The problem of polynomial interpolation is to find a polynomial $y_n(x)$ of degree $\leq n$, which fits the given distinct data exactly, i.e.,

$$y_n(x_i) = y(x_i), \;\; i = 0, 1, 2, \cdots, n.$$

Such a polynomial is called the **interpolating polynomial**.

1.2 Newton's formulae for interpolation

Given the set of $(n+1)$ values $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ of x and y, it is required to find $y_n(x)$, a polynomial of the nth degree such that y and $y_n(x)$ agree at the tabulated points. Let the values of x be equidistance i.e., $x_i = x_0 + ih$, $i = 0, 1, 2, \cdots, n$.

Since $y_n(x)$ is a polynomial of the nth degree, it may be written as

$$\begin{aligned} y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots \\ \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}). \end{aligned} \tag{1.1}$$

Imposing now the condition that y and $y_n(x)$ should agree at the set of tabulated points, we obtain

$$a_0 = y_0; \;\; a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}; \;\; a_2 = \frac{\Delta^2 y_0}{h^2 2!}; \cdots ; \;\; a_n = \frac{\Delta^n y_0}{h^n n!}.$$

Setting $x = x_0 + ph$ and substituting for $a_0, a_1, \cdots, a_n$, Eq. (1.1) gives

$$\begin{aligned} y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \\ \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}\Delta^n y_0, \end{aligned} \tag{1.2}$$

which is **Newton's forward difference interpolation formula** and useful for interpolation **near the beginning** of a set of tabular values.

Remark. Error in the approximation is given by

$$y(x) - y_n(x) \approx \frac{p(p-1)\cdots(p-n)}{(n+1)!}\Delta^{n+1} y(\xi), \;\; x_0 < \xi < x_n.$$

Instead of assuming $y_n(x)$ as in (1.1), if we choose it in the form

$$\begin{aligned} y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \cdots \\ \cdots + a_n(x - x_n)(x - x_{n-1})\cdots(x - x_1), \end{aligned} \tag{1.3}$$

and then impose the condition that y and $y_n(x)$ should agree at the tabulated points $x_n, x_{n-1}, \cdots, x_1, x_0$, we obtain (after some simplification)

$$\begin{aligned} y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots \\ \cdots + \frac{p(p+1)\cdots(p+n-1)}{n!}\nabla^n y_n, \end{aligned} \tag{1.4}$$

where $p = (x - x_n)/h$. This is **Newton's backward difference interpolation formula** and useful for interpolation **near the end** of a set of tabular values.

Remark. Error in the approximation is given by

$$y(x) - y_n(x) \approx \frac{p(p+1)\cdots(p+n)}{(n+1)!}\nabla^{n+1} y(\xi), \;\; x_0 < \xi < x_n.$$

The following examples illustrate the use of these formulae.

Example 1. Find the cubic polynomial which takes the following values:

x	1	3	5	7
$y(x)$	24	120	336	720

Hence, or otherwise, obtain the value of $y(8)$.

Solution: We form the difference table:

x	y	Δ	Δ^2	Δ^3
1	24			
		96		
3	120		**120**	
		216		**48**
5	336		168	
		384		
7	720			

Here $h = 2$. With $x_0 = 1$, we have $x = 1{+}2p$ or $p = (x{-}1)/2$. Substituting this value of p in Eq. (1.2), we obtain

$$\begin{aligned}
y(x) &= y_0 + \frac{x-1}{2}\Delta y_0 + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2!}\Delta^2 y_0 + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{3!}\Delta^3 y_0 \\
&= 24 + \frac{x-1}{2}(96) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2!}(120) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{3!}(48) \\
&= x^3 + 6x^2 + 11x + 6.
\end{aligned}$$

To find $y(8)$, substitute $x = 8$ in $y(x)$, we obtain $y(8) = 990$.

Note: This process of finding the value of y for some value of x outside the given range is called **extrapolation**.

Example 2. The table below gives the values of $\tan x$ for $0.10 \le x \le 0.30$

x	0.10	0.15	0.20	0.25	0.30
y	0.1003	0.1511	0.2027	0.2553	0.3093

Find (i) $\tan 0.12$, and (ii) $\tan 0.26$.

Solution: We form the difference table:

x	y	Δ/∇	Δ^2/∇^2	Δ^3/∇^3	Δ^4/∇^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		**0.0002**
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

(*i*) To find $\tan 0.12$, we have $0.12 = 0.10 + p(0.05)$, which gives $p = 0.4$.

Hence Newton's forward difference interpolation formula (1.2) gives

$$\begin{aligned}
\tan(0.12) &= y_0 + (0.4)\Delta y_0 + \frac{(0.4)(0.4-1)}{2!}\Delta^2 y_0 + \frac{(0.4)(0.4-1)(0.4-2)}{3!}\Delta^3 y_0 \\
&\quad + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!}\Delta^4 y_0 \\
&= 0.1003 + (0.4)(0.0508) + \frac{(0.4)(0.4-1)}{2!}(0.0008) \\
&\quad + \frac{(0.4)(0.4-1)(0.4-2)}{3!}(0.0002) \\
&\quad + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!}(0.0002) \\
&= 0.1205
\end{aligned}$$

(*ii*) To find $\tan 0.26$, we have $0.26 = 0.30 + p(0.05)$, which gives $p = -0.8$.

Hence Newton's backward difference interpolation formula (1.4) gives

$$\begin{aligned}
\tan(0.26) &= y_n + (-0.8)\nabla y_n + \frac{(-0.8)(-0.8+1)}{2!}\nabla^2 y_n \\
&\quad + \frac{(-0.8)(-0.8+1)(-0.8+2)}{3!}\nabla^3 y_n \\
&\quad + \frac{(-0.8)(-0.8+1)(-0.8+2)(-0.8+3)}{4!}\nabla^4 y_n \\
&= 0.3093 + (-0.8)(0.0540) + \frac{(-0.8)(-0.8+1)}{2!}(0.0014) \\
&\quad + \frac{(-0.8)(-0.8+1)(-0.8+2)}{3!}(0.0004) \\
&\quad + \frac{(-0.8)(-0.8+1)(-0.8+2)(-0.8+3)}{4!}(0.0002) \\
&= 0.2662
\end{aligned}$$

Example 3. The following are the number of deaths in four successive ten year age groups. Find the number of deaths at 45-50 and 50-55.

Age group	25-35	35-45	45-55	55-65
Deaths	13229	18139	24225	31496

Solution: The difference table is

Age upto (x)	No. of deaths $f(x)$	Δ	Δ^2	Δ^3
35	13229			
		18139		
45	31368		6086	
		24225		1185
55	55593		7271	
		31496		
65	87089			

Here $h = 10, x_0 = 35, p = \dfrac{x - x_0}{h} = \dfrac{50 - 35}{10} = 1.5$

By Newton's forward difference formula (1.2), we have

$$\begin{aligned}
P(50) &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \\
&= 13229 + 1.5(18139) + \frac{1.5(1.5-1)}{2!}(6086) + \frac{1.5(1.5-1)(1.5-2)}{3!}(1185) \\
&= 42645.6875 \approx 42646
\end{aligned}$$

$\therefore$ Deaths at the age between 45-50 is $y_{50} - y_{45} = 42646 - 31368 = 11278$

Deaths at the age between 50-55 is $y_{55} - y_{50} = 55593 - 42646 = 12947$.

1.3 Central difference interpolation formulae

In the preceding section, we derived and discussed Newton's forward and backward difference interpolation formulae, which are applicable for interpolation near the beginning and end respectively, of tabulated values. We shall, in the present section, discuss the central difference formulae which are most suited for interpolation near the middle of a tabulated set.

1.3.1 Gauss' central difference interpolation formulae

In this section, we will discuss Gauss' forward and backward formulae.

1.3.1.1 Gauss' forward formula

We consider the following difference table 1.1 in which the central ordinate is taken for convenience as y_0 corresponding to $x = x_0$.
The differences used in this formula are red number shown in Table 1.1. The formula is, therefore, of the form

$$y_p = y_0 + G_1\Delta y_0 + G_2\Delta^2 y_{-1} + G_3\Delta^3 y_{-1} + G_4\Delta^4 y_{-2} + G_5\Delta^5 y_{-2} + \cdots \tag{1.5}$$

where $G_1, G_2, \cdots$ have to be determine.

Table 1.1: Gauss' forward difference table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

The y_p on the left side can be expressed in terms of y_0, Δy_0 and higher-order differences of y_0, as follows:

$$\begin{aligned}
y_p &= E^p y_0 \\
&= (1+\Delta)^p y_0 \\
&= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots
\end{aligned}$$

Similarly, the right side of Eq. (1.5) can also be expressed in terms of y_0, Δy_0 and higher-order differences of y_0. We have

$$\begin{aligned}
\Delta^2 y_{-1} &= \Delta^2 E^{-1} y_0 \\
&= \Delta^2 (1+\Delta)^{-1} y_0 \\
&= \Delta^2 (1 - \Delta + \Delta^2 - \Delta^3 + \cdots) y_0 \\
&= \Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \cdots, \\
\Delta^3 y_{-1} &= \Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0 + \cdots, \\
\Delta^4 y_{-2} &= \Delta^4 E^{-2} y_0 = \Delta^4 (1+\Delta)^{-2} y_0 \\
&= \Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \cdots
\end{aligned}$$

Hence, Eq. (1.5) gives the identity

$$\begin{aligned}
&y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \\
&= y_0 + G_1 \Delta y_0 + G_2(\Delta^2 y_0 - \Delta^3 y_0 + \cdots) \\
&\quad + G_3(\Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0 + \cdots) \\
&\quad + G_4(\Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \cdots) \qquad (1.6)
\end{aligned}$$

Equating the coefficients of Δy_0, $\Delta^2 y_0$, etc., on both side of Eq. (1.6), and substituting the obtained values in Eq. (1.5), we get

$$\begin{aligned}
y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} \\
+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots \qquad (1.7)
\end{aligned}$$

where $p = (x - x_0)/h$, which is the **Gauss' forward formula**.

1.3.1.2 Gauss' backward formula

The differences used in this formula are green number shown in Table 1.2. The **Gauss' backward formula** is, therefore, of the form (same solution procedure as in Gauss' forward formula)

$$\begin{aligned}
y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} \\
+ \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \qquad (1.8)
\end{aligned}$$

where $p = (x - x_0)/h$.

Table 1.2: Gauss' backward difference table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

Remark. It is convenient to start with x_0 nearest to x and then introduced x_{-1}, x_1, x_{-2}, x_2, and so on.

Example 4. From the following table, find the value of $e^{1.17}$ using Gauss's forward formula:

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Solution: We have $1.17 = 1.15 + p(0.05)$, which gives $p = 2/5 = 0.4$. Using Gauss' forward formula (1.7), we have

$$\begin{aligned}
e^{1.17} &= 3.1582 + (0.4)(0.1619) + \frac{(0.4)(0.4-1)}{2!}(0.0079) \\
&\quad + \frac{(0.4+1)(0.4)(0.4-1)}{3!}(0.0004) + \frac{(0.4+1)(0.4)(0.4-1)(0.4-2)}{4!}(0) \\
&\quad + \frac{(0.4+2)(0.4+1)(0.4)(0.4-1)(0.4-2)}{5!}(0.0001) \\
&\quad + \frac{(0.4+2)(0.4+1)(0.4)(0.4-1)(0.4-2)(0.4-3)}{6!}(0.0001) \\
&= 3.2221
\end{aligned}$$

The difference table is

x	e^x	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.00	2.7183						
		0.1394					
1.05	2.8577		0.0071				
		0.1465		0.0004			
1.10	3.0042		0.0075		0		
		0.1540		0.0004		0	
1.15	3.1582		0.0079		0		0.0001
		0.1619		0.0004		0.0001	
1.20	3.3201		0.0083		0.0001		
		0.1702		0.0005			
1.25	3.4903		0.0088				
		0.1790					
1.30	3.6693						

1.3.2 Stirling's formula

Taking the mean of Gauss' forward and backward formulae (1.7)-(1.8), we obtain

$$\begin{aligned} y_p = y_0 + p\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} \\ + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \end{aligned} \tag{1.9}$$

which is known as **Stirling's formula**.

1.3.3 Bessel's formula

This is very useful formula for practical interpolation, and is given as

$$\begin{aligned} y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)(p-1/2)}{3!}\Delta^3 y_{-1} \\ + \frac{(p+1)p(p-1)(p-2)}{4!}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots \end{aligned} \tag{1.10}$$

1.3.4 Practical interpolation

In the preceding sections, we have derived some interpolation formulae of great practical importance. A natural question is: Which one of these formulae gives the most accurate result?.

(i) For interpolation at the beginning or end of a table of values, Newton's forward and backward interpolation formulae have to be used respectively.

(ii) For interpolation near the middle of a set of values, the following are choices:

$$\text{Stirling's formula if} \qquad -\frac{1}{4} \leq p \leq \frac{1}{4}$$

and

$$\text{Bessel's formula for} \qquad \frac{1}{4} \leq p \leq \frac{3}{4}.$$

Example 5. Using Stirling's interpolation formula to find y at $x = 32$ from the given table:

x	20	30	40	50
y	512	439	346	243

Solution: Since $x = 32$ lying between 30 and 40. So we take 30 as the origin and $h = 10$. Therefore, $p = (32 - 30)/10 = 0.2 < 1/4$. Hence, we apply Stirling's formula. The difference table is

x	y	Δ	Δ^2	Δ^3
20	512			
		-73		
			-20	
30	439			10
		-93		
			-10	
40	346			
		-103		
50	243			

The Stirling's formula is

$$y_p = y_0 + p\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Therefore

$$\begin{aligned} y_{0.2} &= 439 + 0.2\left(\frac{-93-73}{2}\right) + \frac{0.04}{2}(-20) + 0 \\ &= 422 \end{aligned}$$

1.4 Interpolation with unevenly spaced points

We discuss, in the present section, Lagrange's interpolation and Newton's general interpolation formulae.

1.4.1 Lagrange's interpolation formula

Given the $(n+1)$ points $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ where the values of x need not necessarily be equally spaced. The **Lagrange's interpolating polynomial** passes through given data points is given as:

$$P_n(x) = \sum_{i=0}^{n} \ell_i(x)\, y_i, \qquad y_i = y(x_i) \tag{1.11}$$

where

$$\ell_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

Interchanging x and y in Eq. (1.11), we obtain the **inverse interpolation** as

$$P_n(y) = \sum_{i=0}^{n} \ell_i(y)\, x_i, \qquad x_i = x(y_i) \tag{1.12}$$

where

$$\ell_i(y) = \frac{(y-y_0)(y-y_1)\cdots(y-y_{i-1})(y-y_{i+1})\cdots(y-y_n)}{(y_i-y_0)(y_i-y_1)\cdots(y_i-y_{i-1})(y_i-y_{i+1})\cdots(y_i-y_n)}.$$

Example 6. Certain corresponding values of x and $\log_{10} x$ are $(300, 2.4771)$, $(304, 2.4829)$, $(305, 2.4843)$ and $(307, 2.4871)$. Find $\log_{10} 301$.

Solution: From Eq. (1.11), we have $P_3(x) = \sum_{i=0}^{3} \ell_i(x)\, y_i$, i.e.,

$$\begin{aligned} P_3(301) = \log_{10} 301 &= \frac{(301-304)(301-305)(301-307)}{(300-304)(300-305)(300-307)}(2.4771) \\ &+ \frac{(301-300)(301-305)(301-307)}{(304-300)(304-305)(304-307)}(2.4829) \\ &+ \frac{(301-300)(301-304)(301-307)}{(305-300)(305-304)(305-307)}(2.4843) \\ &+ \frac{(301-300)(301-304)(301-305)}{(307-300)(307-304)(307-305)}(2.4871) \\ &= 1.2739 + 4.9658 - 4.4717 + 0.7106 = 2.4786. \end{aligned}$$

Example 7. If $y_1 = 4, y_3 = 12, y_4 = 19$ and $y_x = 7$, find x.

Solution: Using Eq. (1.12), we have

$$x = \sum_{i=0}^{3} \ell_i(y)\, x_i$$
$$x = \frac{(7-12)(7-19)}{(4-12)(4-19)}(1) + \frac{(7-4)(7-19)}{(12-4)(12-19)}(3) + \frac{(7-4)(7-12)}{(19-4)(19-12)}(4)$$
$$= \frac{1}{2} + \frac{27}{14} - \frac{4}{7} = \frac{26}{14} = 1.86$$

1.4.2 Divided differences and their properties

The Lagrange's interpolation formula has the disadvantage that if another interpolation point were added, then the interpolation coefficient $\ell_i(x)$ will have to be recomputed. We therefore seek an interpolation polynomial which has the property that a polynomial of higher order may be derived from it by simply adding new terms. **Newton's general interpolation formula** is one such formula and it employs what are called **divided differences**. It is our principal purpose in this section to define such differences and discuss their properties.

Let $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ be the given $(n+1)$ data points. Then the divided differences of order $1, 2, \cdots, n$ are defined by the relations:

$$\begin{aligned}
[x_0, x_1] &= \frac{y_1 - y_0}{x_1 - x_0}, \qquad \text{(first divided difference)};\\
[x_0, x_1, x_2] &= \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}, \qquad \text{(second divided difference)};\\
&\vdots\\
[x_0, x_1, \cdots, x_n] &= \frac{[x_1, x_2, \cdots, x_n] - [x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}, \text{ (nth divided difference)}.
\end{aligned} \tag{1.13}$$

Note: The divided differences are symmetrical in their arguments.

Even if the arguments are equal, the divided differences may still have a meaning.

We then set $x_1 = x_0 + h$ so that

$$\begin{aligned}[x_0, x_1] &= \lim_{h\to 0}[x_0, x_0 + h] = \lim_{h\to 0} \frac{y(x_0 + h) - y(x_0)}{h} \\ &= y^{'}(x_0), \quad \text{ if } y(x) \text{ is differenctiable.}\end{aligned}$$

Similarly,

$$\underbrace{[x_0, x_0, \cdots, x_0]}_{(r+1) \text{ argumetnts}} = \frac{y^r(x_0)}{r!}.$$

Now let the arguments be equally spaced so that $x_n = x_0 + nh$, $n = 1, 2, ..$ Then we obtain

$$\begin{aligned}[x_0, x_1] &= \frac{y_1 - y_0}{x_1 - x_0} = \frac{1}{h}\Delta y_0, \\ [x_0, x_1, x_2] &= \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left(\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right) = \frac{1}{h^2\, 2!}\Delta^2 y_0,\end{aligned}$$

and in general,

$$[x_0, x_1, \cdots, x_n] = \frac{1}{h^n\, n!}\Delta^n y_0. \tag{1.14}$$

Remark. If the tabulated function is a polynomial of nth degree, then $\Delta^n y_0$ would be a constant and hence the nth divided difference would also be a constant.

1.4.2.1 Newton's general interpolation formula

By definition, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0},$$

so that

$$y = y_0 + (x - x_0)[x - x_0].$$

Again

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives

$$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1].$$

Substituting this value of $[x, x_0]$ in above equation, we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1]$$

Proceeding in this way, we obtain

$$\begin{aligned} y = y_0 &+ (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \cdots \\ &+ (x - x_0)(x - x_1)\cdots(x - x_n)[x, x_0, x_1, \cdots, x_n] \end{aligned} \tag{1.15}$$

This formula is called **Newton's general interpolation formula with divided differences**.

Example 8. Using the following table find $f(x)$ as a polynomial in x:

x	-1	0	3	6	7
$f(x)$	3	-6	39	822	1611

Solution: The divided difference table is

x	$f(x)$	1st DD	2nd DD	3rd DD	4th DD
-1	3				
		-9			
0	-6		6		
		15		5	
3	39		41		1
		261		13	
6	822		132		
		789			
7	1611				

Using Newton's general interpolation formula (1.15), we get

$$\begin{aligned} f(x) &= 3 + (x + 1)(-9) + (x + 1)(x)(6) + (x + 1)(x)(x - 3)(5) \\ &\quad + (x + 1)(x)(x - 3)(x - 6)(1) \\ &= x^4 - 3x^3 + 5x^2 - 6. \end{aligned}$$

Exercise

1. Form a table of differences for the function $f(x) = x^3 + 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$. Continue the table to obtain $f(6)$ and $f(7)$.

2. Evaluate (a) $\Delta^2 x^3$, (b) $\Delta^2 \cos x$, (c) $\Delta[(x+1)(x+2)]$, (d) $\Delta(\tan^{-1} x)$.

3. Find the missing term in the following:

x	0	5	10	15	20	25	30
y	1	3	-	73	225	-	1153

4. Using Gauss's backward formula, find the value of $f(33)$ given that $f(25) = 1$, $f(30) = 3$, $f(35) = 5$ and $f(40) = 7$.

5. Find a cubic polynomial which fits the data $(-2, -12)$, $(-1, -8)$, $(2, 3)$ and $(3, 5)$.

6. Given $f(x) = 1/x^2$, find the divided differences $[a, b]$ and $[a, b, c]$.

7. For the polynomial $P_n(x) = \sum_{r=0}^{n} a_r x^{n-r}$, show that $\Delta^n P_n(x) = a_0 n! h^n$. Verify the result by preparing the finite difference table of $P_3(x) = 2x^3 + 3x - 1$ by tabulating it for $x = -2(1)3$.

8. From the following table, find the number of students who obtained marks between 60 and 70:

Marks obtained	0-40	40-60	60-80	80-100	100-120
No. of students	250	120	100	70	50

Solution

1. 239, 371; **2.** (a) $6h^2(h+x)$, (b) $\cos(x+2h) - 2\cos(x+h) + \cos x$, (c) $2x+4$, (d) $\tan^{-1} \dfrac{h}{x(x+h)}$; **3.** 17, 551; **4.** **5.** $-\dfrac{1}{15}x^3 - \dfrac{3}{20}x^2 + \dfrac{241}{60}x - \dfrac{39}{10}$; **6.** $-\dfrac{a+b}{a^2b^2}$, $-\dfrac{ab+bc+ca}{a^2b^2c^2}$; **8.** 54.

For the video lecture use the following link

https://youtube.com/channel/UCk9ICMqdkOOGREITx-2UaEw