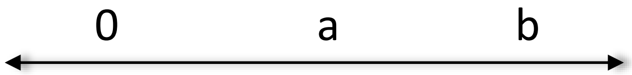
Functions

R: The set of all real numbers

(a , b) = a < x < b is open interval

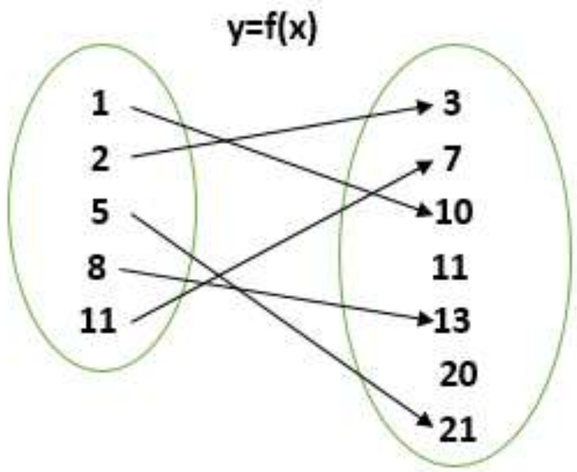
[a , b] = a ≤ x ≤ b is a closed interval



The Function:- A function **((f))** from a set of **D ={ x:x ϵ /R}** to a set **R ={ y:y ϵ R}** is a rule that assigns a unique (single) element **f(x) ϵ R** to each element in **x ϵ D**, denoted by **y= f(x)**. **((f: $\underset{D}{X} \longrightarrow \underset{R}{Y}$))**

The set **D** is called domain of **f** and the set of **R** is called the range of **f**.

Ex:



Df = {1, 2, 5, 8, 11} = Domain
Codomain = {3, 7, 10, 11, 13, 20, 21}
Rf = {3, 7, 10, 11, 13, 20, 21} = Range

The Domain and Range of Root Functions:-

Let y=$\sqrt{f(x)} \mp K$

Df: كل القيم الحقيقية عدا التي تجعل تحت الجذر عدد سالب اي

Df = { X: f(x)≥0}

Rf: لأيجاد قيم **Rf** نعتمد على قيم **Df** أي

1. اذا كانت قيم **Df**غير محددة (ليست محصورة بفترة) هنا يكون **Rf = { y:y ≥K}**
2. اذا كانت قيم **Df** محددة (**a ≤ x ≤ b**) هنا نعوض في الدالة لأيجاد قيم **Rf**.

<u>Ex: find the domain and range if</u>:-

(1) Y = f(x) = $\sqrt{x} - 2$

$\text{sol} = \because\ x - 2 \geq 0 \rightarrow x \geq 2$

$\therefore D_F = \{x : x \geq 2\}$

$\therefore R_F = \{y : y \geq 0\}$

(2) $y = 3 + \sqrt{2 - x}$

$\text{sol} = \because\ 2 - x \geq 0 \rightarrow [-x \geq -2](-1) \rightarrow x \leq 2$

$\therefore D_F = \{x : x \leq 2\}$

$\therefore R_F = \{y : y \geq 3\}$

(3) $y = \sqrt{x^2 - 4}$

$\text{sol} = \because\ x^2 - 4 \geq 0 \rightarrow (x - 2)(x + 2) \geq 0$

$x - 2 = 0 \rightarrow x = 2$

$x + 2 = 0 \rightarrow x = -2$

$(x - 2)(x + 2)$

$\therefore D_F = \{x : x \leq -2 \ or \ x \geq 2\}$

Or $\{-\infty < x \leq -2, 2 \leq x < \infty\}$

Or $(-\infty, -2], [2, \infty)$

$\therefore R_F = \{y : y \geq 0\}$

(4) $y = \sqrt{1 - x^2}$

$\text{sol} = \because\ 1 - x^2 \geq 0 \rightarrow (1 - x)(1 + x) \geq 0$

$1 - x = 0 \rightarrow x = 1$

$1 + x = 0 \rightarrow x = -1$

$(1 - x)(1 + x)$

$\therefore D_F = \{x : -1 \leq x \leq 1\}$

$\therefore R_F = \{y : 0 \leq y \leq 1\}$

The Domain and Range of fraction Functions:-

Let $y = \frac{f(x)}{g(x)}$ → Df= R/ {g(x) =0}

Df: كل القيم عدا التي تجعل المقام = صفر

Rf: نبدل السؤال من y بدلالة x الى x بدلالة y

Ex: find the domain and range if:-

(1) $y = \frac{1}{x}$

Sol: x=0

$\therefore D_F = R/\{0\}$

Rf : $y = \frac{1}{x} \rightarrow xy = 1 \rightarrow x = \frac{1}{y}$

$\therefore R_F = R/\{0\}$

$(2)\ y = \frac{1}{x+3}$

$sol\text{:}\ x + 3 = 0 \rightarrow x = -3$

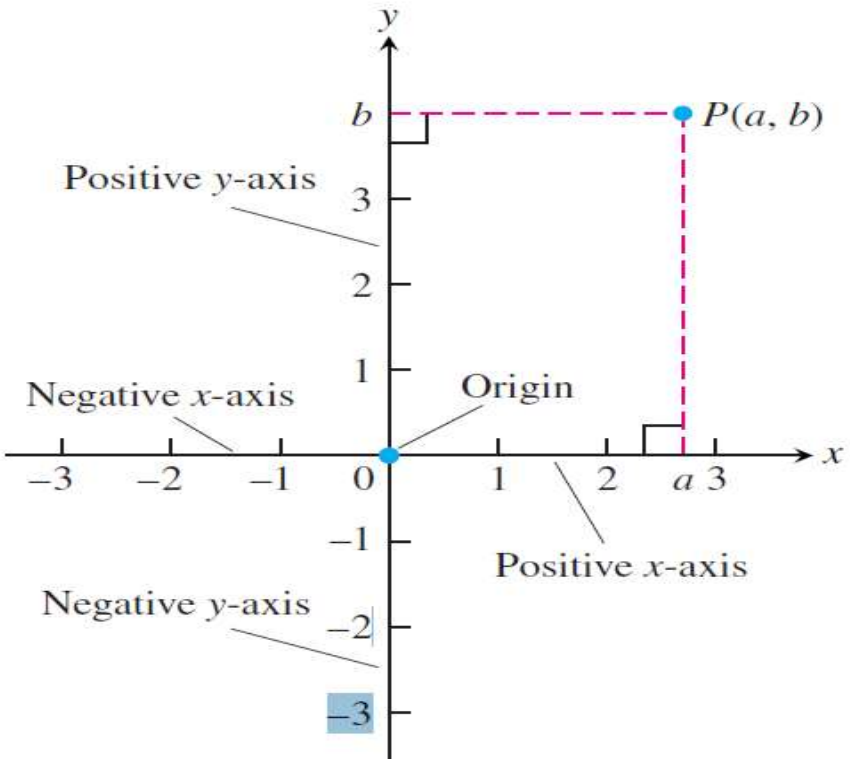
$\therefore D_F = R/\{-3\}$

Rf: $y = \frac{1}{x+3} \rightarrow xy + 3y = 1 \rightarrow xy = 1 - 3y \rightarrow x = \frac{1-3y}{y}$

$\therefore R_F = R/\{0\}$

Graphs of some functions

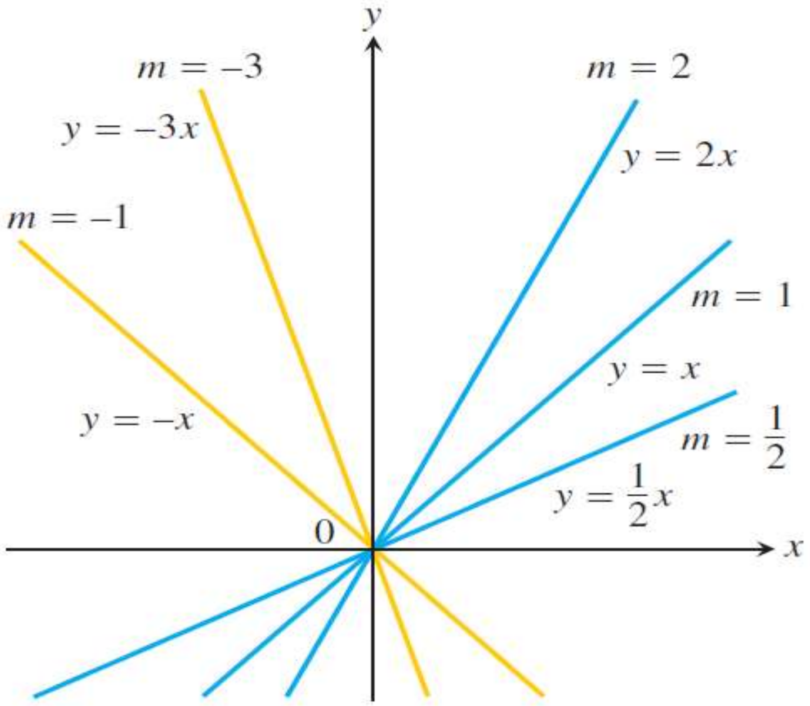
(1) Coordinates



2-Lines (first degree) معادلة المستقيم من الدرجة الأولى

a. Lines passes through origin

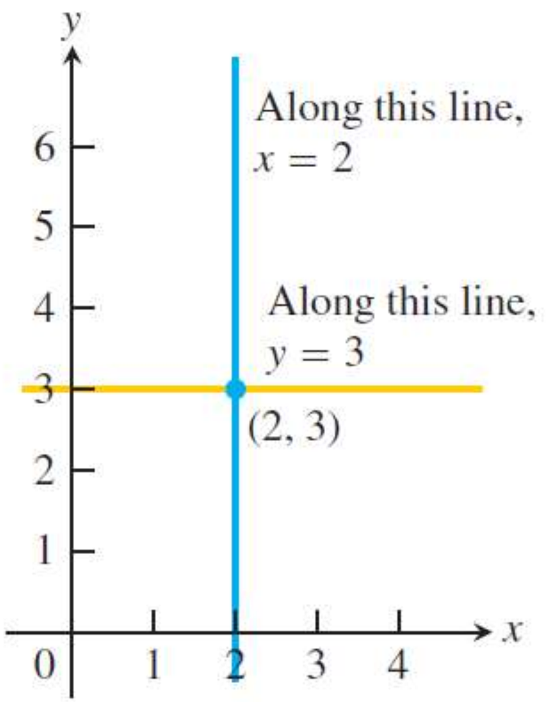
$y = mx$ $\quad$ $m: slope$ $\quad$ $m = \tan\theta$



Blue lines -> $y = mx$ $\quad$ $m > 0$

Yellow lines -> $y = mx$ $\quad$ $m < 0$

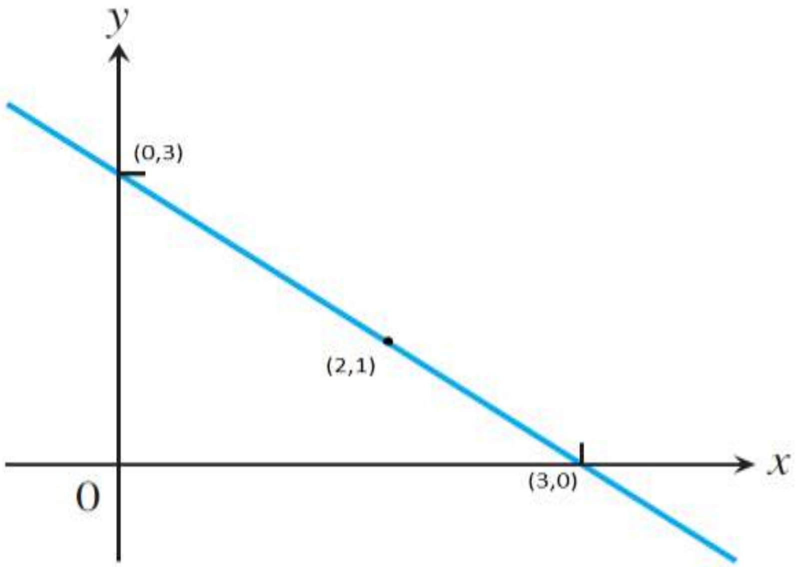
b- Vertical & horizontal lines



C- lines that not passes through origin

$$y = mx + c$$

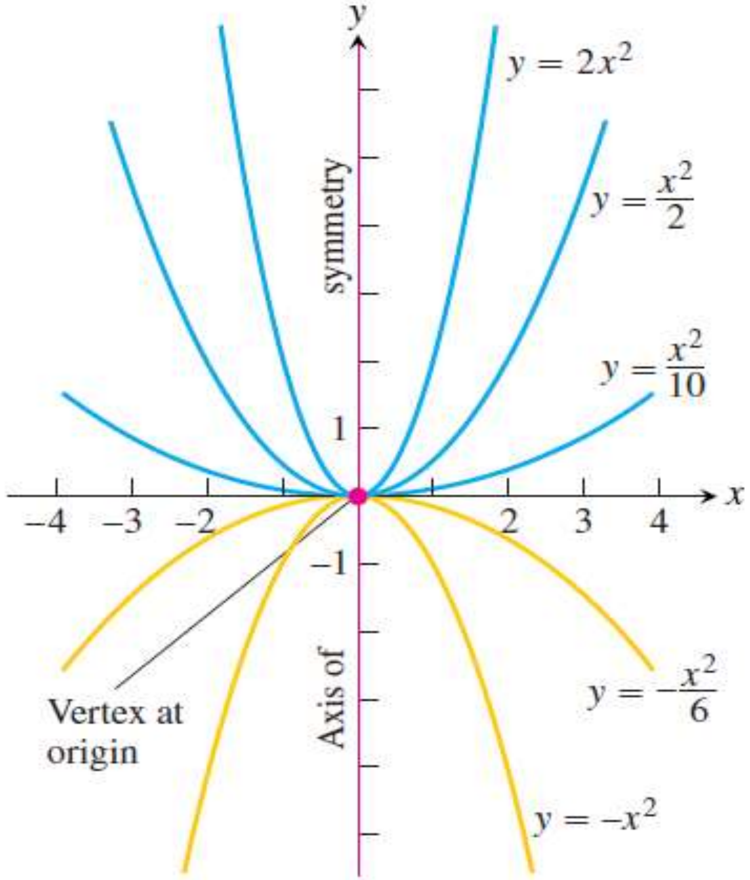
Ex:- $x + 3 = 3 \rightarrow \ y = -x + 3$



3- Parabola(تربيعية في احد المتغيرات) القطع المكافئ

a-with vertices at origin

$y = mx^2 \qquad x = my^2$



4- circle X&Y تربيعية في

شروط معادلة الدائرة :

1-تربيعية في X و Y

2-معامل x^2 يساوي معامل y^2

3-خالية من الحد xy

$$(x-h)^2 + (y-k)^2 = r^2$$

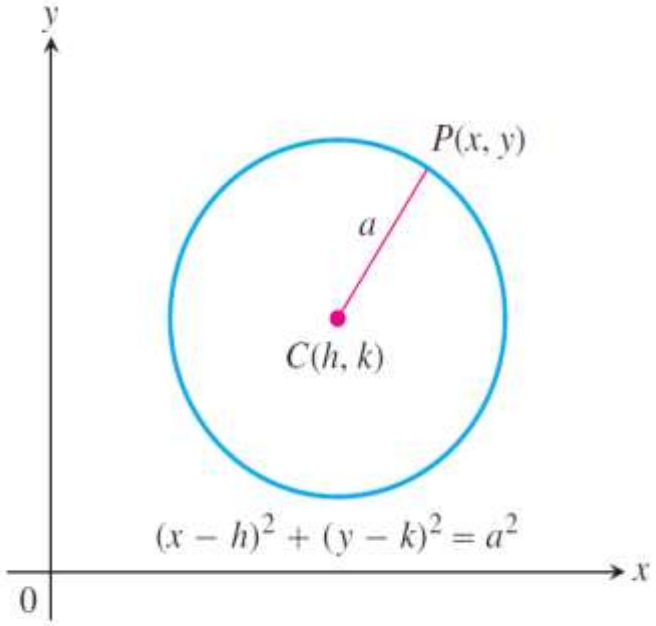
$$x^2 + y^2 + ax + by + c = 0$$

لأيجاد المركز و نصف القطر من المعادلة العامة نستخدم القوانين التالية:-

$$center = (h,k) = \left(\frac{-a}{2}, \frac{-b}{2}\right)$$

$$radius = r = \sqrt{h^2 + k^2 - c}$$

Note: - The circle of radius centered at the origin is:- $x^2 + y^2 = r^2$

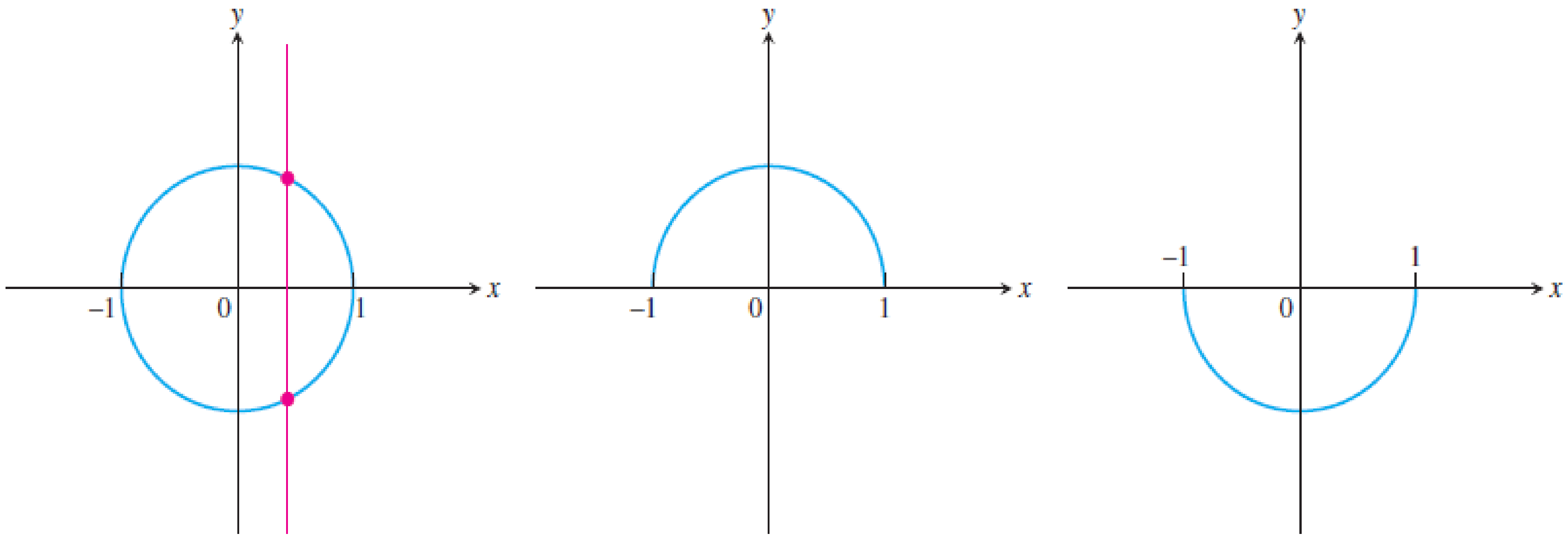


Ex:-Notes:- $r = \sqrt{h^2 + k^2 - c}$

If $h^2 + k^2 - c > 0 \rightarrow circle$

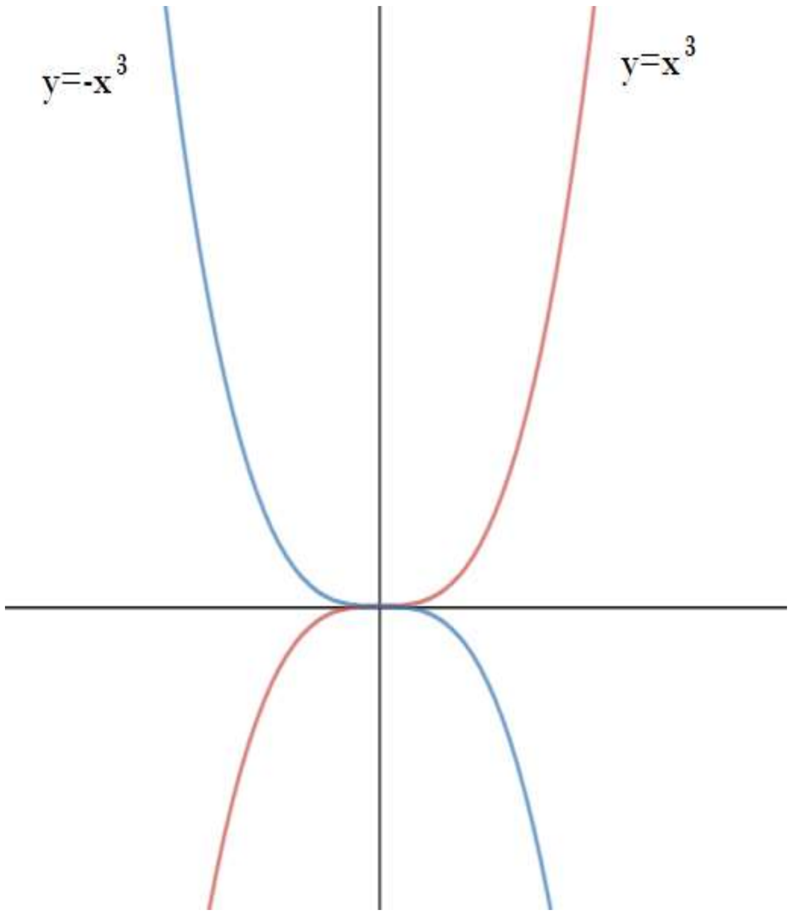
If If $h^2 + k^2 - c = 0 \rightarrow point$

If $h^2 + k^2 - c < 0 \rightarrow$ لاتعني شيء

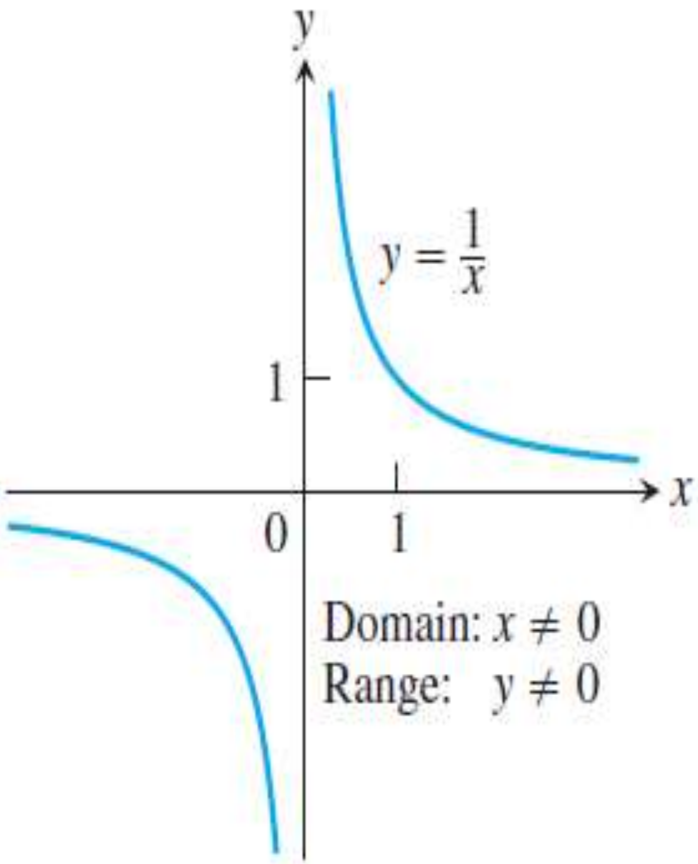


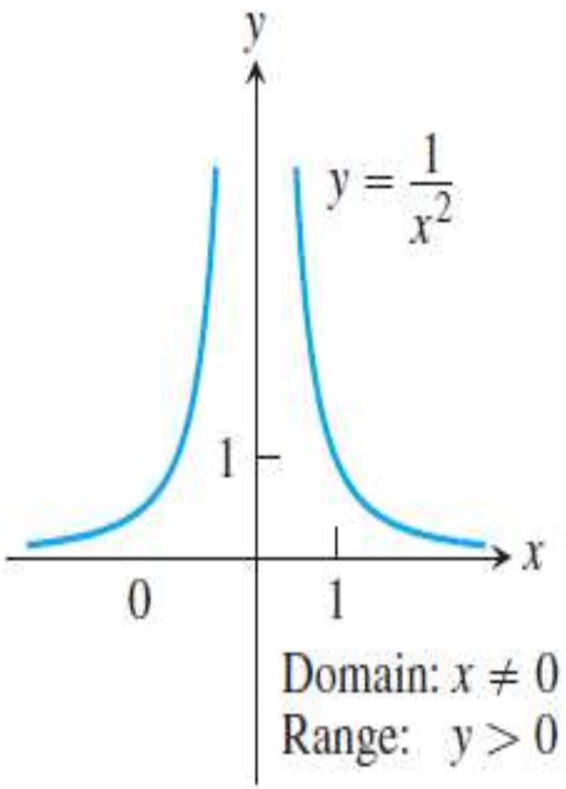
(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

5-cubic functions الدوال التكعيبية



6-fraction functions الدوال الكسرية

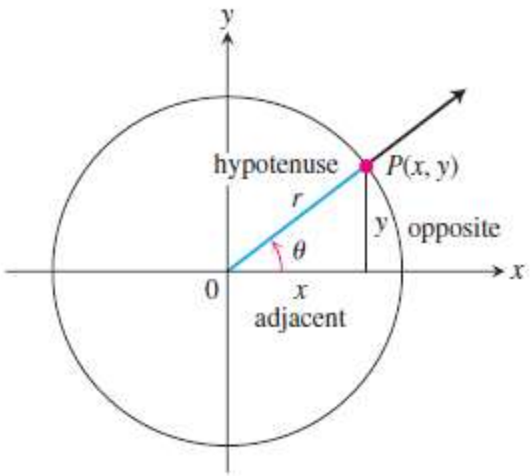




Trigonometry

1. the Sin of θ defined as :

Sin $\theta = \frac{y}{r}$ $\frac{\text{مقابل}}{\text{وتر}}$



2 – The Cos of θ defined as:

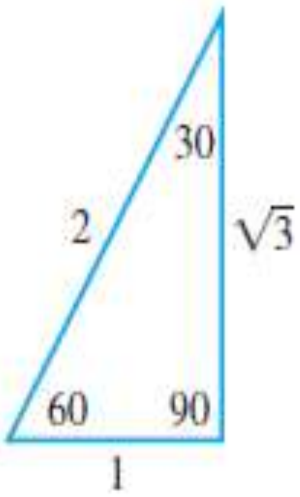
Cos $\theta = \frac{x}{r}$ $\frac{\text{مجاور}}{\text{وتر}}$

3 – The Tan of θ defined as:

Tan $\theta = \frac{\sin\theta}{\cos\theta}$ $\frac{\text{مقابل}}{\text{مجاور}}$

4 – The cot of θ defined as:

Cot θ = $\frac{\cos\theta}{\sin\theta}$ $\frac{x}{y}$

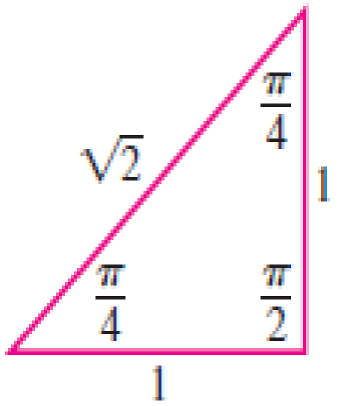


5 – The sec of θ defined as

Sec θ = $\frac{1}{\cos\theta}$ $\frac{r}{x}$

6 – The Csc of θ defined as:

Csc θ = $\frac{1}{\sin\theta}$ $\frac{r}{y}$



Note

(1) $sin^2\theta + cos^2\theta = 1$

(2) $1 + tan^2\theta = sec^2\theta$

(3) $1 + cot^2\theta = csc^2\theta$

Note

$$1^\circ = \frac{\pi}{180} \text{ radian}$$

$$1 \text{ radian} = \frac{180}{\pi} \text{ degree}$$

Ex:

$30^\circ = 30\ x\ \frac{\pi}{180} = \frac{\pi}{60}$, $45^\circ = 45\ x\ \frac{\pi}{180} = \frac{\pi}{4}$

$60^\circ = 60\ x\ \frac{\pi}{180} = \frac{\pi}{3}$, $90^\circ = 90\ x\ \frac{\pi}{180} = \frac{\pi}{2}$

$120^\circ = 120\ x\ \frac{\pi}{180} = 2\ \frac{\pi}{3}$

Note

(1) $\sin(2\pi + \theta) = \sin\theta$, $\cos(2\pi + \theta) = \cos\theta$

(2) $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$

(3) $\sin(2\pi - \theta) = \sin\theta$, $\cos(2\pi - \theta) = -\cos\theta$

(4) $\sin(\pi + \theta) = \sin\theta$, $\cos(\pi + \theta) = -\cos\theta$

*(5) $\sin(\frac{\pi}{2}-\theta)=\cos\theta$, $\sin(\frac{\pi}{2}-\theta)=\sin\theta$

(6) $\sin(\frac{\pi}{2}+\theta)=\cos\theta$, $\sin(\frac{\pi}{2}+\theta)=-\sin\theta$

(7) $\sin(x\pm y)=\sin x\cos y \pm \cos x\sin y$

(8) $\cos(x\pm y)=\cos x\cos y \pm \sin x\sin y$

(9) $\sin A+\sin B=2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$

(10) $\sin A-\sin B=2\sin(\frac{A-B}{2})\cos(\frac{A+B}{2})$

(11) $\cos A+\cos B=2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$

(12) $\cos A-\cos B=-2\sin(\frac{A+B}{2})\sin(\frac{A-B}{2})$

(13) $\sin 2x=2\sin x\cos x$

(14) $\cos 2x=\cos^2x-\sin^2x=1-2\sin^2x=2\cos^2x-1$ مهم

(15) $\sin^2x=\frac{1}{2}(1-\cos 2x)$

$\cos^2x=\frac{1}{2}(1+\cos 2x)$

مهم

$45°=\frac{\pi}{4}$, $225°=5\frac{\pi}{4}$

$90°=\frac{\pi}{2}$, $270°=\frac{3\pi}{2}$

$135°=\frac{3\pi}{4}$, $315°=7\frac{\pi}{4}$

$180°=\pi$, $360°=2\pi$

Define تعريف

If F: IR→IR is a given function then:-

(1) f is called even function if :- متناظرة حول محور الصادات

$$f(-x) = f(x) \text{ For all } x \in Df$$

(1) f is called odd function if :- متناظرة حول محور الاصل

$$f(-x) = -f(x) \text{ For all } x \in Df$$

Examples

(1) $y = f(x) = \rightarrow f(-x) = (-x)^4 = x^4 \; even$

(2) $y = f(x) = x^5 \rightarrow f(-x) = (-x)^5 = x^4 = -x^5 \; odd$

(3) $y = f(x) = 5 \rightarrow f(-x) = 5 = f(x) \quad even$

(4) $y = f(x) = x - 5 \rightarrow f(-x) = -x - 5 \; not\ even\ nor\ odd$

(5) $y = f(x) = sinx$

$f(-x) = sin(-x) = -sinx = -f(x) \quad odd$ حول نقطة

(6) $y = f(x) = cosx$

$f(-x) = cos(-x) = cosx \quad even$

(7) $y = f(x) = tanx$

$$f(-x) = \frac{sin(-x)}{cos(-x)} = \frac{-sinx}{cosx} = -f(x) \quad odd$$

Note

(1) $even \pm even = even$

(2) $odd \pm odd = odd$

(3) $\frac{odd}{odd} = \frac{odd}{odd} = odd \; x \; odd = evev \; x \; even = even$

(4) $odd \; x \; even = even \; x \; odd = odd$

(5) $\frac{odd}{even} = \frac{even}{odd} = odd$

Ex:

(1) $f(x) = \frac{x}{x^2+1} \quad \frac{odd}{even+odd} = \frac{odd}{even} = odd$

(2) $f(x) = x^2 cosx \qquad even \; x \; even = even$

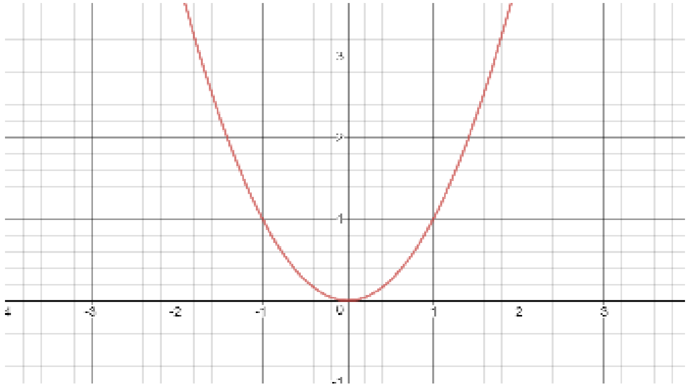
(3) $f(x) = \frac{x}{2 + sinx} \qquad \frac{odd}{even + odd}$

Note

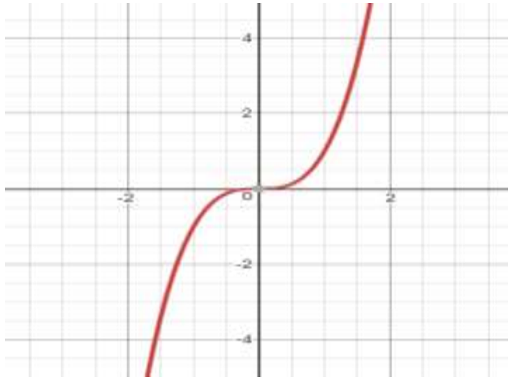
(1) $even \; functions \; are \; Symmetrical \; about \; y - axis$

(2) $odd \; functions \; are \; Symmetrical \; about \; y - origin$

Ex:



Y = x^2 $even$



y = x^3 odd

Define

$$if\ f: x \to y\ \&\ g: y \to z\ then\ the\ composition\ :-$$

$$gof(x) = g\big(f(x)\big)$$

Ex:

$$if\ f(x) = sinx\ ,\ g(x) = \frac{-x}{2}\ :$$

$$find\ gof(x)\ \&\ fog(x)$$

Sol:- $gof(x) = g\big(f(x)\big) = \frac{-f(x)}{2} = \frac{-sinx}{2}$

$$fog(x) = f\big(g(x)\big) = sin\big(g(x)\big) = sin\left(-\frac{x}{2}\right) = -sin\frac{x}{2}$$

Ex:

$$if\ f(x) = x^2 \qquad g(x) = x - 7$$

$$find\ gof\ (x)\ \&\ fog(x)$$

Sol:-

$$g\big(f(x)\big) = g(x^2) = x^2 - 7$$

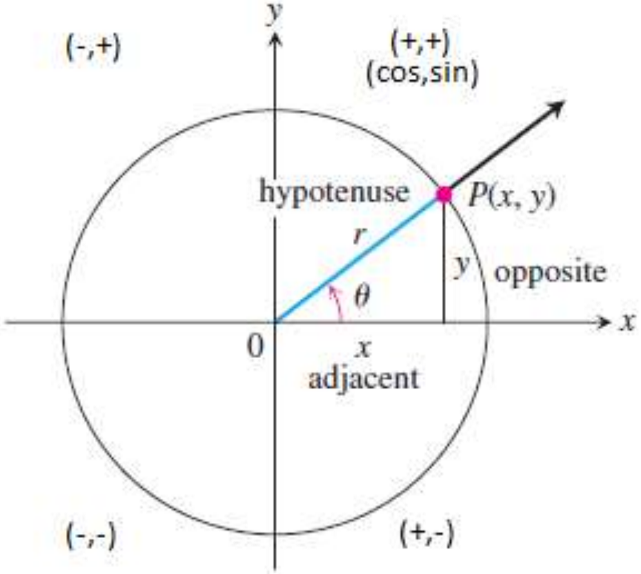
$$f\big(g(x)\big) = f(x-7) = (x-7)^2$$

- **Graphs of Trigonometric functions:-**

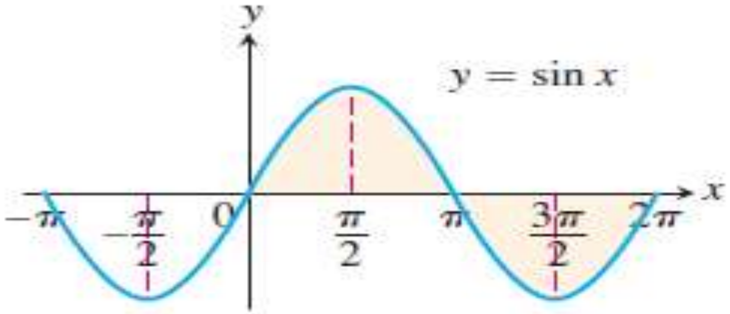
(1) Y=sinx

$$\text{Sin}\theta = \frac{y}{1} \rightarrow y = sin\theta$$

$$\text{Cos}\theta = \frac{x}{1} \rightarrow x = cos\theta$$

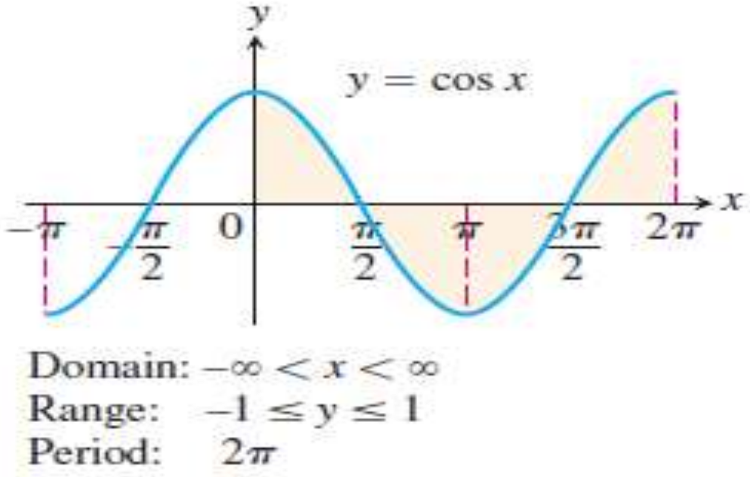


Y=sinx



Domain: $-\infty < x < \infty$
Range: $-1 \leq y \leq 1$
Period: 2π

- **Y=cosx**



Y= tanx

$\Delta OPA\ similer\ \Delta OBQ$

$\frac{BQ}{AP} = \frac{OB}{OA}$

Since

OB=1

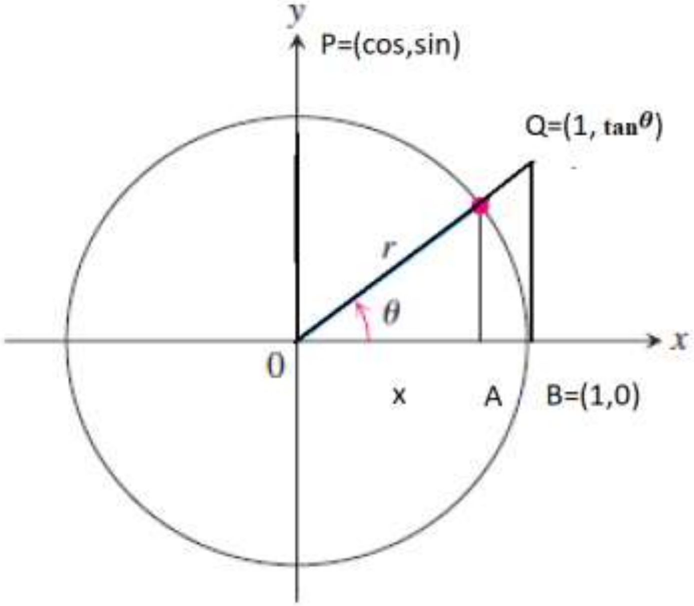
OA=Cosθ , $AP = Sin\theta$

BQ=$\frac{OB}{OA}$, $AP = \frac{1}{COS\theta}$ **, Sinx=tanθ**

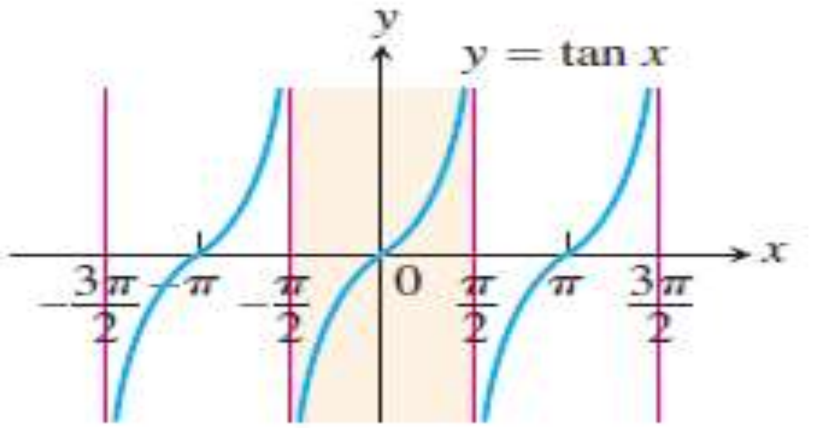
BQ (x,y)

OB=X=1 , Y=BQ= Tanθ

$\therefore Q = (1, TAN)$



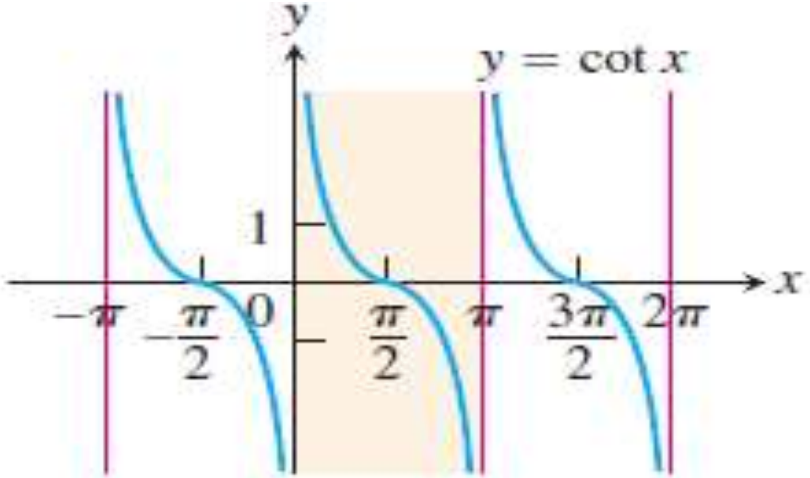
Y= Tanx



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Range: $-\infty < y < \infty$

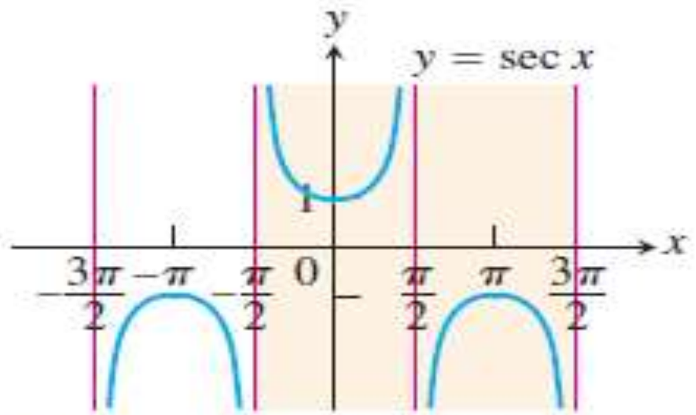
Y=cotx



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$

Range: $-\infty < y < \infty$

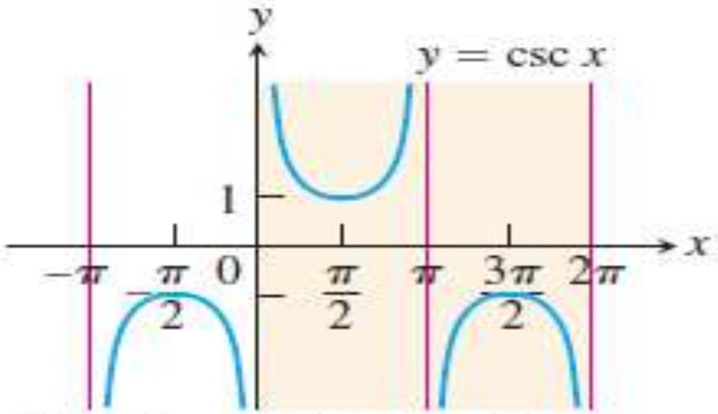
Y =secx



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Range: $y \leq -1$ and $y \geq 1$

Y=cscx



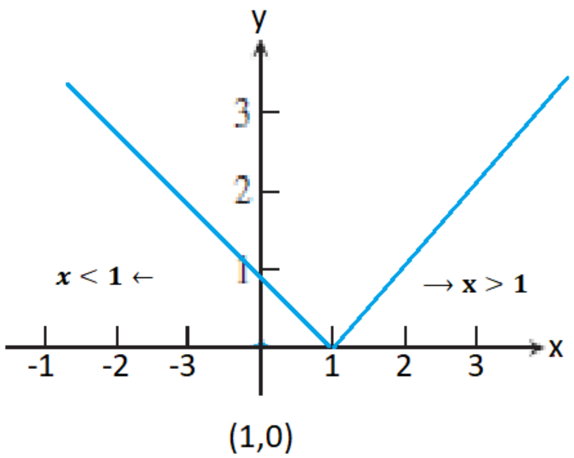
Domain: $x \neq 0, \pm\pi, \pm2\pi, \ldots$

Range: $y \leq -1$ and $y \geq 1$

Absolute value function :-

$$fx = |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

$$Df = R \,, Rf = [0, \infty) = IR^{+}$$



EX: sketch the graph:-

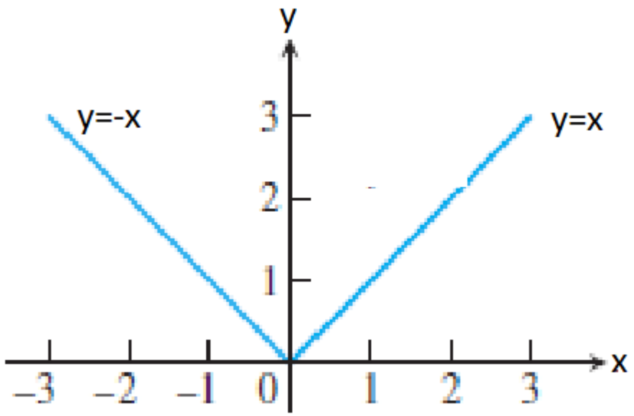
- Y=f(x)=| X-1 |
- Y=f(x)=| 2-x |

نقوم برسم المعادلة بدون قيمة مطلقة ثم
نقلب الرسم في جهة السالب الى الموجب

Sol1

$$F(x) = \begin{cases} (x-1) & x > 1 \\ -(x-1) & x < 1 \end{cases}$$

$$\Rightarrow y = x - 1 \ , \qquad y = 1 - x$$

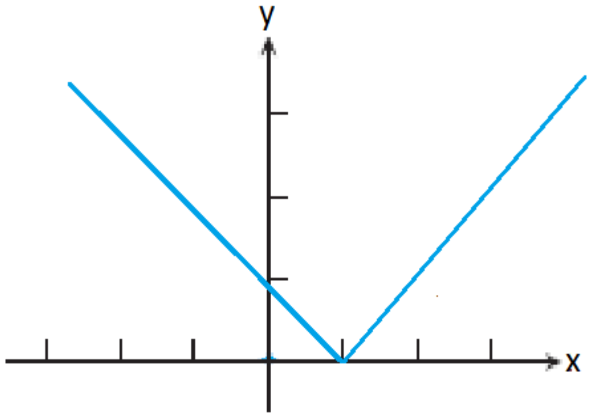


Sol2

$$F(x) = \begin{cases} x-2 & x > 2 \\ 2-x & x < 2 \end{cases}$$

$$\Rightarrow y = x - 2 \ , \qquad y = 2 - x$$

اذا كان الناتج موجب بعد التعويض نبقيه كما
هو واذا سالب نضربه بأشارة سالبة



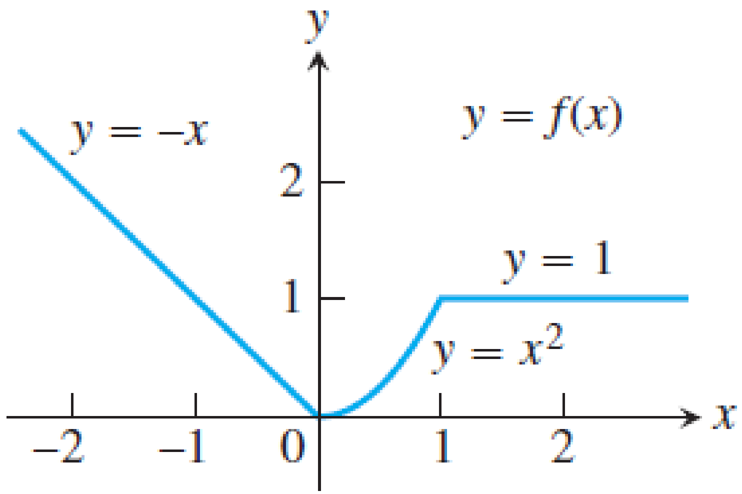
Piecewise-defined function:-

Sometimes a function is described by using different formulas and Different parts of its domain (one example is absolute value fun.)

Ex: - graphing Piecewise-defined function:

$$F(x) = \begin{cases} -x \;, & x < 0 \\ x^2 \;, & 0 \le x \le 1 \\ 1 \;, & x > 1 \end{cases}$$

Is defined on the entire real line but has values given by different formulas depending on the position of x.



Notes

1) $|a\,b|=|a|\,|b|$ or $|a\,b\,c\ldots m|=|a||b||c|\ldots.|m|||$
2) $|a+b| \le |a|+|b|$ or $|a+b+c+\ldots m|\le|a|+|b|+|c|+\ldots+|m|$
3) $|a-b| \ge |a| - |b|$
4) If $|x|\le a \leftarrow\rightarrow -a \le x \le a$
5) If $|x|\ge a \leftarrow\rightarrow -a \ge x \ge a$
6) $|x-c| \le a \leftarrow\rightarrow -a+c \le x \le a+c$
7) $|x-c| \ge a\leftarrow\rightarrow x<-a+c \cup x>a+c$

Ex:- find the value of x which satisfy $\left|\frac{3x+1}{2}\right| < 1$

Sol:

$-1< \frac{3x+1}{2} <1 \rightarrow -2< x3 + 1 <2 \rightarrow -3< 3x <1$

$\rightarrow -1< x < \frac{1}{3}$

Note : if we have a graph of y=f(x) then :-

1) -F(x)=-y ينقلب الرسم على محور السينات
2) F(-x) ينقلب على محورالصادات
3) F(x)+2=y+2 يتحرك الرسم الى الاعلى
4) F(x)-2 يتحرك الرسم الى الاسفل
5) F(x-2) يتحرك الرسم نحو اليمين
6) F(x+2) يتحرك الرسم الى اليسار
7) lF(x)l الرسم تحت محور السينات ينقلب الى محور الصادات

Ex :- sketch the graph of:-

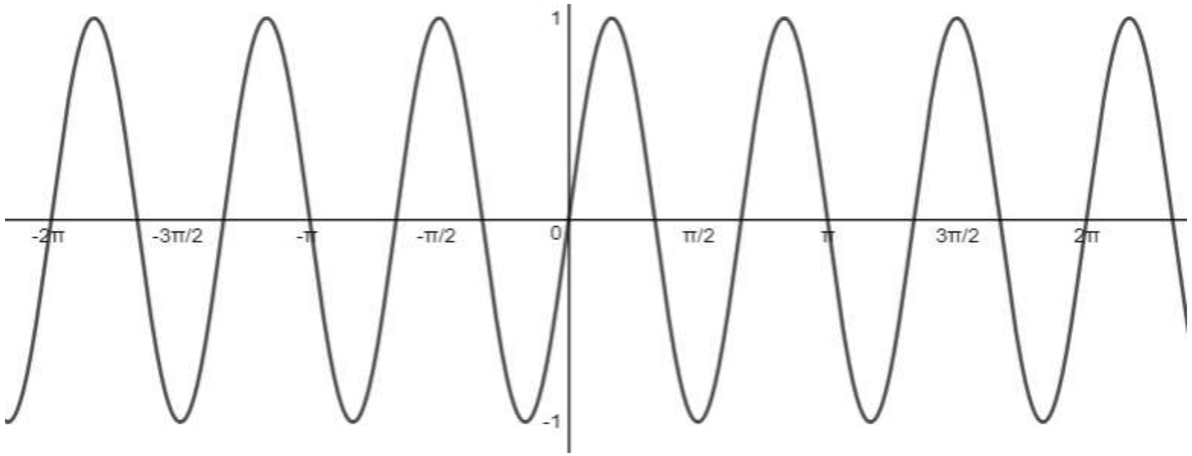
(1) y= 3sinx , (2) y=sin4x , (3) y=3sin4x

ملاحظة 1: العدد المضروب بالدالة المثلثية يكبر حجم الموجة اذا كان اكبر من واحد ويصغرها اذا كان بين صفر وواحد .

ملاحظة 2: العدد المضروب في الزاوية يضاعف عدد الموجات اذا كان اكبر من واحد ويقللها اذا كان بين صفر وواحد .

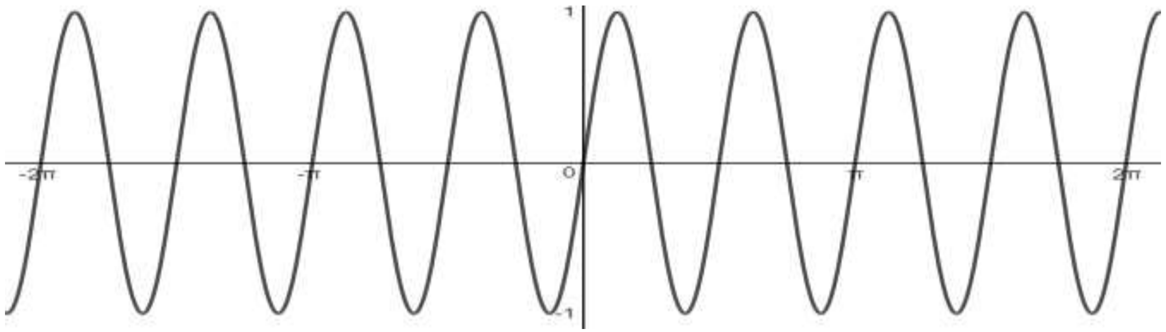
Sol1 :

sin 3x

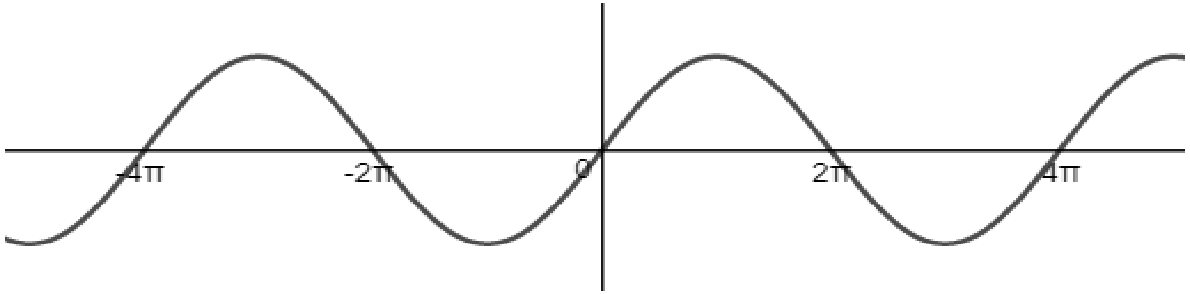


Sol2:

y=sin4x



Y=sin($\frac{1}{2}x$)

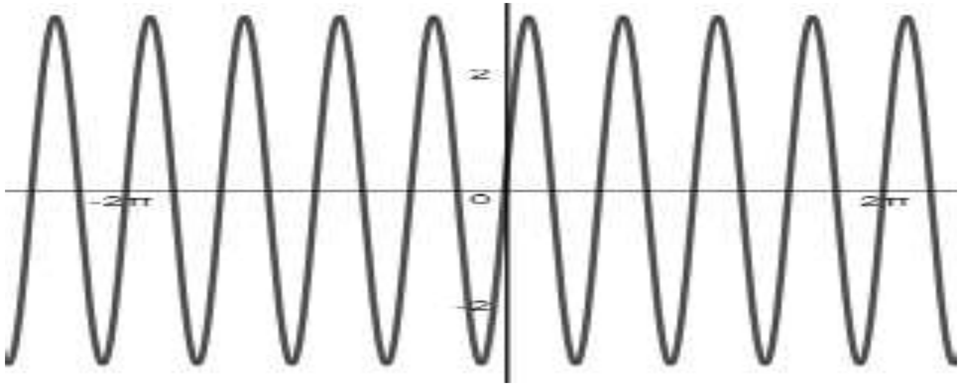


$$\frac{1}{2}\boldsymbol{x} = \boldsymbol{\pi} \ , \boldsymbol{x} = 2\boldsymbol{\pi}$$

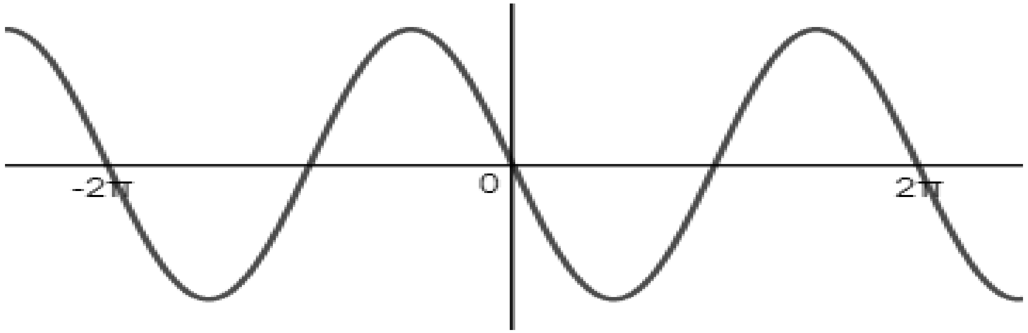
$$\frac{1}{2}\boldsymbol{x} = 2\boldsymbol{\pi} \ , \qquad \boldsymbol{x} = 4\boldsymbol{\pi}$$

Sol3:

Y=3sin4x



Y=sin(x-π)

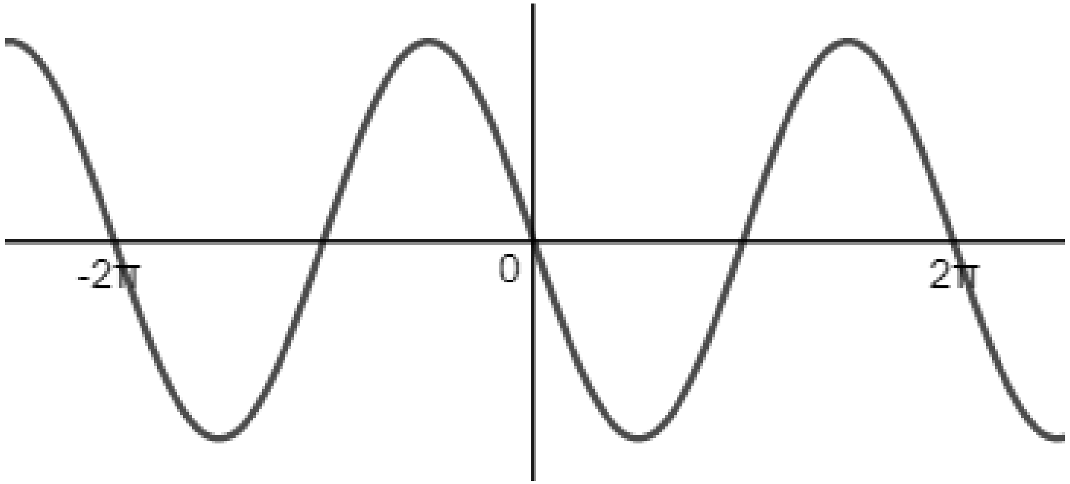


X=$\frac{\pi}{2}$

Y=sin($\frac{\pi}{2} - \pi$) = sin$\frac{-\pi}{2}$

Y=-sin$\frac{\pi}{2}$= -=-1

Y=-sinx

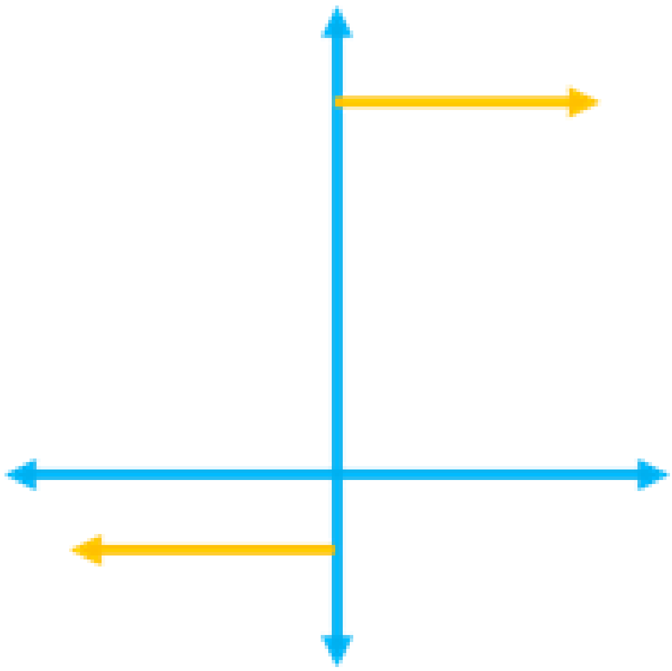


Sin(x-π) = sin –(π-x) = -sin($\pi - x$) = -sinx

Solve problem's / Chapter (1)

Q/ If $f(x) = \begin{cases} 3 & x \geq 0 \\ -1 & x < 0 \end{cases}$ find the Domain, Range and Draw

$D_f = R$, $R_f = \{-1, 3\}$



Q/ find the Domain of $f(x) = \frac{3x^2-4}{\sqrt[3]{x-1}-2}$

$\sqrt[3]{x-1} - 2 = 0 \Rightarrow \sqrt[3]{x-1} = 2 \Rightarrow x - 1 = 8 \Rightarrow x = 9$

$\therefore D_f = \{x : x \in R, x \neq 9\}$

Q/ find the domain and Range for the following functions:

1) $y = x - \frac{1}{x}$

$$D_f = \backslash R/\{0\}$$

To find R_f we can put x as a function of y:

$$y = \frac{x^2 - 1}{x} \Longrightarrow xy = x^2 - 1 \Longrightarrow x^2 - xy - 1 = 0$$

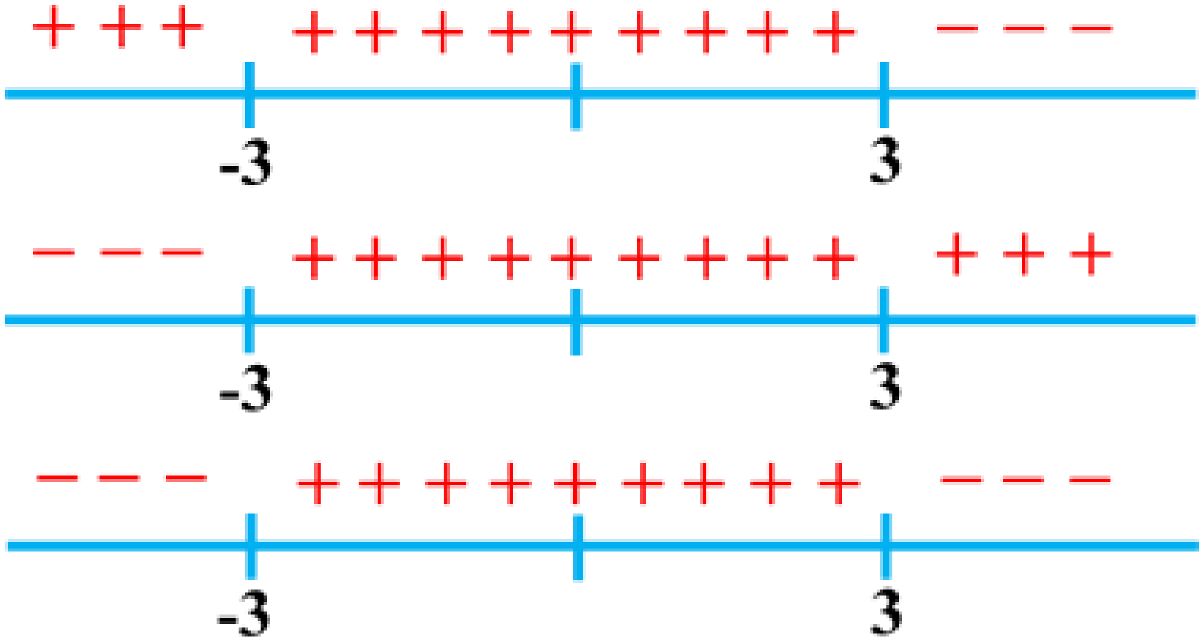
$$\because x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-(-y) \mp \sqrt{y^2 - 4(1)(-1)}}{2(1)} = \frac{y \mp \sqrt{y^2 + 4}}{2}$$

$R = R$ or $(-\infty, \infty)$ or $\{y: -\infty < y < \infty\}$

$$2)\ y = \sqrt{a - x^2}$$

$$a - x^2 = (3 - x)(3 + x)$$

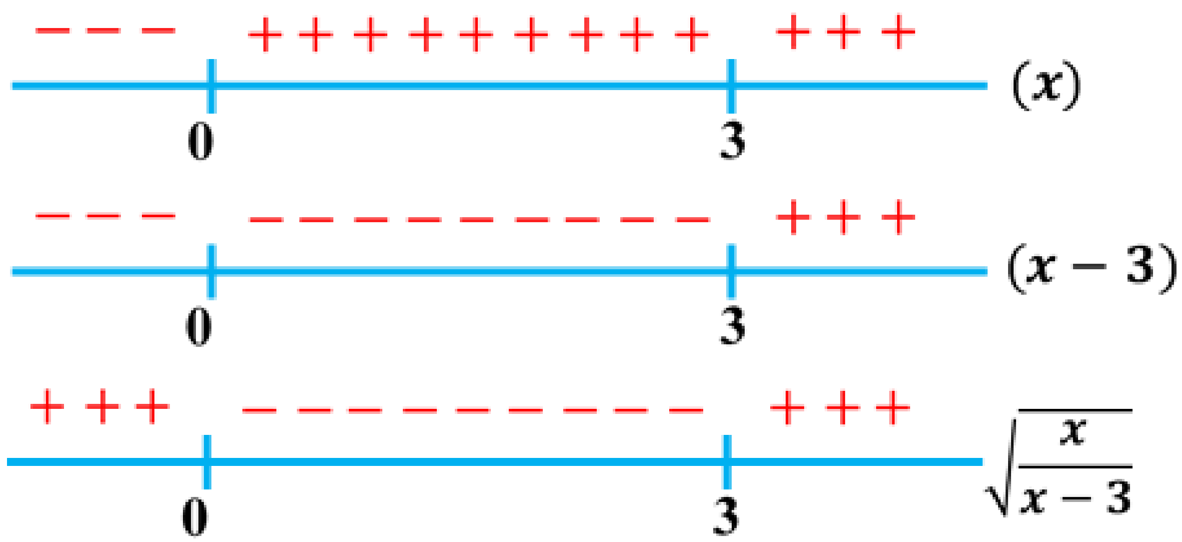


$$\therefore D_f = (-3, 3)$$

$$y = \sqrt{a - x^2} \Longrightarrow y^2 = a - x^2 \Longrightarrow x^2 = a - y^2 \Longrightarrow x = \mp\sqrt{a - y^2}$$

$$\therefore R_f = (-3, 3)$$

$3)\ f(x) = \sqrt{\dfrac{x}{x-3}}$



$(-\infty, 0] \cup [3, \infty)$ / (القيم التي تجعل المقام = صفر)

$\therefore D_f = (-\infty, 0) \cup (3, \infty)$

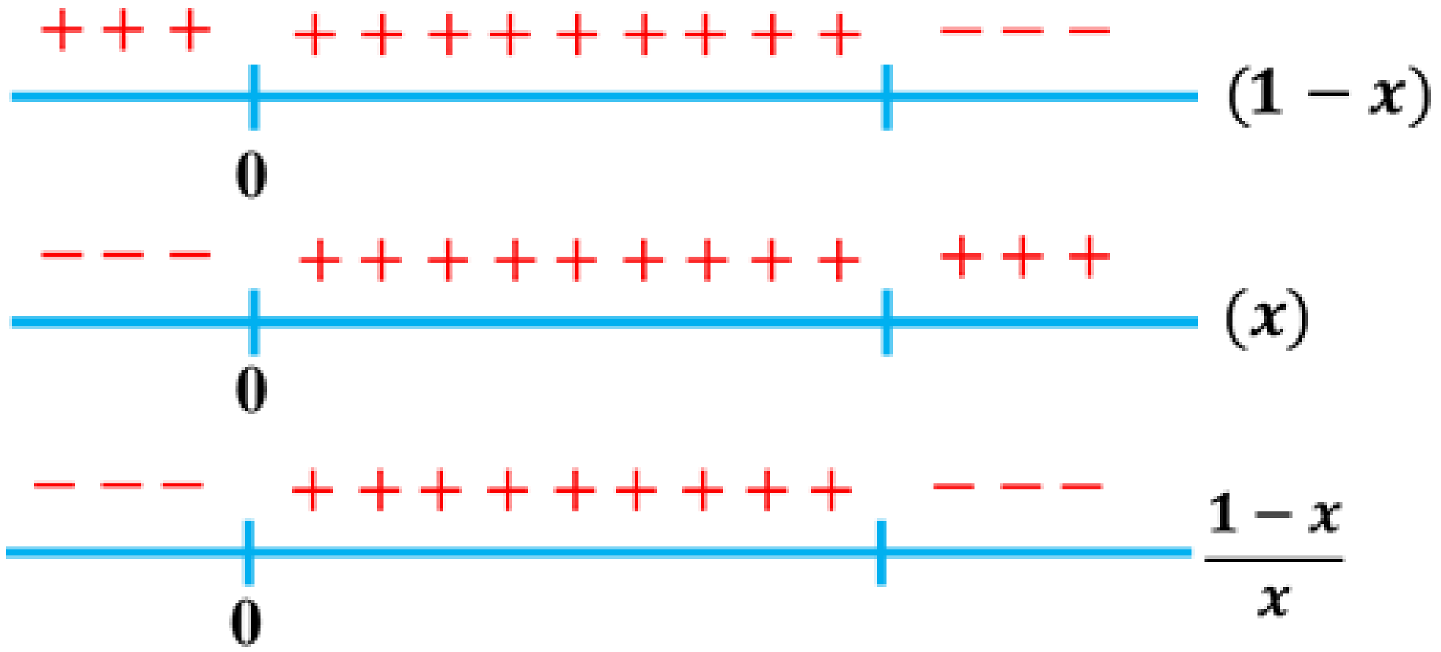
$$y = \sqrt{\frac{x}{x-3}} \Rightarrow y^2 = \frac{x}{x-3} \Rightarrow y^2x - 3y^2 = x$$

$$y^2x - x = 3y^2 \Rightarrow x\left(y^2 - 1\right) = 3y^2 \Rightarrow x = \frac{3y^2}{y^2 - 1}$$

$$y^2 - 1 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \mp 1$$

$\therefore R_f = R/\{1, -1\}$

4) $f(x) = \sqrt{\frac{1}{x} - 1} = \sqrt{\frac{1-x}{x}}$



$\therefore D_f = (0, 1)$

$$y = \sqrt{\frac{1-x}{x}} \Rightarrow y^2 = \frac{1-x}{x} \Rightarrow xy^2 = 1 - x \Rightarrow xy^2 + x = 1 \Rightarrow x\left(y^2 + 1\right) = 1$$

$$x = \frac{1}{y^2 + 1}$$

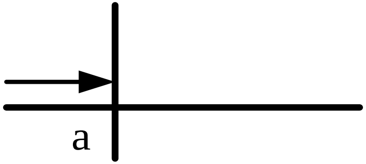
$\therefore R_f = R$

Limits & continuity
الـــغـــاية والاستـــــمـــرارية

Def.:- By $\lim_{x \to a} f(x) = L$ (L is finite)

We means that $f(x)$ approches to (L) as (X) approches to (a)

Def.:- B y $\lim_{x \to a^+} f(x)$ we means that x approches to a from the right



Similarly we define

$\lim_{x \to a^-} f(x)$

Note:-

If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$ then we say that the limit exist and $\lim_{x \to a^+} f(x) = L$

Otherwise the limit does not exist

Ex:- Find $\lim\limits_{x \to a^-} f(x)$, $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to a} f(x)$ if it exist

1) $f(x) = \begin{Bmatrix} 1 & x \geq 2 \\ -1 & x < 2 \end{Bmatrix}$ (a=2)

2) $f(x) = \begin{Bmatrix} x^2 + 1 & x \leq 1 \\ 1 + x & 1 < x < 2 \\ 3 & x \geq 2 \end{Bmatrix}$ (a=1, a=z)

Sol. (1)

$\lim\limits_{x \to 2^+} f(x) = 1$

$\lim\limits_{x \to 2^-} f(x) = -1$

$\because \lim\limits_{x \to 2^+} f(x) \neq \lim\limits_{x \to 2^-} f(x)$

$\therefore$ The limit not exist

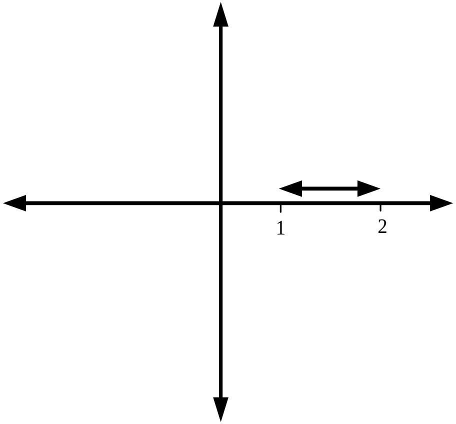
Sol. (2) at a=1

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} (1+x) = 1+1 = 2$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} (x^2+1) = 1^2+1 = 2$$

$$\because \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = 2$$

$\therefore$ limit exist & $\lim_{x\to 1} f(x) = 2$



at a=2

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2} (3) = 3$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (1+x) = 1+2 = 3$$

The limit exist & $\lim_{x\to 2} f(3) = 3$

Theorem: if $\lim_{x\to a} f(x), \lim_{x\to a} g(x)$ exist

1) $\lim_{x\to a} k = k$

2) $\lim_{x\to a} x = a$

3) $\lim_{x\to a} (f(x) \mp g(x)) = \lim_{x\to a} f(x) \mp \lim_{x\to a} g(x)$

4) $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$

5) $\lim_{x\to a} (f(x)g(x)) = \lim_{x\to a} f(x) . \lim_{x\to a} g(x)$

6) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

7) $\lim_{x\to 0^+} = \frac{1}{x} = +\infty$

$\lim_{x\to 0^-} = \frac{1}{x} = -\infty$

$\lim_{x\to \infty} = \frac{1}{x} = 0$

Note: - the following symbol are undefined غير معرفة

$$\begin{cases} \frac{0}{0}, 0^0, 1^\infty, \infty^0, \frac{\infty}{\infty}, \infty - \infty \\ \qquad 0.\infty \end{cases}$$

But we can write: $\infty + \infty = \infty$

Method of find the limits:

1) Algebraic function

أ) التعويض المباشر:

Ex:- $\lim_{x\to 2} f(x) \frac{x^2-1}{x^2-1} = \frac{4+1}{4-1} = \frac{5}{3}$

ب) التحليل والاختصار:

Ex:- $\lim_{x\to 1} f(x) \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \frac{1}{2}$

2) $\lim_{x\to -1} \frac{x^3+1}{x+1} = \lim_{x\to -1} (x^2 - x + 1) = 1 + 1 + 1 = 3$

3) $\lim_{x\to 2} \frac{x^2-4}{\sqrt{x-2}} = \frac{(x-2)(x+2)}{\sqrt{x-2}} = \lim_{x\to 2} (x+2)\sqrt{x-2} = 0$

ج) الضرب بالعامل المنسب: (المرافق)

(√‾‾ -√‾‾) أو (عدد - √‾‾) أو (√‾‾ -عدد)

Ex:- $\lim\limits_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}$

تقترب من عدد وليس من ∞

$$=\lim_{x\to 0}\frac{(\sqrt{x+5}-\sqrt{5})(\sqrt{x+5}+\sqrt{5})}{x(\sqrt{x+5}+\sqrt{5})}$$

$$=\lim_{x\to 0}\frac{\cancel{x}+\cancel{5}-\cancel{5}}{\cancel{x}(\sqrt{x+5}+\sqrt{5})}=\frac{1}{2\sqrt{5}}$$

أ) القسمة على اكبر أس: (1) الدالة المتكونة من بسط ومقام (2) متكونه من متعددات الحدود و

$x\to\infty$ ولاتقترب من عدد

Ex:- $\lim\limits_{x\to\infty}\frac{x^3+2x+1}{2x^4+x^2+1}$

$$=\lim_{x\to\infty}\frac{\frac{x^3+2x+1}{x^4}}{\frac{2x^4+x^2+1}{x^4}}=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{2}{x^3}+\frac{1}{x^4}}{2+\frac{1}{x^2}+\frac{1}{x^4}}$$

$$=\frac{\frac{1}{\infty}+\frac{2}{\infty}+\frac{1}{\infty}}{2+\frac{1}{\infty}+\frac{1}{\infty}}=\frac{0+0+0}{2+0+0}=\frac{0}{2}=0$$

أ. الـــدالة متكـــونة من بسط ومقام

ب. تتكون من متعـــــــددات حـدود

جـ. $x\to\infty$ ولا تقـــترب من عدد

Ex:- $\lim\limits_{x\to\infty}\frac{3x^2+2x+1}{2x+5}$

$$=\lim_{x\to\infty}\frac{3+\frac{2}{x}+\frac{1}{x^2}}{\frac{2}{x}+\frac{2}{x^2}}=\frac{3+0+0}{0+0}=\frac{3}{0}=\infty$$

Ex:- $\lim\limits_{x\to\infty}\frac{x^3+2x+1}{3x^3+4x+2}$

$$=\lim_{x\to\infty}\frac{1+\frac{2}{x^2}+\frac{1}{x^3}}{3+\frac{4}{x^2}+\frac{5}{x^3}}=\frac{1+0+0}{3+0+0}=\frac{1}{3}$$

Ex:- $\lim\limits_{x\to\infty}\frac{5x^5+2x^4+3x^2+x+1}{2x^5+4x^2-x^2}=\frac{5}{2}$

Ex:- $\lim_{x\to\infty}\left(\sqrt{x^2+4x+1}-\sqrt{x^2+7}\right)$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+4x+1}-\sqrt{x^2+7}\right)\left(\sqrt{x^2+4x+1}+\sqrt{x^2+7}\right)}{\left(\sqrt{x^2+4x+1}+\sqrt{x^2+7}\right)}$$

$$=\lim_{x\to\infty}\frac{x^2+4x+1-\left(\sqrt{x^2+7}\right)}{\sqrt{x^2+4x+1}+\sqrt{x^2+7}}=\lim_{x\to\infty}\frac{4x-6}{\sqrt{x^2+4x+1}+\sqrt{x^2+7}}$$

$$=\lim_{x\to\infty}\frac{\frac{4x-6}{x}}{\frac{\sqrt{x^2+4x+1}+\sqrt{x^2+7}}{x}}$$

$$=\lim_{x\to\infty}\frac{4-\frac{6}{x}}{\sqrt{\frac{x^2+4x+1}{x^2}}+\sqrt{\frac{x^2+7}{x^2}}}$$

$$=\lim_{x\to\infty}\frac{4-\frac{6}{x}}{\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{7}{x^2}}}=\frac{4}{2}=2$$

H.W $\lim_{x\to\infty}\frac{\sqrt{x^2+1}+\sqrt{x^2-4}}{x}$

أ) القســـمة الطــــويلة:

1) الدالة المكونة من بسط ومقام على شكل متعددات (حدود فقط)

2) درجة البسط أكبر من درجة المقام

3) جذور المقام هي جذور البسط (مايجعل المقام صفر يجعل البسط صفر)

4) $x \to a$ ولاتكون $x \to \infty$

Ex:- $\lim\limits_{x\to 2} \frac{x^5-32}{x-2}$

$= \lim\limits_{x\to 2}(x^4+2x^3+4x^2+8x+16)$

$= 16+16+16+16+16$

$=80$

$$\begin{array}{r|l} & x^4+2x^3+4x^2+8x+16 \\ \hline x-2 & x^5-32 \\ & \underline{\mp x^5 \pm 2x^4} \\ & 2x^4-32 \\ & \underline{\mp 2x^4 \pm 4x^3} \\ & 4x^3-32 \\ & \underline{\mp 4x^3 \pm 8x^2} \\ & 8x^2-32 \leftarrow x \\ & \underline{\mp 8x^2 \pm 16x} \\ & 16x-32 \\ & \underline{\mp 16x \pm 32} \\ & 0 \quad 0 \end{array}$$

Limit of Trigonometric Functions: غايةُ الــدوال المـــثلـــية

$$\lim_{x\to 0}\frac{sinkx}{kx}=1\ , \qquad \lim_{x\to 0}\frac{kx}{sinkx}=1$$

$$\lim_{x\to 0}\frac{tankx}{kx}=1\ , \qquad \lim_{x\to 0}\frac{kx}{tankx}=1$$

$$\lim_{x\to 0}\frac{1-cosx}{x}=1$$

Ex:- Find limit of the following functions:-

1) $$\lim_{x\to 0}\frac{sin3x}{7x}=\frac{1}{7}\lim_{x\to 0}\frac{sin\,3x}{7x}$$

$$=\frac{3}{7}\lim_{x\to 0}\frac{sin3x}{3x}=\frac{3}{7}$$

2) $$\lim_{x\to 0}\frac{7x}{3\tan 2x}=\frac{3}{7}\lim_{x\to 0}\frac{x}{\tan\ 2x}$$

$$=\frac{3}{7}\cdot\frac{1}{2}\lim_{x\to 0}\frac{2x}{\tan 2x}=\frac{7}{6}$$

3) $$\lim_{x\to 0}\frac{\sin 2x}{x^2+3x}=\lim_{x\to 0}\frac{\sin\ 2x}{x(x+3)}$$

$$=\lim_{x\to 0}\frac{\sin 2x}{x}\cdot\lim_{x\to 0}\frac{1}{x+3}$$

$$=2\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\left(\frac{1}{0+3}\right)=2\left(\frac{1}{3}\right)=\frac{2}{3}$$

$$4)\ \lim_{x\to 0}\frac{\tan 3x}{\sin 7x} \qquad \div x$$

$$=\lim_{x\to 0}\frac{\frac{tan3x}{x}}{\frac{sin7x}{x}}=\lim_{x\to 0}\frac{3\frac{tan3x}{3x}}{7\frac{\sin 7x}{7x}}=\frac{3}{7}$$

$$5)\ \lim_{x\to 0}\frac{sin^2 2x-xtan\,3x}{x^2+tan^2x} \qquad \div x^2$$

$$\lim_{x\to 0}\frac{\frac{sin^2 2x}{x^2}-\frac{xtan3x}{x^2}}{\frac{x^2}{x^2}+\frac{tan^2x}{x^2}}=\lim_{x\to 0}\frac{\frac{sin^2 2x}{x^2}-\frac{tan3x}{x}}{1+\frac{tan^2x}{x^2}}$$

$$=\lim_{x\to 0}\frac{\left(2\frac{\sin 2x}{2x}\right)^2-3\frac{tan3x}{3x}}{1+\left(\frac{tanx}{x}\right)^2}=\frac{2^2-3}{1+1}=\frac{1}{2}$$

4) $\lim\limits_{x \to 0} \frac{1-\cos 2x}{1-\cos x} = \lim\limits_{x \to 0} \frac{2\sin^2 x}{2\sin^2 \frac{1}{2}x} \qquad \div x^2$

$$= \lim_{x \to 0} \frac{\left(\frac{\sin^2 x}{x^2}\right)}{\left(\frac{\sin^2 \frac{1}{2}x}{x^2}\right)} = \lim_{x \to 0} \frac{\left(\frac{\sin x}{x}\right)^2}{\left(\frac{\sin \frac{1}{2}x}{x}\right)^2}$$

$$= \lim_{x \to 0} \frac{\left(\frac{\sin x}{x}\right)^2}{\left(\frac{1}{2}\frac{\sin \frac{1}{2}x}{\frac{1}{2}x}\right)^2} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4$$

5) $\lim\limits_{x \to 0} \frac{1-\cos 4x}{x^2} = \lim\limits_{x \to 0} \frac{2\sin^2 2x}{x^2}$

$$= 2\lim_{x \to 0} \left(\frac{\sin 2x}{x}\right)^2 = 2\lim_{x \to 0} \left(\frac{2\sin 2x}{2x}\right)^2 = 2(2)^2 = 8$$

Special Cases:

Find the limit of:-

1) $\lim\limits_{x \to \pi} \frac{\sin x}{\pi - x} = \frac{0}{0}$ غــير ممكــن

Sol.

Let $y = \pi - x \to x = \pi - y$

$x \to \pi$

$y \to 0$

$$\therefore \lim_{y \to 0} \frac{\sin(\pi - y)}{y} = \lim_{y \to 0} \frac{\sin\pi\cos y - \cos\pi\sin y}{y}$$

$$= \lim_{y \to 0} \frac{\sin y}{y} = 1$$

2) $\lim\limits_{x\to\frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2}-x} = \frac{0}{0}$ **غير ممكن**

Sol.

Let $y = \frac{\pi}{2} - x \to x = \frac{\pi}{2} - y$

$x \to \frac{\pi}{2}\,, y \to 0$

$$\lim_{y\to 0} \frac{\cos\left(\frac{\pi}{2} - y\right)}{y} = \lim_{y\to 0} \frac{\cos\frac{\pi}{2}\cos y + \sin\frac{\pi}{2}\sin y}{y}$$

$$= \lim_{y\to 0} \frac{\sin y}{y} = 1$$

3) $\lim\limits_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

Sol.

Let $y = \frac{1}{x} \to x = \frac{1}{y} \to x \to \infty, y \to 0$

$$\lim_{y\to 0} \frac{1}{y} \sin y = \lim_{y\to 0} \frac{\sin y}{y} = 1$$

4) $\lim\limits_{x\to 0}\frac{1-cos2x}{2x}=\lim\limits_{x\to 0}\frac{2sin^2x}{2x}$

$$=\lim_{x\to 0}\frac{sinx}{x}.sinx$$

$$=\lim_{x\to 0}\frac{sinx}{x}.\lim_{x\to 0}sinx$$

$$=1.(\sin 0)=0$$

5) $\lim\limits_{x\to \pi}\frac{1-sin\left(\frac{\pi}{2}\right)}{\pi-x}$

Sol.

Let $y=\pi-x \to x=\pi-y$

$x\to\pi\ ,\ y\to 0$

$$\lim_{y\to 0}\frac{1-sin\left(\frac{\pi-y}{2}\right)}{y}=\lim_{y\to 0}\frac{1-sin\left(\frac{\pi}{2}-\frac{y}{2}\right)}{y}$$

$$=\lim_{y\to 0}\frac{1-\left(sin\frac{\pi}{2}cos\frac{y}{2}-cos\frac{\pi}{2}sin\frac{y}{2}\right)}{y}$$

$$=\lim_{y\to 0}\frac{1-cos\frac{y}{2}}{y}=\lim_{y\to 0}\frac{2sin^2\frac{y}{4}}{y}$$

$$=2\lim_{y\to 0}\frac{sin\frac{y}{4}}{y}.\lim_{y\to 0}sin\frac{y}{4}$$

$$=2\lim_{y\to 0}\frac{sin\left(\frac{y}{4}\right)}{y}.(0)$$

$$=0$$

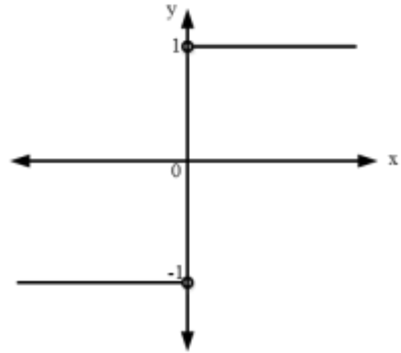
Ex: $\boldsymbol{f(x) = \frac{x}{|x|}}$ (one sided limits)

$D_f = IR/\{0\}$

$\lim_{x \to o^+} \frac{x}{|x|} = \lim_{x \to o^+} \frac{x}{x} = 1 = L_1$

$\lim_{x \to o^-} \frac{x}{|x|} = \lim_{x \to o^-} \frac{x}{x} = -1 = L_2$

So $f(x)$ does not have a (two sided) limit at 0

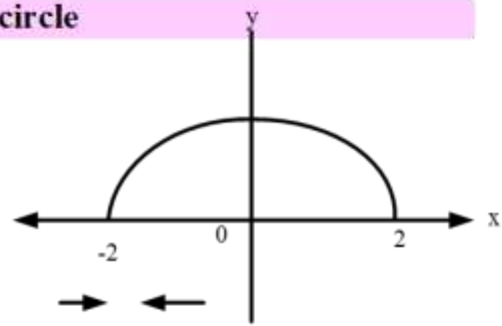


Ex: One Sided limits for a semicircle

$f(x) = \sqrt{4 - x^2}$, $D_f = [-2,2]$

$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$

$\lim_{x \to +2^-} \sqrt{4 - x^2} = 0$



The function does not have a Left –hand limit at x=-2

Or a right-hand limit at x=2

It does not have ordinary two sided limits at either -2 or 2

Vertical Asymptotes: - الخـــطـــوط المقـــــاربة العـــموديـــــة

Unlike polynomials, rational functions may have numbers at which they are not defined. Near such point , many (but not all) rational functions have graphs that approximate a vertical line called a vertical asymptote.

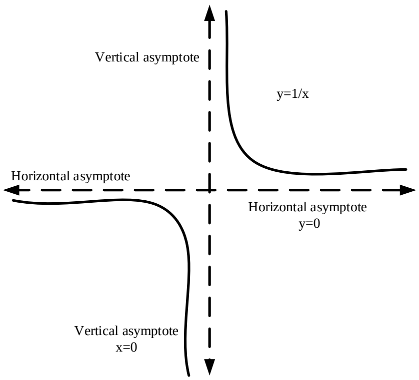
Def.:- A line x=a is a vertical asymptote of the graph of a function y = f(x) if either

$$\lim_{x \to a+} f(x) = \mp\infty \quad or \quad \lim_{x \to a-} f(x) = \mp\infty$$

Ex: $f(x) = \frac{1}{x}\ , x \neq 0$

$$\lim_{x \to 0+} \frac{1}{x} = \infty \quad and \quad \lim_{x \to 0-} \frac{1}{x} = -\infty$$

We say that the line x = 0 (the y-axis) is a vertical asymptote of the graph of $y = \frac{1}{x}$ where the denominator is zero at x = 0 and the function is undefined there.



Horizontal Asymptotes: - الخـــطــوط المقـــاربة الافــقـــيــة

If the distance between the graph of a function and some fixed line approaches zero as ∝ point on the graph moves increasgly far from the origin, we say that the graph approaches the line asymptotically and that the line is an asymptote of the graph.

Looking at $(x) = \frac{1}{x}$, $x \neq 0$, we observe that the x-axis is an asymptote of the curve on the right because

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

And on the left because

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

We say that the x- axis is a horizontal asymptopte of the graph of $f(x) = \frac{1}{x}$

<u>Def.:-</u> A line y=b is a horizontal asymptote of the graph of a function y = f(x) if either $\lim_{x \to \infty} f(x) = b$

Or

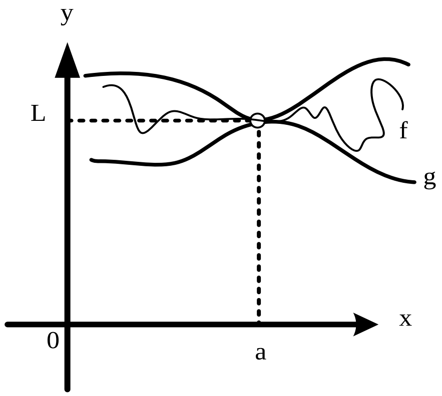
$$\lim_{x \to -\infty} f(x) = b$$

The sandwich theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing a, except possibly at x $=$ a itself. Suppose also that

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

Then $\lim_{x \to a} f(x) = L$



The graph of f is sandwiched between the graph of g and h

<u>Ex:</u> Appling the sandwich Theorem, given that

$$1-\frac{x^2}{4} \le u(x) \le 1+\frac{x^2}{2} \quad \text{for all} \quad x \neq 0$$

Find $\lim_{x \to 0} u(x)$

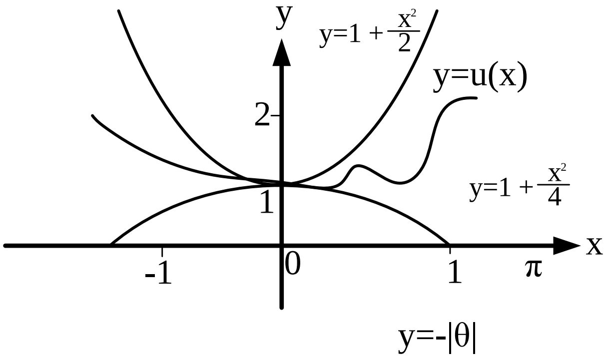
<u>Sol.</u> Since

$$\lim_{x \to 0}\left(1-\left(\frac{x^2}{4}\right)\right)=1 \quad \text{and}$$

$$\lim_{x \to 0}\left(1+\left(\frac{x^2}{2}\right)\right)=1$$

The sandwich Th. implies that

$$\lim_{x \to 0} u\,(x)=1$$



Any function u(x)

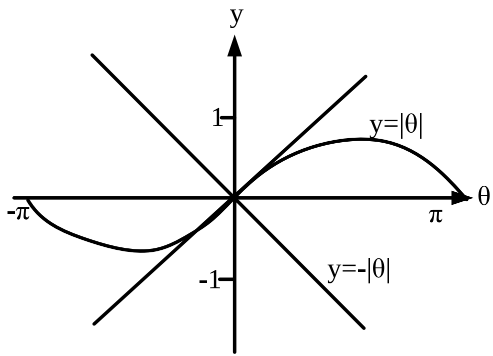
Whose graph lies in the region between

Y=1+(x^2/2) and y=1-(x^2/4) has limit 1 as x$\longrightarrow$ 0

Ex: Appling the sandwich Theorem, to confirm

$$\lim_{\theta \to 0} \sin\theta = 0$$

Sol.



$\because\ -\theta \le \sin\theta \le \theta \quad \forall\ \theta$

$\because\ \lim_{\theta \to 0}(-|\theta|) = \lim_{\theta \to 0}(|\theta|) = 0$

$\because\ \lim_{\theta \to 0} \sin\theta = 0$ by the sandwich th.

Ex: prove that $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

Proof:

$\because -1 \le \sin x \le 1 \qquad \div x$

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

نأخذ $\lim_{x \to \infty}$ لجميع الاطراف

$$\lim_{x \to \infty} \frac{-1}{x} \le \lim_{x \to \infty} \frac{\sin x}{x} \le \lim_{x \to \infty} \frac{1}{x}$$

$$0 \le \lim_{x \to \infty} \frac{\sin x}{x} \le 0$$

$$\therefore \lim_{x \to \infty} \frac{\sin x}{x} = 0$$

The continuity:-

Def.:- a function $y = f(x)$ is to be continues at a point a if :-

1) f (a) exists (i.e. a is in the domain of f)

2) $\lim_{x \to a} f(x)$ exists (i.e. f has a limit at x ⟶ a)

3) $\lim_{x \to a} f(x) = f(a)$

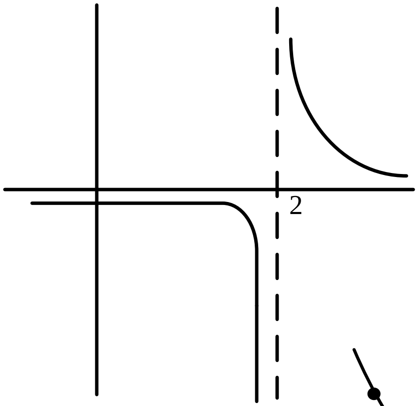
Otherwise the function is discontinues

Example: Test the continuity of

1) $f(x) = \frac{1}{x-2} \quad at \quad x = 2$

$\because f(2)$ does not exist

$\therefore f$ is discontinues at x=2

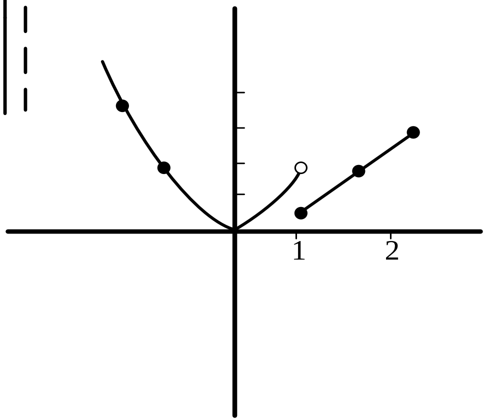


2) $f(x) = \begin{cases} x^2 & x < 1 \\ \frac{x}{2} & x \geq 1 \end{cases} \; at \;\; a = 1$

$\because \lim_{x \to 1^+} f(x) = \frac{1}{2} \neq \lim_{x \to -1} f(x) = 1$

$\therefore \lim_{x \to 1} f(x)$ dose not exists

$\Rightarrow f$ is discontinues at x = 1



1) $f(x) = \frac{1}{x}$ is continues at x = 2 because

(1) $f(2) = \frac{1}{2}$, $(\mathbf{2})\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$, $(\mathbf{3})\lim_{x \to 2} \frac{1}{x} = f(2) = \frac{1}{2}$

But $f(x)$ is discontinues at x = 0 because $f(0) = \infty$ is not exist

Remark: - if a function is continues at all points of an interval$a \leq x \leq b$, then it is said to be continuous at that interval.

Ex:- $f(x) = \frac{1}{x}$ is continuous in $(-\infty, 0)$ and $(0, \infty)$ but it is discontinues at $(-\infty, \infty)$ and (-5, 5) , (-3, 3) , (-2,4), (-1,3),…

Theorem: - if the function f(x) & g(x) are continuous at x = a, then all of the following combinations are continuous at x = a.

1. $f(x) \mp g(x)$

2. $f(x).\, g(x)$

3. $kg\ (x)$ $\quad (k\ is\ \ any\ number)$

4. Furthermore, if $g(a) \neq 0$, then $\frac{f(x)}{g(x)}$ is continuous at x = a

Ex: **the following functions are continuous at every value of**

$x: f(x) = x^{14} + 20x^4$, $g(x) = 5x(2-x) + \frac{1}{x^2+1}$

The following function is continues at every value of x except x = 5 and x = -2

$$h(x) = \frac{x+3}{x^2-3x-10} = \frac{x-3}{(x-5)(x+2)}$$

Step function: - [the greatest integer function$y = [x]$]

Def: - For each real number x, the value $y = [x]$ is the greatest interger that is less than or equal to x ($\leq x$)

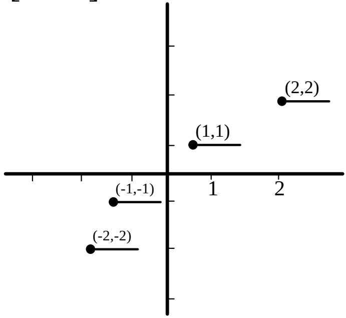
$[0] = 0$, $[0.1] = 0$, $[0.33] = 0$, $[0.9] = 0$

$[1] = 1$, $[0.5] = 1$, $[1.9] = 1$

$[2] = 2$, $[2.1] = 2$, $[3] = 3$, $[3.3] = 3$

$[-0.1] = -1$, $[-0.5] = -1$, $[-1.7] = -2$, $[-1.9] = -2$

Note that $D_f = R$, $R_f = integer\ no. = z$



Now: show whether the step function $y = [x]$ is continuous or not at x =1, x=2

at x = 1

$f(1) = [1] = 1 \quad \lim_{x \to 1^+} f(x) = 1$

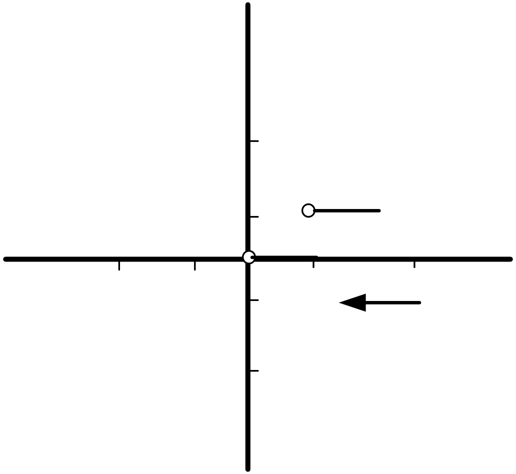
$\lim_{x \to 1^-} f(x) = 0$

at x = 2

$f(2) = [2] = 2 \quad \lim_{x \to 2^+} f(x) = 2$

$\lim_{x \to 2^-} f(x) = 1$

$\therefore$ The fun. Is discontinuous at x = 1 & x = 2



Solved Problem's

Evaluate the limit in each of the following:

1. $\lim_{x \to 2} 5x = 5 \lim_{x \to 2} x = 5.2 = 10$

2. $\lim_{x \to 2} (2x + 3) = 2 \lim_{x \to 2} x + \lim_{x \to 2} 3 = 2.2 + 3 = 7$

3. $\lim_{x \to 2} (x^2 - 4x + 1) = 4 - 8 + 1 = -3$

4. $\lim_{x \to 3} \frac{x-2}{x+2} = \frac{\lim_{x \to 3} (x-2)}{\lim_{x \to 3} (x+2)} = \frac{1}{5}$

5. $\lim_{x \to -2} \frac{x^2-4}{x^2+4} = \frac{4-4}{4+4} = 0$

6. $\lim_{x \to 4} \sqrt{25 - x^2} = \sqrt{\lim_{x \to 4} (25 - x^2)} = \sqrt{9} = 3$

Note: D not assume from these problems that $\lim_{x \to a} f(x)$ is invariably $f(a)$

1. $\lim_{x \to -5} \frac{x^2-25}{x+5} = \lim_{x \to -5}(x-5) = -10$

Evaluate the limit in each of the following:

1. $\lim_{x \to 4} \frac{x-4}{x^2-x-12} = \lim_{x \to 4} \frac{x-4}{(x+3)(x-4)} = \lim_{x \to 4} \frac{1}{x+3} = \frac{1}{7}$

The division by $x-4$ before passing to the limit is valid $x \neq 4$ as $x \to 4$; hence $x-4$ is never zero

2. $\lim_{x \to 3} \frac{x^3-27}{x^2-9} = \lim_{x \to 3} \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} = \lim_{x \to 3} \frac{x^2+3x+9}{x+3} = \frac{9}{2}$

3. $\lim_{h \to 3} \frac{(x+h)^2-x^2}{h} = \lim_{h \to 0} \frac{x^2+2hx+h^2-x^2}{h} = \lim_{h \to 0} \frac{2hx+h^2}{h} = \lim_{h \to 0} (2x+h) = 2x$

4. $\lim_{x \to 2} \frac{4-x^2}{3-\sqrt{x^2+5}} = \lim_{x \to 2} \frac{(4-x^2)(3+\sqrt{x^2+5})}{(3-\sqrt{x^2+5})(3+\sqrt{x^2+5})} = \lim_{x \to 2} \frac{(4-x^2)(3+\sqrt{x^2+5})}{4-x^2}$

$$= \lim_{x \to 2}\left(3+\sqrt{x^2+5}\right) = 6$$

5. $\lim_{x \to 1} \frac{x^2+x-2}{(x-1)^2} = \lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)^2} = \lim_{x \to 1} \frac{x+2}{x-1} = \infty; \mathit{no\ limit\ exists}$

In the following, interpret $\lim_{x \to \pm\infty}$ as an abbreviation for $\lim_{x \to +\infty}$ or $\lim_{x \to -\infty}$.

Evaluate the limit by first dividing numerator and by the highest power of x present and then using $\lim_{x \to \infty} \frac{1}{x} = 0$

1. $\lim_{x \to \infty} \frac{3x-2}{9x+7} = \lim_{x \to \infty} \frac{3-2/x}{9+7/x} = \frac{3-0}{9+0} = \frac{1}{3}$
2. $\lim_{x \to \infty} \frac{6x^2+2x+1}{6x^2-3x+4} = \lim_{x \to \infty} \frac{6+2/x+1/x^2}{6-3/x+4/x^2} = \frac{6+0+0}{6-0+0} = 1$
3. $\lim_{x \to \infty} \frac{x^2+x-2}{4x^3-1} = \lim_{x \to \infty} \frac{1/x+1/x^2-2/x^3}{4-1/x^3} = \frac{0}{4} = 0$
4. $\lim_{x \to \infty} \frac{2x^3}{x^2+1}$: $\lim_{x \to \infty} \frac{2}{1/x+1/x^3} = -\infty$; not limit exists

   $$\lim_{x \to \infty} \frac{2}{1/x + 1/x^3} = +\infty; \text{not limit exists}$$
5. Given $f(x) = x^2 - 3x$, find $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$:

   Since $f(x) = x^2 - 3x, we\ have\ f(x+h) = (x+h)^2 - 3(x+h) and$

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{(x^2+2hx+h^2-3x-3h)-(x^2-3x)}{h} = \lim_{h \to 0} \frac{2hx+h^2-3h}{h}$$

$$= \lim_{h \to 0}(2x+h-3) = 2x-3$$

Evaluate

(a) $\lim\limits_{x \to 2}(x^2 - 4x)$ **(b)** $\lim\limits_{x \to -2}(x^3 + 2x^2 - 3x - 4)$ **(c)** $\lim\limits_{x \to 1}\frac{(3x-1)^2}{(x-1)^3}$

(d) $\lim\limits_{x \to 0}\frac{3^x-3^{-x}}{3^x+3^{-x}}$ **(e)** $\lim\limits_{x \to 2}\frac{x-1}{x^2-1}$ **(f)** $\lim\limits_{x \to 2}\frac{x^2-4}{x^2-5x+6}$

(g) $\lim\limits_{x \to -1}\frac{x^2+3x-2}{x^2+4x+3}$ **(h)** $\lim\limits_{x \to 2}\frac{x-2}{x^2-4}$ **(i)** $\lim\limits_{x \to 2}\frac{x-2}{\sqrt{x^2-4}}$

(j) $\lim\limits_{x \to 2}\frac{\sqrt{x-2}}{x^2-4}$ **(k)** $\lim\limits_{h \to 0}\frac{(x+h)^3-x^3}{h}$ **(l)** $\lim\limits_{x \to 1}\frac{x-1}{\sqrt{x^2+3}-2}$

Ans. $(a) - 4; (b)0;\ (c)\frac{1}{2};\ (d)0; (e)\frac{1}{2}; (f) -$

$4; (g)\frac{1}{2}\ ; (i)0\ ; (j)\infty, not\ limit; (k)3x^2; (l)2$

Evaluate the limit in each of the following:-

1. $\lim\limits_{x \to \frac{3\pi}{2}} \frac{\sin(2x-\pi)}{x-\frac{3\pi}{2}}$

Let $y = x - \frac{3\pi}{2} \to x = y + \frac{3\pi}{2}$

$x \to \frac{3\pi}{2}$

$x \to 0$

$$\lim_{y \to 0} \frac{\sin\left(2\left(y+\frac{3\pi}{2}\right)-\pi\right)}{y} = \lim_{y \to 0} \frac{\sin(2y+2\pi)}{y}$$

$$= \lim_{y \to 0} \frac{\sin 2y \cos 2\pi + \cos 2y \sin 2\pi}{y}$$

$$= \lim_{y \to 0} \frac{\sin 2y}{y} = 2 \quad \lim_{y \to 0} \frac{\sin 2y}{y} = 2$$

$$2.\ \lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x$$

Let $y = \frac{\pi}{2} - x \quad \to x = \frac{\pi}{2} - y$

$$x \to \frac{\pi}{2}$$

$$y \to 0$$

$$= \lim_{y \to 0} y.\tan\left(\frac{\pi}{2} - y\right) = \lim_{y \to 0} y.\frac{\sin\left(\frac{\pi}{2} - y\right)}{\cos\left(\frac{\pi}{2} - y\right)}$$

$$= \lim_{y \to 0} y.\frac{\sin\frac{\pi}{2}\cos y - \cos\frac{\pi}{2}\sin y}{\cos\frac{\pi}{2}\cos y + \sin\frac{\pi}{2}\sin y}$$

$$= \lim_{y \to 0} y.\frac{\cos y}{\sin y}$$

$$= \lim_{y \to 0} \frac{y}{\sin y}.\lim_{y \to 0} \cos y = 1$$

3. $\lim_{x\to 0} \frac{1-\sqrt{\cos x}}{x^2}$

$$= \lim_{x\to 0} \frac{1-\sqrt{\cos x}}{x^2} = \lim_{x\to 0} \frac{1-\sqrt{\cos x}}{x^2} \cdot \frac{1+\sqrt{\cos x}}{1+\sqrt{\cos x}}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x^2\left(1+\sqrt{\cos x}\right)}$$

$$= \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x^2\left(1+\sqrt{\cos x}\right)}$$

$$= \lim_{x\to 0} \frac{2*\frac{1}{4}\sin^2\frac{x}{2}}{\frac{x^2}{4}} * \lim_{x\to 0} \frac{1}{\left(1+\sqrt{\cos x}\right)}$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{\pi}{2}}{\frac{x}{2}}\right)^2 * \lim_{x\to 0} \frac{1}{\left(1+\sqrt{\cos x}\right)}$$

$$= \frac{1}{2} = \frac{1}{1+1} = \frac{1}{4}$$

3. $\lim\limits_{x \to 0} \frac{1-\cos^2 9x}{9x^2}$

$$\lim_{x \to 0} \frac{1-\cos^2 9x}{9x^2} = \lim_{x \to 0} \frac{(1-\cos^2 9x)(1+\cos^2 9x)}{9x^2}$$

$$= \lim_{x \to 0} \frac{9\sin^2 9x}{9 * 9x^2} * \lim_{x \to 0}(1+\cos^2 9x)$$

$$= 9\lim_{x \to 0}\left(\frac{\sin 9x}{9x}\right)^2 * \lim_{x \to 0}(1+\cos^2 9x)$$

$$= 9 * 2 = 18$$

4. $\lim\limits_{x \to \frac{\pi}{2}} \frac{2x-\pi}{\cos x}$

Let $y = 2x-\pi \;\to x = \frac{y+\pi}{2}$ $, x \to \frac{\pi}{2}$ $\quad y \to 0$

$$\lim_{y \to 0} \frac{y}{\cos\left(\frac{y+\pi}{2}\right)} = \lim_{y \to 0} \frac{y}{\cos\left(\frac{\pi}{2}+\frac{y}{2}\right)}$$

$$\lim_{y \to 0} \frac{y}{-\sin\left(\frac{y}{2}\right)} = -2\lim_{y \to 0} \frac{\frac{y}{2}}{\sin\left(\frac{y}{2}\right)} = -2$$

By $\left(\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\right)$

1) $\lim\limits_{x \to 0} \frac{cot\ 4x}{cot\ 2x}$

$$\lim_{x \to 0} \frac{cot\ 4x}{cot\ 2x} = \lim_{x \to 0} \frac{tan\ 2x}{tan\ 4x} = \lim_{x \to 0} \frac{2\,\frac{tan\ 2x}{2x}}{4\,\frac{tan\ 4x}{4x}} = \frac{2(1)}{4(1)} = \frac{1}{2}$$

2) $\lim\limits_{x \to 0} \frac{xsinx}{1-cosx} = \lim\limits_{x \to 0} \frac{xsin(1+cosx)}{(1-cosx)(1+cosx)}$

$$= \lim_{x \to 0} \frac{xsinx(1 + cosx)}{1 - sin^2x}$$

$$= \lim_{x \to 0} \frac{x}{sinx} . \lim_{x \to 0} (1 + cosx)$$

$$= (1). (1 + 1) = 2$$

Is $\boldsymbol{f(x) = \begin{cases} x^2 + 2x - 3 & x \geq 1 \\ 3x - 3 & x < 1 \end{cases}}$ **Continuous at x=1**

Sol:

1) $f(x) = x^2 + 2x - 3$

$f(1) = 1 + 2 - 3 = 0 \neq \infty \therefore$ Exists

2) $\lim_{x \to 1^+} (x^2 + 2x - 3) = 0$

$\lim_{x \to 1^-} (3x - 3) = 0 \neq \infty \therefore$ Exists

3) $\because f(1) = \lim_{x \to 1} f(x)$

$\therefore f(x)$ Continuous at x=1

Is $\boldsymbol{f(x) = \begin{cases} x^2 + ax + 3 & x \geq 1 \\ 2x + 7 & x < 1 \end{cases}}$ **Continuous at x=1**

Find the value of (a)

Sol:

$\because f(x)$ cont. at x=1

$\therefore f(1) = \lim_{x \to 1} f(x)$

1) $f(x) = x^2 + ax + 3 \to f(1) = 1 + a + 3 = a + 4$

2) $\lim_{x \to 1} = (2x + 7) = 2 + 7 = 9$

$a + 4 = 9$

$a = 5$

Chapter (3): Differentiation:

If (f) is a continuous function on $[a, b]$, then if $\lim_{\Delta x \to 0} f(x + \Delta x) - f(x)$ exist at a point $x \in (a, b)$ we call this limit artful derivative of (f) at x and denoted ly $f'(x)$ or $\frac{d\,f(x)}{dx}$ or $\frac{dv}{dx}$ or y'

Note: The derivative of a function (f) at the point $(x = a)$ is the slope of the tangent line to the curve of (f) at the point $(a, f(a))$

$$\frac{dy}{dx} = slope = \lim \theta = \frac{\Delta y}{\Delta x}$$

Ex: (1) find the definition of $f(x)$ to find the derivative of $f(x) = \sqrt{x}$

(2) find the equation of the line tangent to the cawed of $f(x) = \sqrt{x}$ at $x = 1$

(1) $\left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$

$$\lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left(\sqrt{x+\Delta x} - \sqrt{x}\right)\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x \sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

(2) $f(1) = \sqrt{1} = 1$

Tangent point $= (1, 1)$

$$m = slop = derivative\ at\ (1,1)$$

$$= \frac{1}{2\sqrt{x}}\Big|(1,1) = \frac{1}{2}$$

$$y - y_1 = m(x - x_1) \Longrightarrow y - 1 = \frac{1}{2}(x - 1)$$

Note: The equation of the normal line is $y - 1 = 2(x - 1)$

Theorem:

(1) $\frac{d}{dx}(sin) = \cos x$

(2) $\frac{d}{dx}(cos) = -\sin x$

(3) $\frac{d}{dx}(tan) = \sec^2 x$

(4) $\frac{d}{dx}(cot) = -\csc^2 x$

(5) $\frac{d}{dx}(sec) = \sec x \,.\, \tan x$

(6) $\frac{d}{dx}(csc) = -\csc x \,.\, \cot x$

Proof (1):

Let $y = f(x) = \sin x$

$$\frac{dy}{dx} = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \longrightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \longrightarrow 0} \frac{\sin x \cos h - \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \longrightarrow 0} \frac{\sin x\,(1 - \cos h) + \cos x \sin h}{h}$$

$$= -\sin \lim_{h \longrightarrow 0} \frac{(1 - \cos h)}{h} + \cos x \lim_{h \longrightarrow 0} \frac{\sin h}{h} = \cos x$$

Theorem:

(1) $\frac{d}{dx}(K) = 0$

(2) $\frac{d}{dx}(x^n) = -\sin x$

(3) $\frac{d}{dx}(f(x) \mp g(x)) = \frac{d}{dx}\big(f(x)\big) \mp \frac{d}{dx}\big(g(x)\big)$

(4) $\frac{d}{dx}(K\, f(x)) = K \frac{d}{dx}\big(f(x)\big)$

(5) $\frac{d}{dx}(f(x).\, g(x)) = f(x) \frac{d}{dx}\big(g(x)\big) + g(x) \frac{d}{dx}\big(f(x)\big)$

(6) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}\big(f(x)\big) - f(x)\frac{d}{dx}\big(g(x)\big)}{\big(g(x)\big)^2}$

(7) $\frac{d}{dx}(u^n) = n\, u^{n-1} \frac{du}{dx}$

Higher order derivative:

1) Second derivative is:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Symbol $\frac{d^2y}{dx^2} = y'' = f''(x)$

2) Third derivative is:

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

Symbol $\frac{d^3y}{dx^3} = y''' = f'''(x)$

3) Fourth derivative is:

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right)$$

Symbol $\frac{d^4y}{dx^4} = y^{(4)} = f(x)^{(4)}$

Ex: find $\frac{d^4y}{dx^4}$ if:

$$y = x^6 - 3x^4 + \cos x$$

$$\frac{dy}{dx} = 6x^5 - 12x^3 - \sin x$$

$$\frac{d^2y}{dx^2} = 30x^4 - 36x^2 - \cos x$$

$$\frac{d^3y}{dx^3} = 120x^3 - 72x + \sin x$$

$$\frac{d^4y}{dx^4} = 360x^2 - 72 + \cos x$$

Theorem: (chain rule)

If $y = f(t)$ and $t = g(x)$ then:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dx}{dt}$$

Theorem: (parametric rule)

1) If $y = f(t)$ and $x = g(t)$ then:

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$(2) \quad y'' = \frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx'}{dt}} \qquad\qquad y'' = \frac{dy'}{dt} = \frac{\frac{d}{dt}(y')}{\frac{dx}{dt}}$$

$$(3) \quad y''' = \frac{d^3y}{dx^3} = \frac{\frac{dy''}{dt}}{\frac{dx''}{dt}}$$

Ex: Find $\frac{dy}{dx}$ if $y = \sin t \ \& \ x = x^3 + t$

$$\frac{dy}{dx} = \cos t \ \& \frac{dt}{dx} = 3x^2 + 1$$

$$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx} = \cos t \ \times \left(3x^2 + 1\right)$$

$$= \cos\left(x^3 + x\right) \times \left(3x^2 + 1\right)$$

عند اشتقاق الدالة المثلثية نشتق اعتيادياً بحسب القوانين السابقة مضروباً في مشتقة الزاوية.

Ex: If $y=\left(t^2+1\right)^4$ & $x=t^2+5$ find $\frac{d^2y}{dx^2}$

$$\frac{dy}{dx}=4\left(t^2+1\right)^3(2t)\ \ \&\ \frac{dx}{dt}=2t$$

$$y'=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{4\left(t^2+1\right)^3(2t)}{2t}=4\left(t^2+1\right)^3$$

$$\frac{dy'}{dx}=12\left(t^2+1\right)^2(2t)\ \ \&\ \frac{dx}{dt}=2t$$

$$y''=\frac{d^2y}{dx^2}=\frac{\frac{dy'}{dt}}{\frac{dx'}{dt}}=\frac{12\left(t^2+1\right)^2(2t)}{2t}=12\left(t^2+1\right)^2$$

Ex: If $x = t - t^2$, $y = t - t^3$ find $\frac{d^2y}{dx^2}$

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - 3t^2}{1 - 2t}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx'}{dt}} = \frac{\frac{(1 - 2t)(-6t) - \left(1 - 3t^2\right)(-2)}{(1 - 2t)^2}}{1 - 2t}$$

Ex: find $\frac{dy}{dx}$ if $y = 2t^2 - t$ and $t = x^{-1} + 2$

$$\frac{dy}{dx} = 4t - 1 \quad \& \quad \frac{dt}{dx} = (-1)x^{-2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = (4t - 1)(-1)x^{-2}$$

$$= \left(4\left(x^{-1} + 2\right) - 1\right)\left(-x^{-2}\right) = (4x^{-1} + 7)(-x^{-2})$$

Ex: find $\frac{dy}{dx}$ if $y = t^2$, $x = \frac{1}{1-t}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{\frac{(1-t).0 - 1(1)}{(1-t)^2}} = \frac{2t}{\frac{1}{(1-t)^2}} = 2t(1-t)^2$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx'}{dt}} = \frac{2t.\big(2(1-t)(1)\big) + (1-t)^2}{\frac{1}{(1-t)^2}}$$

$$= \left[4t(1-t) + 2(1-t)^2\right].(1-t)^2$$

$$= (1-t)^3[4t + 2(1-t)]$$

$$= (1-t)^3[4t + 2 - 2t]$$

$$= (1-t)^3[2t + 2] = 2(t+2)\,(1-t)^3$$

Ex: find $\frac{dy}{dx}$ if $y = t^3 \quad , \quad x = t^2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2}{2t} = \frac{3}{2}t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx'}{dt}} = \frac{\frac{3}{2}}{2t} = \frac{3}{4}t$$

Ex: find $\frac{dy}{dx}$ if:

1) $y = \tan\left(x^2 + x\right) \longrightarrow y' = \sec^2(x^2 + x) \times (2x + 1)$

2) $y = \sec\left(\frac{3x+7}{3}\right) \longrightarrow y' = \sec\left(\frac{3x+7}{3}\right).\tan\left(\frac{3x+7}{3}\right)\left(\frac{3}{3}\right)$

3) $y = \cot\left(\frac{x}{x+1}\right) \longrightarrow y' = \csc^2\left(\frac{x}{x+1}\right)\left(\frac{(x+1)-x}{(x+1)^2}\right)$

4) $y = \csc(x\sin x) \longrightarrow y' = -\csc(x\sin x)\cot(x\sin x)$

$[x\cos x + \sin x]$

5) $y = \cos\left(x^2 + 1\right) \longrightarrow y' = -\sin\left(x^2 + 1\right)(2x)$

$$6)\ y = \sin^3 x^2 \longrightarrow y' = 3\sin^2 x^2 \cos x^2\,(2x)$$

$$7)\ y = \tan^4\left(x^2+1\right) \longrightarrow y' = 4\tan^3\left(x^2+1\right)\sec^2\left(x^2+1\right)(2x)$$

$$8)\ y = \cos^3 x \longrightarrow y' = 3\cos^3 x\,(-\sin x)$$

$$9)\ y = \sqrt{\sec\left(x(\cos x)\right)} \longrightarrow y' = \frac{\sec(x\cos x)\tan(x\cos x)\,[-x\sin x + \cos x]}{2\sqrt{\sec(x\cos x)}}$$

$$10)\ y = x^3\cot^4\left(x^2+x\right) \longrightarrow y' = x^3.\left(4\cot^3\left(x^2+x\right).\left(-\csc^2\left(x^2+x\right)\right).(2x+1)\right)$$
$$+\cot^4\left(x^2+x\right)(3x^2)$$

$$11)\ y = \frac{x}{\sin^2(x^3)} \longrightarrow y' = \frac{\sin^2\left(x^2\right).(1) - x.2\sin\left(x^3\right).\cos\left(x^3\right).\left(3x^2\right)}{\sin^4\left(x^3\right)}$$

L' Hopital's Rule:

Theorem:

suppose that $f(a) = g(a) = 0$ or $f(a) = g(a) = \infty$ and $f'(a).\, g'(a)$ exist such that $g'(a) \neq 0$ then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)}{g'(a)}$$

Ex: $\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{\sin x}{1} = \frac{0}{1} = 0$

Ex: $\lim_{x \to 0} \frac{\sin^2 x}{x^2} = \lim_{x \to 0} \frac{2 \sin x \cos x}{2x} = \lim_{x \to 0} \frac{\sin^2 x + \cos^2 x}{1} = \frac{0+1}{1} = 1$

Ex: $\lim_{x \to 0} \frac{x \cos x - x}{\sin x - x} = \lim_{x \to 0} \frac{-x \sin x + \cos x - 1}{\cos x - 1} = \lim_{x \to 0} \frac{-x \cos x - \sin x - \sin x}{-\sin x}$

$$= \lim_{x \to 0} \frac{+x \sin x - \cos x - 2 \cos x}{-\cos x} = \frac{-1-2}{-1} = 3$$

Implicit differentiation:

Let $f(x,y)=0$ (1) to find $\frac{dy}{dx}$ we differentiate both aides of (1) with respect to x treating (y)as a function of (x) then we find $\frac{dy}{dx}$ this method is called implicit differentiation.

Ex: find $\frac{dy}{dx}$ if:

1) $x^2+y^2=5$

شروط الاشتقاق الضمني

1) نشتق الطرفين بالنسبة الى x ونعامل y كدالة بالنسبة الى x (عند اشتقاق المقدار الموجود في y نضربه في y').

2) ننقل الحدود التي تحتوي على المشتقة (y') الى الطرف الايسر والبقية الى الطرف الأيمن.

3) بواسطة عمليات رياضية ممكن إيجاد (y').

$$2x+2yy'=0$$

$$2yy'=-2x \Rightarrow y'=\frac{-2x}{2y}=\frac{-x}{y}$$

$$2)\ \tan\left(xy^2\right) + (y + x)^2 = 6$$

$$\sec^2\left(xy^2\right)\left[2xyy' + y^2\right] + 2(y + x)(y' + 1) = 0$$

$$2xy\sec^2\left(xy^2\right)y' + 2(y + x)y' = -2(y + x) - y^2\sec^2\left(xy^2\right)$$

$$y' = \frac{-2(y + x) - y^2\sec^2\left(xy^2\right)}{2xy\sec^2(xy^2) + 2(y + x)}$$

Example: find the equation of tangent and normal for the case:

$$x^2 + \cos(xy) = y \quad \text{at} \quad (1, 0)$$

$$2x - \sin(xy)\,[xy' + y] = y'$$

$$2x - x\sin(xy)\,y' - y\sin(xy) = y'$$

$$y' + x\sin(xy)\,y' = 2x - y\sin(xy)$$

$$y'(1 + x\sin(xy)) = 2x - y\sin(xy)$$

$$y' = \frac{2x - y\sin(xy)}{1 + x\sin(xy)} = \frac{2 - 0}{1 + 0} = 2$$

The tangent line: $y - 0 = 2(x - 1)$

The normal line: $y - 0 = \frac{-1}{2}(x - 1)$

Ex: find $\frac{dy}{dx}$ if:

1) $x^3 + x^2y + y^3 = x^2 + 2y^2 + xy$

$$3x^2 + x^2y' + 2yx + 3y^2y' = 2x + 4yy' + xy' + y$$

$$\therefore\ x^2y' + 3y^2y' - 4yy' - xy' = 2x + y - 2yx$$

$$y'\left[x^2 + 3y^2 - 4y - x\right] = 2x + y - 2yx$$

$$y' = \frac{2x + y - 2yx}{x^2 + 3y^2 - 4y - x}$$

$$2)\ \sqrt{(x^2+y^2)^3} = xy + x^3 - 7$$

$$(x^2+y^2)^{\frac{3}{2}} = xy + x^3 - 7$$

$$\frac{3}{2}(x^2+y^2)^{\frac{1}{2}}(2x+2yy') = xy' + y + 3x^2$$

$$3(x^2+y^2)^{\frac{1}{2}}(x+yy') = xy' + y + 3x^2$$

$$3x(x^2+y^2)^{\frac{1}{2}} + 3yy'(x^2+y^2)^{\frac{1}{2}} = xy' + y + 3x^2$$

$$3yy'(x^2+y^2)^{\frac{1}{2}} - xy' = y + 3x^2 - 3x(x^2+y^2)^{\frac{1}{2}}$$

$$y'\left[3y(x^2+y^2)^{\frac{1}{2}} - x\right] = y + 3x^2 - 3x(x^2+y^2)^{\frac{1}{2}}$$

$$y' = \frac{y + 3x^2 - 3x(x^2+y^2)^{\frac{1}{2}}}{3y(x^2+y^2)^{\frac{1}{2}} - x}$$

CHAPTER 4

APPLICATIONS OF DERIVATIVES

DEFINITIONS Absolute Maximum, Absolute Minimum

Let f be a function with domain D. Then f has an absolute maximum value on D at a point c if

$$f(x) \leq f(c) \quad \text{for all x in D}$$

and an absolute minimum value on D at c if

$$f(x) \geq f(c) \quad \text{for all x in D.}$$

THEOREM 1 The Extreme Vale Theorem

If f is continuous on a closed interval [a, b], then f attains both absolute maximum value M and an absolute minimum value m in [a, b]. That is, there are numbers x1 and x2 in [a, b] with $f(x1) = m$, $f(x2) = M$, and $m \leq f(x) \leq M$ for every other x in [a, b] (Figure 4.3).

DEFINITIONS Local Maximum, Local Minimum

A function f has a local maximum value at an interior point c of its domain if

$f(x) \leq f(c)$ for all x in some open interval containing c.

A function f has a local minimum value at an interior point c of its domain if

$f(x) \geq f(c)$ for all x in some open interval containing c.

THEOREM 2 The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$f'(c) = 0$.

Graphing of function using First and Second derivatives:

Facts and Definition :

1. If the first derivative ƒ is positive (+), then the function ƒ is increasing.
2. If the first derivative ƒ is negative (-), then the function ƒ is decreasing.
3. If the second derivative ƒ is positive (+), then the function ƒ is concave up.
4. If the second derivative ƒ is negative (-), then the function ƒ is concave down.

5. If the point x=a determines a **relative maximum** for function ƒ, if ƒ is continuous at x=a, and the first derivative ƒ' is positive (+) for x < a and negative (-) for x > a. The point x=a determine an **absolute maximum** for function ƒ, it corresponds to the largest y-value in the range of ƒ.

___+__+____a____-__-___

↗ Relative Maximum ↘

6. The point x =a determines a **relative minimum** for function ƒ, if ƒ is continuous at x=a, and the first derivative ƒ' is negative (-) for x < a and positive (+) for x>a. The point x=a determine an **absolute minimum** for function ƒ, it corresponds to the smallest y-value in the range of ƒ.

___-__-____a____+__+___

↘ Relative Minimum ↗

7. The point x=a determines an inflection point for function ƒ, if ƒ is continuous at x=a, and the second derivative ƒ'' is negative for
x < a and positive for x > a or positive for x < a and negative for x > a.

- ____+___+______a______-___-____ U ∩ ____-___-______a______+___+____ ∩ U

Strategy for graphing y=f(x):

1. Identify the domain of f and any symmetries the curve may have

- نجد اوسع مجال للداله ماعدا التي تجعل المقام يساوي صفر
- نجد نقاط التناظر مع نقطه الاصل او مع محور الصادات وهي اما تكون داله زوجيه او فرديه

2. Identify any asymptotes

نجد المحاذيات:
1- المحاذي العامودي vertical asymptotes: هو العدد الذي يصفر المقام x=a
2-المحاذي الافقي horizontal asymptotes: يمكن استخراجه إذا البسط عدد y = 0
*إذا كان البسط عدد ثابت فيعني أن المحاذي الافقي يساوي صفر
*درجة البسط تساوي درجة المقام هنا المحاذي الافقي يساوي (معامل أكبر أس في البسط على معامل أكبر أس في المقام).
3- المحاذي المائل oblique asymptotes: يستخدم اذا كانت درجة البسط أكبر من درجة المقام

3. Find ƒ(x): The critical points of the ƒ and identify the function's behavior at each one, when the curve is increasing and where it's decreasing.

4. Find ƒ''(x), the points of infliction, if any occur, and determine the servility of the curve.

نجد المشتقة الثانية ونستخرج مناطق التقعر والتحدب ونقط الانقلاب إن وجدت.

5. Plot key points, such as the intercepts and the points found in step 3 and 4, and then sketch it.

EXAMPLE 7 Using the Graphing Strategy

Sketch the graph of f(x) = $\frac{(x+1)^2}{1+x^2}$

Solution

The domain of f is (-∞, ∞) and there are no symmetries about either axis or the origin (Section 1.4).

Find f' and f''

f(x) = $\frac{(x+1)^2}{1+x^2}$

x-intercept at x=-1,

y-intercept (y=1) at x=0

f'(x) = $\frac{(1+x^2)\,.2x(x+1)^2-(x+1)^2.\,2x}{(1+x^2)^2}$

= $\frac{2(1-x^2)}{(1+x^2)^2}$

Critical Points:

x=-1, x=1

f'' (x) = $\frac{(1+x^2)^2\,.2(-x)-2(1-x^2)[2(1+x^2).\,2x]}{(1+x^2)^4}$

= $\frac{4x(x^2-3)}{(1+x^2)^3}$

After some algebra

3. Behavior at critical points. The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (step 2) since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$ yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$ yielding a relative maximum by the Second Derivative Test. We will see in Step 6 that both are absolute extrema as well.

4. Increasing and decreasing. We see that on the interval $(-\infty,-1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.

5. Inflection points. Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}$, 0, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(-\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus each point is a point on inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, and concave up again on $(\sqrt{3}, \infty)$.

6. *Asymptotes*. Expanding the numerator of f(x) and then dividing both numerator and denominator by x^2 gives

$$f(x) = \frac{(x+1)^2}{1+x^2} \quad = \quad \frac{x^2+2x+1}{1+x^2}$$ Expanding denominator

$$= \frac{1+\frac{2}{x}+\frac{1}{x^2}}{\frac{1}{x^2}+1}$$ Dividing by x^2

We see that f(*x*) → 1+ as *x* → ∞ and that f(*x*) → 1- as *x* → -∞, Thus, the line *y* = 1 is a horizontal asymptotes.

Since f decreases on (-∞, -1) and then increases on (-1, 1), we know that f(-1)=0 is a local minimum. Although f decreases on (1, ∞), it never crosses the horizontal asymptote *y* = 1 on that interval (-∞, -1), approaching it from below. Therefore, there are no vertical asymptotes (the range of f is 0 ≤ *y* ≤ 2).

7. The graph of f is sketched in Figure 4.31. Notice how the graph is concave down as it approaches the horizontal asymptote y = 1 as x→-∞, and concave up in its approach to y = 1 as x→ ∞.

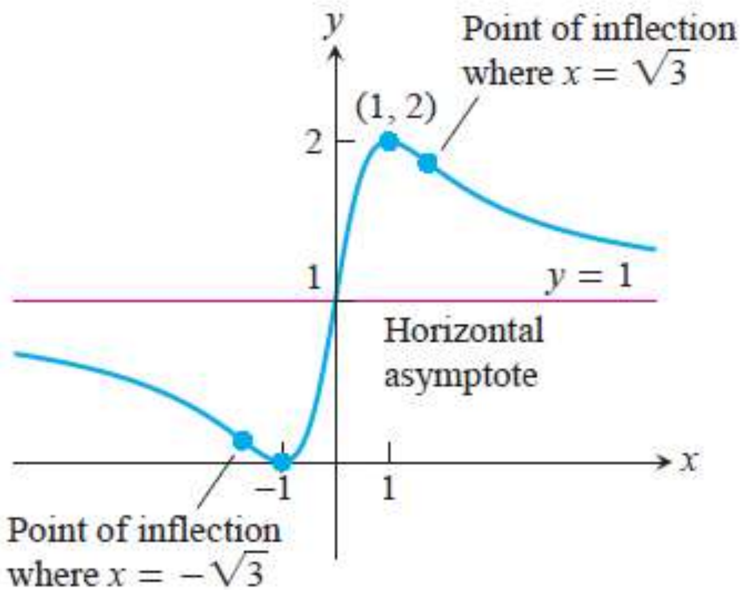


FIGURE 4.31 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 7).

Sketch the Following Functions:

1. f(x) = $\frac{x^2}{x^2-1}$

Df = R/{1, -1}

Asymptotes.

Vertical x=$\pm$1

Horizontal y=1

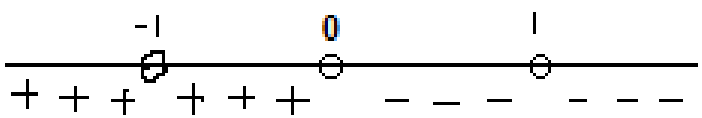
$$f'(x) = \frac{(x^2-1)(2x)-[x^2(2x)]}{(x^2-1)^2}$$

$$= \frac{2x^3-2x-2x^3}{(x^2-1)^2}$$

$$f'(x) = -\frac{2x}{(x^2-1)^2}=0$$

-2x=0

x=0



نأخذ نقاط التي تصغر مقان أيضاً تعويض بالمشتقة

f increase on (-∞, -1), (-1, 0)

f decrease on (0, 1), (1, ∞)

at x=0

f(0)= y=0 →(0, 0) in max point

f''(x)= $\frac{\left(x^2-1\right)^2(-2)+2x[2\left(x^2-1\right)2x]}{(x^2-1)^4}$

f''(x) = $\frac{-2x^2+2+8x^2}{(x^2-1)^3}$

= $\frac{6x^2+2}{(x^2-1)^3}$

$6x^2+2=0$

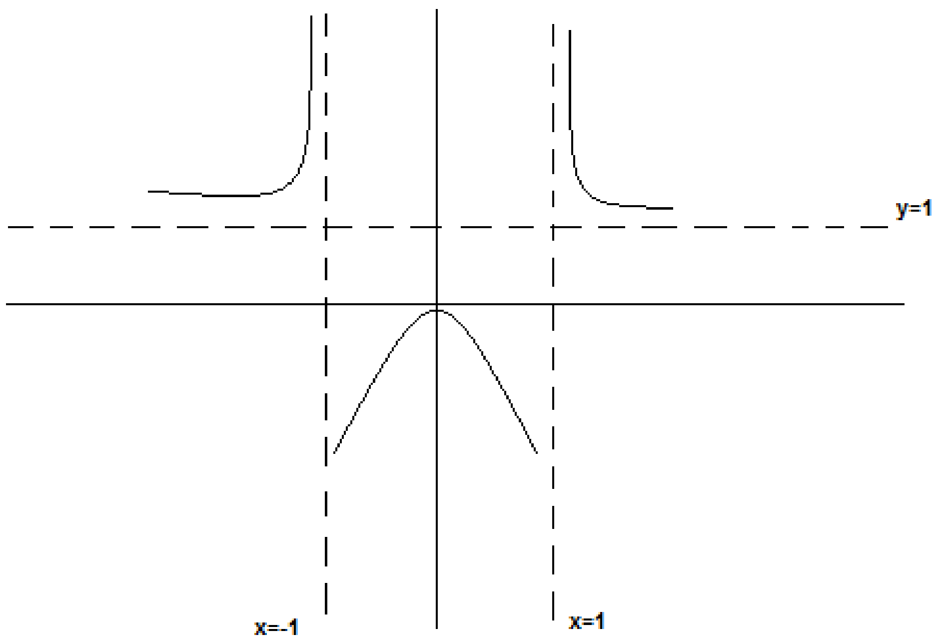
$6x^2 = -2$ يهمل

$-1 \qquad\qquad 1$

$+ + + \quad - - - \quad + + +$

ƒ is concave up on (-∞, -1), (1, ∞)

ƒ is concave down on (-1, 1)



2. f(x)= $\frac{4-x^3}{x^2}$

Solution:

يوجد محاذي افقي اذا كان فقط درجة بسط تساوي درجة مقام واذا كان درجة البسط أكبر من المقام يكون محاذي مائل ولا يوجد افقي

Df= R/{0}

Asymptotes.

Vertical: x=0

There is no horizontal asymptote

Y = - x is the oblique asymptote

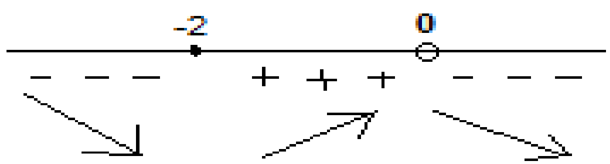
$$f'(x) = \frac{x^2(-3x^2)-(4-x^3)(2x)}{x^4}$$

$$= \frac{-3x^4-8x+2x^4}{x^4}$$

$$= \frac{-x^4-8x}{x^4}$$

$$= \frac{x(-x^3-8)}{x^4} \rightarrow -x^3 - 8 = 0$$

$$x^3 = -8 \rightarrow x = 2$$



f is increased on (-2, 0)

f is decreased on (-∞, -2), (0, ∞)

at x = -2 f(-2) = y = $\frac{4+8}{4}$

$\rightarrow \frac{12}{4} = 3$

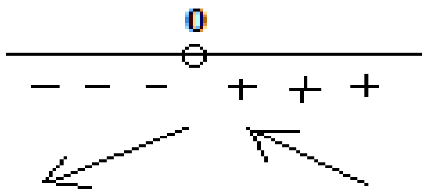
(-2, 3) is the minimum point.

f''(x) = $\frac{x^3(-3x^2)-[(-x^3-8)3x^2]}{x^6}$

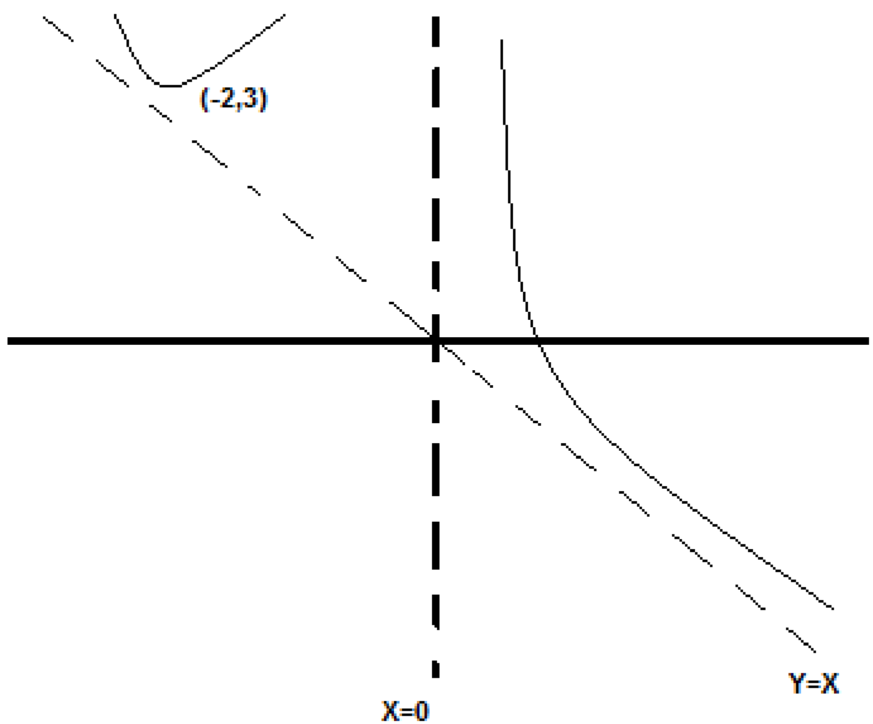
f''(x) = $\frac{-3x^5-[-3x^5-24x^2]}{x^6}$

= $\frac{-3x^5+3x^5+24\,x^2}{x^6}$

= $\frac{24}{4} \neq 0$



f is concave up on (-∞, ∞)



Rolle's Theorem:

If ƒ is continuous on [a, b] differentiable on (a, b) and ƒ (a)= ƒ (b)=0 then there is at least one number c in (a, b) such that f'(c)=0

Example: Find the two intercepts of the function ƒ(x)=$x^2 - 3x + 2$ and show that ƒ'(x)=0 at some point between the two intercepts

Solation: By setting ƒ(x) =0 we have $x^2 - 3x + 2 = 0$

(x-1)(x-2)=0

x=1 & x=2

the ƒ(1)= ƒ(2)=0

and since ƒ is differentiable everywhere, Rolle's theorem guarantees the existence of some c in the interval (1,2) such that ƒ'(c)=0 , to find c we solve the equation

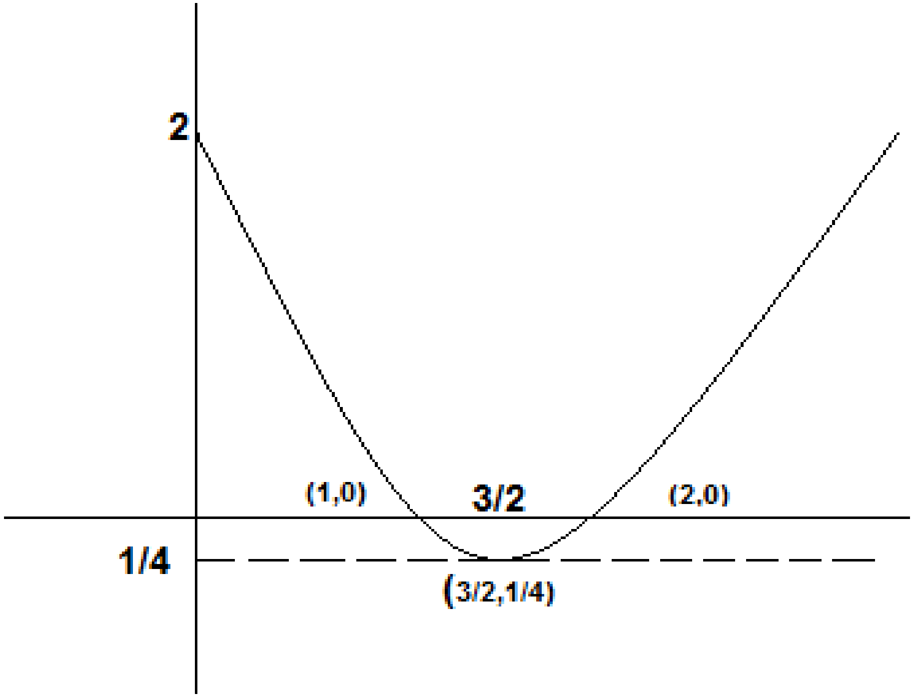
ƒ'(c)=2x-3=0

2x=3

x=$\frac{3}{2}$

Since $\frac{3}{2}$ is in the interval (1, 2) and $f'(\frac{3}{2}) = 0$

we have $c=\frac{3}{2}$



The Mean Value Theorem:

If f is continuous on [a, b] and differentiable on (a, b), then there is a number exists c in (a, b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Example: Given f(x) = 3 – $\frac{6}{x}$, find all c in the interval (2, 6) such that

$$f'(c) = \frac{f(6)-f(2)}{6-2}$$

Solation:

$$f'(c) = \frac{f(6)-f(2)}{6-2}$$

$$= \frac{\left(3-\frac{6}{6}\right)-(3-\frac{6}{2})}{4}$$

$$= \frac{1}{2}$$

and since f $satifies\ the\ conditions$ of the Mean value theorm there exists at least are C in (2, 6) such that $f'(c)$= 1/2

solving the equation

$f'(x) = \frac{6}{x^2} \rightarrow \frac{6}{x^2} = \frac{1}{2}$

$\rightarrow x^2 = 12 \rightarrow x = \pm\sqrt{12}$

Finding in the interval (2, 6) we choose c =$\pm\sqrt{12}$