First, apply the quotient rule

$$f(x) = \frac{u(x)}{v(x)}$$
$$= \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

$$y_1 = \frac{e^{x_1}}{e^{x_1} + e^{x_2} + e^{x_3} + e^{x_4}}$$

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial(\frac{e^{x_1}}{e^{x_1}+e^{x_2}+e^{x_3}+e^{x_4}})}{\partial x_1}$$

$$\frac{\partial y_1}{\partial x_1} = \frac{e^{x_1}(e^{x_1} + e^{x_2} + e^{x_3} + e^{x_4}) - e^{x_1}(e^{x_1})}{(e^{x_1} + e^{x_2} + e^{x_3} + e^{x_4})^2}$$

Cancel the above 2 terms

Now, we want to reshape this remaining as y * (1-y)

Factorize the numerator and denominator to directly gets the y part

$$\implies \frac{\partial y_1}{\partial x_1} = \frac{e^{x_1}(e^{x_2} + e^{x_3} + e^{x_4})}{(e^{x_1} + e^{x_2} + e^{x_3} + e^{x_4})^2}$$

$$\implies \frac{\partial y_1}{\partial x_1} = (\frac{e^{x_1}}{e^{x_1} + e^{x_2} + e^{x_3} + e^{x_4}})(\frac{e^{x_2} + e^{x_3} + e^{x_4}}{e^{x_1} + e^{x_2} + e^{x_3} + e^{x_4}})$$

Clearly the first part is y.

How to convert the second to 1 - y? Add to the numerator: e^{x1} - e^{x1} and rearrange

$$\implies \frac{\partial y_1}{\partial x_1} = y_1\left(\frac{e^{x_1} + e^{x_2} + e^{x_3} + e^{x_4} - e^{x_1}}{e^{x_1} + e^{x_2} + e^{x_3} + e^{x_4}}\right)$$

Split the numerator into 2 parts to get 1 - y

$$\implies \frac{\partial y_1}{\partial x_1} = y_1\left(1 - \frac{e^{x_1}}{e^{x_1} + e^{x_2} + e^{x_3} + e^{x_4}}\right)$$

$$\implies \frac{\partial y_1}{\partial x_1} = y_1(1 - y_1)$$

Similarly,

$$\frac{\partial y_1}{\partial x_2} = \frac{\partial\left(\frac{e^{x_1}}{e^{x_1}+e^{x_2}+e^{x_3}+e^{x_4}}\right)}{\partial x_2}$$

$$\implies \frac{\partial y_1}{\partial x_2} = \frac{-e^{x_1}e^{x_2}}{(e^{x_1} + e^{x_2} + e^{x_3} + e^{x_4})^2}$$

$$\implies \frac{\partial y_1}{\partial x_2} = -\left(\frac{e^{x_1}}{e^{x_1} + e^{x_2} + e^{x_3} + e^{x_4}}\right)\left(\frac{e^{x_2}}{e^{x_1} + e^{x_2} + e^{x_3} + e^{x_4}}\right)$$

$$\implies \frac{\partial y_1}{\partial x_2} = -y_1 y_2$$

Images source

- Note his softmax derivative implementation is clearly wrong (next task)