Req 2

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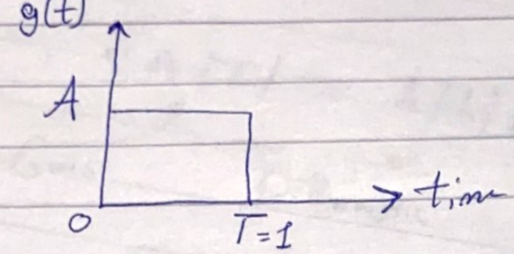


$\because g(t)_1 = -g(t)_0$

if "0" was sent

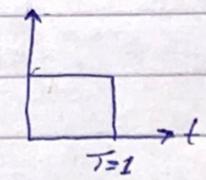
if "1" was sent

$y(t) = \begin{cases} g_0(T)_1 + n(T) & \Rightarrow \text{"1" was sent} \\ g_0(T)_0 + n(T) & \Rightarrow \text{"0" was sent} \end{cases}$

So, if we know what found at $y(t)$, thus we can derive the probability of error

Case 1 (MF)

$\because$ MF with unit energy.

$h(t) = Kg(T-t) =$  $= Kg(t)_1$

Has energy $K^2 \times 1$ and thus

$\boxed{K = 1}$

$\therefore g_0(T)_1 = \int_{-\infty}^{\infty} g(\tau)\, h(t-\tau)\Big|_{t=T} d\tau$

$= \int_{-\infty}^{\infty} g(\tau)_1\, h(T-\tau)\, d\tau$

$\because h(t) = g(T-t)$

$\therefore g(t) = h(T-t)$

$\therefore g_0(T)_1 = \int_{-\infty}^{\infty} g(\tau)^2\, d\tau = \boxed{1}$

if "0" was sent:-

~~$g_0(T)_0$~~

$$g_0(T)_0 = \int_{-\infty}^{\infty} g(\tau)_0 \, h(T-\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} -g(\tau)_1 \, h(\tau)_1 \, d\tau = \boxed{-1}$$

$$\Rightarrow n(T) = h(t) * w(t) \quad \Leftarrow \text{AWGN}$$

$$= \int_{-\infty}^{\infty} w(\tau) \, h(T-\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} w(\tau) \cdot g(\tau)_1 \, d\tau$$

$$= \int_0^1 w(\tau) \, d\tau$$

$\therefore$ $n(T_p)$ is a Continous Sum of random Gaussian random variables and thus is a gaussian random variable it self.

$$M_n(T_p) = E[n(T)] = E\left[\int_0^1 w(\tau) \, d\tau\right]$$

$$= \int_0^1 \underbrace{E[w(\tau)]}_{0} \, d\tau = 0$$

$$\therefore \sigma^2_{n(t)} = \text{var}[n(T)]$$

$$= E[n^2(T)] - \underbrace{(M_{n(t)})^2}_{0}$$

$$= E[n^2(T)] = R_n(0)$$

$$= \int_{-\infty}^{\infty} S_n(f) \, df = \int_{-\infty}^{\infty} S_w(f) \, |H(f)|^2 \, df$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 \, df$$

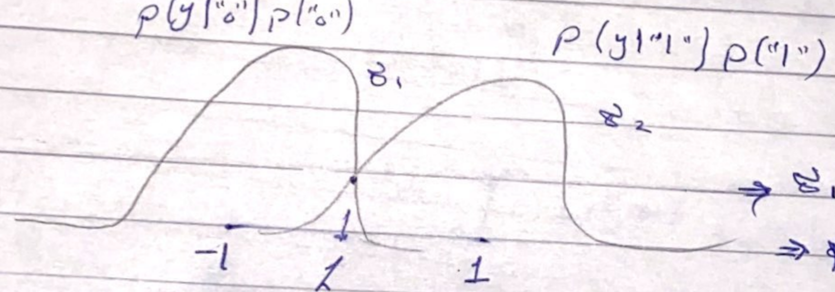
$$= \frac{N_0}{2} \int_{-\infty}^{\infty} \underbrace{|h(t)|^2 \, dt}_{1} = \boxed{\frac{N_0}{2}}$$

$$\therefore\; y(T) = \begin{cases} 1 + n(T) & \text{"1 was sent"} \\ -1 + n(T) & \text{"0 was sent"} \end{cases}$$

$$\therefore\; y(T) \mid \text{"1"} \sim \mathcal{N}\left(\mu = 1,\; \sigma = \sqrt{\tfrac{N_0}{2}}\right)$$

$$\therefore\; y(T) \mid \text{"0"} \sim \mathcal{N}\left(\mu = -1,\; \sigma = \sqrt{\tfrac{N_0}{2}}\right)$$

$\Rightarrow$



$\Rightarrow \sigma_1 = \sigma_2 = \sqrt{\frac{N_0}{2}}$

$\Rightarrow P(\text{"1"}) = P(\text{"0"}) = \boxed{\frac{1}{2}}$

$$\lambda_{opt} = \frac{-1+1}{2} = \boxed{0}$$

$$\therefore\; P(\text{error}) = P(y > \lambda \mid \text{"0"})\,P(\text{"0"}) + P(y < \lambda \mid \text{"1"})\,P(\text{"1"})$$

$$= \frac{1}{2}\left(\underbrace{P(y > \lambda \mid \text{"0"}) + P(y > \lambda \mid \text{"0"})}_{2}\right)$$

$$= P(y > \lambda \mid \text{"0"})$$

$$\therefore\; y \mid \text{"0"} \sim \mathcal{N}\left(\mu = -1,\; \sigma = \sqrt{\tfrac{N_0}{2}}\right)$$

let $z = \frac{y - \mu}{\sigma} \Rightarrow z \mid \text{"0"} \sim \mathcal{N}(\mu = 0, \sigma = 1)$

$$\therefore\; P\left(z > \frac{\lambda + 1}{\sqrt{\frac{N_0}{2}}} \;\middle|\; \text{"0"}\right)$$

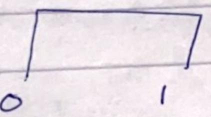
$\therefore$ Q-function:$\rightarrow$ $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}}\,dt$, thus, $P\left(z > \frac{\lambda+1}{\sqrt{\frac{N_0}{2}}}\right) =$

$$Q\left(\frac{\lambda + 1}{\sqrt{\frac{N_0}{2}}}\right) = Q\left(\sqrt{\frac{2}{N_0}}\right) = Q\left(\frac{1}{\sigma}\right)$$

④

(b)

$$y(T) = \begin{cases} +1 + w(t) & \text{"1" sent} \\ -1 + w(t) & \text{"0" sent} \end{cases}$$

$\because g(t)_1 =$  $\therefore g(T)_1 = \boxed{1}$

$\because g(t)_0 = -g(t)_1 \qquad \therefore g(T)_0 = \boxed{-1}$

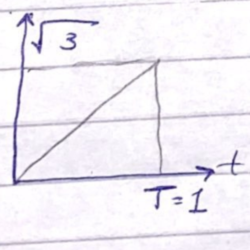
$g(t) = g_o(t) \qquad$ as $\Rightarrow g(t) = g(t) * \delta(t)$

$\because w(T) \sim N(\mu = 0, \ \delta = \sqrt{\frac{N_0}{2}})$

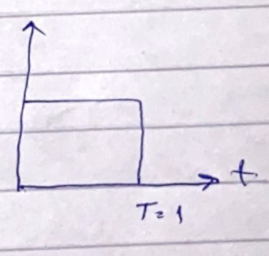
Hence,

$$P(e) = Q\left(\sqrt{\frac{2}{N_0}}\right) = Q\left(\frac{1}{\delta}\right)$$

(c) $h(t) =$



$g(t)_1 =$

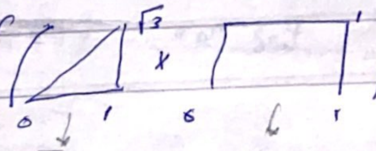


$g(t)_0 = -g(t)_1$

$$\therefore y(T) = \begin{cases} g_o(T)_1 + n(T) \\ g_o(T)_0 + n(T) \end{cases}$$

$g_o(T)_1 = g(t)_1 * h(t)|_{t=T}$

$= \int_{-\infty}^{\infty} h(\tau) g(T-\tau)\, d\tau$

$= \int_0^1$  $d\tau = \boxed{\frac{\sqrt{3}}{2}}$

$\frac{\sqrt{3}}{2} \times 1$

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$\cdot g_o(T)_0 = \frac{-\sqrt{3}}{2}$ (by homogenity of Convolution)

$\because n(T) = h(t) * w(t)|_T$

Gaus — Deterministic — Gaussian random

$= \int_{-\infty}^{\infty} w(\tau)\, h(T-\tau)\, d\tau$

thus, $\mu_{n(t)} = E[n(T)] = E[\int_{-\infty}^{\infty} w(\tau) h(T-\tau) d\tau]$

$= \int_{-\infty}^{\infty} h(T-\tau)\, E[w(\tau)]\, d\tau$

deterministic — $\boxed{0}$

$= \boxed{0}$

$\delta^2_{n(t)} = E[n^2(T)] - \mu^2_{n(t)}$

$= \int_{-\infty}^{\infty} S_n(f)\, df$

$= \int_{-\infty}^{\infty} S_w(f)\, |H(f)|^2\, df$

$= \frac{N_0}{2} \int_0^1 |h(t)|^2\, dt$

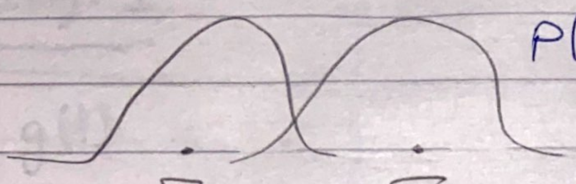
$= \frac{N_0}{2} \int_0^1 (\sqrt{3}\, t)^3\, dt = \frac{N_0}{2}$

$y(T) = \frac{\sqrt{3}}{2} + n(T)$ "1" sent

$\frac{-\sqrt{3}}{2} + n(T)$ "0" sent

$\Rightarrow y|"1" \sim N(\mu = \frac{\sqrt{3}}{2}, \delta = \sqrt{\frac{N_0}{2}})$, $y|"0" \sim N(\mu = -\frac{\sqrt{3}}{2}, \delta = \sqrt{\frac{N_0}{2}})$

$p(y|\text{"0"})p(\text{"0"})$



$p(y|\text{"1"})p(\text{"1"})$

$-\frac{\sqrt{3}}{2}$ $\quad$ $+\frac{\sqrt{3}}{2}$

- $p(\text{"0"}) = p(\text{"1"}) = \boxed{\frac{1}{2}}$
- $\sigma_1 = \sigma_2 = \sqrt{\frac{N_0}{2}}$

$\lambda_{opt} = \boxed{0}$ by Symmetry

$\therefore p(\text{error}) = p(y > \lambda | \text{"0"})$

$$= p\left(Z > \frac{\lambda + \frac{\sqrt{3}}{2}}{\sqrt{\frac{N_0}{2}}}\right) \qquad Z = \frac{y-\mu}{\sigma}$$

$$= Q\left(\sqrt{\frac{3}{2N}}\right) = Q\left(\frac{\sqrt{3}}{2} \cdot \frac{1}{\sigma}\right)$$