Synchronization of Type 1 Phase Oscillators on Directed Acyclic Graphs

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Abstract

We study the effects of directionality on synchronization of dynamical networks. Performing the linear stability analysis and the numerical simulation of the Kuramoto model in directed networks, we show that balancing in- and out-degrees of all nodes enhances the synchronization of sparse networks, especially in networks with high clustering coefficient and homogeneous degree distribution. Furthermore, by omitting all the feedback loops, we show that while hierarchical directed acyclic graphs are structurally highly synchronizable, their global synchronization is too sensitive to the choice of natural frequencies and is strongly affected by noise.

Methods

The Kuramoto model [1] consists of a collection of N coupled phase oscillators, θ_i, having natural frequencies ω_i distributed with a given probability density and its time evolution is given by:

$$\dot{\theta}_i = \omega_i + \frac{\kappa}{N}\sum_{j=1}^{N} a_{ij}\Big[u\sin(\theta_j-\theta_i) + (1-u)\frac{(1-cos(\theta_j-\theta_i))}{2}\Big],$$

where κ shows the overall coupling strength. To quantify the degree of synchrony, the phase order parameter is defined as $r(t) = \frac{1}{N}\langle|\sum_{i=1}^{N} e^{i\theta_i(t)}|\rangle$, where $0 \leq r \leq 1$ measures the phase coherence of oscillators. The value $r=1$ shows the fully synchronized state, while $r=0$ indicates the incoherent solution. Here, $\langle\dots\rangle$ indicates averaging over different network realizations and initial conditions. The time average of r after achieving a steady state is represented by R. In our simulations, the initial values of θ_i are randomly drawn from a uniform distribution in interval $[0, 2\pi]$, and natural frequencies are identical.

Conclusion

The purpose of this report is to investigate the effect of structure on synchronization of type I and II oscillators [2, 3] and to find an optimum structure for type I oscillators to synchronize. We show that feed forward loops help the synchrony of homogeneous type I oscillators. This is also true for networks with hubs (Scale free networks). Actually the information flow start from hubs in these networks.

References

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Lyapunov spectrum analysis on motifs and network properties

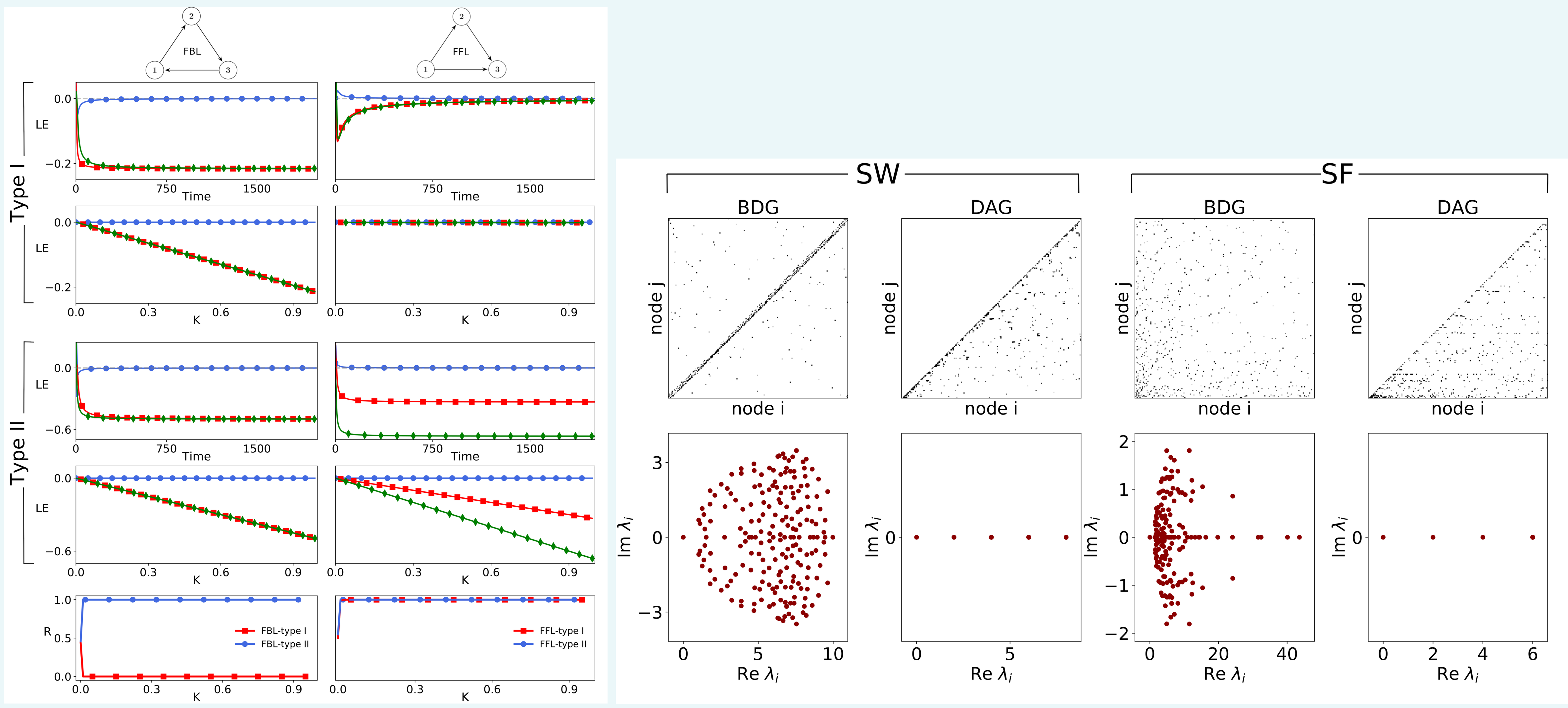

Figure 1: (LEFT PANEL) Synchronization stability of FBL and FFL motifs. Lyapunov exponents and order parameter versus time and coupling for identical type I or type II phase ossillators connected by FBL (Left column) and FFl (right column) loops.
(RIGHT PANEL) Properties of two different directed networks. Adjacency matrix (top) and Eigenvalues of the Laplacian matrix (bottom), for small world and scale free Balanced Directed Graphs (BDGs) and Directed Acyclic Graphs(DAGs) with N=200 nodes.

Lyapunov spectrum analysis and synchronizations of networks

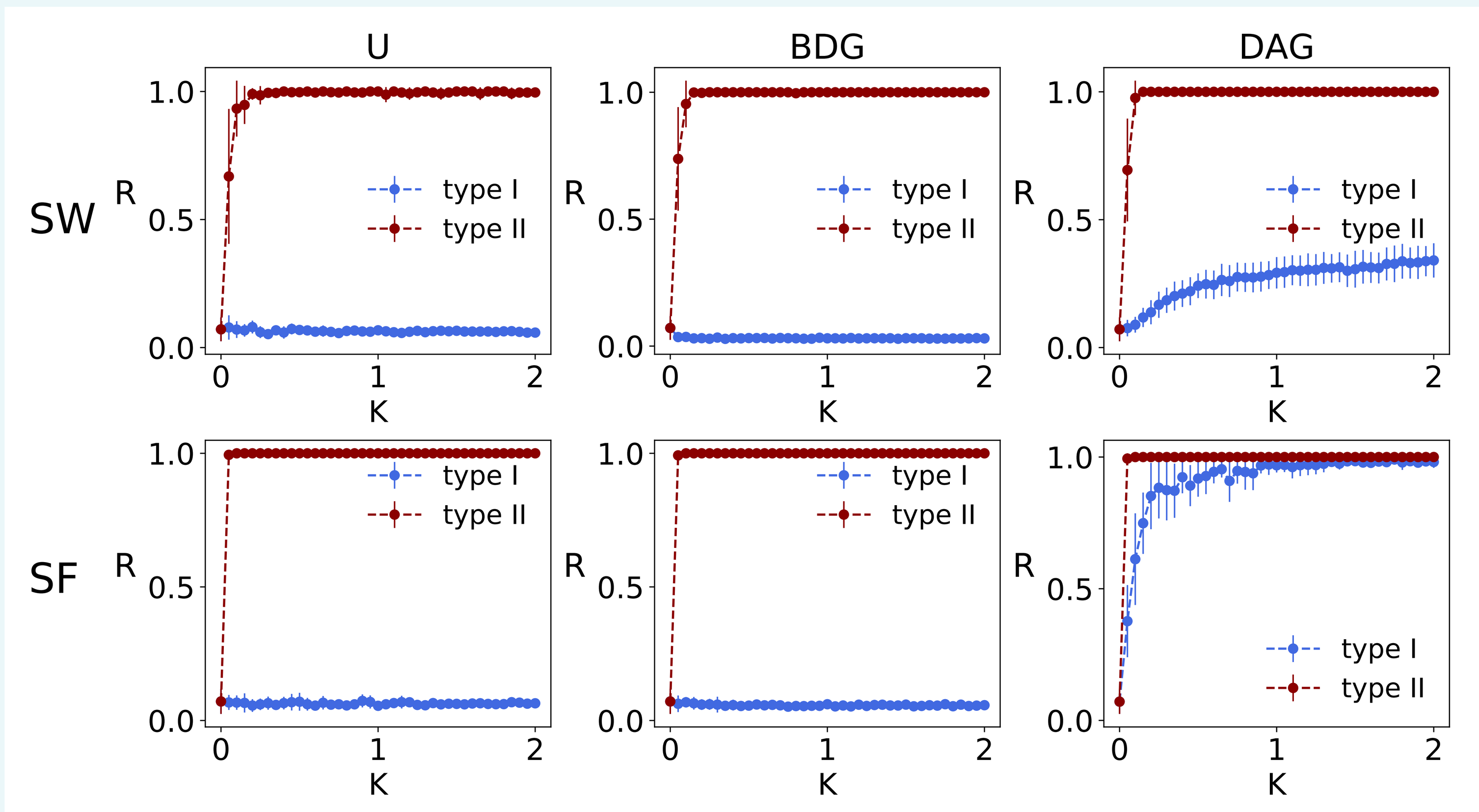

Figure 2: Comparison of phase synchronizability of identical type I and type II oscillators on different network structures. Stationary order parameter versus coupling strength of the small-world (top) and scale-free (bottom) networks with different link directionalities (U(Undirected), BDG(Balanced Directed graph) and DAG(directed Acyclic graph)). Here, $N = 200$ and the results averaged over 30 network realizations and random initial phases.

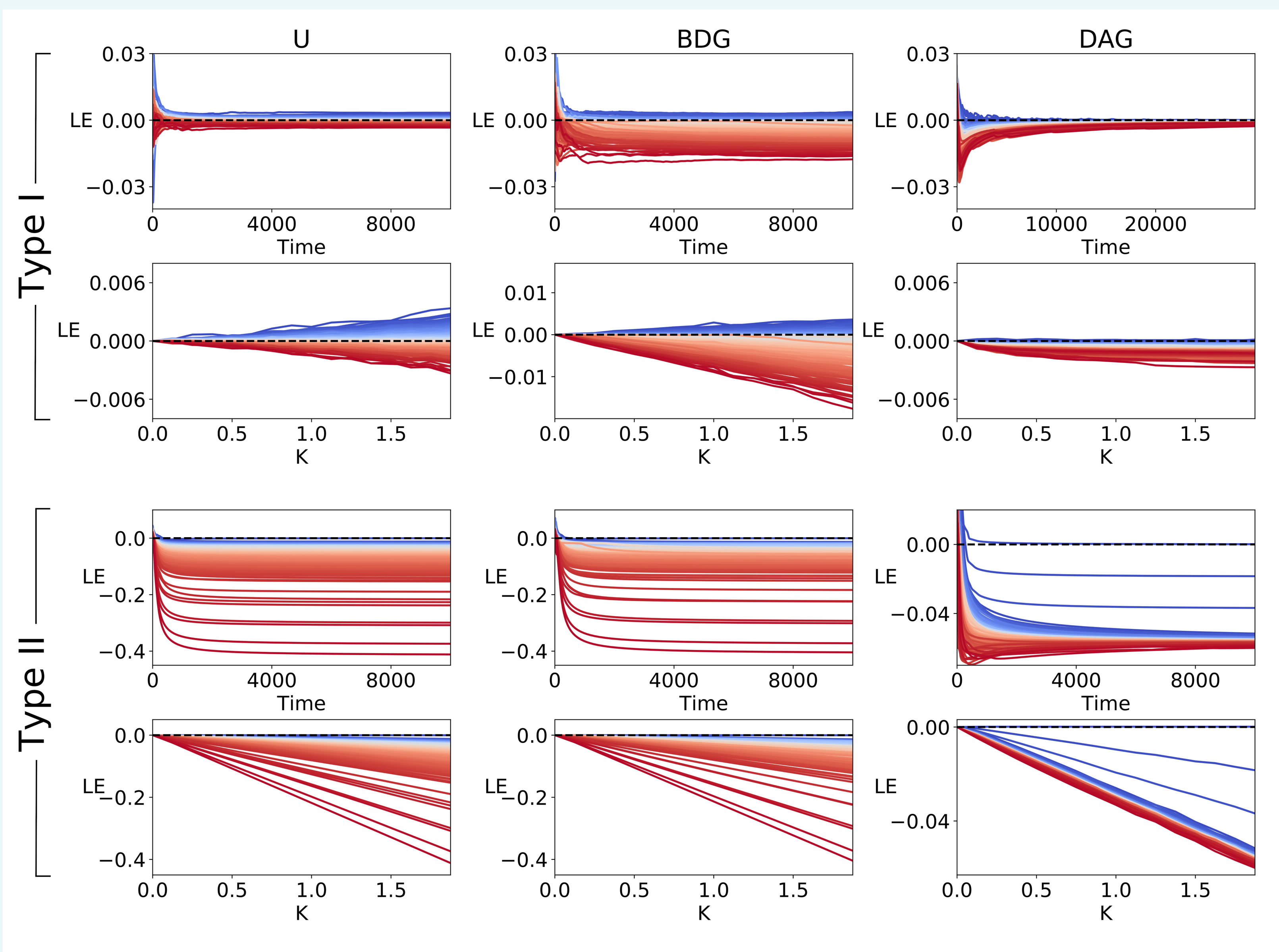

Figure 3: Comparison of synchronizability of identical type I and type II oscillators on scale-free networks with different directionalities based on Lyapunov exponents . Lyapunov exponents versus time and coupling strength for type I (top) or type II(bottom) phase oscillators situated on undirected graphs, BDGs and DAGs, with **N=200** nodes.